Title: Curvature in Noncommutative Geometry

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Abstract: After the seminal work of Connes and Tretkoff on the Gauss-Bonnet theorem for the noncommutative 2-torus and its extension by Fathizadeh and myself, there have been significant developments in understanding the local differential geometry of these noncommutative spaces equipped with curved metrics. In this talk, I will review a series of joint works with Farzad Fathizadeh in which we compute the scalar curvature for curved noncommutative tori and prove the analogue of Weyl's law and Connes' trace theorem. Our final formula for the curvature matches precisely with the one computed independently by A. Connes and H. Moscovici. I will then report on our recent work on the computation of scalar curvature for noncommutative 4-tori (which involves intricacies due to violation of the KÃ¤hler condition). We show that metrics with constant curvature are extrema of the analogue of the Einsteinâ€“Hilbert action. A purely noncommutative feature in these works is the appearance of the modular automorphism from Tomitaâ€“Takesaki theory of KMS states in the final formulas for the curvature.

Scalar Curvature, Gauss-Bonnet Theorem and Einstein-Hilbert Action for Noncommutative Tori

Masoud Khalkhali
joint work with Farzad Fathizadeh

Perimeter Institute, May 2014

What is curvature?

Classical geometry: R^i_{jkl}, R_{ij}, R.

Einstein-Hilbert action: $\int_M R\, dvol$.

Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$

Chern-Weil theory: $\mathrm{Tr}(e^{\Omega})$ $\Omega^i_j = R_{ijkl} dx^k \wedge dx^l$.

Curvature in NCG

Connection-Curvature formalism of Connes in 1981 (NC Chern-Weil theory):

$$\nabla : E \to E \otimes_A \Omega^1 A, \qquad \nabla \in \mathrm{End}_{\mathbb{C}}(E \otimes_A \Omega A)$$

$$\nabla^2 \in \mathrm{End}_{\Omega A}(E \otimes_A \Omega A) = \mathrm{End}_A(E) \otimes_A \Omega A.$$

Any cyclic cocycle $\varphi : A^{\otimes(2n+1)} \to \mathbb{C}$ defines a closed graded trace $\int_\varphi : \Omega A \to \mathbb{C}$. Can define $\int_\varphi \mathrm{Tr}(e^\Omega)$, etc. But won't discuss it here.

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How to define the scalar curvature of a spectral triple (A, H, D)?
This is also answered by Connes since late 1980's and is based on ideas of spectral geometry. But computing it in concrete examples is only acheived in the last few years!

A spectral triple is a NC Riemannian manifold. It is tempting to think that one might be able to define a Levi-Civita type connection for a spectral triple and then define the curvature of this connection. For many reasons this does not work in NCG in general.

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Spectral geometry

- (M, g) = closed Riemannian manifold. Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

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Heat trace asymptotics

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t,x,y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t,x,x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x,\triangle)+a_1(x,\triangle)t+a_2(x,\Delta)t^2+\cdots)$$

- $a_i(x,\triangle)$,   Seeley-De Witt-Gilkey coefficients.

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- Theorem: $a_i(x, \triangle)$ are universal polynomials in the curvature tensor $R = R^1_{jkl}$ and its covariant derivatives:

$$
\begin{aligned}
a_0(x,\triangle) &= 1 \\
a_1(x,\triangle) &= \frac{1}{6}S(x) \qquad \text{scalar curvature} \\
a_2(x,\triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
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Spectral Triples

Noncommutative geometric spaces are described by spectral triples:

$$(\mathcal{A}, \mathcal{H}, D),$$

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \qquad (*\text{-representation}),$$

$$D = D^* : Dom(D) \subset \mathcal{H} \to \mathcal{H},$$

$$D\,\pi(a) - \pi(a)\,D \in \mathcal{L}(\mathcal{H}).$$

Examples.

$$\big(C^\infty(M), L^2(M,S), D = \text{Dirac operator}\big).$$

$$\Big(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), \frac{1}{i}\frac{\partial}{\partial x}\Big).$$

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Noncommutative 2-Torus $A_\theta = C(\mathbb{T}^2_\theta)$

It is the universal C^*-algebra generated by U and V s.t.

$$U^* = U^{-1},$$

$$V^* = V^{-1},$$

$$VU = e^{2\pi i\theta} UV,$$

where $\theta \in \mathbb{R}$ is fixed.

