Title: Critical Behavior of the Classical XY-model on Fractal Structures

Date: May 07, 2014 04:50 PM

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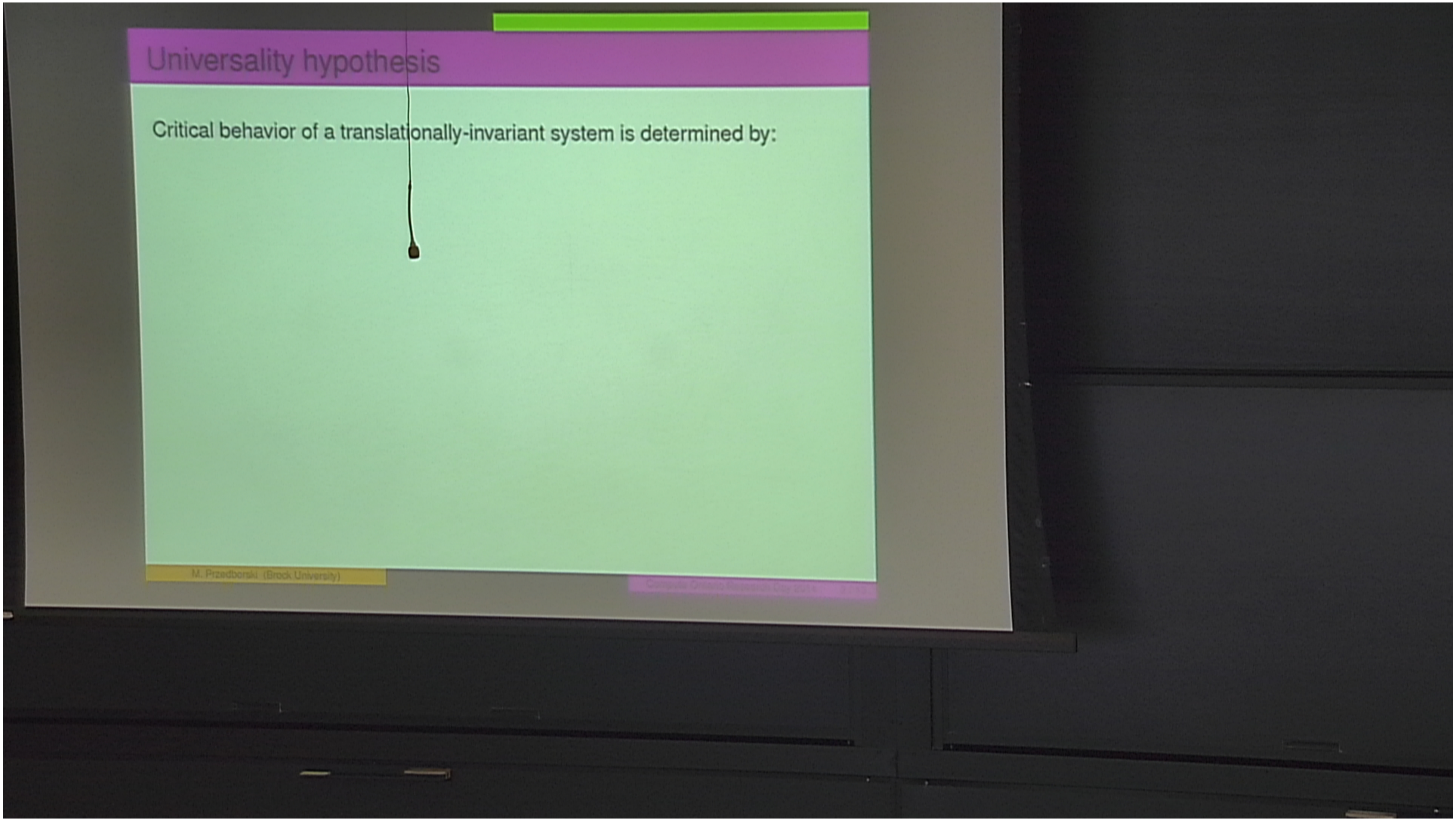
Abstract: There has been considerable interest in determining whether the universality hypothesis extends to systems which are of non-integer dimension or to systems which are scale invariant (fractals). Specifically research into these types of systems is concerned with determining the relevance of topological properties to their critical phenomena. We have performed Monte Carlo simulations for the XY model on three fractal lattices with different topological properties: the Sierpinski pyramid Menger sponge and Sierpinski carpet. We will give an overview of our results and show that while some properties such as the order of ramification are important in determining the critical behavior of these structures the fractal dimension is not.


