

Title: What's wrong with Goldstone?

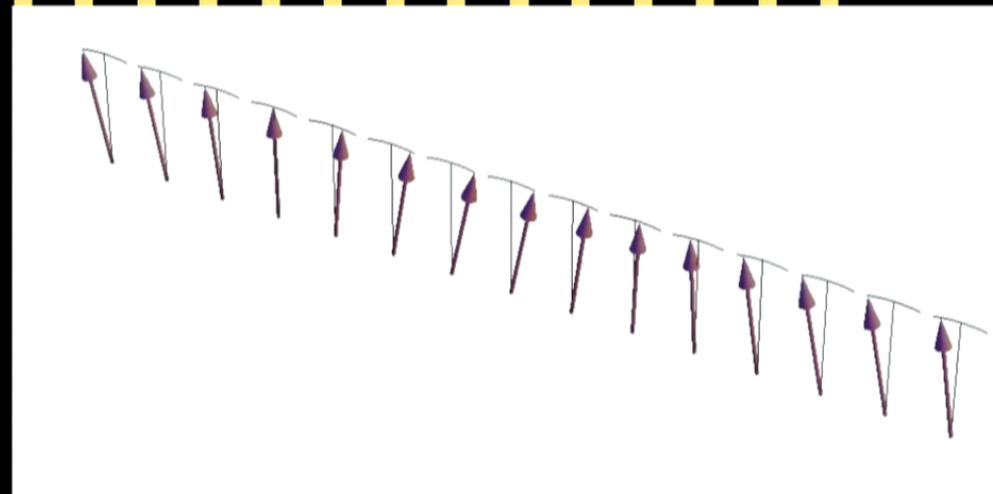
Date: May 28, 2014 11:30 AM

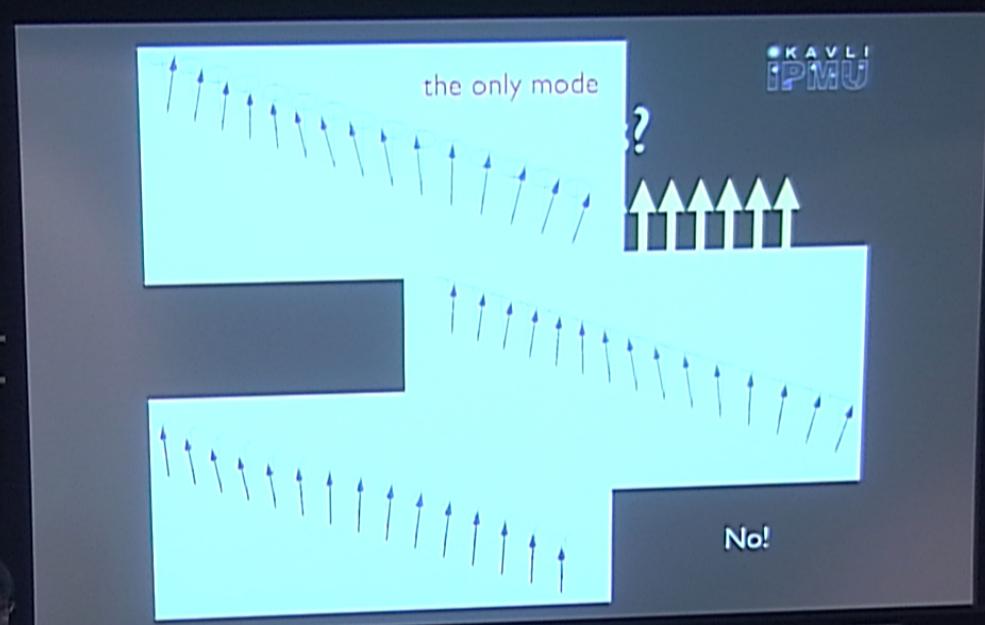
URL: <http://pirsa.org/14050106>

Abstract: <span>Spontaneous Symmetry Breaking is a very universal concept applicable for a wide range of subjects: crystal, superfluid, neutron stars, Higgs boson, magnets, and many others. Yet there is a variety in the spectrum of gapless excitations even when the symmetry breaking patterns are the same. We unified all known examples in a single-line Lagrangian of the low-energy effective theory.</span>



# two NGBs?



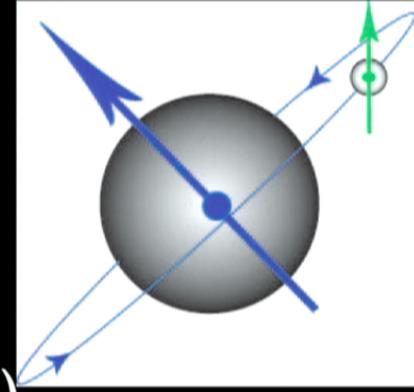


A man in a black t-shirt with 'KAVLI IPMU' printed on it stands at a podium, gesturing towards the screen.





# spinor BEC



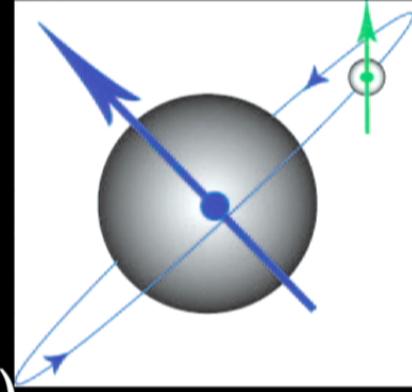
- BEC of  $F=1$  atoms (ferromagnetic)
- $\text{SO}(3) \times \text{U}(1) \rightarrow \text{SO}(2)$

$$\psi = \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} = \begin{pmatrix} R_x & I_x \\ R_y & I_y \\ R_z & I_z \end{pmatrix}$$

$$\langle \psi \rangle = v \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = v \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$



# spinor BEC



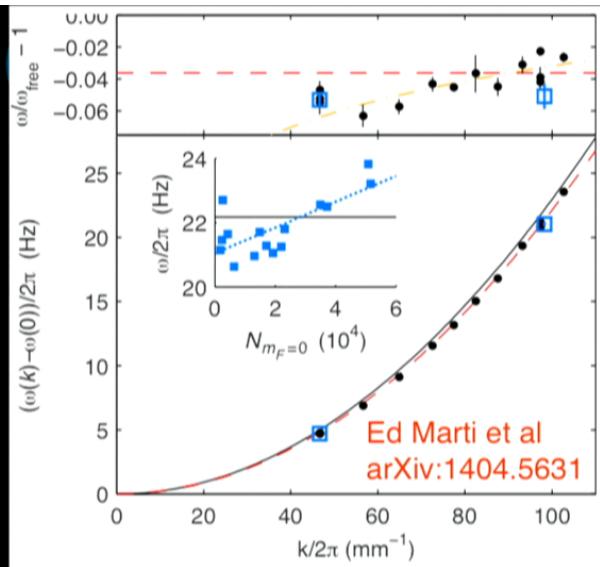
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- $G/H = \mathbb{RP}^3$

