

Title: Black holes and Boyle's Law --- the thermodynamics of Einstein's greatest mistake

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Abstract: <span>The thermodynamics of black holes will be reviewed and recent developments incorporating pressure into the first law described. The asymptotically AdS Kerr metric has a van der Waals type critical point with a line of first order phase transitions terminating at a critical point with mean field exponents. The phase structure and stability of black holes in higher dimensions will also be described. </span>

# Black holes and Boyle's law — the thermodynamics of Einstein's greatest blunder

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Perimeter Institute  
20th May, 2014

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Black-hole thermodynamics



# Black holes and Boyle's law — the thermodynamics of Einstein's greatest blunder

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# Outline

Review of black hole thermodynamics

Temperature and entropy

Laws of thermodynamics

Hawking radiation

Pressure and enthalpy

Black hole enthalpy

Black hole volume

Equation of state

Carnot cycles

The first law

Critical behaviour

Thermodynamic stability

Local stability

de Sitter space-time

Conclusions

# Temperature and entropy

- Schwarzschild black hole,  
event horizon:  $r_h = 2G_N M$   $(c = 1)$
- Area:  $A = 16\pi G_N M^2$
- Surface gravity:  $\kappa = \frac{1}{4G_N M}$
- Entropy:  $S \propto \frac{A}{\ell_{Pl}^2}$ , Bekenstein (1972)  $(\ell_{Pl}^2 = \hbar G_N)$
- Temperature,  $T = \frac{\kappa \hbar}{2\pi}$  Hawking (1974)

Hawking temperature

$$T = \frac{\hbar}{8\pi G_N M}$$

Solar mass black hole:  $T = 6 \times 10^{-8} K$



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# Laws of black hole thermodynamics

## Zeroth Law

Surface gravity  $\kappa$  is constant on the event horizon

Internal energy  $U(S)$ ,  $T = \frac{\partial U}{\partial S}$ : identify  $U(S) = M \Rightarrow dM = T dS$ .

## First Law

$$dM = T dS + \Omega dJ + \Phi dQ$$

With  $S \propto \frac{A}{\hbar} \propto \frac{16\pi M^2}{\hbar}$  and  $T = \frac{\hbar}{8\pi M}$ :  $S = \frac{1}{4} \frac{A}{\hbar}$ .  $(G_N = 1)$

## Second Law

Area never decreases (classically)

Solar mass black hole:  $S \approx 10^{78}$

## Third Law

Cannot achieve  $\kappa = 0$  in a finite number of steps

For Kerr black holes  $\kappa \rightarrow 0$  as  $J^2 \rightarrow M^4$ ,  $S = \ln \mathcal{W} \Rightarrow \mathcal{W} \approx 10^{4 \times 10^{77}}$

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# Hawking Radiation

- $T = \frac{\hbar}{8\pi M}$ , black hole radiates thermal energy.  
As  $M$  goes down  $T$  goes up, radiates even more!
- Heat capacity  $\frac{dM}{dT} = -\frac{\hbar}{8\pi T^2} < 0$ .
- Radiates with power  $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{M^2}$   
Lifetime  $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$ ,  $M \sim 10^{12} \text{ kg} \Rightarrow \tau \sim 10^{10} \text{ years}$ .
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# Black hole enthalpy

First Law of thermodynamics

$$dU = T dS - P dV$$

- Include cosmological constant  $\Lambda$ , contributes pressure  $P$  and energy density  $\epsilon = -P = \frac{\Lambda}{8\pi}$
- Thermal energy

$$U = M + \epsilon V = M - PV \Rightarrow M = U + PV$$

$U = U(S, V)$ : Legendre transform

Enthalpy

$$M = U + PV = H(S, P)$$

Henneaux+Teitelboim (1984);  
Kastor, Ray+Traschen [0904.2765].

# Thermodynamic Volume

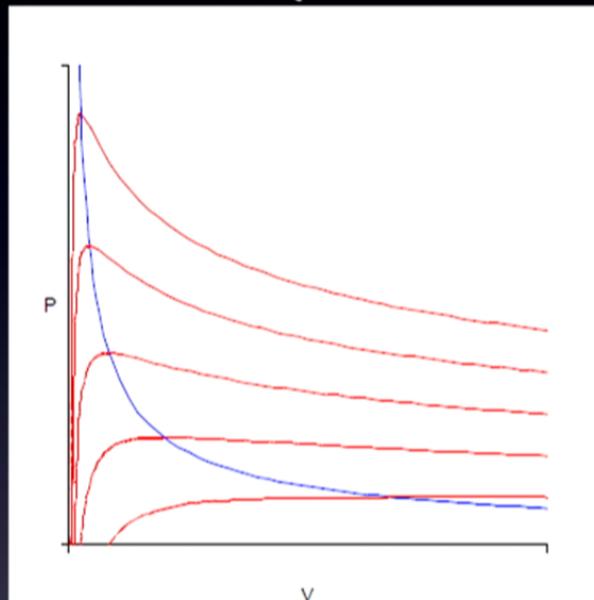
- Define the **thermodynamic volume**  $(P \geq 0, \Lambda \leq 0)$

