Title: On the role of the extra kinetic term coupling Æ› in classical HoÅ™ava-Lifshitz gravity

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Abstract: We study the classical constraint algebra of HoÅ™ava-Lifshitz gravity, where due to the breaking of 4d diffeomorphism symmetry, there is a new dimensionless coupling absent in GR and whose role is not yet clear. Starting from two apparently contradictory results, we show how the role of the extra coupling differs between the projectable and non-projectable versions of the theory. In particular, we see how in the latter, it gives rise to a non-trivial constraint algebra, akin to the conditions seen in the CMC gauge of GR.

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On the role of the extra kinetic term coupling λ in classical Hořava-Lifshitz gravity

Luís Pires

IMAPP, Radboud University, Nijmegen, Netherlands

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Hořava-Lifshitz gravity[1] (HLG) is a proposal for a theory of quantum gravity (QG)

- Usual metric degrees of freedom,
- Standard quantum field theory methods,
  - Unitarity (not more than second order time derivatives),
  - Perturbative renormalizability (absent in GR).

[1] P. Hořava: *Quantum gravity at a Lifshitz point*, arXiv:0901.3775v2 [hep-th];

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Outline

1. Introduction
   - Hořava-Lifshitz gravity
   - Hamiltonian formulation of GR
2. Hamiltonian formulation of λ-R models
   - Non-projectable version
   - Projectable version
3. Conclusion

Hořava-Lifshitz gravity I - Anisotropic scaling and foliation

UV fixed point with anisotropic scaling

- FP where solutions of the theory respect anisotropic scaling

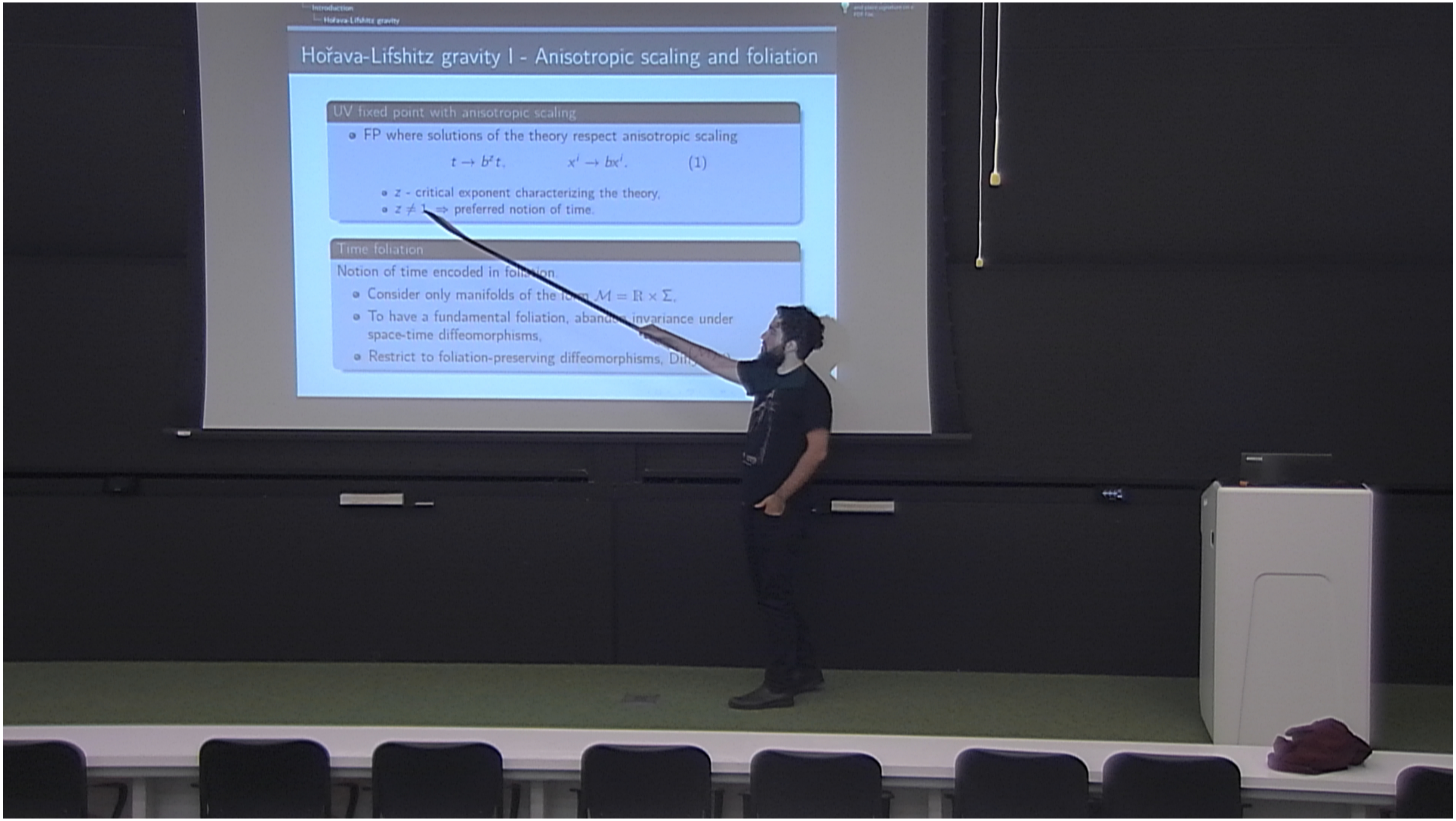
$$t \to b^z t, \qquad x^i \to b x^i. \tag{1}$$

  - z - critical exponent characterizing the theory,
  - $z \neq 1 \Rightarrow$ preferred notion of time.

