Title: (John Imbrie) A Rigorous Result on Many-Body Localization

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Abstract: I will discuss a proof of many-body localization for a one-dimensional spin chain with random local interactions. The proof depends on a physically reasonable assumption that limits the amount of level attraction in the system. I construct a sequence of local rotations that completely diagonalizes the Hamiltonian and exhibits the local degrees of freedom.

On Many-Body Localization for Quantum Spin Chains

John Imbrie

May 14, 2014

Overview

A one-dimensional spin chain with random local interactions exhibits many-body localization. The proof depends on a physically reasonable assumption that limits the amount of level attraction in the system. The construction uses a sequence of local unitary transformations to diagonalize the Hamiltonian and connect the exact many-body eigenfunctions to the original basis vectors.

See arxiv:1403.7837

Many of the ideas were worked out in collaboration with Tom Spencer. Many thanks to David Huse, who encouraged us to work on this problem (in 2010!).

Does Many-Body Localization really occur?

Does the phenomenon of localization persist in a more realistic model with interacting particles? This question was raised in Anderson's original 1958 paper, and subsequent work in the physics literature supports the idea. (Fleishman, Anderson 1980, Giamarchi, Schulz 1987, Santos Rigolin, Escobar 2004, Gornyi, Mirlin, Polyakov 2005, Basko, Aleiner, Altschuler 2006, Znidaric, Prosen, Prelovsek 2008, Pal & Huse 2010, ...)

My goal here is to discuss a proof of MBL, modulo an assumption on level statistics.

Model

Random field, random transverse field, random exchange Ising model on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^{K} h_i S_i^z + \sum_{i=-K}^{K} \gamma_i S_i^x + \sum_{i=-K-1}^{K} J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable.

Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

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Assumption: Limited level attraction

Assumption **LLA**(ν, c). *Consider the Hamiltonian H in boxes of size n. Its eigenvalues satisfy*

$$P\left(\min_{\alpha\neq\beta}|E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu c^n,$$

for all $\delta > 0$ and all n.

Note 1: One expects Poisson statistics (ν=1) or repulsive statistics ($\nu > 1$, like GOE), but not level attraction ($\nu < 1$).

Note 2: **LLA**(ν, c) is needed only for a single value of δ, exponentially small in n.

Results

Assume **LLA**(ν, c). Then MBL holds as follows:

(i) Existence of a labeling system for eigenstates by spin/metaspin configurations, with metaspins needed only on a dilute collection of resonant blocks. (Spin variables used to label basis vectors can also be used to label the exact eigenstates, but the correspondence becomes somewhat arbitrary in resonant regions, so we use the term "metaspin" instead.)

(ii) Faster-than-power-law decay of the probability of resonant blocks, which implies their diluteness. (This is critical to the whole concept of a labeling system—without it the labeling system would lose its meaning.)

(iii) Diagonalization of H via a sequence of local rotations defined via convergent graphical expansions with exponential bounds. (Locality means that graphs depend only on the random variables in their immediate vicinity.)

(iv) Bounds establishing closeness of expectations of local observables in any eigenstate to their naïve ($\gamma = 0$) values, when observables are not in resonant regions. (This makes precise the idea that eigenstates resemble the basis vectors.)

(v) Almost sure convergence of local energy differences and expectations of local observables as $\Lambda \longrightarrow \mathbb{Z}$.

(vi) Exponential decay of truncated expectations, except on a set of rapidly decaying probability. (This shows the exponential loss of entanglement with distance for the subsystems associated with the observables.)

(vii) Faster-than-power-law decay of averaged truncated expectations.

Theorem

(c.f. (iv),(vi),(vii) above) Assume **LLA(ν,c)** *for some fixed ν, c. Then there exists a $\kappa > 0$ such that for γ sufficiently small,*

$$\mathbb{E}\,\mathrm{Av}_\alpha\,|\langle S_0^z\rangle_\alpha| = 1 - O(\gamma^\kappa), \qquad (1)$$

where $\langle\cdot\rangle_\alpha$ denotes the expectation in the eigenstate α, and Av_α denotes an average over α. Furthermore, for any i, j,

$$\max_\alpha |\langle \mathcal{O}_i; \mathcal{O}_j\rangle_\alpha| \le \gamma^{|i-j|/3} \text{ with probability } 1-(\gamma^\kappa)^{1+c_3(\log(|i-j|/8\vee 1))^2},$$

