Title: TBA

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Abstract:

On the local integrals of motion for MBL systems

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see arXiv next days

**on leave from ICTP, Trieste*

Integrals of motion for MBL systems

The MBL phase is endowed with *local* integrals of motion which are pseudo-spin operators (Huse, Oganesyan, Serbyn, Papic, Abanin, Imbrie etc.)

The goals:

a) show IOM follow from a BAA-*like* construction

b) in some approximation the criterion for existence of the IOM is the same as the existence of the MBL phase in BAA

c) present a new calculation of the rare events probability

Integrals of motion for MBL systems

Strategy:

1) Define LIOM perturbatively
2) Write down equations
3) Approximate equations
4) Find the solution and study the radius of convergence

This gives the critical interaction strength

Integrals of motion for MBL systems

Non interacting, infinite disorder:

$$n_i = c_i^\dagger c_i$$

together with $c_i,\ c_i^\dagger,\ \mathbb{I}_i$ form a linear basis for the 4^N dimensional space of operators

$$O_i = a_0 \mathbb{I}_i + a_1 c_i^\dagger c_i + a_2 c_i^\dagger + a_3 c_i$$

$a_0 = \mathrm{Tr}(O_i \mathbb{I}_i) - \mathrm{Tr}(O_i c_i^\dagger c_i)$
etc.

Integrals of motion for MBL systems

At finite disorder

$$H_0 = \sum_{i \in V} -J(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \epsilon_i c_i^\dagger c_i$$

We can again find local conserved quantities

$$n_\alpha = \sum_{i,j \in V} \phi_a^*(i)\phi_a(j) c_i^\dagger c_j$$

$$c_a = \sum_{i \in V} \phi_a(i) c_i$$

$$c_a^\dagger = \sum_{i \in V} \phi_a^*(i) c_i^\dagger$$

assuming all eigenfunctions are localized

$$\phi_a(i)\phi_a(j) \sim e^{-c_a(|i-x_a|+|x_a-j|)} \leq e^{-c_a|i-j|}$$

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Integrals of motion for MBL systems

Having binary (pseudospin) IOM is good, you can label the eigenstates. But might be difficult to construct.

However, the important aspect is the locality of the IOM

$$\mathrm{Tr}(I_a(c_i^\dagger c_i)(c_j^\dagger c_j)) \sim e^{-|i-j|/b}$$

Moreover, given a set of LIOM we can* perturbatively build pseudo-spin LIOM

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A model hamiltonian (BAA)

Let us introduce interaction

$$H = H_0 + \lambda \sum_{a,b,c,d} U_{ab,cd} c_a^\dagger c_b^\dagger c_c c_d$$

assume all single particle states *a* are localized

δ_ξ is the single-particle level spacing

$$U_{ab,cd} = \begin{cases} \delta_\xi & \text{if } |\epsilon_a - \epsilon_d|, |\epsilon_b - \epsilon_c| \leq \delta_\xi \\ 0 & \text{otherwise} \end{cases}$$

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A model hamiltonian (BAA)

$$H = H_0 + \lambda \sum_{a,b,c,d} U_{ab,cd} c_a^\dagger c_b^\dagger c_c c_d$$

We want to find $n_a \to I_a(\lambda)$ with $[I_a, H] = 0$

I_a local IOM

This means that dressed particles are still localized

That these exist for sufficiently small λ is the nature of MBL

Integrals of motion for BAA

One can solve perturbatively

$$I_a = n_a + \lambda I_a^{(1)} + \lambda^2 I_a^{(2)} + \lambda^3 I_a^{(3)} + \ldots$$

$$H = H_0 + \lambda U$$

Imposing order by order

$$[I_a, H] = 0$$

we find

$$[H_0, I_a^{(n)}] = -[U, I_a^{(n-1)}]$$

which can be written as

$$I_a^{(n)} = -\mathrm{Ad}_{H_0}^{-1}([U, I_a^{(n-1)}])$$

can be solved assuming I_a is symmetric in the basis of H_0

An ansatz for the IOM

We make the ansatz

$$I_a = n_a + \sum_{N\geq 1} \sum_{|\mathcal{I}|=|\mathcal{J}|=N} A_{\mathcal{I},\mathcal{J}} O_{\mathcal{I},\mathcal{J}}$$

with $A_{\mathcal{I},\mathcal{J}}$ complex numbers and

$$O_{\mathcal{I},\mathcal{J}} = \prod_{b\in\mathcal{I}} c_b^\dagger \prod_{a\in\mathcal{J}} c_a$$