Time foliation

Notion of time encoded in foliation.

- Consider only manifolds of the form $\mathcal{M} = \mathbb{R} \times \Sigma$,
- To have a fundamental foliation, abandon invariance under space-time diffeomorphisms,
- Restrict to foliation-preserving diffeomorphisms, $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$.



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HLG IV - Potential term

Potential term of the action

The general form of the action is

$$S = S_K + S_V = S_K + \int dt \int d^3x \sqrt{g}\, N\mathcal{V}(g_{ij}, N)$$

- $\mathcal{V}(g_{ij}, N)$ contains all spatial invariants with dimension equal to or lower than the kinetic term.
- Since $[K^2] = 2d$, up to 6th order spatial derivatives are allowed.

Invariants in 3 + 1 dimensions

Here's a list of terms that should be included [a]

- 6th order in spatial derivatives:
  - $R^3, \quad RR_{ij}R^{ij}, \quad R^i{}_j R^j{}_k R^k{}_i, \quad R\nabla^2 R, \quad \nabla_k R_{ij} \nabla^k R^{ij}$
- 4th, 2nd and 0th order:
  - $R_{ij}R^{ij}, R^2, \qquad R, \qquad 1$

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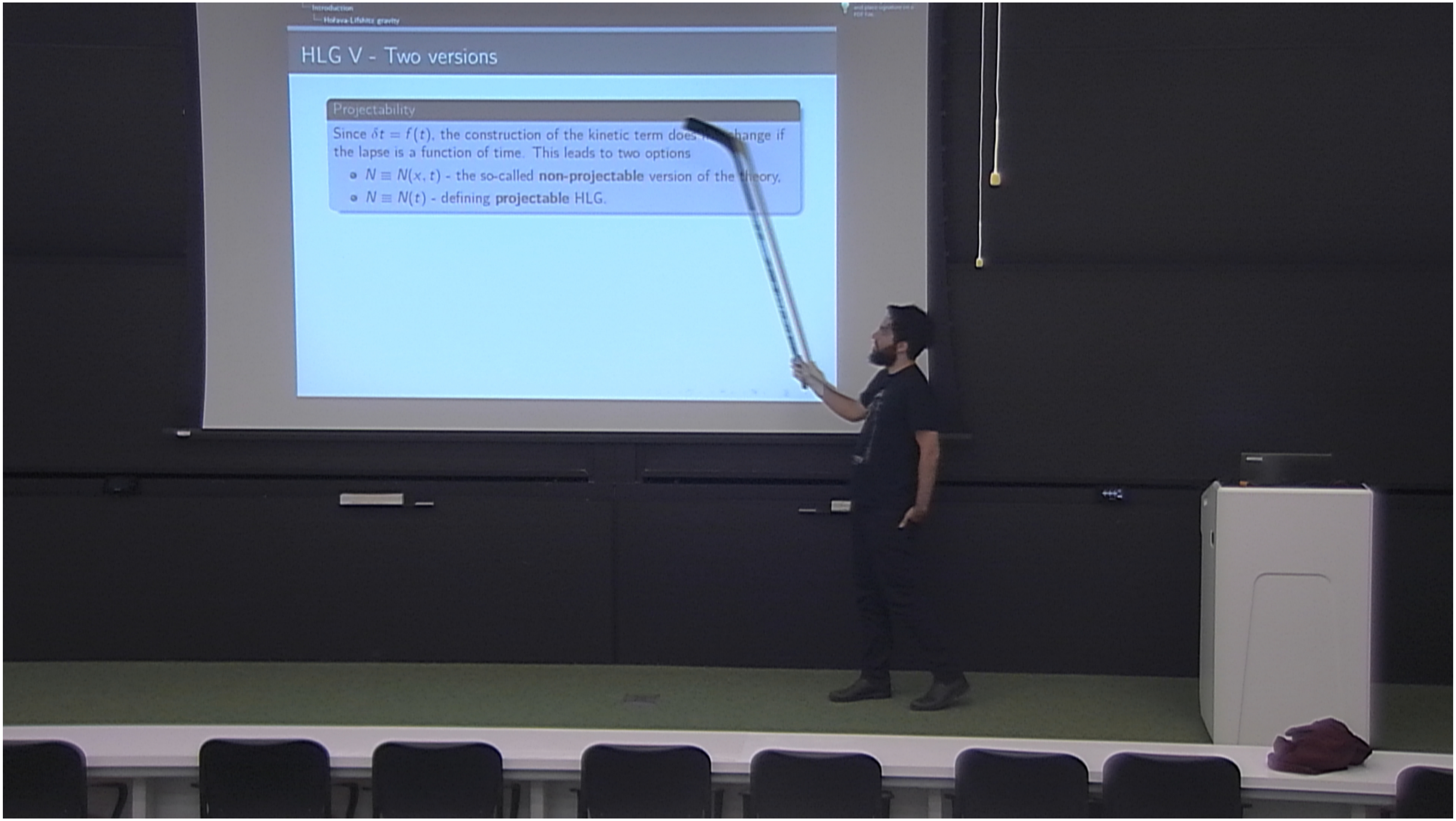
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Hořava-Lifshitz gravity
HLG V - Two versions
Projectability
Since δt = f(t), the construction of the kinetic term does
the lapse is a function of time. This leads to two options
N ≡ N(x, t) - the so-called non-projectable version of the theory.
N ≡ N(t) - defining projectable HLG.