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- $G/H = \mathbb{R}\mathbb{P}^3$
- 3 broken generators
- 1 NGB with  $E^\infty p$
- 1 NGB with  $E^\infty p^2$

Ed Marti et al  
arXiv:1404.5631

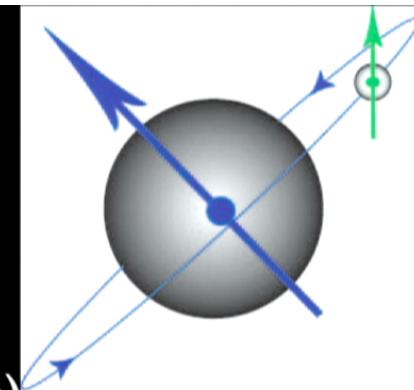
# nor BEC

atoms (ferromagnetic)

$\rightarrow \text{SO}(2)$

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# Heisenberg models

- anti-ferromagnet  $H = +J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j$  2 NGBs  
 $\langle 0|J_z^0|0\rangle = 0$   $E \propto p$



- ferromagnet  $H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j$  1 NGB  
 $\langle 0|J_z|0\rangle = -i\langle 0|[J_x, J_y]|0\rangle \neq 0$   $E \propto p^2$



$J_x$  and  $J_y$  canonically conjugate to each other cf.  $[x, p] = i \hbar$   
describing a single degree of freedom together



# General formula

- Define a commutator among broken generators

$$\rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle$$

- $n_B = 1/2 \text{ rank } \rho$  counts the number of canonically conjugate pairs (Type-B)
  - each pair describes one d.o.f.
- the remainder  $n_A = n_{BG} - 2n_B$ 
  - stand-alone NGB d.o.f. (Type-A)

$$\xrightarrow{\text{generically}} E \propto p^2$$

$$\xrightarrow{\text{generically}} E \propto p$$

$$n_{NGB} = n_A + n_B = n_{BG} - \frac{1}{2} \text{rank} \rho$$

conjectured by Watanabe and Brauner



$$n_{NGB} = n_{BG}$$

K A V L I  
IPMU  
rank

# Applications

example	coset space	BG	NGB	rank $\rho$	theorem
anti-ferromagnet	$O(3)/O(2)$	2	2	0	$2=2-0$
ferromagnet	$O(3)/O(2)$	2	1	2	$1=2-1$
superfluid $^4\text{He}$	$U(1)$	1	1	0	$1=1-0$
superfluid $^3\text{He}$ B phase (in magnetic field)	$O(3) \times O(3) \times U(1)/O(2)$	4	4	0	$4=4-0$
BEC ( $F=0$ )	$O(2) \times O(3) \times U(1)/O(2)$	4	3	2	$3=4-1$
BEC ( $F=1$ ) polar	$U(1)$	1	1	0	$1=1-0$
BEC ( $F=1$ ) ferro	$O(3) \times U(1)/SO(2)$	3	3	0	$3=3-0$
3-comp. Fermi liquid	$U(3)/U(2)$	5	3	4	$3=5-2$
neutron star	$U(1)$	1	1	0	$1=1-0$
kaon cond. ( $\mu=0$ )	$U(2)/U(1)$	3	3	0	$3=3-0$
kaon cond. ( $\mu \neq 0$ )	$U(2)/U(1)$	3	2	2	$2=3-1$
crystal	$R^3/Z^3$	3	3	0	$3=3-0$
(in magnetic field)	$R^3/Z^3$	3	2	2	$2=3-1$

# Formalism –Internal Symmetries–

H.Watanabe and HM  
letter arXiv:1203.0609  
full paper arXiv:1402.7066



# Low- $E$ Effective L



# Low- $E$ Effective L

- consider  $\pi^a(x)$  fields:  $\mathbb{R}^{3,1} \rightarrow G/H$  (“pions”)
- Write action  $S = \int d^4x L(\boldsymbol{\pi}, \partial\boldsymbol{\pi})$   
which is **G-invariant**



# Low- $E$ Effective L

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- Write action  $S = \int d^4x L(\boldsymbol{\pi}, \partial \boldsymbol{\pi})$   
which is **G-invariant**
- expand in **powers of derivative**, keep low  
orders (often up to the second order)

$$\mathcal{L}_{\text{eff}} = g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b$$



# spectrum



$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$



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- around the origin  $c_a(\pi) = c_a(0) + \frac{1}{2}c_{ab}\pi^b + O(\pi^2)$



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- around the origin  $c_a(\pi) = c_a(0) + \frac{1}{2}c_{ab}\pi^b + O(\pi^2)$
- in the subspace where  $c_{ab}$  is invertible,  $L = p\dot{q} - H$   
 $[\pi^a, \pi^b] = -i(c^{-1})^{ab}$
- the  $c_a$  term dominates over  $g_{ab}$  term  $E \gg E^2$
- broken Noether currents  $j_a^0 = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\pi}^b} \delta_a \pi^b = c_{ba} \pi^b$   
 $[j_a^0, Q_b] = -ic_{ca}c_{db}(c^{-1})^{cd} = ic_{ab}$
- Namely, for  $\rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle$   $c_a \dot{\pi}^a \approx \frac{1}{2} \rho_{ab} \pi^b \dot{\pi}^a$
- when  $c_a$  present,  $E \propto p^2$ , otherwise  $E \propto p$  !