The geometry of the Kronecker foliation $dy = \theta dx$ on the ordinary torus $\mathbb{R}^2/\mathbb{Z}^2$ is closely related to the structure of this algebra.

A representation of A_θ:

$$U\xi(x) = e^{2\pi i x}\xi(x), \qquad V\xi(x) = \xi(x+\theta), \qquad \xi \in L^2(\mathbb{R}).$$

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Action of $\mathbb{T}^2 = (\frac{\mathbb{R}}{2\pi\mathbb{Z}})^2$ on A_θ and Smooth Elements

- $$\alpha_s : A_\theta \to A_\theta, \quad s \in \mathbb{R}^2,$$
$$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n, \qquad m, n \in \mathbb{Z}.$$

- $$A_\theta^\infty := \{a \in A_\theta; \qquad s \mapsto \alpha_s(a) \text{ is smooth from } \mathbb{R}^2 \text{ to } A_\theta\}$$
$$= \Big\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \in A_\theta; \quad (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \Big\}.$$

- $$\delta_j = \frac{\partial}{\partial s_j} \Big|_{s=0} \alpha_s \;: A_\theta^\infty \to A_\theta^\infty.$$

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Conformal Structure on A_θ (Connes)

The Dolbeault operators associated with $\tau \in \mathbb{C}, \Im(\tau) > 0$ are

$$\partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)},$$
$$\bar{\partial} = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(0,1)}.$$

The conformal structure represented by τ is encoded in

$$\psi(a,b,c) = -\varphi_0\big(a\,\partial(b)\,\bar{\partial}(c)\big), \qquad a,b,c \in A_\theta^\infty,$$

which is a positive Hochschild cocycle.

A Spectral Triple $(A_\theta^\infty, \mathcal{H}, D)$

$$\mathcal{H} := \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

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Conformal Geometry of $\mathbb{T}^2_\theta$ with $\tau = i$ (Cohen-Connes, late 80's)

Let

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \qquad \text{be the eigenvalues of } \partial_\varphi^* \partial_\varphi,$$

and

$$\zeta(s) = \sum \lambda_j^{-s}, \qquad \Re(s) > 1.$$

Then

$$\zeta(0) + 1 =$$

$$\varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big),$$

where

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1+u^{1/2})\mathcal{L}_2(u) + (1+u^{1/2})^2\mathcal{L}_3(u),$$

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Scalar Curvature for $(A_\theta^\infty, \mathcal{H}, D)$

It is the unique element $R \in A_\theta^\infty$ such that

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$$\zeta_a(s) := \text{Trace}(a\, |D|^{-2s}), \qquad \text{Re}(s) \gg 0.$$

Equivalently, consider small-time heat kernel expansions:

$$\text{Trace}(a\, e^{-tD^2}) \sim \sum_{n \geq 0} B_n(a, D^2)\, t^{\frac{n-2}{2}}, \qquad a \in A_\theta^\infty.$$

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Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^2$

Theorem. (Connes-Moscovici; Fathizadeh-Kh.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, R is equal to