We impose $[I_a, H] = 0$ and we get a set of equations for $A_{\mathcal{I},\mathcal{J}}$

An ansatz for the IOM

Once we get the $A_{\mathcal{I},\mathcal{J}}$ we need to see if the series we are constructing converges

$$P\left(\sum_{N=1}^{\infty}\sum_{|\mathcal{I}|=|\mathcal{J}|=N}|A_{\mathcal{I},\mathcal{J}}|^2\right)=1$$

So we need to find the distribution of the amplitudes and count the number of terms in the sum

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Equations for the IOM

We get very complicated, linear equations on operator space

$$
\begin{aligned}
&\sum_{n=1}^{N} \left(\epsilon_{j_n} - \epsilon_{i_n}\right) A^{(\alpha)}_{\jmath,\jmath} + \boxed{\lambda} \sum_{m<l} \sum_{\gamma<\delta} \left\{ U_{\gamma\delta, j_m j_l} A^{(\alpha)}_{\jmath, \tilde{\jmath}^{\gamma\delta}_{ml}} - U_{i_m i_l, \gamma\delta} A^{(\alpha)}_{\tilde{\jmath}^{\gamma\delta}_{ml}, \jmath} \right\} + \\
&+ \lambda \sum_{m<l} \sum_{n} \sum_{\gamma} \left\{ U_{i_n \gamma, j_m j_l} A^{(\alpha)}_{\tilde{\jmath}_n, \tilde{\jmath}^{\gamma}_{ml}} - U_{i_m i_l, j_n \gamma} A^{(\alpha)}_{\tilde{\jmath}^{\gamma}_{ml}, \tilde{\jmath}_n} \right\} + \\
&+ \boxed{\lambda} \delta_{N,2} U_{i_1 i_2, j_1 j_2} \left(-\delta_{\alpha i_1} - \delta_{\alpha i_2} + \delta_{\alpha j_1} + \delta_{\alpha j_2}\right) = 0
\end{aligned}
\tag{21}
$$

We need to simplify!

Simplified equations for the IOM

The interaction $A_{\mathcal{I},\mathcal{J}}$ connects the amplitudes at different locations on operator space

If $\lambda = 0$ then $A_{a,a} = 1$

Consider only the "forward approximation" in which

$$(\mathcal{I},\mathcal{J}) \to (\mathcal{I}',\mathcal{J}')$$

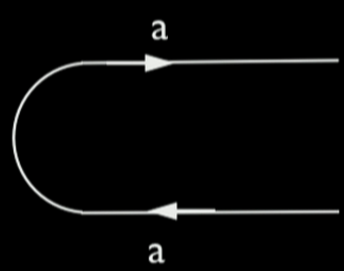
only if they differ by exactly one index

one more particle and one more hole

Diagrammatic for the IOM

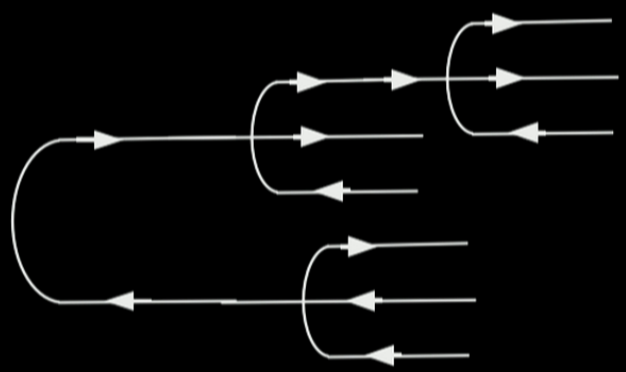
This means that we are considering diagrams like