Universality hypothesis
Critical behavior of a translationally-invariant system is determined by:
M. Przedborski (Brock University)

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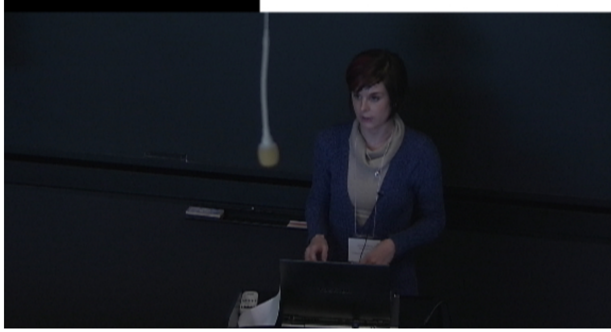
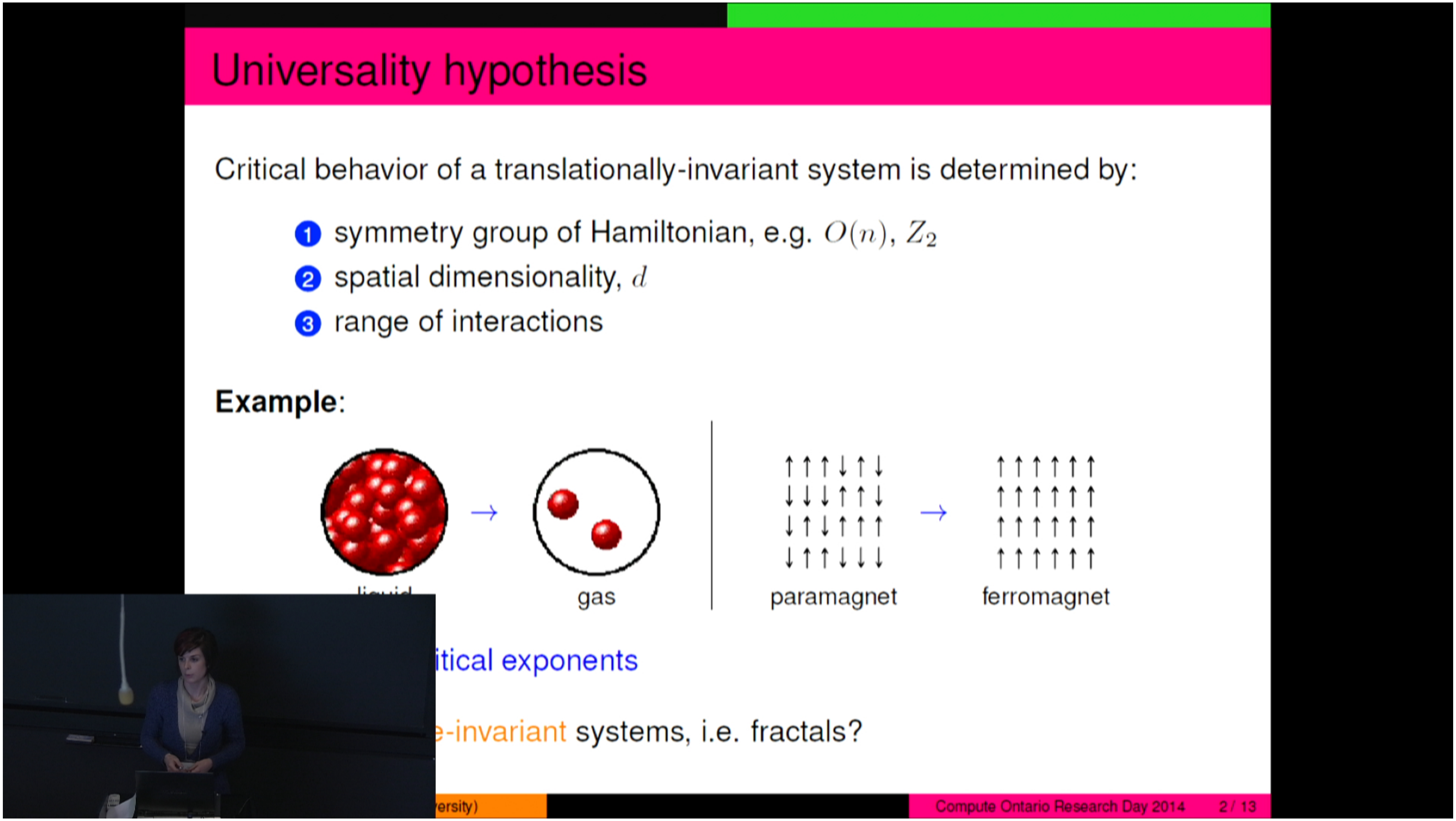
3. range of interactions