$$V = \left( \frac{\partial H}{\partial P} \right)_S$$

- AdS Schwarzschild:  $V = \frac{4\pi}{3} r_h^3$   
Kastor, Ray+Traschen [0904.2765]  
Higher dimensions:  $V = \frac{\Omega_d}{d+1} r_h^d$   
BPD [1009.5022]
- AdS Myers-Perry Cvetic, Gibbons+Kubizňák [1012.2888]
- Charged Kerr  
 $\text{sgn}(J) > 0: V = \frac{4\pi r_h^3}{3} S - \omega r_h^2$ : isentropic or isovolumetric

# Equation of state

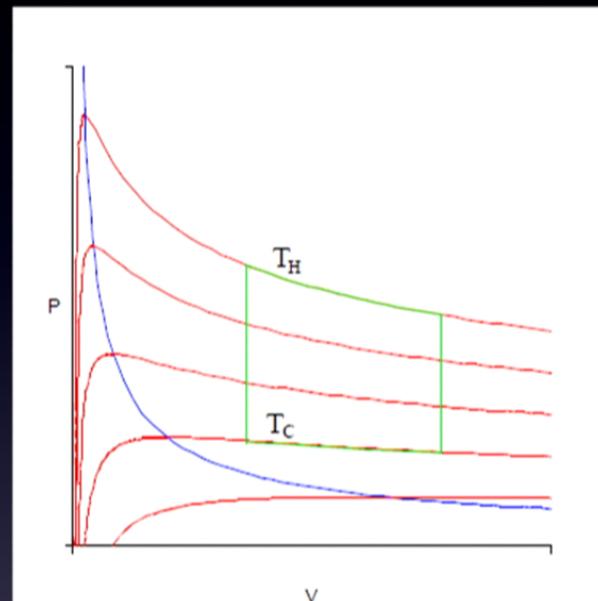
$$\text{Schwarzschild: } T(V, P) = \frac{\hbar}{4\pi} \left\{ \left( \frac{3V}{4\pi} \right)^{-\frac{1}{3}} + 8\pi P \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} \right\}$$



# Carnot cycles

ISENTROPIC  $\Leftrightarrow$  ISOVOLUMETRIC: Carnot cycles

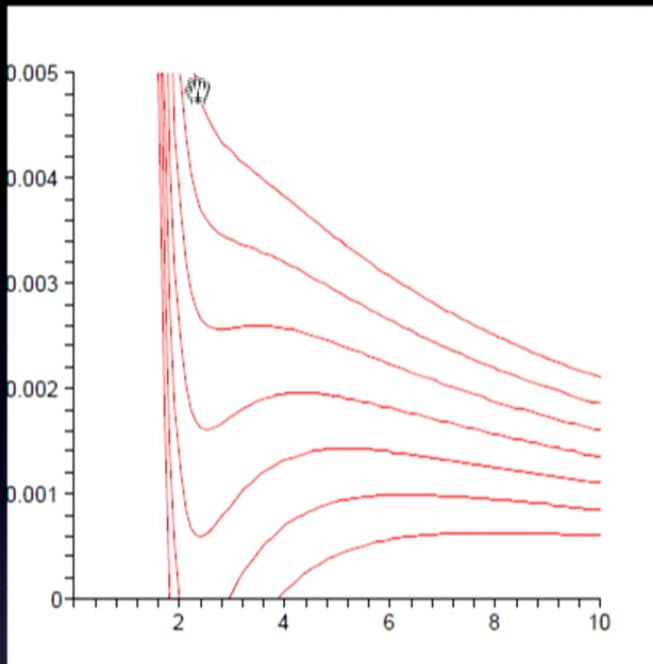
C.V. Johnson [1404.5982]



Isentropic change,  $P^\gamma/V = \text{const} \rightarrow \gamma$  is infinite ( $J = 0$ )