# Bottomline

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- SSB leads to gapless excitations (NGBs)
- Lorentz invariance:  $n_{\text{NGB}}=n_{\text{BG}}$ ,  $E=c\rho$
- w/o Lorentz invariance:  
• Type A:  $\rho_{ab}=0$ , typically  $E \propto \rho$   
• Type B:  $\rho_{ab} \neq 0$ , typically  $E \propto \rho^2$ 
  - $n_{\text{NGB}}=n_A+n_B$
  - $n_{\text{BG}}=n_A+2n_B$
- explicit effective Lagrangian  $\Rightarrow$  interactions
- underlying partially symplectic geometry

For  $\text{SO}(3)/\text{SO}(2)=S^2$ ,  $\mathcal{L}_{\text{eff}} = \frac{s}{2} \frac{n_y \dot{n}_x - n_x \dot{n}_y}{1 + n_z} - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$



# physics

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- What is  $c_a(\pi)$ ?
- it defines one-form  $c_1 = c_a(\pi) d\pi^a$  on  $G/H$
- $L$  must be  $G$ -invariant up to a surface term

$$\mathcal{L}_{V_i} c = i_{V_i} dc + d(i_{V_i} c) = \boxed{de_i} + d(i_{V_i} c) \quad de_i = i_{V_i} \omega$$

- the Noether current picks up surface term

$$j_i^0 = -\bar{g}_{ab} h_i^a \dot{\pi}^b + e_i$$

- in the ground state = stationary:

$$\langle 0 | j_i^0 | 0 \rangle = e_i(0)$$



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# geometry

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$$\mathcal{L}_{V_i} c_1 = d\chi$$

- its exterior derivative is  $G$ -invariant

$$\omega_2 = dc_1$$

$$\mathcal{L}_{V_i} \omega_2 = d\mathcal{L}_{V_i} c_1 = d^2 \chi = 0$$

- Namely,  $G/H$  is endowed with a  $G$ -invariant closed two-form  $\omega_2$  (may be degenerate)



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Darboux's theorem:  $\omega_2 = \sum dp_i \wedge dq_i$   
*presymplectic structure*



# stability@ $T=0$ in $d+1$ dim

$$\mathcal{L}_{\text{eff}} = \bar{g}_{ab} \dot{\pi}^a \dot{\pi}^b - g_{ab} \nabla_i \pi^a \nabla^i \pi^b$$

- Type A:

*à la Hořava-Lifshitz*



# stability@ $T=0$ in $d+1$ dim

- Type A:

- scaling

$$\mathcal{L}_{\text{eff}} = \bar{g}_{ab} \dot{\pi}^a \dot{\pi}^b - g_{ab} \nabla_i \pi^a \nabla^b \pi^b$$

$$\vec{x}' = a\vec{x}, \quad t' = at$$

- interaction

$$\pi'^a(a\vec{x}, at) = a^{(1-d)/2} \pi^a(\vec{x}, t)$$

$$\nabla_i \pi^a \nabla_i \pi^b \pi^c \sim a^{-(d-1)/2}$$

- IR free for  $d \geq 2$  (no SSB in  $d=1$ )

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# stability@ $T=0$ in $d+1$ dim

- Type A:
  - scaling  $\mathcal{L}_{\text{eff}} = \bar{g}_{ab}\dot{\pi}^a\dot{\pi}^b - g_{ab}\nabla_i\pi^a\nabla\pi^b$   
 $\vec{x}' = a\vec{x}, t' = at$   
 $\pi'^a(a\vec{x}, at) = a^{(1-d)/2}\pi^a(\vec{x}, t)$
  - interaction  $\nabla_i\pi^a\nabla_i\pi^b\pi^c \sim a^{-(d-1)/2}$
  - IR free for  $d \geq 2$  (no SSB in  $d=1$ )
- Type B:
  - scaling  $\mathcal{L}_{\text{eff}} = \rho_{ab}\pi^a\dot{\pi}^b - g_{ab}\nabla_i\pi^a\nabla\pi^b$   
 $\vec{x}' = a\vec{x}, t' = a^2t$   
 $\pi'^a(a\vec{x}, a^2t) = a^{-d/2}\pi^a(\vec{x}, t)$
  - interaction  $\nabla_i\pi^a\nabla_i\pi^b\pi^c \sim a^{-d/2}$

*à la Hořava-Lifshitz*



# explicit construction

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- for compact semi-simple case, we found closed expressions

$$U(\pi) = e^{i\pi^a(x)T^a}$$

$$gU = U' h'(\pi', g)$$



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- for compact semi-simple case, we found closed expressions

$$U(\pi) = e^{i\pi^a(x)T^a}$$

$$gU = U'h'(\pi', g)$$

$$\omega = U^{-1}dU = \sum T_k \omega^k$$

$$g_{ab}(\pi) = g_{cd}(0) \omega_a^c(\pi) \omega_b^d(\pi)$$



# General Geometry

$$\begin{array}{ccc} G/H & \longleftarrow & F \\ \downarrow \pi & & \\ B & & \end{array}$$



# General Geometry

closed  $G$ -inv

$$d c_1 = \pi^* \omega_2$$

symplectic  
homogeneous

$$\omega_2$$

$$G/H \longleftrightarrow F$$

$$\downarrow \pi$$

$$B$$

$$\omega_2 = \frac{1}{2} \rho_{ab} d\pi^a \wedge d\pi^b + O(\pi)^3 \quad \rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle$$



# General Geometry

closed  $G$ -inv

$$d c_1 = \pi^* \omega_2$$

symplectic  
homogeneous  $\omega_2$

$$G/H \xleftarrow{\pi} B$$

Type A  
 $E \propto p$

$$\omega_2 = \frac{1}{2} \rho_{ab} d\pi^a \wedge d\pi^b + O(\pi)^3 \quad \rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle$$



# central extension

$$\mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + \bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - g_{ab}(\pi)\nabla_i\pi^a\nabla_i\pi^b$$

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$$de_i = i_{V_i} \omega$$

$$\mathcal{L}_{V_j} de_j = \mathcal{L}_{V_j} i_{V_i} \omega = i_{[V_j, V_i]} \omega + i_{V_i} \mathcal{L}_{V_j} \omega = f_{ji}{}^k i_{V_k} \omega = f_{ji}{}^k de_k$$

$$\mathcal{L}_{V_j} e_j = f_{ji}{}^k e_k + z_{ji}$$

- If  $H^2(\mathfrak{g}) \neq 0$ , a central extension  $z_{ji} \neq 0$  possible
- impossible for semi-simple  $\mathfrak{g}$



# Examples: $n_{BG}=2$

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- $S^2$ :  $\text{SO}(3)$ -invariant closed two-form is

$$\omega_2 = d(\cos \theta) \wedge d\phi$$

$$dc_1 = \frac{1}{2} d \frac{n_y dn_x - n_x dn_y}{1 + n_z} = d[(-1 + \cos \theta) d\phi]$$

- the leading terms are

$$\mathcal{L}_{\text{eff}} = \frac{s}{2} \frac{n_y \dot{n}_x - n_x \dot{n}_y}{1 + n_z} - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

- ferromagnet with one dof,  $E \propto p^2$



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- But if  $c_1$  absent, need to consider 2nd term

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \dot{n}_i \dot{n}_i - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

- anti-ferromagnet with two dof,  $E \propto p$



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- $\mathbb{R}^2 = \mathbb{C}$ : invariant closed two-form is

$$\omega_2 = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$$
$$c_1 = \frac{i}{2} z d\bar{z}$$