n_a



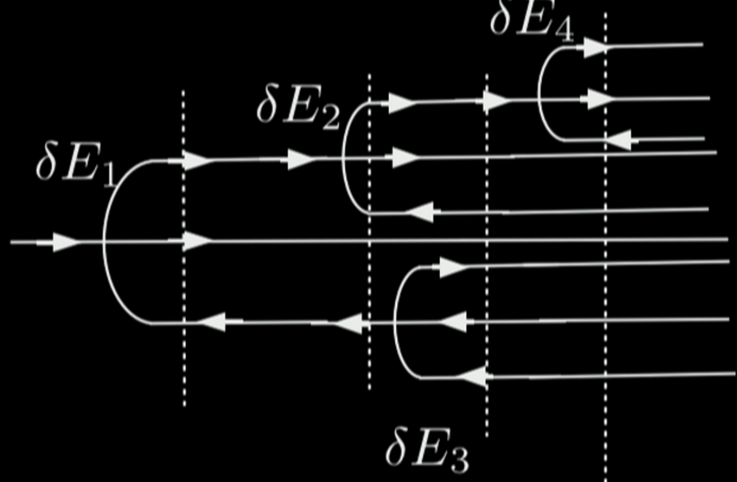
to 0-th order

and, for example:
at 3-rd order



Diagrammatic for the IOM

These are the same diagram that are considered in the ImSCBA by BAA

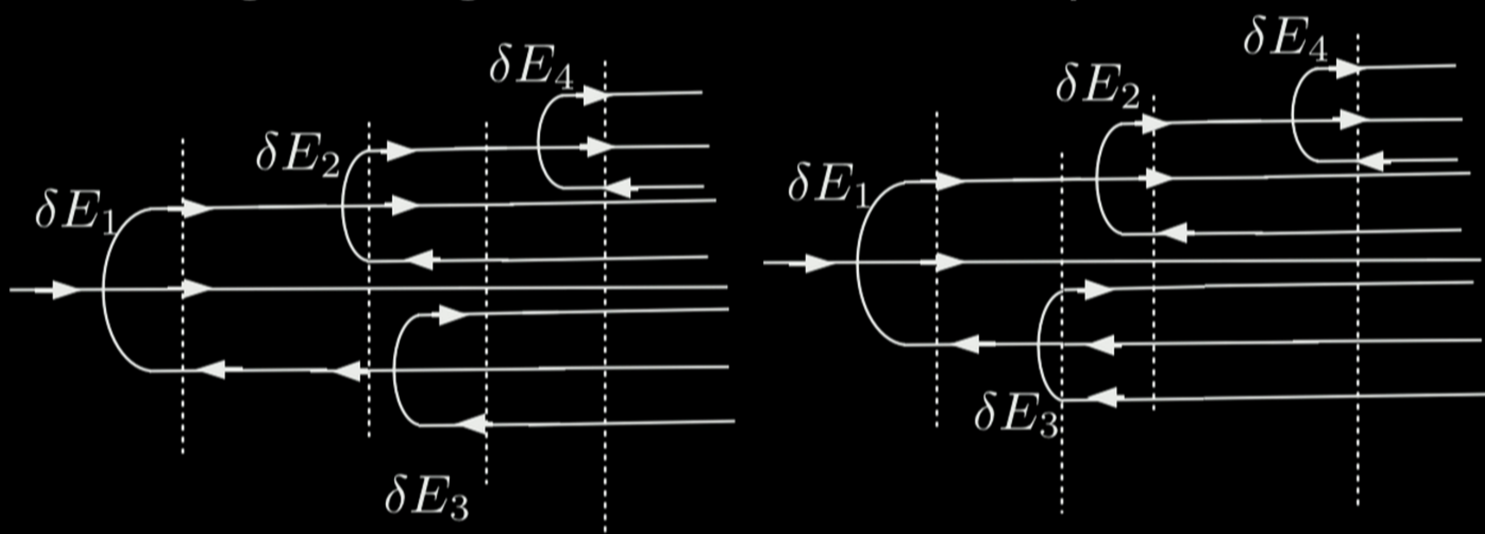


$$\delta E_1 = (\epsilon_a - \epsilon_b + \epsilon_c - \epsilon_d)^{(1)}$$

$$A = \frac{\lambda\delta_\xi}{E+\delta E_1}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2+\delta E_3}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_2+\delta E_3+\delta E_4}$$