$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$

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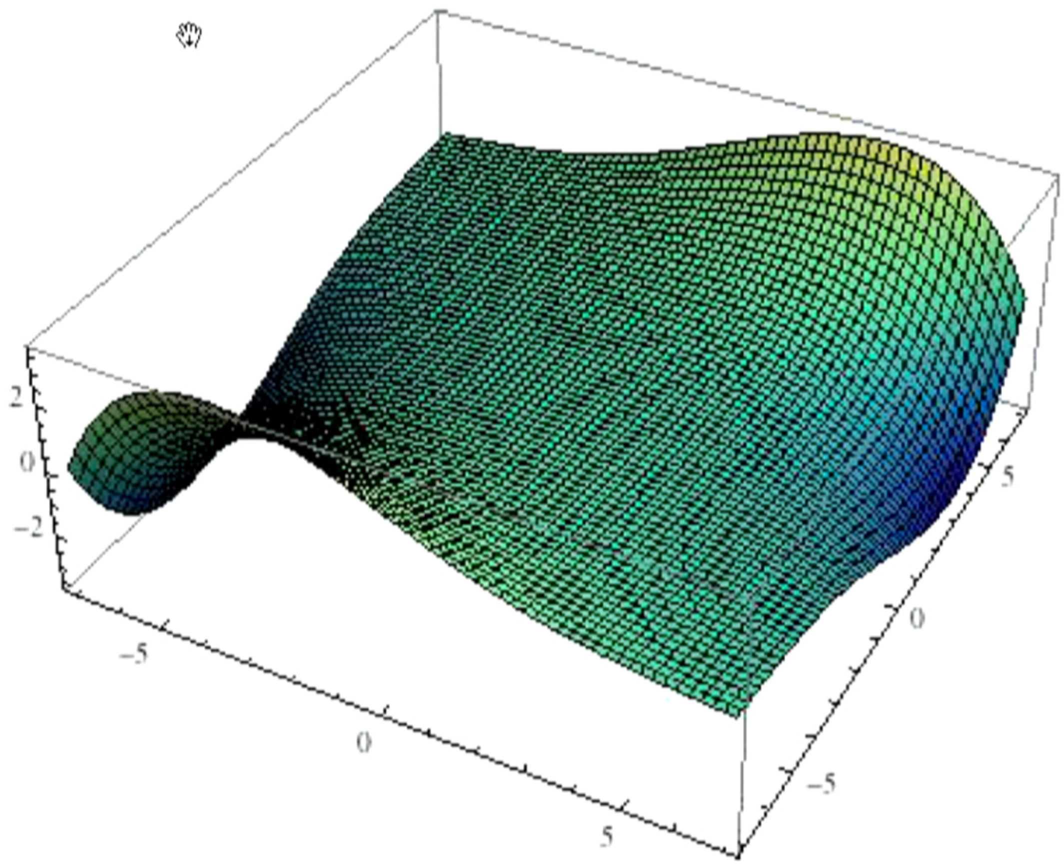
$R_2(s,t) =$

$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



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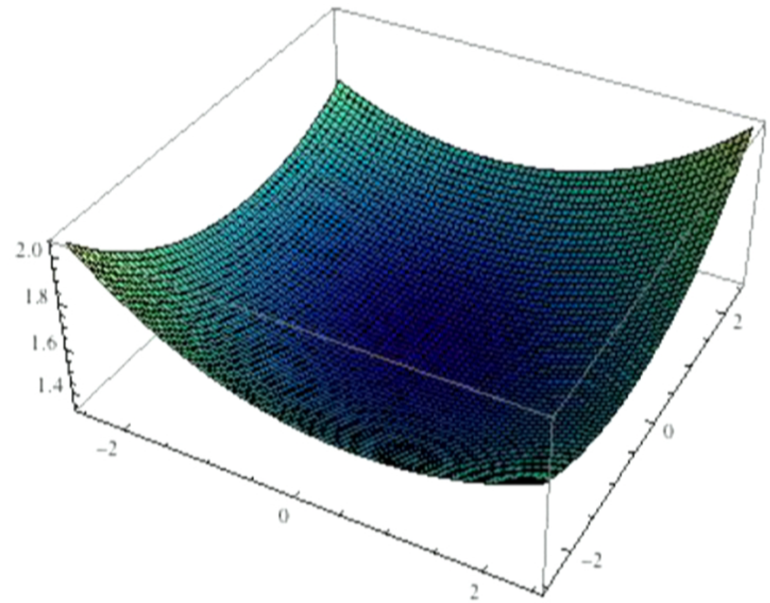
$$W(s,t) = \frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^2$

Theorem. (Connes-Moscovici; Fathizadeh-Kh.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, R is equal to

$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$

$$+R_2(\nabla,\nabla)\Big(\delta_1(\frac{h}{2})^2 + |\tau|^2\,\delta_2(\frac{h}{2})^2 + \Re(\tau)\,\big\{\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})\big\}\Big)$$

$$+i\,W(\nabla,\nabla)\Big(\Im(\tau)\,[\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})]\Big).$$

Noncommutative 4-Torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal C^*-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