- the leading terms are

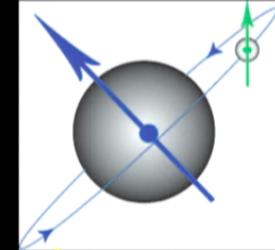
$$\mathcal{L}_{\text{eff}} = \frac{i}{2} z \dot{\bar{z}} - \frac{1}{2m} \vec{\nabla} z \vec{\nabla} \bar{z}$$

- free non-rel particle with one dof,  $E \propto p^2$
- or 2d lattice in  $B$ , with one dof,  $E \propto p^2$

$$[z(x), \bar{z}(y)] = -i\delta(x - y) \quad \text{central extension } H^2(\mathbb{R}^2) \neq 0$$



# Examples: $n_{BG}=3$



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- $\text{SO}(3) \times \text{U}(1)/\text{SO}(2) = \mathbb{R}\mathbb{P}^3 = S^3/\mathbb{Z}_2$
- spinor BEC ferromagnetic phase:

$$\psi = v \frac{e^{i\theta}}{\sqrt{2}(1 + \bar{z}z)} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \quad \psi^\dagger \psi = v^2$$

$$\psi^* i \dot{\psi} = v^2 \left( -\dot{\theta} + i \frac{z^* \dot{z} - \dot{z}^* z}{1 + z^* z} \right)$$

- one type-B with  $E \propto p^2$
- one type-A with  $E \propto p$

Hopf map  $\mathbb{R}\mathbb{P}^3 \rightarrow S^2$  down to a symplectic homogeneous  $S^2$



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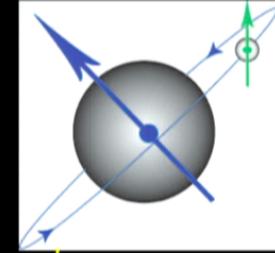
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- $\text{SO}(3) \times \text{U}(1)/\text{SO}(2) = (S^2 \times S^1)/\mathbb{Z}_2$
- spinor BEC polar phase:

$$\psi = v e^{i\theta} \vec{n} \quad \vec{n}^2 = 1$$

$$\psi^* i \dot{\psi} = v^2 i \dot{\theta} \approx 0$$

- three type-A NGBs with  $E \propto p$
- vanishing presymplectic structure





# Examples: $n_{BG}=3$

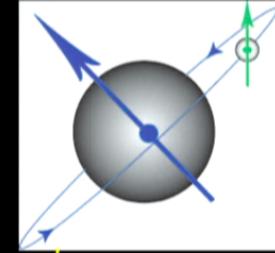
$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- $\text{SO}(3) \times \text{U}(1)/\text{SO}(2) = (S^2 \times S^1)/\mathbb{Z}_2$
- spinor BEC polar phase:

$$\psi = v e^{i\theta} \vec{n} \quad \vec{n}^2 = 1$$

$$\psi^* i \dot{\psi} = v^2 i \dot{\theta} \approx 0$$

- three type-A NGBs with  $E \propto p$
- vanishing presymplectic structure





# no-go theorem

- Not every NGBs can be paired as Type-B
- $SU(3)/U(1)^2$ : Kähler and symplectic

Type-A	Type-B	$n_A+2n_B=6$
6	0	6
4	1	6
2	2	6
0	3	6



# Classification of presymplectic structures

- Borel (1954):  $G$  compact semi-simple,  $T \subset G$  a torus,  $U$  centralizer of  $T$ , then  $G/U$  Kähler
  - Note Kähler manifolds are symplectic
  - For a given  $G/H$ , find all  $U \supset H$
  - Project  $G/H$  to  $G/U$ , with fiber  $U/H$
  - pull back symplectic form on  $G/U$  to  $G/H$
- If  $G$  is not semi-simple, it has  $U(1)^k$  factors, and possible central extensions are

$$\dim H^2(\mathfrak{u}(1)^k) = \frac{1}{2}k(k-1)$$

$n_A$	$n_B$	$U$
30	0	.
20	5	$SU(5) \times U(1)$
14	8	$SU(4) \times SU(2) \times U(1)$
12	9	$SU(4) \times U(1)^2$
12	9	$SU(3)^2 \times U(1)$
8	11	$SU(3) \times SU(2) \times U(1)^2$
6	12	$SU(3) \times U(1)^3$
6	12	$SU(2)^3 \times U(1)^2$
4	13	$SU(2)^2 \times U(1)^3$
2	14	$SU(2) \times U(1)^4$
0	15	$U(1)^5$

TABLE III. Possible number of type-A and type-B NGBs for  $SU(6)/U(1)^5$ .

$n_A$	$n_B$	$U$
40	0	.
24	8	$SO(8) \times U(1)$
20	10	$U(5)$
14	13	$SO(6) \times U(2)$
12	14	$SO(6) \times U(1)^2$
12	14	$U(4) \times U(1)$
10	15	$SO(4) \times U(3)$
8	16	$U(3) \times U(2)$
6	17	$SO(4) \times U(2) \times U(1)$
6	17	$U(3) \times U(1)^2$
4	18	$SO(4) \times U(1)^3$
4	18	$U(2)^2 \times U(1)$
2	19	$U(2) \times U(1)^3$
0	30	$U(1)^5$

TABLE IV. Possible number of type-A and type-B NGBs for  $SO(10)/U(1)^5$ .

$n_A$	$n_B$	$U \subset SO(11)$	$U \subset Sp(5)$
50	0	.	.
32	9	$SO(9) \times U(1)$	$Sp(4) \times U(1)$
20	15	$SO(7) \times U(2)$	$Sp(3) \times U(2)$
20	15	$U(5)$	$U(5)$
18	16	$SO(7) \times U(1)^2$	$Sp(3) \times U(1)^2$
14	18	$SO(5) \times U(3)$	$Sp(2) \times U(3)$
14	18	$SO(3) \times U(4)$	$Sp(1) \times U(4)$
12	19	$U(4) \times U(1)$	$U(4) \times U(1)$
10	20	$SO(5) \times U(2) \times U(1)$	$Sp(2) \times U(2) \times U(1)$
8	21	$SO(5) \times U(1)^3$	$Sp(2) \times U(1)^3$
8	21	$SO(3) \times U(3) \times U(1)$	$Sp(1) \times U(3) \times U(1)$
8	21	$U(3) \times U(2)$	$U(3) \times U(2)$
6	22	$SO(3) \times U(2)^2$	$Sp(1) \times U(2)^2$
6	22	$U(3) \times U(1)^2$	$U(3) \times U(1)^2$
4	23	$SO(3) \times U(2) \times U(1)^2$	$Sp(1) \times U(2) \times U(1)^2$
4	23	$U(2)^2 \times U(1)$	$U(2)^2 \times U(1)$
2	24	$SO(3) \times U(1)^4$	$Sp(1) \times U(1)^4$
2	24	$U(2) \times U(1)^3$	$U(2) \times U(1)^3$
0	25	$U(1)^5$	$U(1)^5$

TABLE V. Possible number of type-A and type-B NGBs for  $SO(11)/U(1)^5$  and  $Sp(5)/U(1)^5$ .