Diagrammatic for the IOM

For a given diagram need to sum over permutations



$$A = \frac{\lambda\delta_\xi}{E+\delta E_1}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_3}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_3+\delta E_2}\frac{\lambda\delta_\xi}{E+\delta E_1+\delta E_3+\delta E_2+\delta E_4}$$

Diagrammatic for the IOM

For a given diagram need to sum over permutations

$$A = \sum_P A(E, \{\epsilon\}, P)$$

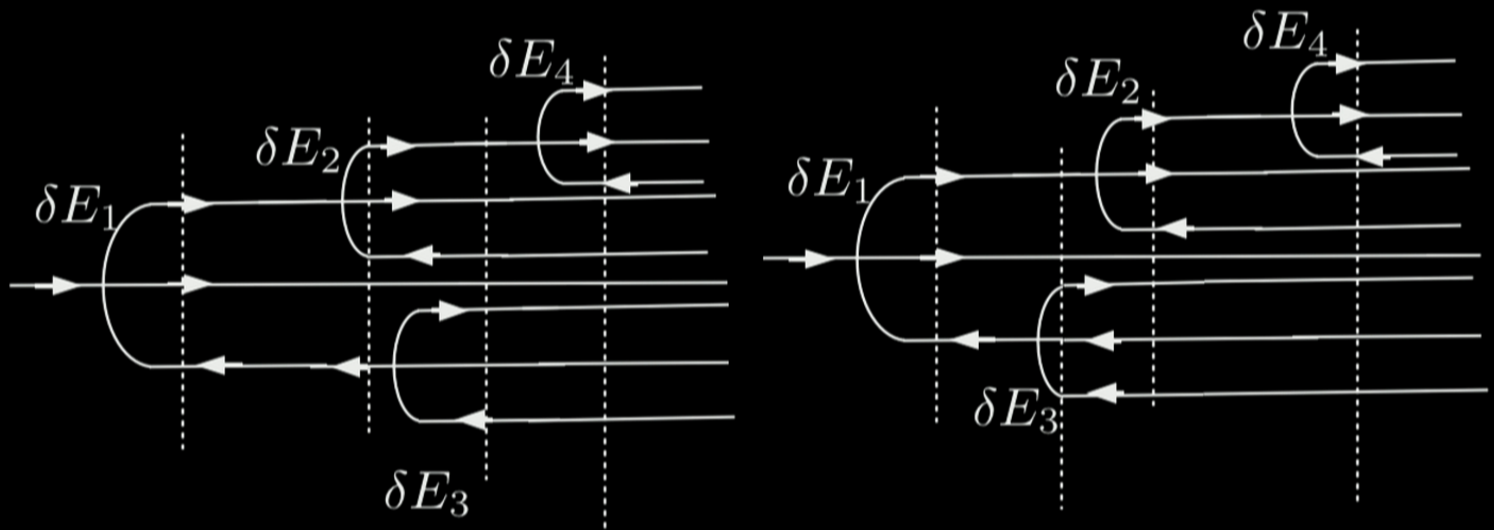
this might led one to think that the rare event in which $A > 1$ can come independently from each term in the sum therefore giving rise to a resonance probability

$$P \sim N!(\lambda/c)^N$$

which would give no MBL phase

Diagrammatic for the IOM

For a given diagram need to sum over permutations



$$A = \frac{\lambda \delta_\xi}{E + \delta E_1} \frac{\lambda \delta_\xi}{E + \delta E_1 + \delta E_3} \frac{\lambda \delta_\xi}{E + \delta E_1 + \delta E_3 + \delta E_2} \frac{\lambda \delta_\xi}{E + \delta E_1 + \delta E_3 + \delta E_2 + \delta E_4}$$

Diagrammatic for the IOM

But the terms in the sum are not independent, on the contrary, so only a few of them can be resonant