for some constant $c_3 > 0$. Here $\langle \mathcal{O}_i; \mathcal{O}_j\rangle_\alpha \equiv \langle \mathcal{O}_i\mathcal{O}_j\rangle_\alpha - \langle \mathcal{O}_i\rangle_\alpha\langle \mathcal{O}_j\rangle_\alpha$, with $\mathcal{O}_i$ any operator formed from products of $S^x_{i'}$ or $S^z_{i'}$, for i' near i. Finally,

$$\mathbb{E}\,\mathrm{Av}_\alpha|\langle \mathcal{O}_i; \mathcal{O}_j\rangle_\alpha| \le (\gamma^\kappa)^{1+c_3(\log(|i-j|/8\vee 1))^2}.$$

All bounds are uniform in Λ.

No thermalization

Consider infinite temperature, so Av_α is a uniform weighting of eigenstates. Then with thermalization,[1] averages of eigenstate expectations of S_0^z should go to zero as $\Lambda \longrightarrow \infty$. This would contradict (1) above.

Another consequence of (iv) is that essentially all of the eigenstates have a nonuniform spatial distribution of energy, which persists for all time. So in a basic sense, there is no transport in the system.

[1] Eigenstate Thermalization Hypothesis: Deutsch 1991, Srednicki 1994

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KAM Method

We obtain a complete diagonalization of the H by successively eliminating low-order off-diagonal terms. The process runs on a sequence of length scales $L_k = (15/8)^k$, and off-diagonal elements of order γ^m, $m \in [L_k, L_{k+1})$ will be eliminated in the k^{th} step.

The orthogonal rotations that accomplish this can be written as a convergent power series, provided nonresonant conditions are satisfied. Resonant regions are diagonalized as blocks in quasidegenerate perturbation theory.

For 1-body Anderson (new proof of exponential localization): Imbrie, Spencer 2013 (unpublished—useful intro to these ideas)

References

Quasiperiodic 1-body: Bellisard, Lima, Scoppola, Testard 1983, Chulaevsky, Sinai 1991, Eliasson 1997

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Resonances

Perturbation theory works if there are gaps between eigenvalues. Initially, the only off-diagonal term is $\gamma_i S_i^x$, which is local. We only need to worry about single-flip resonances (for the moment). Let

$$\Delta E_i \equiv E(\sigma) - E(\sigma^{(i)}) = 2\sigma_i(h_i + J_i\sigma_{i+1} + J_{i-1}\sigma_{i-1}).$$

We say that the site i is resonant if $|\Delta E_i| < \varepsilon \equiv \gamma^{1/20}$.

Then for nonresonant sites the ratio $\gamma_i/\Delta E_i$ is $\leq \gamma^{19/20}$.

Resonant sites have probability ε and so form a dilute set of regions where perturbation theory breaks down.

Observe that $[A, H_0] = -J^{\mathrm{per}}$:

$$[A, H_0]_{ij} = \frac{J^{\mathrm{per}}_{ij} E_j - E_i J^{\mathrm{per}}_{ij}}{E_i - E_j} = -J^{\mathrm{per}}_{ij}.$$

Then $[A, H] = -J^{\mathrm{per}} + [A, J]$, and so

$$\begin{aligned} H^{(1)} &= e^{A} H e^{-A} = H + [A, H] + \frac{[A, [A, H]]}{2!} + \dots \\ &= H_0 + J^{\mathrm{res}} + J^{\mathrm{per}} - J^{\mathrm{per}} + [A, J] + \frac{[A, -J^{\mathrm{per}} + [A, J]]}{2!} + \dots \\ &= H_0 + J^{\mathrm{res}} + \frac{1}{2}[A, J^{\mathrm{per}}] + [A, J^{\mathrm{res}}] + \dots \\ &= H_0 + J^{\mathrm{res}} + J^{(1)}. \end{aligned}$$

Reference: Datta, Fernandez, Fröhlich, 1996, 1999

Perturbation Theory

Let $H = H_0 + \mathcal{J}$ with H_0 diagonal and $\mathcal{J}$ off-diagonal. Put

$$\mathcal{J} = J^{\mathrm{res}} + J^{\mathrm{per}}$$

where J^{res} contains terms internal to resonant blocks. Then put

$$A_{\sigma\tilde{\sigma}} = \frac{J^{\mathrm{per}}_{\sigma\tilde{\sigma}}}{E_\sigma - E_{\tilde{\sigma}}}.$$

First order perturbation theory can be implemented by using e^{-A} for a basis change (preserves norm).