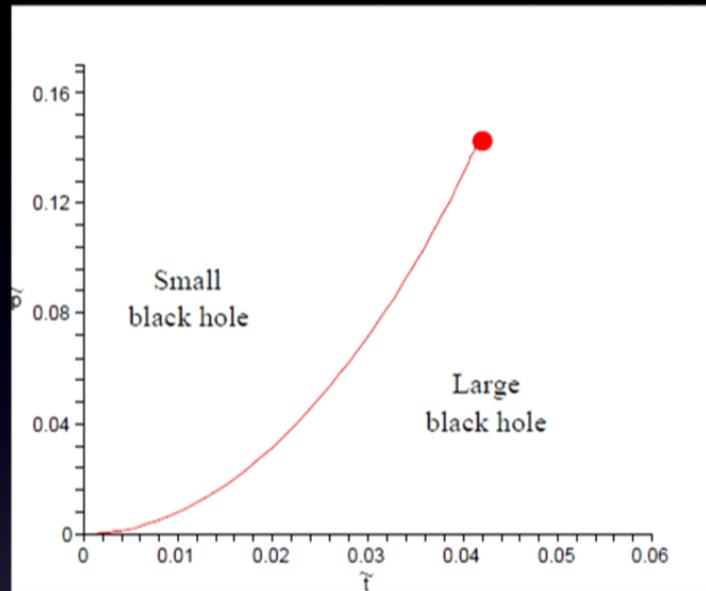


# Rotating black holes: AdS Kerr



- $T = \text{const}, P/J$  versus  $V^{\frac{1}{3}}/J^{\frac{1}{2}}$  (dimensionless).
- Waals gas      Caldarelli, Gognola+Klemm [hep-th/9908022];  
Kubizňák+Mann [arXiv:1205.0559]; BPD [1106.6260], [1209.1272]

# Phase diagram

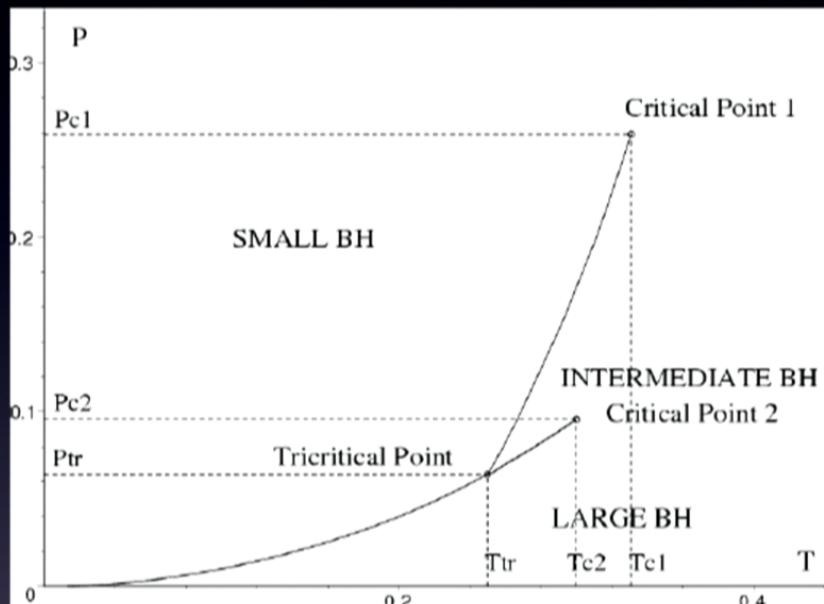


- Line of first order phase transitions, with a critical point
- Latent heat:  $L = T \Delta S = M_{large} - M_{small}$
- Clapeyron equation:  $\frac{dP}{dT} = \frac{\Delta S}{\Delta V}$

# $T - P$ phase diagram

## Higher dimensions

- 6-Dimensions: two angular momenta  $J_1$  and  $J_2$ ,  
phase diagram depends on the ratio  $q = \frac{J_1}{J_2}$



Altimirano, Kubizňák, Mann+Sherkatghanad, [1401.2586]

- Examples of re-entrant phase transitions

Altimirano, Kubizňák+Mann, [1306.5756]

Black-hole thermodynamics

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# Local thermodynamic stability

## Ordinary thermodynamics

- Usually entropy is a function of extensive variables  
 $S(U, V, N) = S(X_a)$ .
- Local thermodynamic stability requires that  $S$  is a **concave** function,  $-\frac{\partial S}{\partial x_a \partial x_b}$  is positive definite.
- Alternatively  $U(S, V, N)$  is **convex**.
- Intensive variables are the Legendre transforms:  $x^A = \frac{\partial U}{\partial X_A}$ .
- Scale lengths:  $\lambda^d U(S, V, N) = U(\lambda^d S, \lambda^d V, \lambda^d N) \Rightarrow$

$$U = TS - PV + \mu N \quad (\text{Gibbs-Duhem relation})$$

- The entropy is independent of the intensive variables

$$\Xi(T, P, \mu) = U - TS + PV - \mu N = 0.$$

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- The complete Legendre transform of the function

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¶

- The complete Legendre transform of  $U$  vanishes,

$$\Xi(T, P, \mu) = U - TS + PV - \mu N = 0.$$

# Local thermodynamic stability

## Black hole thermodynamics

- What is extensive and what intensive for black holes?