For example the *N* terms

$$\Sigma = \frac{1}{\mathcal{E}_0(\mathcal{E}_0+\tilde{\mathcal{E}})(\mathcal{E}_0+\tilde{\mathcal{E}}+\mathcal{E}_1)\cdots(\mathcal{E}_0+\tilde{\mathcal{E}}+\cdots+\mathcal{E}_n)} + \frac{1}{\mathcal{E}_0(\mathcal{E}_0+\mathcal{E}_1)(\mathcal{E}_0+\mathcal{E}_1+\tilde{\mathcal{E}})\cdots(\mathcal{E}_0+\mathcal{E}_1\cdots+\mathcal{E}_n)} + \cdots + \frac{1}{\mathcal{E}_0(\mathcal{E}_0+\mathcal{E}_1)(\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2)\cdots(\mathcal{E}_0+\mathcal{E}_1\cdots+\tilde{\mathcal{E}})} \qquad (30)$$

using an integral representation

$$\Sigma = \frac{1}{\tilde{\mathcal{E}}}\frac{1}{\mathcal{E}_0(\mathcal{E}_0+\mathcal{E}_1)(\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2)\cdots(\mathcal{E}_0+\mathcal{E}_1+\cdots+\mathcal{E}_n)} - \frac{1}{\tilde{\mathcal{E}}}\frac{1}{(\mathcal{E}_0+\tilde{\mathcal{E}})(\mathcal{E}_0+\tilde{\mathcal{E}}+\mathcal{E}_1)(\mathcal{E}_0+\tilde{\mathcal{E}}+\mathcal{E}_1+\mathcal{E}_2)\cdots(\mathcal{E}_0+\tilde{\mathcal{E}}+\mathcal{E}_1+\cdots+\mathcal{E}_n)} \qquad (32)$$

2 terms

Diagrammatic for the IOM

This integral representation

$$\Sigma = \lim_{\epsilon\to 0}\int \frac{d\omega_1 d\omega_2 \delta\left(\omega_1+\omega_2-\mathcal{E}_0\right)}{(\omega_1-i\epsilon)(\omega_1+\tilde{\mathcal{E}}-i\epsilon)(\omega_2-i\epsilon)(\omega_2+\mathcal{E}_1-i\epsilon)\cdots(\omega_2+\mathcal{E}_1+\cdots+\mathcal{E}_n-i\epsilon)} \quad (31)$$

Closing the contour above gives N terms and below gives 2 terms only

$$N! \to b^N$$

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Diagrammatic for the IOM

How many diagrams?

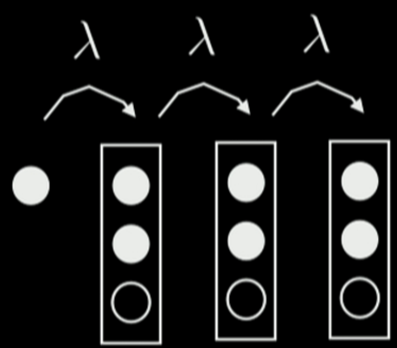
Topology

$$D^{(n)} = \sum_{n1+n2+n3=n} D^{(n_1)} D^{(n_2)} D^{(n_3)}$$

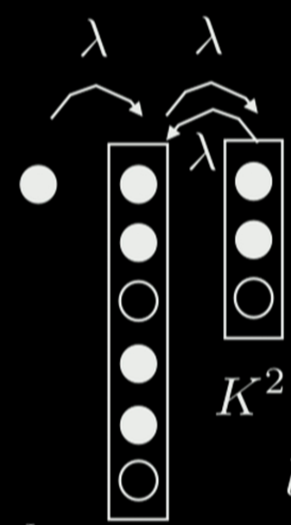
$$D^{(n)} = \binom{3n}{n} \frac{1}{2n+1} \sim \sqrt{\frac{3}{16\pi}} n^{-3/2} \left(\frac{27}{4}\right)^n$$

Diagrammatic for the IOM

Assignments of *a,b,c,d* indices



$K^3 \quad l = 3\xi$



$K^2(K-1)$

$l = \xi$

$$K = \int_{-J}^{J} \frac{d\epsilon_c}{\delta_\xi} \int_{-J}^{J} \frac{d\epsilon_b}{\delta_\xi} \int_{-J}^{J} \frac{d\epsilon_d}{\delta_\xi} \Theta(\delta_\xi - |\epsilon_a - \epsilon_b|)\Theta(\delta_\xi - |\epsilon_c - \epsilon_d|) \simeq \frac{J}{\delta_\xi}$$