## List of possible $U$ for $G$ with rank=5

$n_A$	$n_B$	$U$
30	0	.
20	5	$SU(5) \times U(1)$
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6	22	$U(3) \times U(1)^2$	$U(3) \times U(1)^2$
4	23	$SO(3) \times U(2) \times U(1)^2$	$Sp(1) \times U(2) \times U(1)^2$
4	23	$U(2)^2 \times U(1)$	$U(2)^2 \times U(1)$
2	24	$SO(3) \times U(1)^4$	$Sp(1) \times U(1)^4$
2	24	$U(2) \times U(1)^3$	$U(2) \times U(1)^3$
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## List of possible $U$ for $G$ with rank=5



$$n_{NGB} = n_{BG}$$

1 K A V L I  
2 IRMO rank

# Applications

example	coset space	BG	NGB	rank $\rho$	theorem
anti-ferromagnet	$O(3)/O(2)$	2	2	0	$2=2-0$
ferromagnet	$O(3)/O(2)$	2	1	2	$1=2-1$
superfluid $^4\text{He}$	$U(1)$	1	1	0	$1=1-0$
superfluid $^3\text{He}$ B phase (in magnetic field)	$O(3) \times O(3) \times U(1)/O(2)$	4	4	0	$4=4-0$
BEC ( $F=0$ )	$O(2) \times O(3) \times U(1)/O(2)$	4	3	2	$3=4-1$
BEC ( $F=1$ ) polar	$U(1)$	1	1	0	$1=1-0$
BEC ( $F=1$ ) ferro	$O(3) \times U(1)/SO(2)$	3	3	0	$3=3-0$
3-comp. Fermi liquid	$U(3)/U(2)$	5	3	4	$3=5-2$
neutron star	$U(1)$	1	1	0	$1=1-0$
kaon cond. ( $\mu=0$ )	$U(2)/U(1)$	3	3	0	$3=3-0$
kaon cond. ( $\mu \neq 0$ )	$U(2)/U(1)$	3	2	2	$2=3-1$
crystal (in magnetic field)	$R^3/Z^3$	3	3	0	$3=3-0$
	$R^3/Z^3$	3	2	2	$2=3-1$



# Low-dimensions

- There are additional terms one can write up to two derivatives in low dimensions
- |+|:  $\tilde{c}_a(\pi)\nabla_x\pi^a + \tilde{g}_{ab}(\pi)\dot{\pi}^a\nabla_x\pi^b + \tilde{b}_{ab}(\pi)\dot{\pi}^a\nabla_x\pi^b$
- 2+|:  $-\frac{1}{2}b_{ab}(\pi)\epsilon^{rs}\nabla_r\pi^a\nabla_s\pi^b$

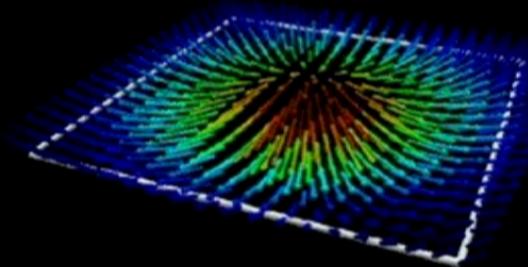
# Non-commuting momenta for topological solitons in 2+1 dimensions

$$[P_x, P_y] \propto N_{\text{topological}}$$

H.Watanabe and HM, arXiv:1401.8139



# skyrmion



- Consider a Heisenberg ferromagnet
- On a two-dimensional plane, non-trivial maps  $\mathbb{R}^2 \rightarrow S^2$  classified by  $\pi_2(S^2) = \mathbb{Z}$
- skyrmion has moduli:
  - translations in  $x$  and  $y$  directions
  - dilation
  - rotation
- derive effective Lagrangian for moduli
- momenta don't commute!

$$[P_x, P_y] = i\hbar \ 4\pi s N_{\text{skyrmion}}$$



# Derivation I

- Effective Lagrangian
- All spins up at infinity
- canonical commutator
- Noether charges for translations
- commutator has a surface term that we normally ignore
- it is precisely the winding number!

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \frac{n_y \dot{n}_x - n_x \dot{n}_y}{1 + n_z} - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & s(1 + \cos \theta) \dot{\phi} \\ & - f^2 ((\vec{\nabla} \theta)^2 + \sin^2 \theta (\vec{\nabla} \phi)^2)\end{aligned}$$

$$[s \cos \theta(x), \phi(y)] = -i\hbar \delta^2(x - y)$$

$$P_i = \int d^2x \ s(1 + \cos \theta) \nabla_i \phi$$

$$\begin{aligned}
[s \cos \theta(x), \phi(y)] &= -i\hbar \delta^2(x-y) \\
P_i &= \int d^2x \ s(1+\cos \theta) \nabla_i \phi \\
[P_1, P_2] &= \int d^2x d^2y \ [s(1+\cos \theta) \nabla_1 \phi(x), s(1+\cos \theta) \nabla_2 \phi(y)] \\
&= -i\hbar \int d^2x d^2y \ [\nabla_2^y \delta(x-y) \nabla_1^x \phi(x) s(1+\cos \theta)(y) \\
&\quad - \nabla_2^x \delta(x-y) \nabla_2^y \phi(y) s(1+\cos \theta)(x)] \\
&= i\hbar s \int d^2x d^2y \ [\nabla_1^x \phi(x) \nabla_2^y \cos \theta(y) \\
&\quad - \nabla_2^y \phi(y) \nabla_2^x \cos \theta(x)] \delta(x-y) \\
&= i\hbar s \int d^2x \ [\nabla_1 \phi \nabla_2 \cos \theta - \nabla_2 \phi \nabla_1 \cos \theta] \\
&= i\hbar 4\pi s N_{\text{winding}}
\end{aligned}$$



# Derivation II

$$\mathcal{L}_{\text{eff}} = \frac{s}{2} \frac{n_y \dot{n}_x - n_x \dot{n}_y}{1 + n_z} - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