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Example:
gas
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Universality hypothesis

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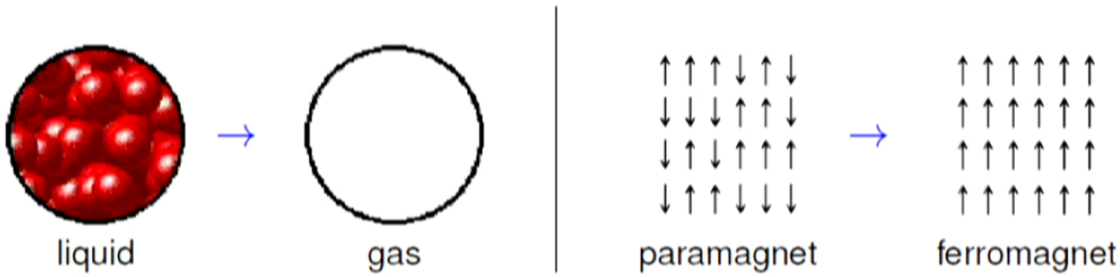
Example:



$\Rightarrow$ same critical exponents

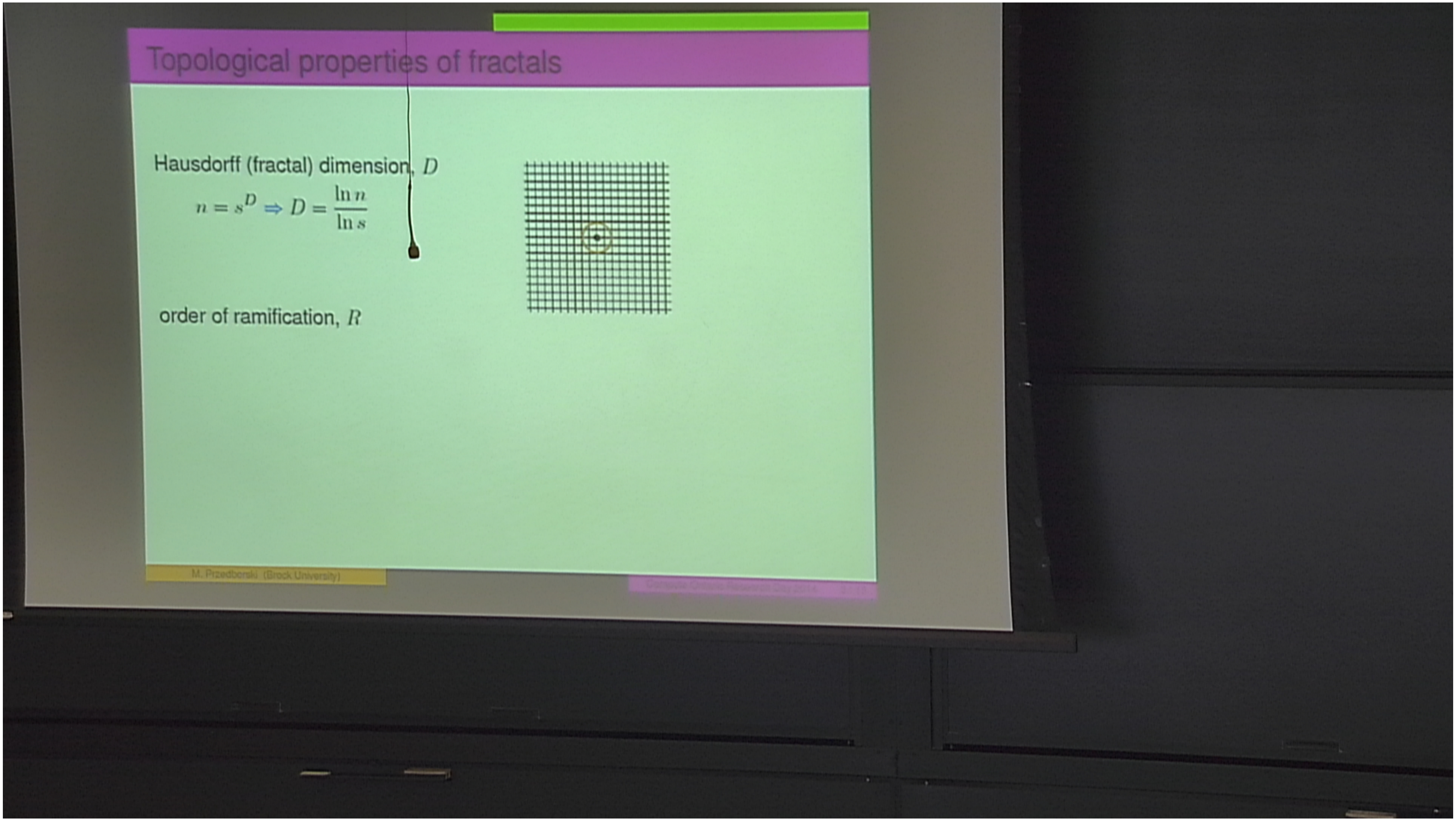
What about scale-invariant systems, i.e. fractals?