Ballistic particles go further and have also more phase space

Diagrammatic for the IOM

So we need to compute

$$\sum_{\mathcal{I},\mathcal{J}} |A_{(\mathcal{I},\mathcal{J})}|^2 \sim \max_{\text{graphs}} |A_{(\mathcal{I},\mathcal{J})}|^2$$

this is because the probability is long-tailed

$$P(A_{(\mathcal{I},\mathcal{J})}) \sim A^{-2}$$

So the last step is to find *P(A)*

Probability distribution of *A*

We assume gaussian distribution $\delta E_i = x_i$

$$P(x) = \frac{1}{\sqrt{2\pi}\delta_\xi} e^{-\frac{x^2}{2\delta_\xi^2}}$$

$$A = \prod_{n=1}^{N} \frac{\lambda \delta_\xi}{S_n} \qquad S_n = \sum_{i=1}^{n} x_i$$

Need to look at $Y = -\sum_{n=1}^{N} \log |S_n|$

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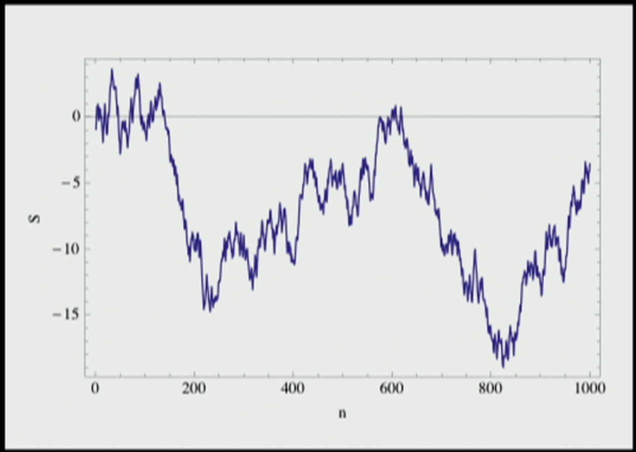
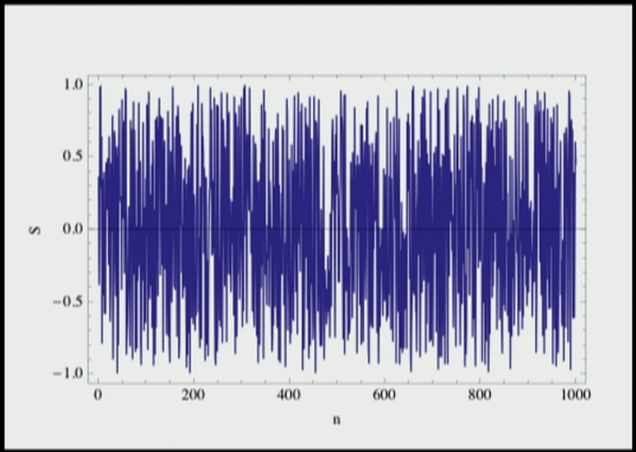
Probability distribution of *A*

The series converges if there is a *z<1*

$$P\left[\max_{(27K/4)^N\, extractions} (\lambda\delta_\xi)^N e^Y < z^N\right] \to 1$$