Renormalized Hamiltonian:

$$H^{(1)} = \Omega^{\mathrm{tr}} H \Omega$$

Properties of new Hamiltonian:

After the change of basis:

J^{per} is gone

J^{res} is still there

$J^{(1)}$ is quadratic and higher order in γ, containing terms of the form $(\mathrm{Ad}A)^n J$, with $n = 1, 2, \ldots$, where $(\mathrm{Ad}A)B \equiv [A, B]$.

$A(i)$ commutes with $A(j)$ or $J(j)$ if $|i - j| > 1$.

$J^{(1)}$ is now long-range: it has a random walk expansion exhibiting exponential decay away from the resonant blocks.

We also perform exact rotations O in "small" resonant blocks to diagonalize the Hamiltonian there.

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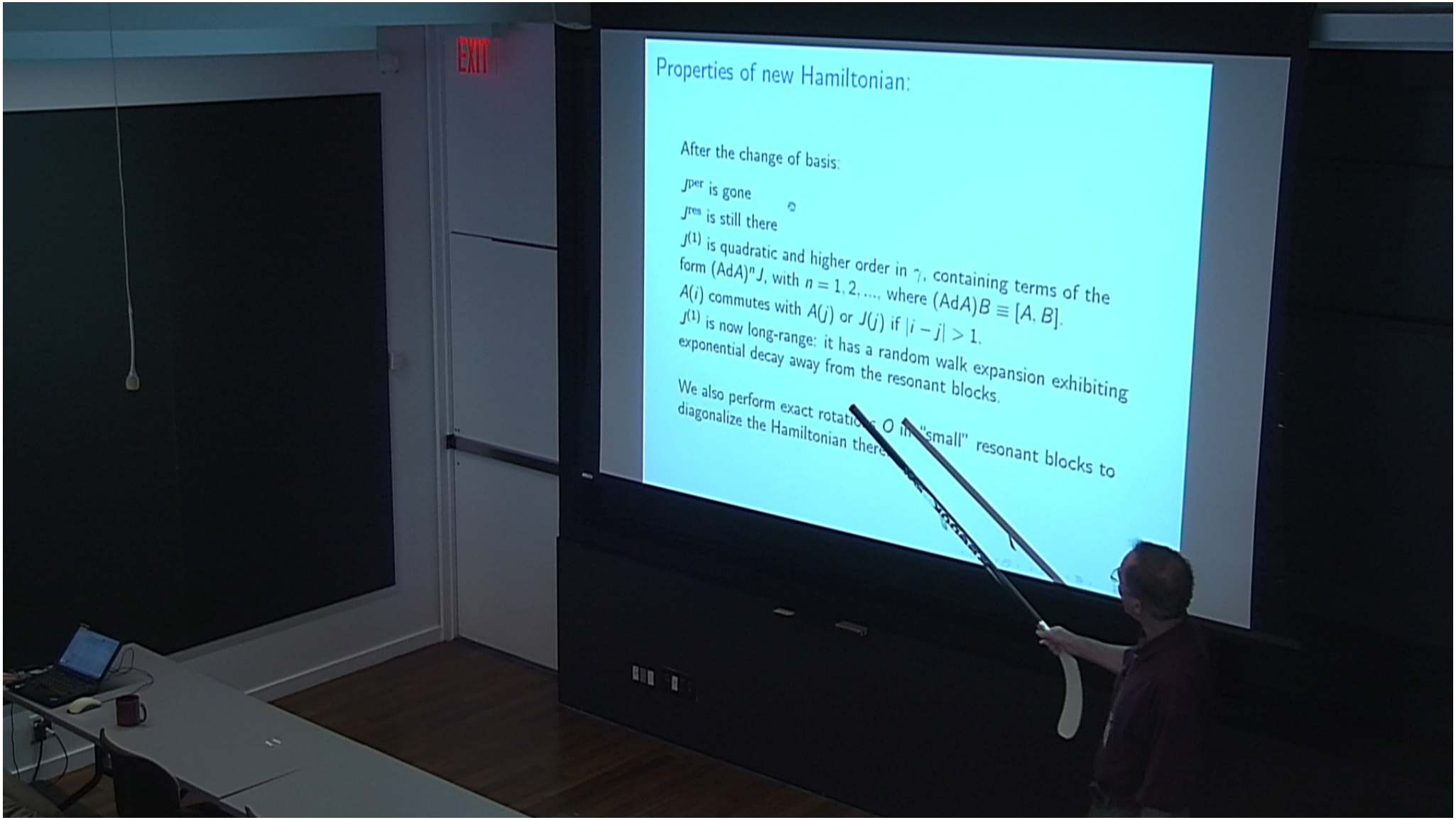
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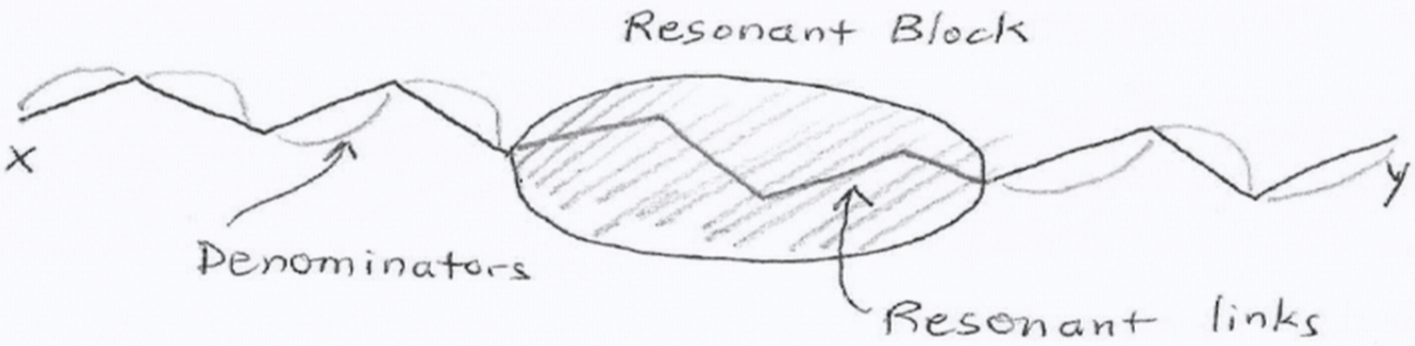
Exponential decay