HLG V - Two versions

Projectability

Since $\delta t = f(t)$, the construction of the kinetic term does not change if the lapse is a function of time. This leads to two options

- $N \equiv N(x, t)$ - the so-called **non-projectable** version of the theory,
- $N \equiv N(t)$ - defining **projectable** HLG.

Generalized Wheeler-DeWitt metric

The action we study, which will be referred to as λ-R model can be written as

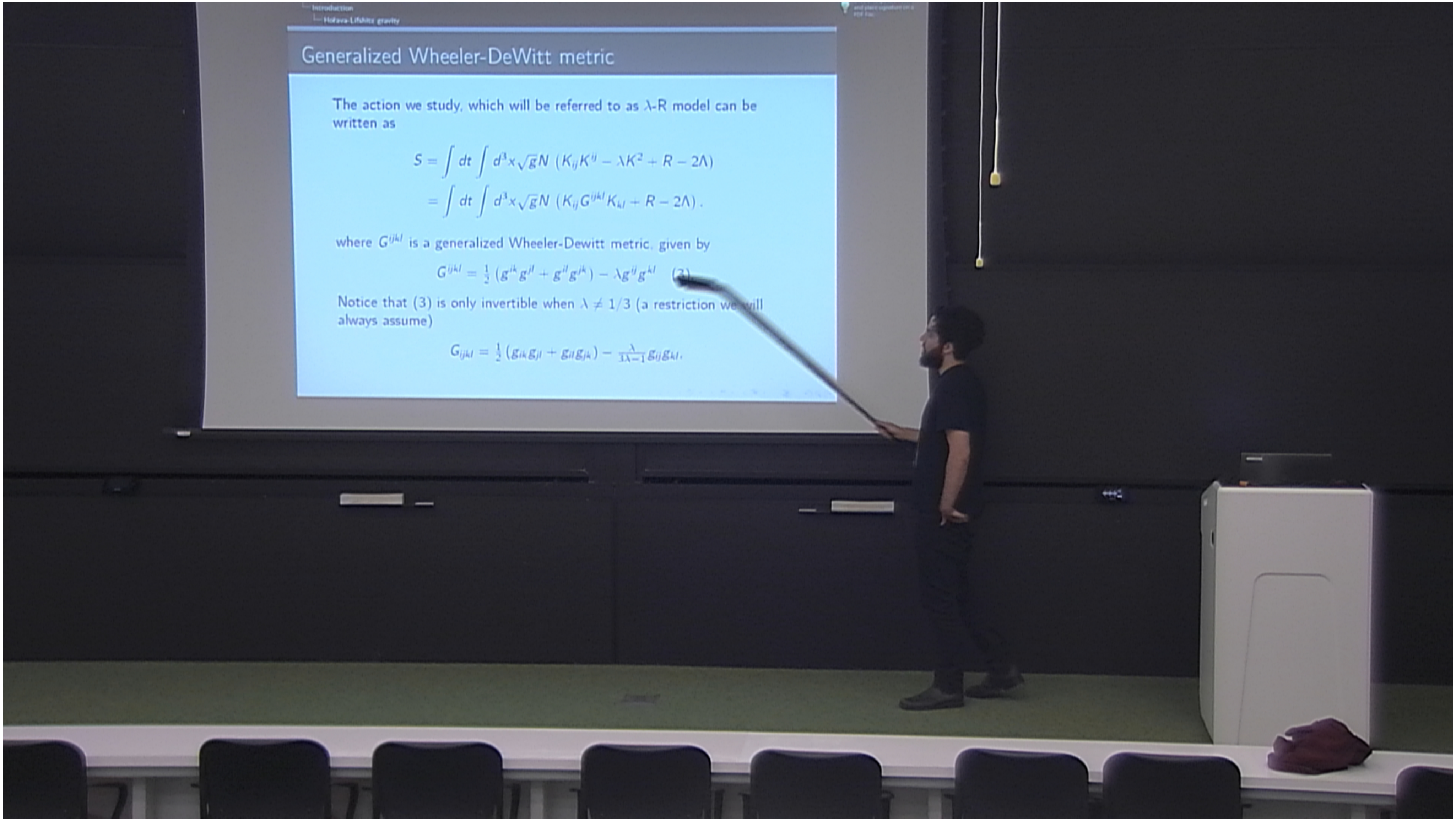
$$S = \int dt \int d^3x \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 + R - 2\Lambda \right)$$
$$= \int dt \int d^3x \sqrt{g} N \left(K_{ij} G^{ijkl} K_{kl} + R - 2\Lambda \right),$$

where G^{ijkl} is a generalized Wheeler-Dewitt metric, given by

$$G^{ijkl} = \frac{1}{2} \left(g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl} \quad (3),$$

Notice that (3) is only invertible when $\lambda \neq 1/3$ (a restriction we will always assume)

$$G_{ijkl} = \frac{1}{2} \left(g_{ik} g_{jl} + g_{il} g_{jk} \right) - \frac{\lambda}{3\lambda - 1} g_{ij} g_{kl},$$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An apparent contradiction in the existing literature

λ implies different physics

- In 90's, Giulini and Kiefer[a], studied this model.
  - Computed the acceleration of the three-volume and showed it to change sign when $\lambda < 1/3$,
    $$\ddot{V} = -\frac{2}{3\lambda - 1}\int d^3x\sqrt{g}\,(R - 3\Lambda)$$
  - In the same context, obtained stronger cosmological constraints on λ.