Thermodynamic Variable	Dimension
Mass, $M$	$D - 3$
Entropy, $S$ (area)	$D - 2$
Angular momenta, $J^i$	$D - 2$
Volume, $V$	$D - 1$
Electric Charge, $Q$	$D - 3$
Temperature, $T$	-1
Angular velocity, $\Omega_i$	-1
Pressure, $P$ ( $\Lambda$ )	-2
Electric potential, $\Phi$	0

- Define extensive variables as those that depend on space-time dimension  $D$ .



# Local thermodynamic stability

## Necessary condition

- Intensive variables:  $x_A = \frac{\partial U}{\partial X^A}$ .
- Let  $\mathcal{D}^B = d_B X^B (\Sigma_B)$  then the Smarr relation is

$$U = \sum_B \mathcal{D}^B x_B.$$

- Take  $\mathcal{D}^A \frac{\partial}{\partial X^A}$ :

$$U_{AB} \mathcal{D}^A \mathcal{D}^B = -\frac{(D-2)}{(D-3)^2} (TS + \Omega \cdot J) + 2 \frac{(D-1)}{(D-3)^2} PV$$

(define  $U_{AB} = \frac{\partial^2 U}{\partial X^A \partial X^B}$ , positive definite for stability)

- A **necessary** condition for complete local thermodynamic stability is

$$PV > \frac{1}{2} \frac{(D-2)}{(D-1)} (TS + \Omega \cdot J) > 0 \quad \text{BPD [1403.1507]}$$

# Local thermodynamic stability

Necessary and sufficient conditions for local stability:

- the specific heat at constant  $\Omega_i$  is positive,

$$C_\Omega = T \left. \frac{\partial S}{\partial T} \right|_{\Omega, P} > 0$$

- the isentropic moment of inertia tensor

$$\mathcal{I}^{ij} = \left. \frac{\partial J^i}{\partial \Omega_j} \right|_{S, P} = \left. \frac{\partial J^j}{\partial \Omega_i} \right|_{S, P}$$

is a positive matrix;

- the adiabatic compressibility is positive,

$$\kappa_{S, P} = - \left. \frac{1}{V} \frac{\partial V}{\partial P} \right|_{J^i, S} > 0$$

( $\kappa_{S, P} > 0$  for AdS Myers-Perry metrics in  $D$  space-time dimensions, [BPD arXiv:1308.5403])



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# Asymptotically de Sitter

**BPD, D. Kastor, D. Kubizňák, R.B. Mann and J. Traschen [1301.5926]**

- $P = -\frac{\Lambda}{8\pi} < 0$ .
- Two event horizons: black hole  $r_{bh}$ ; cosmological,  $r_c$ .
- Two different temperatures,  $T_{bh} \neq T_c$ , in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$ .  
$$V_{bh} = \left. \frac{\partial M}{\partial P} \right|_{S_{bh}, J}, V_c = \left. \frac{\partial M}{\partial P} \right|_{S_c, J}.$$
- $V_{bh} = \frac{r_{bh} A_{bh}}{3} + \frac{4\pi}{3} \frac{J^2}{M}, \quad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3} \frac{J^2}{M}$ .

For fixed  $V_c$  the cosmological horizon entropy,  $S_c = \frac{A_c}{4}$ , is maximized by Schwarzschild- de Sitter space-time ( $J = 0$ ).

- Volume between horizons:  $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh} A_{bh}}{3}$ .
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# Conclusions

- $\Lambda \neq 0 \Rightarrow P dV$  term in black hole 1st law.
- Black hole mass is identified with **enthalpy**,  $H(S, P)$ .
- “Thermodynamic” volume:  $V = \left(\frac{\partial H}{\partial P}\right)_S$ .
- $P > 0$  is a necessary condition for complete local thermodynamic stability.
- First Law:  $dW = -TdS - \Omega dJ - \Phi dQ + PdV$ ,  
 $PdV$  Term can affect Penrose processes.

Critical points, van der Waals type equation of state,  
mean field exponents, triple points,  
re-entrant phase transitions, Clapeyron equation.

- Additional terms in the first law for black holes

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- Critical points, van der Waals type equation of state,  
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- Adiabatic compressibility for Myers-Perry,  $0 \leq \kappa_{S,P} < \infty$ .