Estimates on decay of correlations follow similarly.

We get exponential decay either from the small probability of resonances or from the random walk expansion in $\Omega = e^{-A}$.



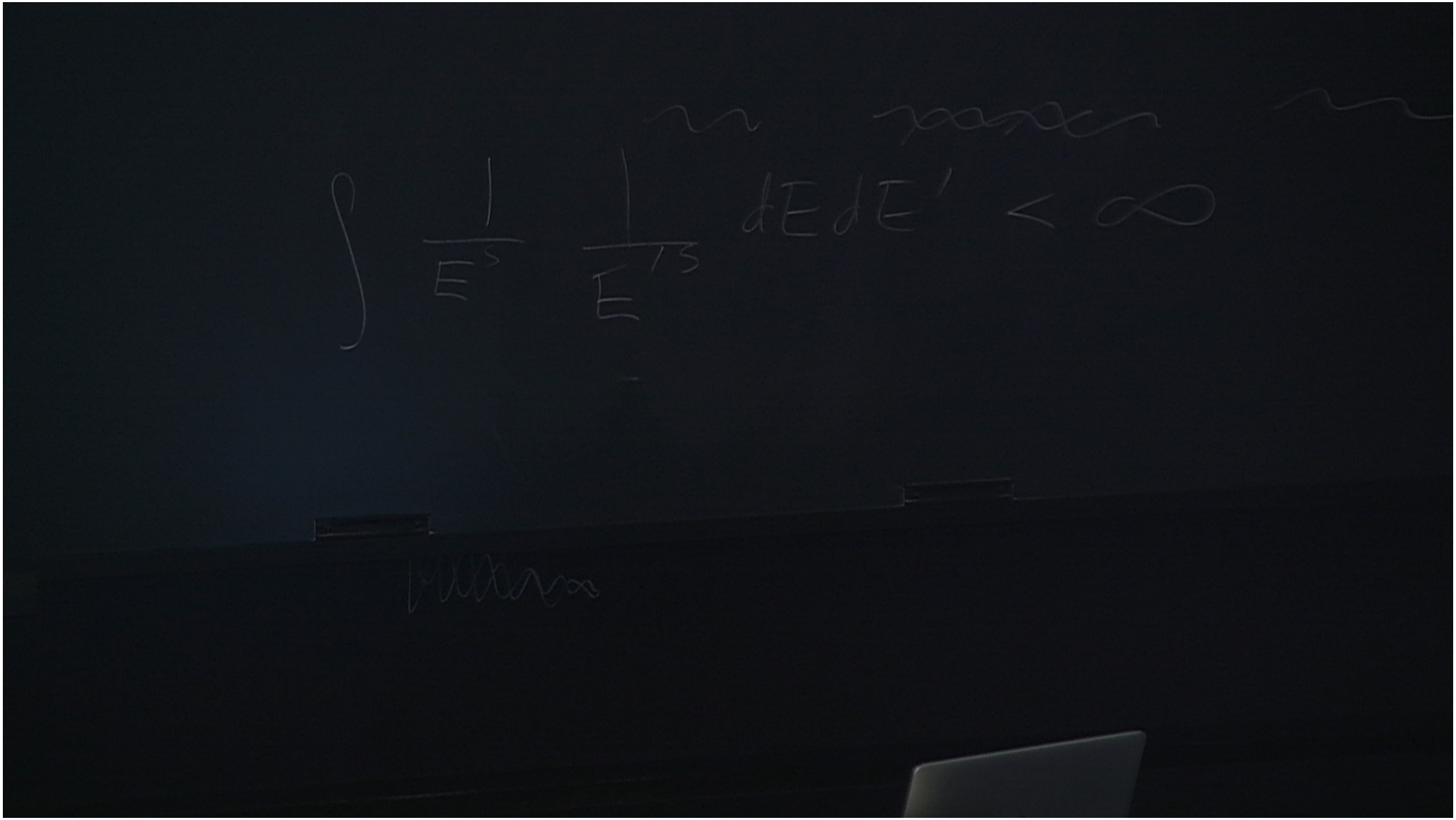
Continue the process on a sequence of length scales $L_k = (15/8)^k$