Topological properties of fractals

Hausdorff (fractal) dimension, D

$$n = s^D \Rightarrow D = \frac{\ln n}{\ln s}$$

order of ramification, R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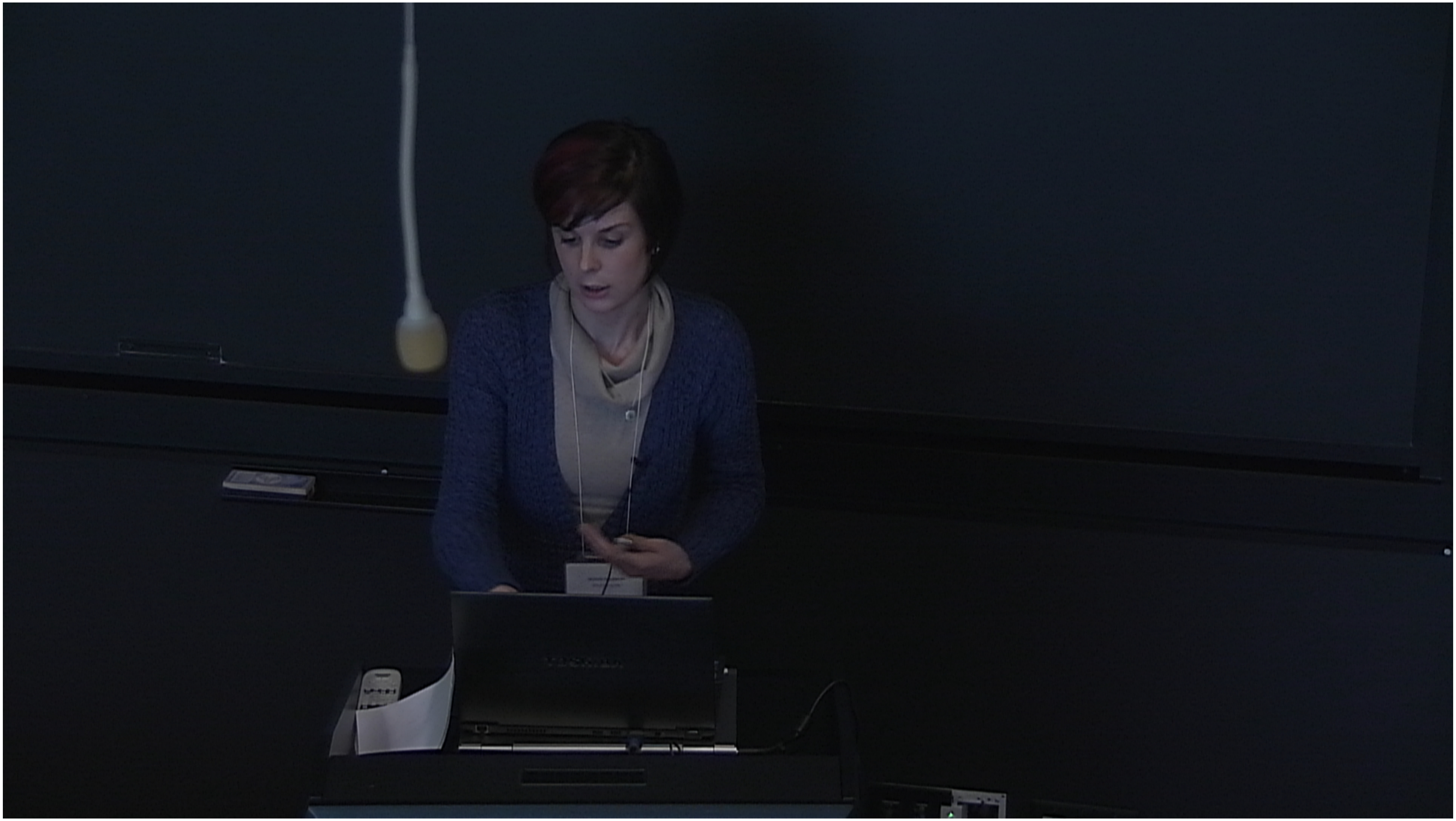