- Write down the soliton solution with moduli parameters
- any holomorphic map is a soliton solution
- plug them into the effective Lagrangian
- obtain the effective quantum mechanics for the solitons

$$\vec{n} = \frac{(z + \bar{z}, i(z - \bar{z}), 1 - \bar{z}z)}{1 + \bar{z}z}$$

$$\mathcal{L}_{\text{eff}} = s \frac{i(\bar{z}\dot{z} - \dot{\bar{z}}z)}{1 + \bar{z}z} - f^2 \frac{\vec{\nabla} \bar{z} \cdot \vec{\nabla} z}{(1 + \bar{z}z)^2}$$

$$z = z(w - w_0)$$

$$\begin{aligned} L_{\text{eff}} &= si(\bar{w}_0 \dot{w}_0 - \dot{\bar{w}}_0 w_0) \int d^2 w \frac{\vec{\nabla} \bar{z} \cdot \vec{\nabla} z}{(1 + \bar{z}z)^2} + O(w_0)^3 \\ &= 4\pi N si(\bar{w}_0 \dot{w}_0 - \dot{\bar{w}}_0 w_0) + O(w_0)^3 \end{aligned}$$



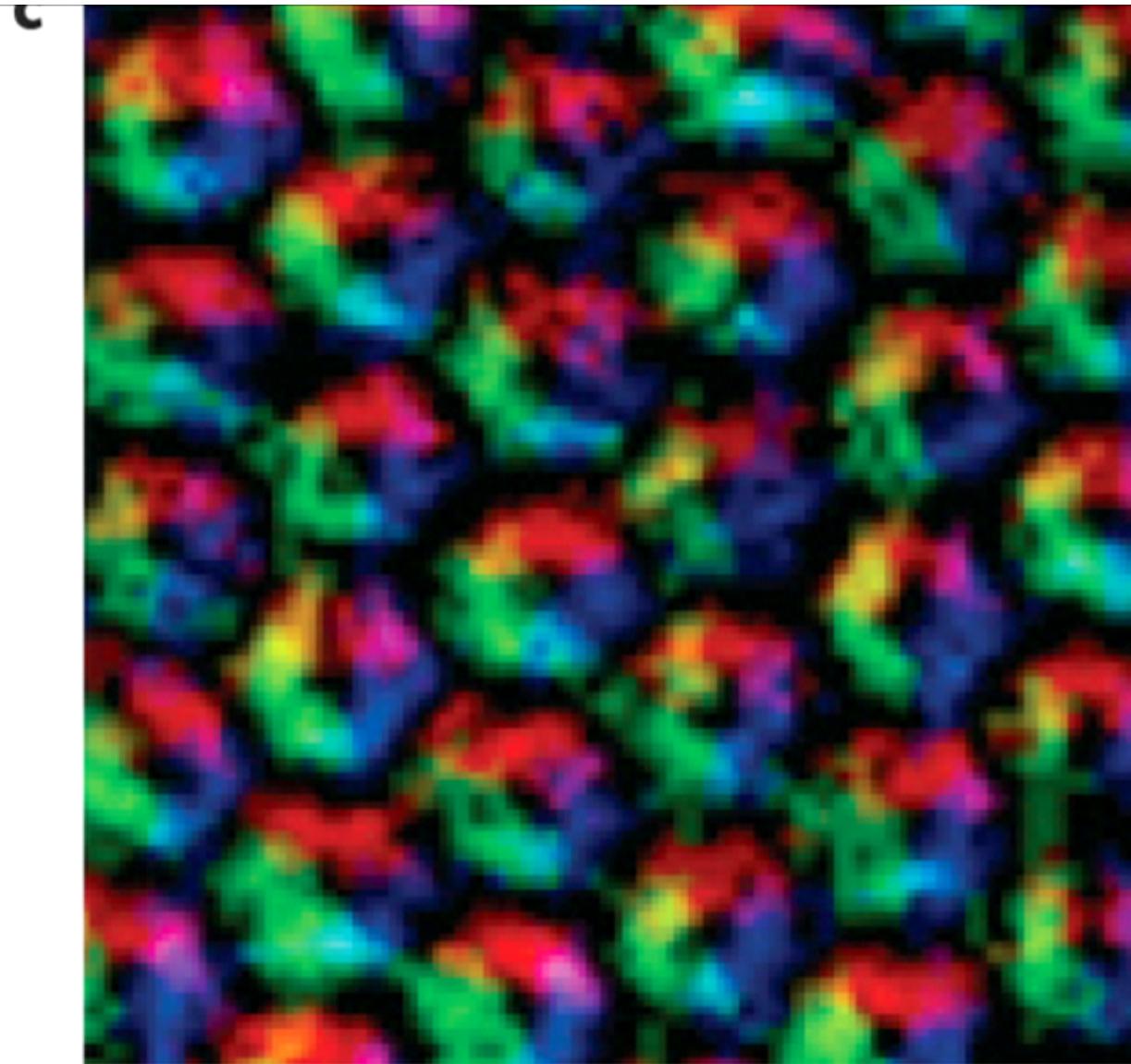
# consequence

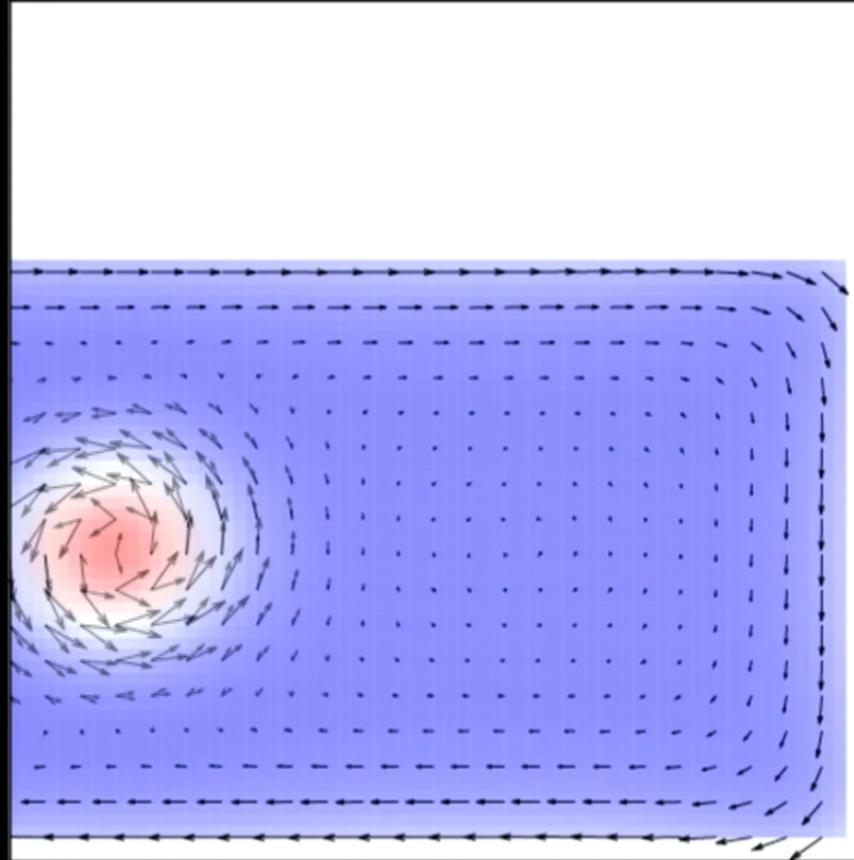
- If you push a skyrmion, it moves *sideways* called Magnus force

$$L = \frac{1}{2}(x\dot{y} - y\dot{x}) - Fx$$

$$\dot{y} - F = 0$$

- skyrmion lives in a “magnetic field” without external fields
- the same happens to vortices in superfluids