$J^{(k)}$ is a sum of graphs $J^{(k)}_{\sigma\tilde{\sigma}}(g)$

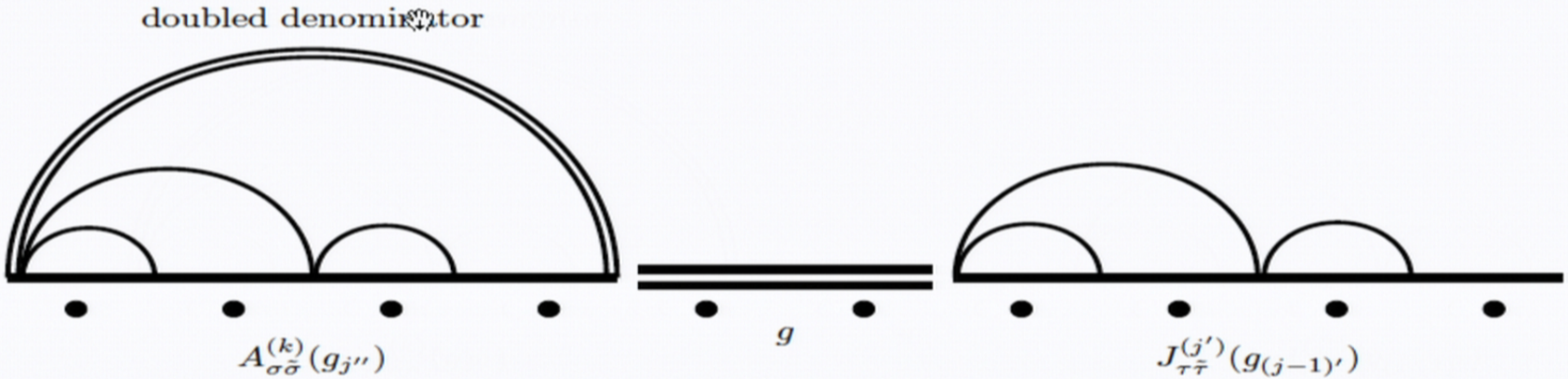
g is resonant if $A^{(k)}_{\sigma\tilde{\sigma}}(g) \equiv \frac{J^{(k)}_{xy}(g)}{E^{(k)}_{\sigma} - E^{(k)}_{\tilde{\sigma}}}$ is larger than $(\gamma/\varepsilon)^{|g|}$.

Energies $E^{(k)}$ are the diagonal elements of $H^{(k)}$: they are renormalized by interactions up to the k^{th} scale.

We need to maintain uniform exponential decay on the probability that g is resonant, because we need to check an exponential number of pairs of states for resonance.