[a] D. Giulini, and C. Kiefer: *Wheeler-DeWitt metric and the attractivity of gravity*, http://arxiv.org/abs/gr-qc/9405040.

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- Recently, Bellorin and Restuccia [a] worked out the Hamiltonian formulation of the non-projectable theory.
  - For asymptotically flat Σ, λ induces a tertiary constraint ($K = 0$),
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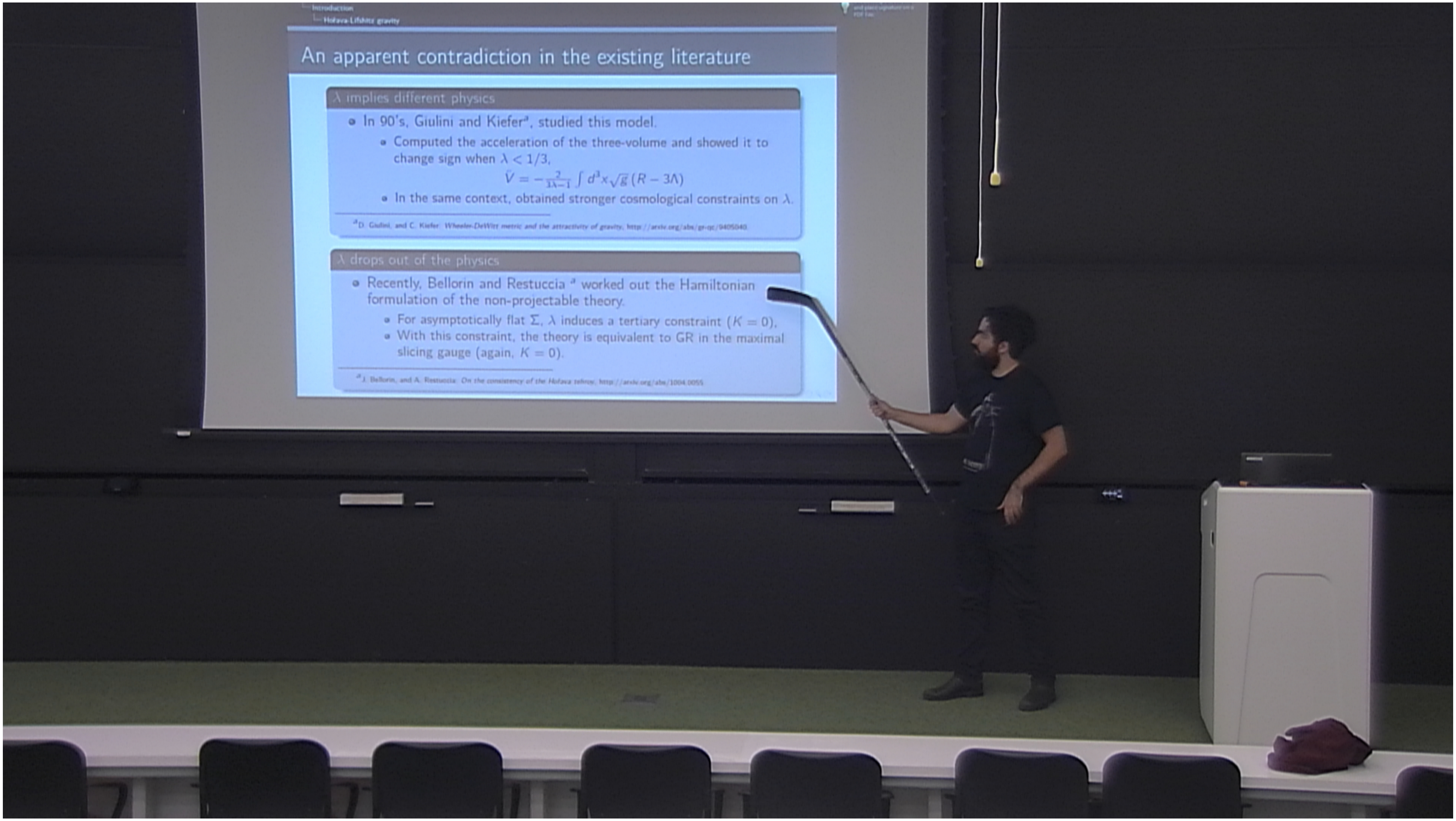