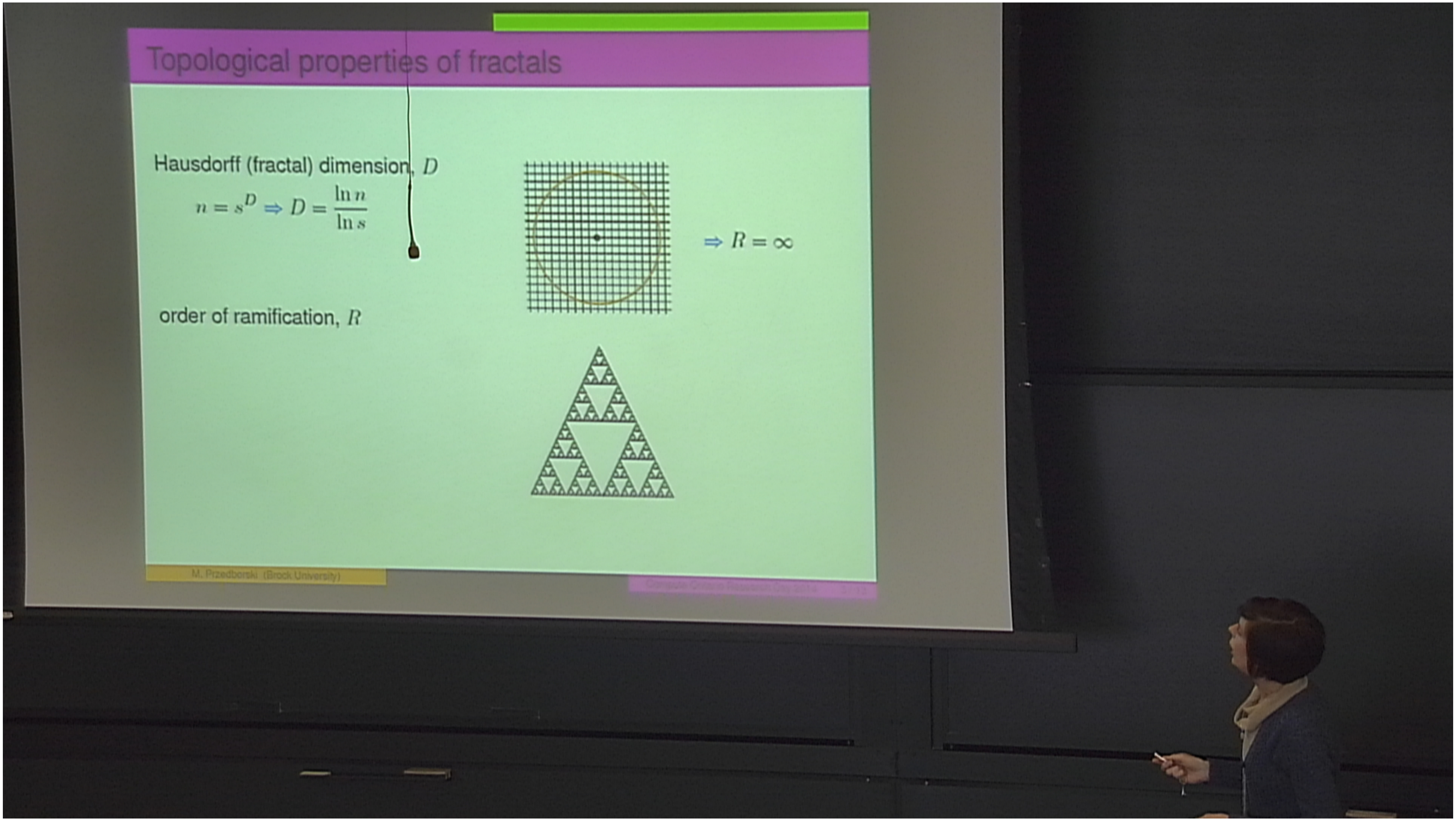
Topological properties of fractals
Hausdorff (fractal) dimension, D
$n = s^D \Rightarrow D = \frac{\ln n}{\ln s}$
$\Rightarrow R = \infty$