$\int \frac{1}{E^5} \frac{1}{E^{13}} dE dE' < \infty$

Diagrams on the k^{th} scale



Loops

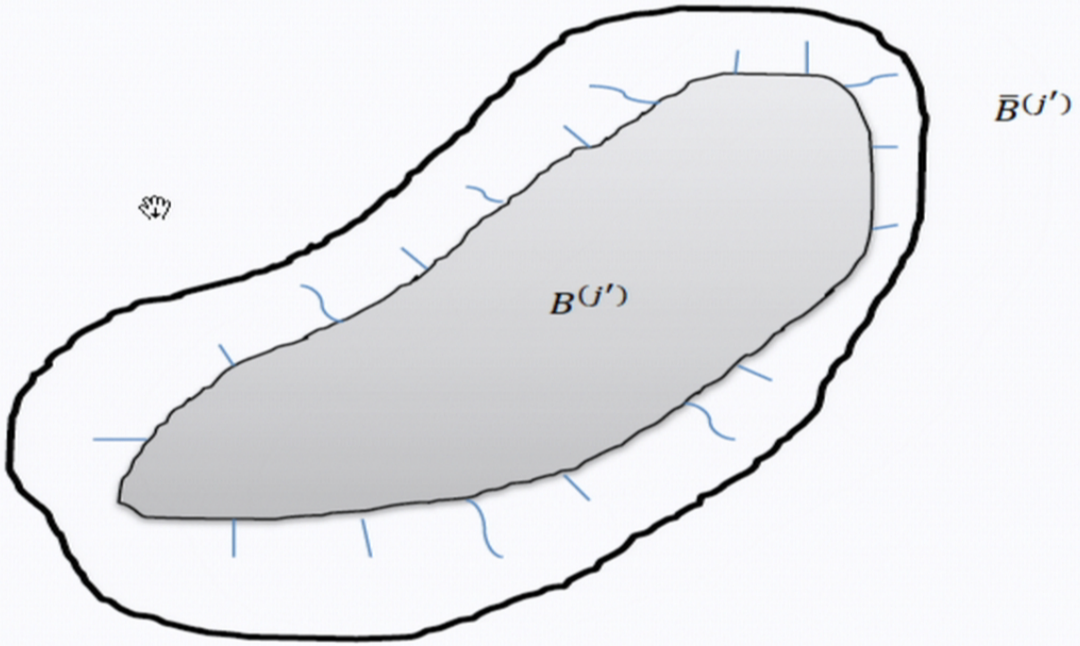
If g returns to previously visited sites, then the denominator graph has duplicated or non-independent denominators. We have integrals like $\int_{|h|\geq\varepsilon} \frac{1}{h^2} d\lambda(h) \leq \varepsilon^{-1}$, which lead to negative powers of ε. This weakens the bound on the probability that the graph is resonant.

If a graph is not too much longer than the distance traveled, it will have at least $1/4$ of its length free of looping problems, and then

$$P\left(|A(g)| \geq (\gamma/\varepsilon)^{|g|}\right) \leq \varepsilon^{s|g|/4}.$$

Graphs that are quite a bit longer than the distance traveled need to be resummed, but the extra links ensure exponential decay in the distance (probabilistic bounds not needed). Resummation leads to slower than quadratic convergence: $L_k = (15/8)^k$.

Large Blocks and Number Divergences



Large blocks need a collar (width proportional to the length) so that they interact only through diagrams of sufficiently high order. Exponential decay of diagrams beats the exponential number of states in the block.

Block-Block Resonances

How to control the probability of resonance between blocks?

Can one show that energy differences within a block vary with the randomness?

This is still an open problem. But with Assumption **LLA**(ν, c), we have a minimum level spacing, with high probability. Then since all three terms of H are random, one can show that energy differences do vary with the randomness.

Summary:
Dealing with Divergences of Perturbation Theory

1. Resonances:

Cutoff $\Delta E > \varepsilon^{L_k}$ tied to a sequence of length scales

2. Sum over scales:

a. Exponential length scales implies interactions $\sim \gamma^{L_k} = \gamma^{(15/8)^k}$ (KAM or Newton's method scaling) controls combinatorics of repeatedly inserting one expansion into another

b. Bounds uniform in k from expectations of fractional moments

c. Loops: Diagrams with too many returns spoil (b), but provide bonus decay, which then prevents the degeneration of decay

3. Number divergences from resonant regions:

a. Collar ensures that they interact only at sufficiently high order

b. Must ensure resonant regions do not "percolate"