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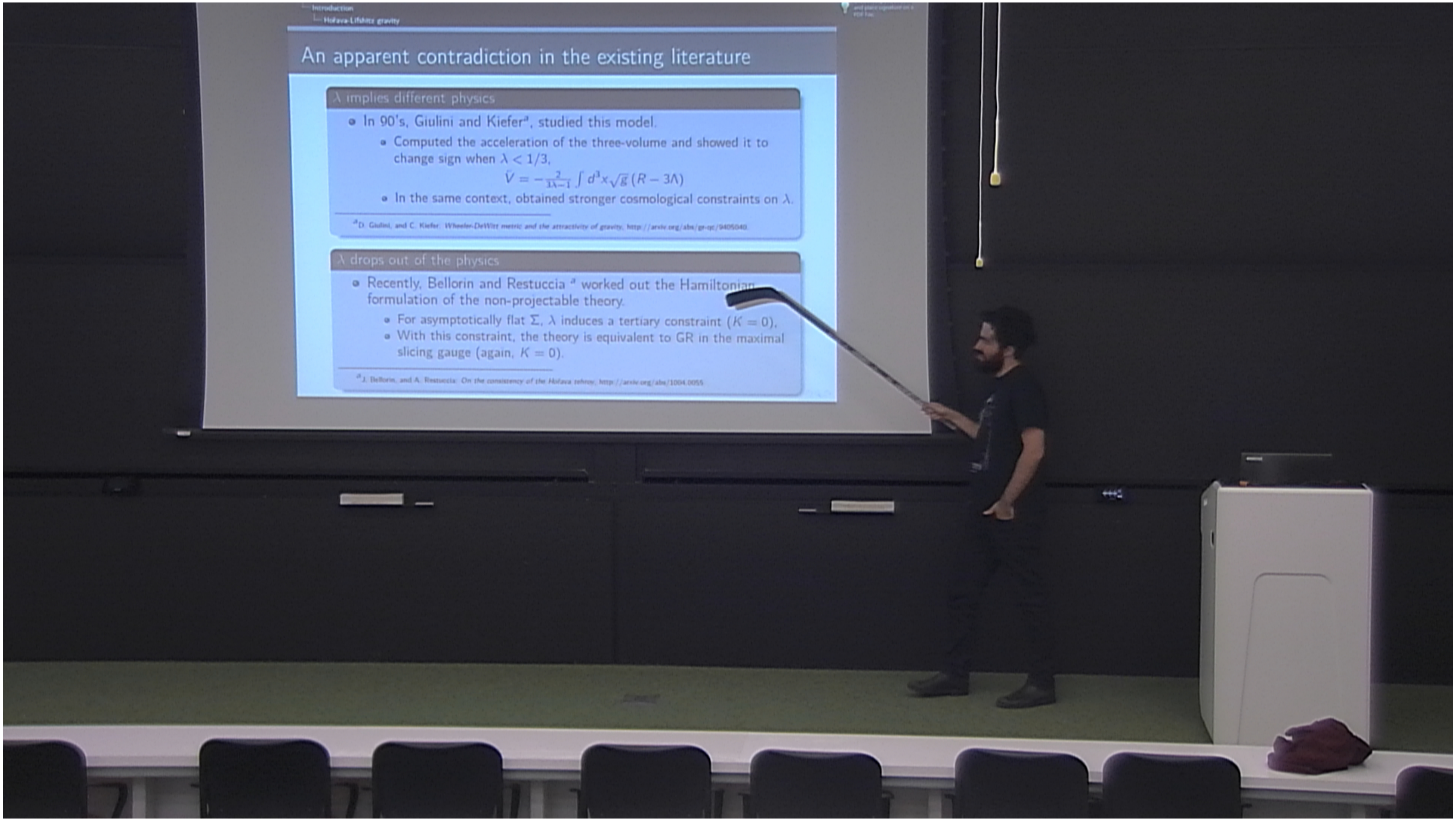
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Constrained systems I

To keep this quick, consider only classical mechanics.

- Take some Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$,
- Define momenta $p^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$.
- Ideally, invert the relation and write $\dot{q}^i\left(q_j, p^j\right)$.
  - Not fully possible when the Hessian, $W_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}$ is not invertible.
  - The dependent relations form the primary constraints of the theory, $\phi_i = 0$.
- Taking into account the invertible relations and the constraints, write the total Hamiltonian

$$H_t = p^i \dot{q}_i(q, p) - \mathcal{L}(q, p) + \alpha^i \phi_i,$$

where the α^j are Lagrange multipliers.

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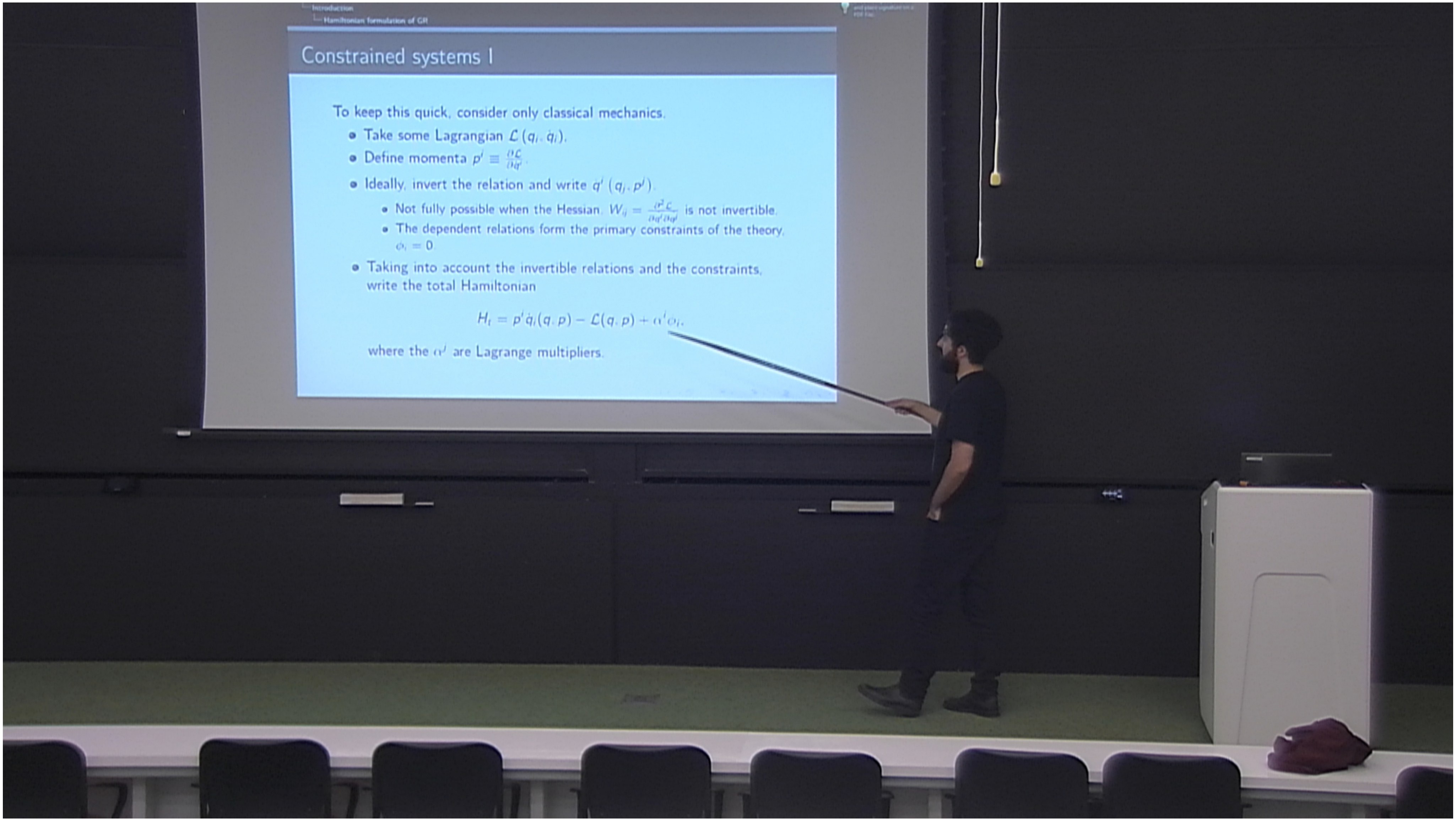
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Constrained systems II