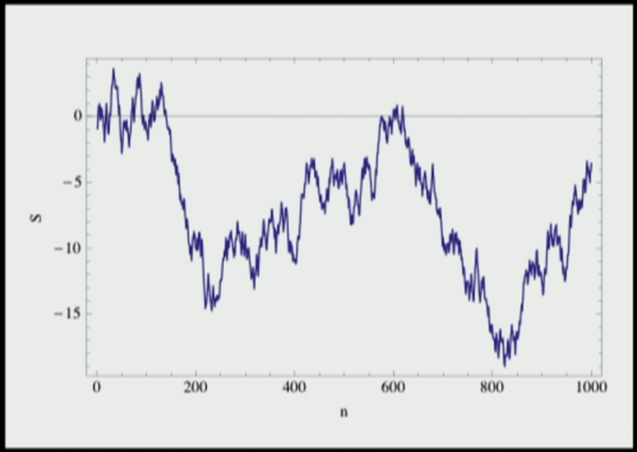


many body



single particle

Probability distribution of *A*



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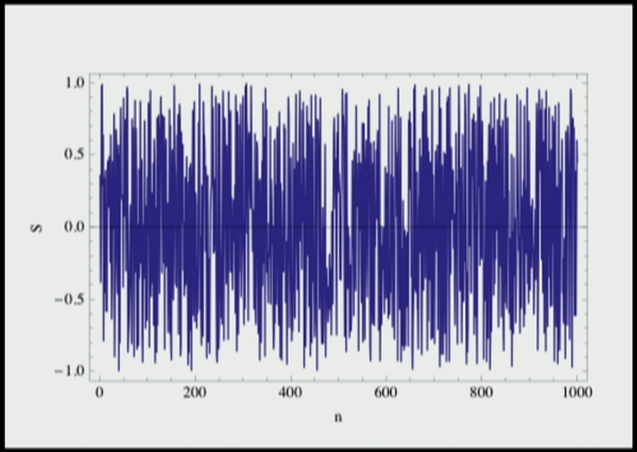

single particle

$$A_{\mathrm{typ}} \sim \prod_n \frac{1}{n} = \frac{1}{\sqrt{N!}}$$

Probability distribution of *A*

We find that

$$G_N(k) = \left(\frac{\delta_\xi^{2k}}{2\pi} \right)^{N/2} \langle \psi' | \mathcal{H}^N | \psi \rangle.$$

$$\mathcal{H}_{n,m} = \frac{\Gamma(\frac{1+k}{2} + m + n)}{\sqrt{\Gamma(1+2m)}\sqrt{\Gamma(1+2n)}}$$

So we need to find the largest eigenvalue of *H*

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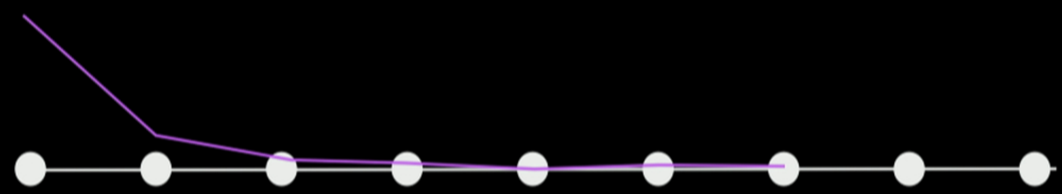
Probability distribution of A

Ironically, the largest eigenstate of H is localized



the potential at $m=0$ diverges for $k=-1$

$$\mu_{\max} = \Gamma\left(\frac{1+k}{2}\right) + \frac{\pi^2}{36}(k+1) + 0.0405(k+1)^2 + \ldots$$

Probability distribution of *A*

$$P(Y) = \int_B \frac{dk}{2\pi i} e^{kY} G_N(k)$$

$$P(Y) = \int \frac{dk}{2\pi i} e^{N(y+\log(\mu))} \simeq \left(\frac{Y}{y_0 N} \right)^N e^{-Y/y_0 (1-\gamma/(2(Y/N)^2)}$$

MBL criterion is:

$$P(Y < Y_c)^{(27K/4)^N} \simeq \exp\left(-(27K/4)^N P(Y > Y_c)\right)$$

1

0

Radius of convergence

Putting all together we get a radius of convergence for the series defining the LIOM

$$\frac{27e}{\sqrt{8\pi}}\frac{J}{\delta_\xi}\lambda_c \log(\lambda_c) = 1$$

Practically the same condition for MBL in BAA

$$I_a = n_a + \sum_{N\geq 1}\sum_{|\mathcal{I}|=|\mathcal{J}|=N} A_{\mathcal{I},\mathcal{J}} O_{\mathcal{I},\mathcal{J}}$$

exist with probability 1 for $\lambda < \lambda_c$

Conclusions

- We can define IOM as a perturbative series in interaction
- We can study the probability of convergence of said series
- This involves studying topologies and real-space configurations of MB wf
- This gives, in the fwd approximation, the same diagrams of BAA and the same convergence radius