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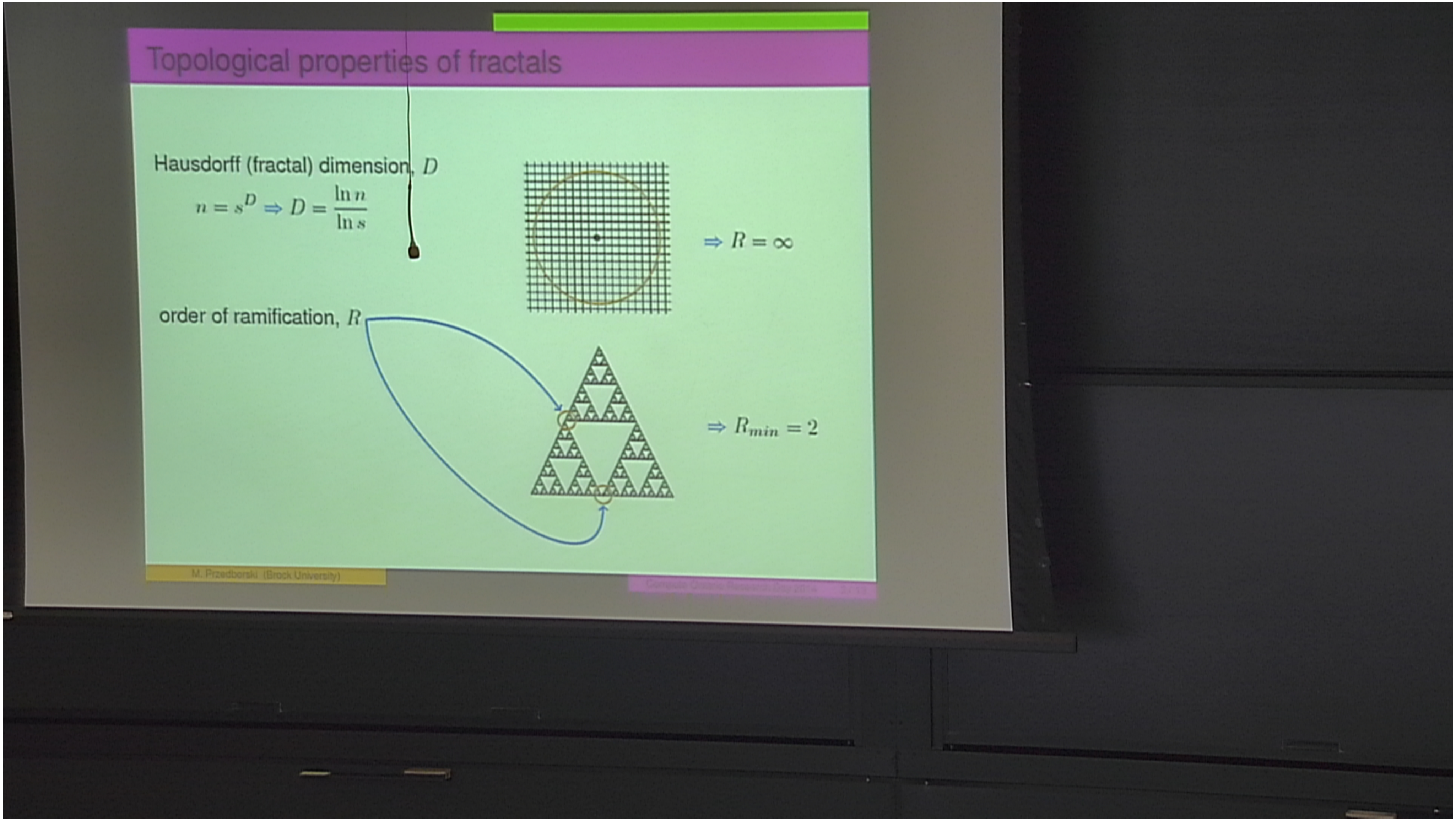
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Hausdorff (fractal) dimension, D