- Using the Poisson bracket structure, impose $\dot{\phi}_i = \{\phi_i, H_t\} \approx 0$:
  - If it is trivial, no further steps are required.
  - If it imposes a condition on phase-space variables, then it is a secondary constraint.
  - If it is an equation for a Lagrange multipliers, it just determines it.
    - Note: $\approx$ means "equal, up to a linear combination of constraints".
- Repeat the process until no new constraints are generated.
- Let $M = \{\phi_i, \phi_j\}$, where the $\{\phi_i\}_{(i=1,\cdots n)}$ includes all n constraints.
  - $\mathrm{Rank}(M) = m$,
  - Number of first class constraints $\mathcal{C}_1 = n - m$.
  - Number of second class constraints $\mathcal{C}_2 = m$.
- The number of physical degrees of freedom, $\mathcal{N}$ is given by

  $$\mathcal{N} = \frac{1}{2}\left(\mathcal{P} - 2\mathcal{C}_1 - \mathcal{C}_2\right)$$

  with $\mathcal{P}$ the dimension of the phase space

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GR I - ADM action and definition of momenta

Einstein-Hilbert action

In ADM variables, the Einstein-Hilbert action reads

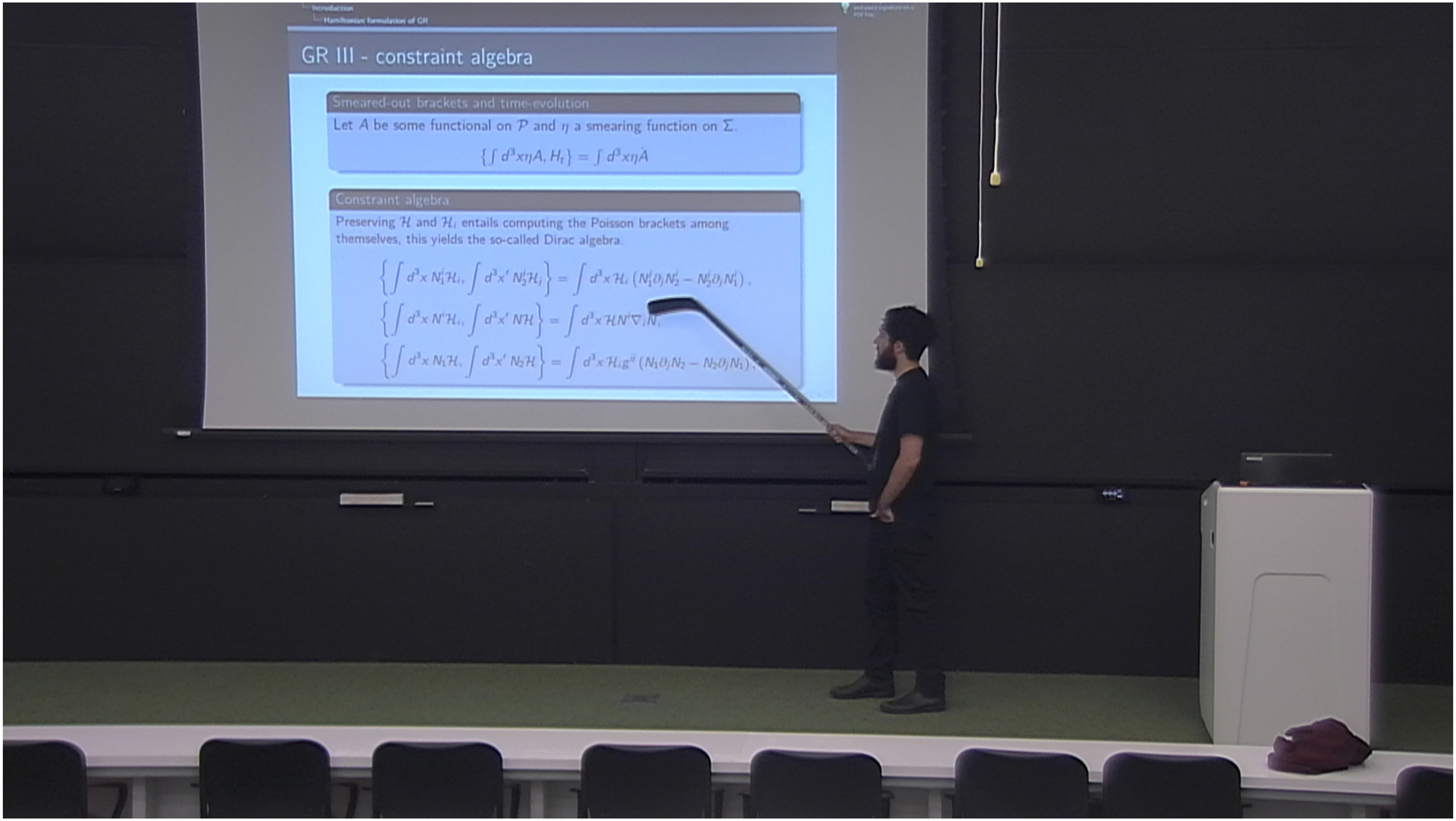
$$S = \int dt \int d^3x \sqrt{g} N \left(K_{ij}K^{ij} - K^2 + R - 2\Lambda \right)$$
$$= \int dt \int d^3x \sqrt{g} N \left(K_{ij}\mathcal{G}^{ijkl}K_{kl} + R - 2\Lambda \right),$$