Iwasaki, Mochizuki, Nagaosa, Nature Nanotech 8, 742 (2013)



# consequence

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# General

- Consider any compact Kähler manifold  $K$  as the target space
- allows for a topological soliton  $H_2(K) \neq 0$
- holomorphic maps  $\mathbb{C} \rightarrow K$  solve EoM
- Use symplectic structure on  $K$  for Type-B NGBs
- consider moduli for translations for  $x$  &  $y$
- They don't commute!
- very similar to *central extension for extended supersymmetry by magnetic charge*

$$\{Q_i^\alpha, Q_j^\beta\} = \epsilon_{ij}\epsilon^{\alpha\beta}Z$$

# redundancies

H.Watanabe and HM, arXiv:1302.4800



# Noether constraints

- They can be understood as a consequence of Noether constraints  $\int d^d x \sum_a c_a(x) j_a^0(x) |0\rangle = 0$
- For broken symmetries, we have  $\langle \pi_b | j_a^0(x) |0\rangle \neq 0$
- then they are linearly redundant

$$\begin{aligned} 0 &= \sum_b |\pi_b\rangle \langle \pi_b| \int d^d x \sum_a c_a(x) j_a^0(x) |0\rangle \\ &= \sum_b |\pi_b\rangle \int d^d x c_a(x) \langle \pi_b | j_b^0(x) |0\rangle \end{aligned}$$



# Examples

- crystal: translations and rotations are both spontaneously broken
- they are both generated by the energy-momentum tensor  $R^{0i} = \epsilon_{ijk}x^j T^{0k}$
- would-be NGBs for rotations are the same excitations as those for translations (phonons)



# Examples

- Ginzburg-Landau theory

$$V = -\mu\psi^*\psi + \lambda(\psi^*\psi)^2$$

- $G=U(1), H=0$

- ${}^4\text{He}$  superfluid

- scalar BEC  $\langle 0|\psi|0\rangle \neq 0$

- $U(1)$   $\psi(\vec{x}, t) \rightarrow e^{i\theta}\psi(\vec{x}, t)$

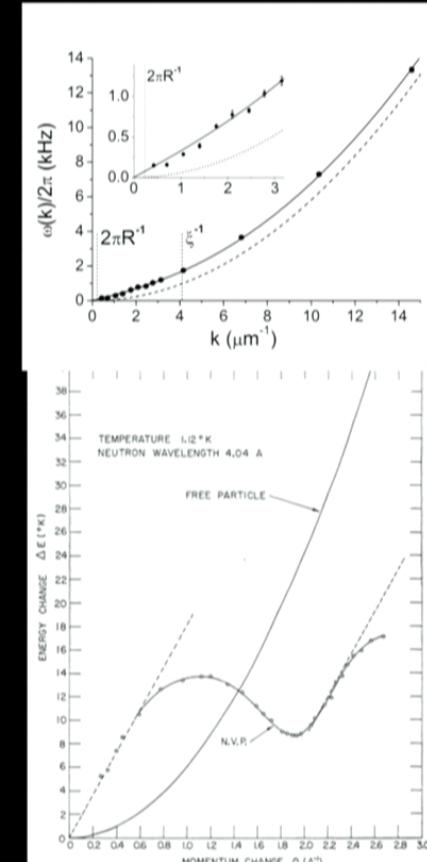
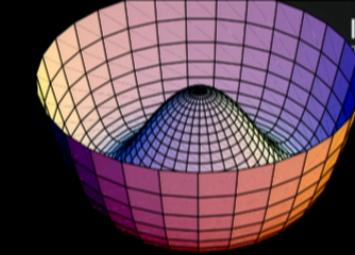
- Galilean boost

$$\psi(\vec{x}, t) \rightarrow e^{i(m\vec{x}\cdot\vec{x} - \frac{1}{2}m\vec{v}^2 t)}\psi(\vec{x} - \vec{v}t, t)$$

- both broken  $n_{BG}=1+3=4$

$$B^{i\mu} = tT^{i\mu} - mx^i j^\mu$$

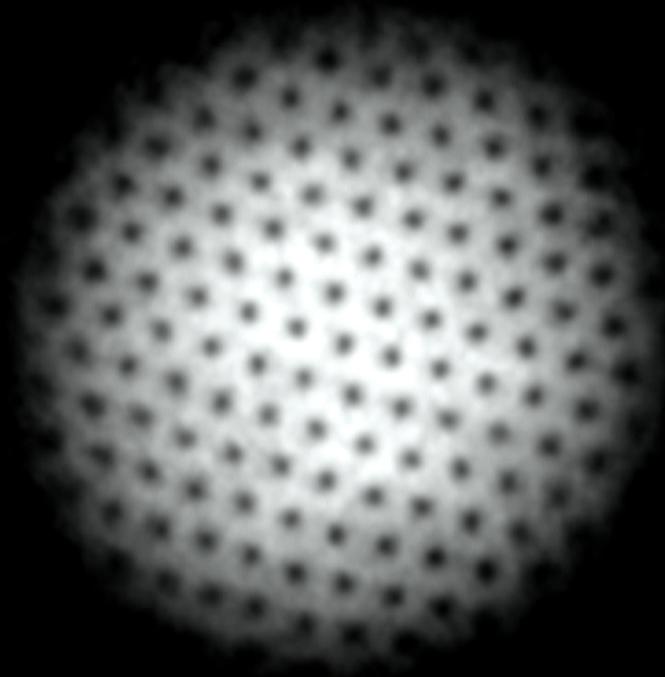
$\Rightarrow$  no separate NGBs for Galilean boosts