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Action of $\mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ on $C(\mathbb{T}^4_\theta)$

$$\mathbb{R}^4 \ni s \mapsto \alpha_s \in \mathsf{Aut}\Big(C(\mathbb{T}^4_\theta)\Big),$$

$$\alpha_s(U^m) := e^{is\cdot m}\, U^m, \quad U^m := U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}, \quad m_j \in \mathbb{Z}.$$

$$\delta_j = \frac{\partial}{\partial s_j}\Big|_{s=0} \alpha_s \; : C^\infty(\mathbb{T}^4_\theta) \to C^\infty(\mathbb{T}^4_\theta),$$

$$\begin{aligned} \delta_j(U_k) &:= U_k \qquad \text{if} \qquad k = j, \\ &:= 0 \qquad \text{if} \qquad k \neq j. \end{aligned}$$

Scalar Curvature for $\mathbb{T}_\theta^4$

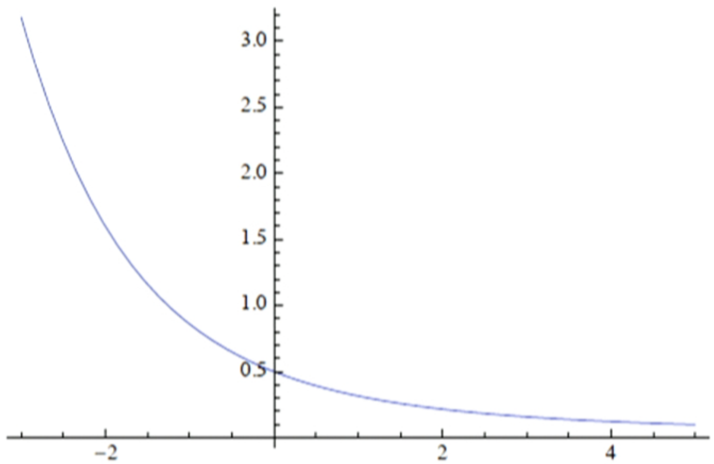
It is the unique element $R \in C^\infty(\mathbb{T}_\theta^4)$ such that

$$\mathrm{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\, R), \qquad a \in C^\infty(\mathbb{T}_\theta^4),$$

where

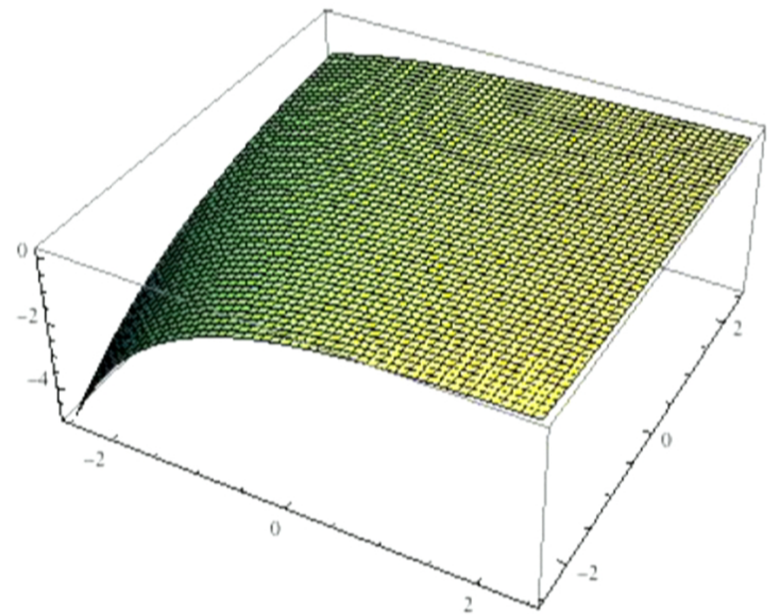
$$\zeta_a(s) := \mathrm{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



$$H(s,t) = \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^3\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^2}{80}+O\left(t^3\right)\right)$$

$$+s^2\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^2}{144}+O\left(t^3\right)\right)+O\left(s^3\right).$$



Einstein-Hilbert Action for $\mathbb{T}^4_\theta$

Theorem. (Fathizadeh-Kh.) We have the local expression (up to a factor of π^2)

$$\varphi_0(R) = \frac{1}{2}\sum_{i=1}^{4}\varphi_0\Big(e^{-h}\delta_i^2(h)\Big) + \sum_{i=1}^{4}\varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\,\delta_i(h)\Big).$$

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