where $\mathcal{G}^{ijkl} = G^{ijkl}\big|_{\lambda=1}$.

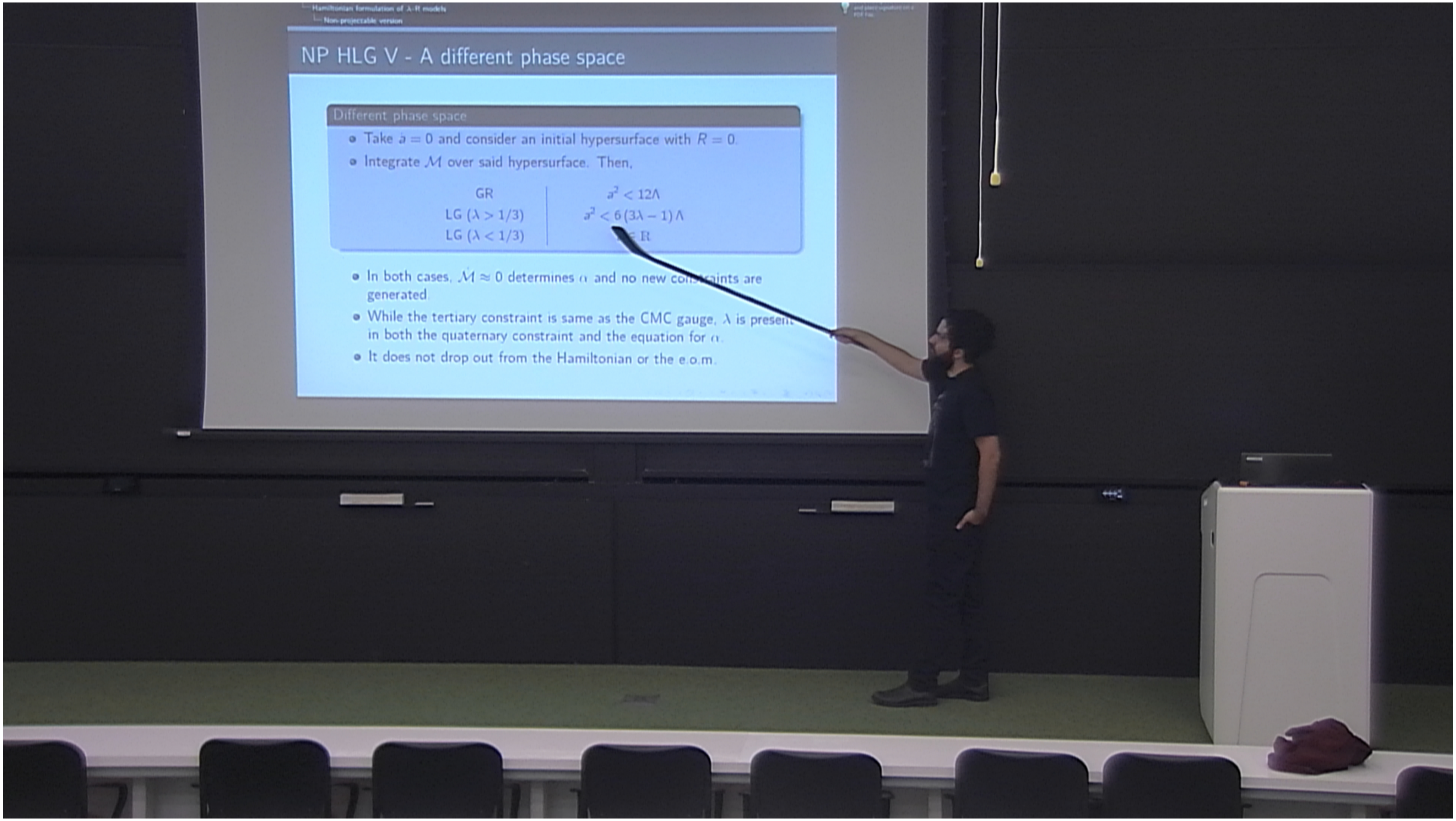
Momenta and primary constraints

The only non-vanishing momentum is that of the spatial metric.