# vortex lattice

- rotate a (2d) BEC
- vortices form a triangular lattice
- broken:  $U(1), P_{x,y}, J_z$
- only one Type-A NGB with  $E \propto p^2$
- called Tkachenko mode  
 $T^{0i} = m j^i - 2m\Omega\epsilon^{ij} x^j j^0$

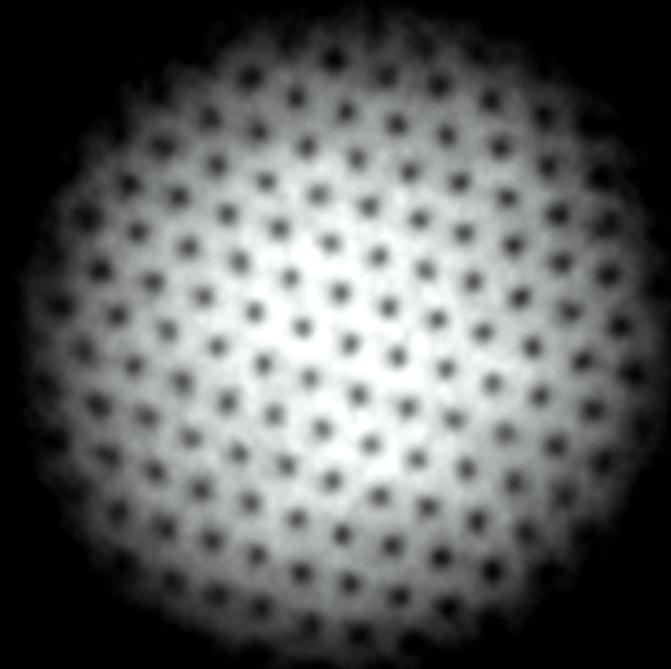




# vortex lattice

- translation of the lattice causes the phase shift

$$\theta \rightarrow \theta + 2m\vec{a} \cdot \vec{\Omega} \times \vec{x}$$



$$T^{0i} = m j^i - 2m\Omega\epsilon^{ij}x^j j^0$$

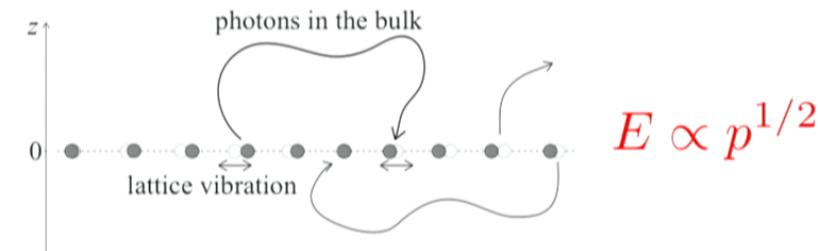
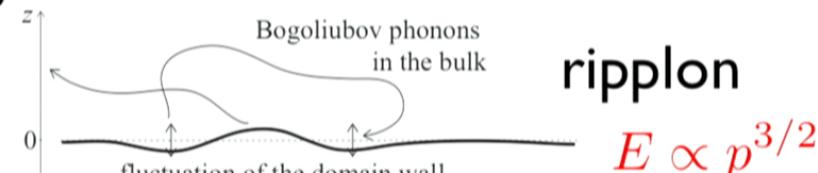
we have a precise effective Lagrangian for this

# Fractional dispersion relation

H.Watanabe and HM, arXiv:1403.3365

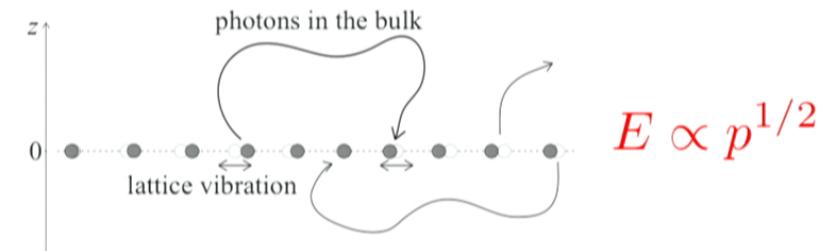
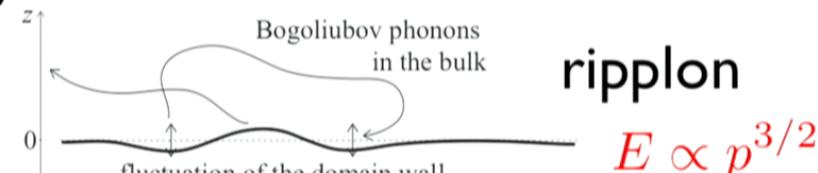
# fractal

- interface between two superfluids
  - integrate out bulk phonons
- 2D Wigner crystal
  - integrate out the Coulomb force



# fractal

- interface between two superfluids
  - integrate out bulk phonons
- 2D Wigner crystal
  - integrate out the Coulomb force





# Conclusion

- age-old subject, yet still a lot to learn!
- for internal symmetries, precise counting rule and dispersion relation of NGBs finally known
- underlying geometry: *presymplectic structure*
- **complete classification** of possibilities
- **central extension** to translations of solitons
- **fractional dispersion** relations
- redundancies make sense for spacetime symmetries
- exact predictions on *massive NGBs* à la BPS
- consistent EBH mechanism w/ type-B NGB

## Field Theories with «Superconductor» Solutions.

J. GOLDSTONE

CERN - Geneva

(ricevuto l'8 Settembre 1960)

**Summary.** — The conditions for the existence of non-perturbative type «superconductor» solutions of field theories are examined. A non-covariant canonical transformation method is used to find such solutions for a theory of a fermion interacting with a pseudoscalar boson. A covariant renormalisable method using Feynman integrals is then given. A «superconductor» solution is found whenever in the normal perturbative-type solution the boson mass squared is negative and the coupling constants satisfy certain inequalities. The symmetry properties of such solutions are examined with the aid of a simple model of self-interacting boson fields. The solutions have lower symmetry than the Lagrangian, and contain mass zero bosons.

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### 1. - Introduction.

This paper reports some work on the possible existence of field theories with solutions analogous to the Bardeen model of a superconductor. This possibility has been discussed by NAMBU<sup>(1)</sup> in a report which presents the general ideas of the theory which will not be repeated here. The present work merely considers models and has no direct physical applications but the nature of these theories seems worthwhile exploring.

The models considered here all have a boson field in them from the beginning. It would be more desirable to construct bosons out of fermions and this type of theory does contain that possibility<sup>(1)</sup>. The theories of this paper have the dubious advantage of being renormalisable, which at least allows one to find simple conditions in finite terms for the existence of «supercon-

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<sup>(1)</sup> Y. NAMBU: Enrico Fermi Institute for Nuclear Studies, Chicago, Report 60-21.