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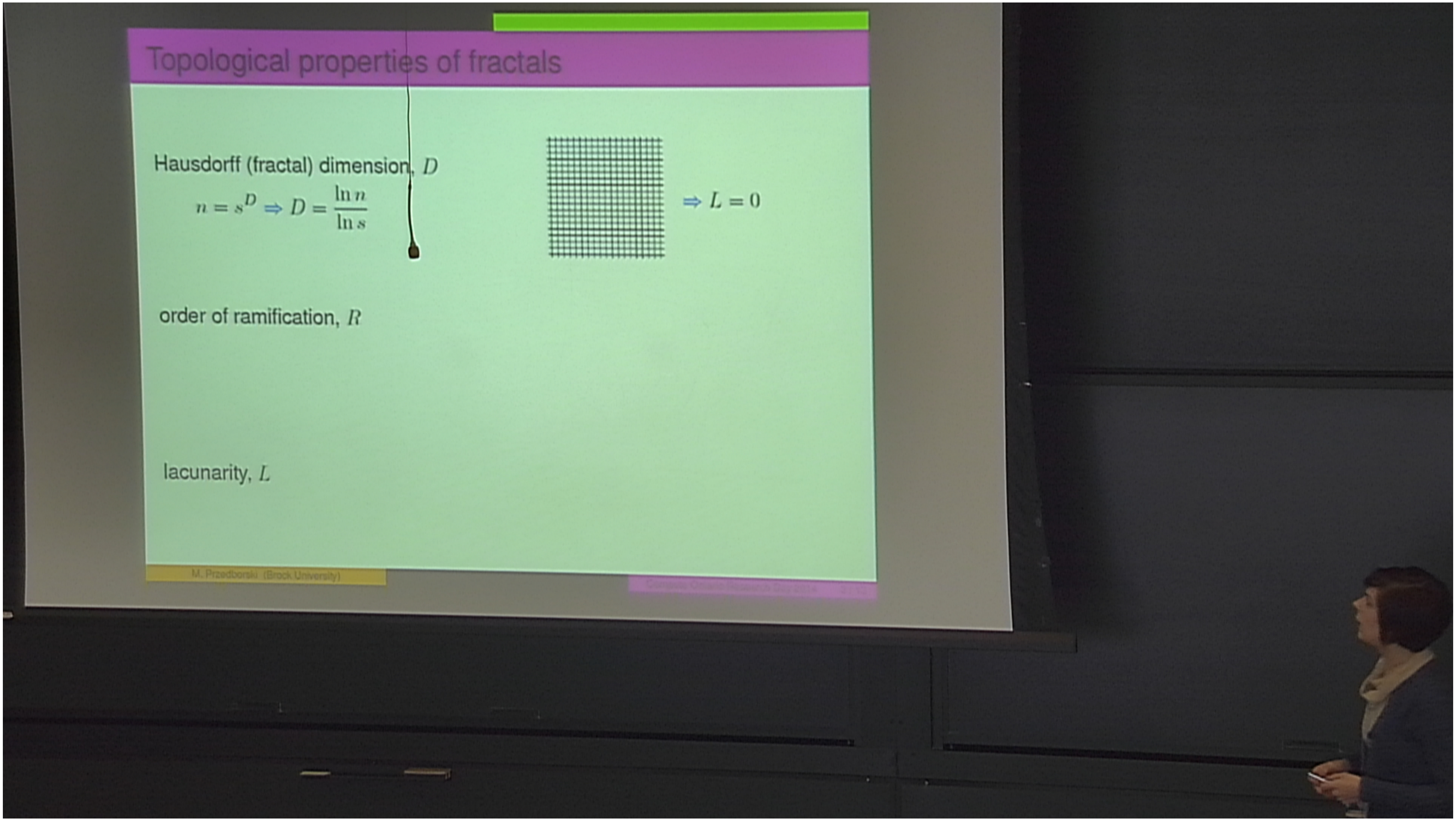
order of ramification, R
$\Rightarrow R = \infty$
$\Rightarrow R_{min} = 2$
M. Przedborski (Brock University)