$$\pi^{ij} \equiv \frac{\delta S}{\delta \dot{g}^{ij}} = \sqrt{g}\mathcal{G}^{ijkl}K_{kl}$$
$$\phi \equiv \frac{\delta S}{\delta \dot{N}} = 0, \qquad \phi_i \equiv \frac{\delta S}{\delta \dot{N}^i} = 0$$



Introduction
Hamiltonian formulation of GR
GR III - constraint algebra
Smeared-out brackets and time-evolution
Let A be some functional on $\mathcal{P}$ and η a smearing function on Σ.
$\{\int d^3x\eta A, H_t\} = \int d^3x\eta\dot{A}$
Constraint algebra
Preserving $\mathcal{H}$ and $\mathcal{H}_i$ entails computing the Poisson brackets among themselves, this yields the so-called Dirac algebra.
$\left\{\int d^3x\, N_1^i\mathcal{H}_i, \int d^3x'\, N_2^j\mathcal{H}_j\right\} = \int d^3x\, \mathcal{H}_i \left(N_1^j\partial_j N_2^i - N_2^j\partial_j N_1^i\right),$
$\left\{\int d^3x\, N^i\mathcal{H}_i, \int d^3x'\, N\mathcal{H}\right\} = \int d^3x\, \mathcal{H} N^i\nabla_i N,$
$\left\{\int d^3x\, N_1\mathcal{H}, \int d^3x'\, N_2\mathcal{H}\right\} = \int d^3x\, \mathcal{H}_i g^{ij} \left(N_1\partial_j N_2 - N_2\partial_j N_1\right),$


Hamiltonian formulation of λ-R models
Non-projectable version
NP HLG V - A different phase space
Different phase space
Take $\dot{a} = 0$ and consider an initial hypersurface with $R = 0$.
Integrate $\mathcal{M}$ over said hypersurface. Then,
GR | $a^2 < 12\Lambda$
LG ($\lambda > 1/3$) | $a^2 < 6(3\lambda - 1)\Lambda$
LG ($\lambda < 1/3$)
In both cases, $\dot{\mathcal{M}} \approx 0$ determines α and no new constraints are generated.
While the tertiary constraint is same as the CMC gauge, λ is present in both the quaternary constraint and the equation for α.
It does not drop out from the Hamiltonian or the e.o.m.

NPHLG VI - Degrees of freedom and acceleration of V

First and second class constraints

- There are 6 first class constraints - $\phi_i \approx 0$ and $\mathcal{H}_i \approx 0$,
  - $\mathcal{H}_i \to \mathcal{H}_i + \phi \nabla_i N$ to take into account N dependence of $\mathcal{M}$.
- 4 second class constraints, $\mathcal{H} \approx 0$, $\pi - a(t)\sqrt{g} \approx 0$, $\phi \approx 0$ and $\mathcal{M} \approx 0$
- $\Rightarrow$ there are two physical degrees of freedom, just like in GR.

Deviations from GR are not caused by extra degrees of freedom!

Projectable HLG I - Integrated constraint and acceleration

Integrated Hamiltonian constraint

- Since $N = N(t)$, the same holds for its momentum, ϕ.
- Preservation of $\phi \approx 0$ in time yields an integrated constraint, namely

$$\int d^3x \mathcal{H} \approx 0$$

- As we have seen, as long as the smearing is space-independent, no tertiary constraint arises.
- Consequently, we are free to work with $N = 1$ and $N^i = 0$.

Acceleration of the three-volume

- The conditions listed are those used by Kiefer and Giulini to obtain

$$\ddot{V} = -\frac{2}{3\lambda - 1} \int d^3x \sqrt{g} \, (R - 3\Lambda)$$

- All the treatment is general for the projectable case and so are the observational constraints derived from cosmological considerations.

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Conclusion

- We have analyzed the Hamiltonian formulation of models described by

$$S = \int dt \int d^3x \sqrt{g} N \left(K_{ij}K^{ij} - \lambda K^2 + R - 2\Lambda \right),$$

- The results of Kiefer and Giulini turn out to be a characteristic feature of the projectable theory.
- In the non-projectable setting:
  - $\dot{\mathcal{H}} \approx 0$ yields a tertiary constraint, $\nabla_i \pi \approx 0$,
    - for compact Σ, we obtain the CMC gauge condition $\pi = a(t)\sqrt{g}$,
    - since $\pi \neq 0$, λ does not drop out of the theory and we do not recover GR.
    - for $\dot{a} = 0$, we showed how $\lambda < 1/3$ impacts the phase space,
    - preserving the tertiary constraint fixes N ($\mathcal{M} \approx 0$),
    - $\dot{\mathcal{M}} \approx 0$ determines the foliation (also λ-dependent).
    - The theory only has two local degrees of freedom.

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    - The theory only has two local degrees of freedom.
  - For both compact and asymptotically flat Σ, the tertiary constraint matches the gauge of GR to which shape dynamics is equivalent.
    - It would be interesting to research possible connections between HLG and shape dynamics.

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Thank you!

Bibliography:

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- R.Loll, and L. Pires: *On the role of the extra kinetic term coupling λ in classical Hořava-Lifshitz gravity*, to appear.

MAXIMUM VOLUME YIELDS MAXIMUM RESULTS