Topological properties of fractals
Hausdorff (fractal) dimension, D
$n = s^D \Rightarrow D = \frac{\ln n}{\ln s}$
$\Rightarrow L = 0$
order of ramification, R
lacunarity, L
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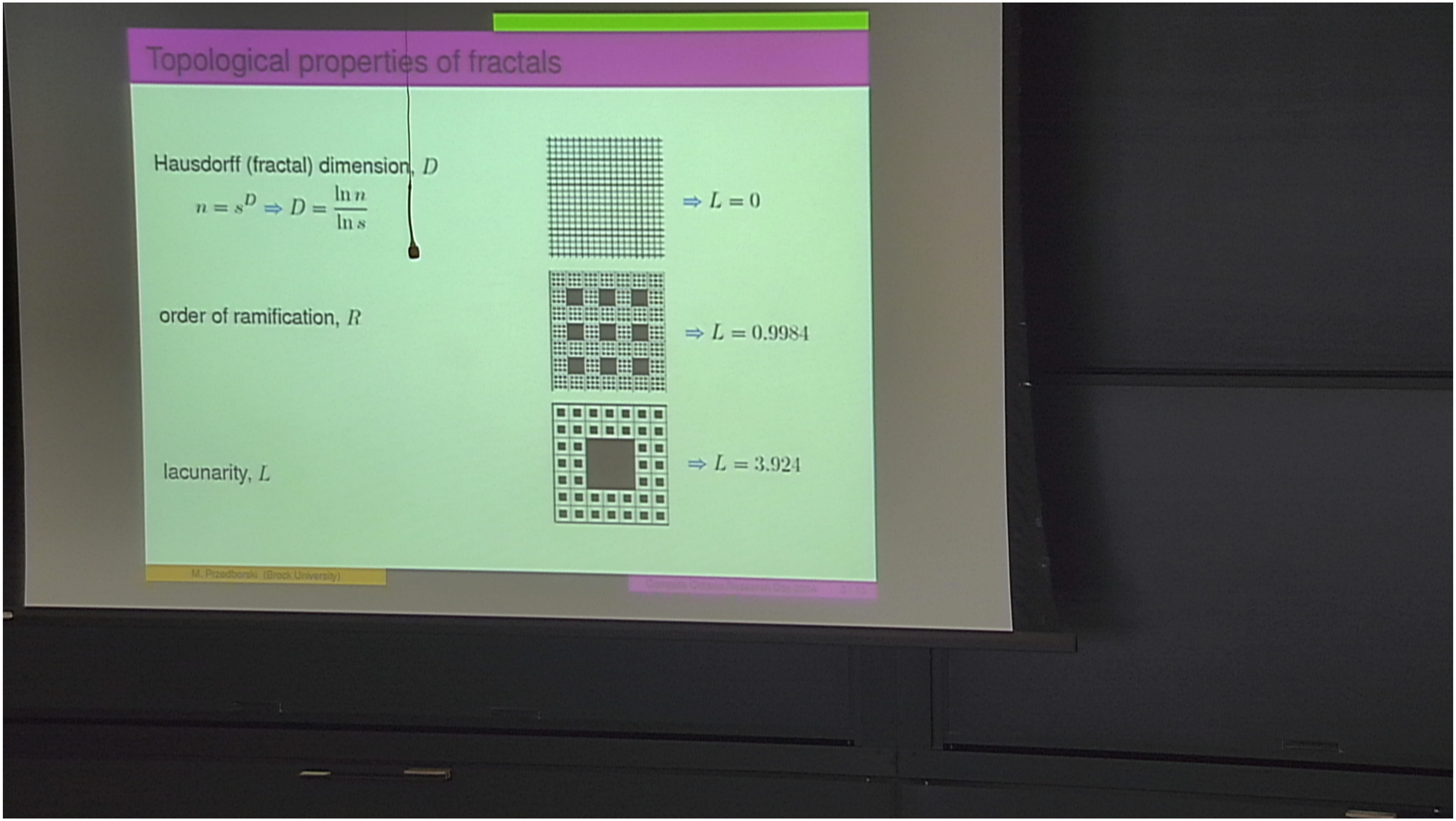
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lacunarity, L
$\Rightarrow L = 0$
$\Rightarrow L = 0.9984$
$\Rightarrow L = 3.924$
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Previous research

(Gefen *et al.*, 1980, 1984): RG techniques to examine Ising model, discrete Z_2 symmetry, on multiple fractal structures

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XY model

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Configuration of $2d$ spins, oriented in xy-plane within $[0, 2\pi)$

Hamiltonian of the system:

$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

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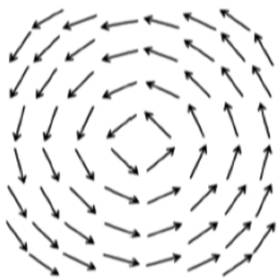
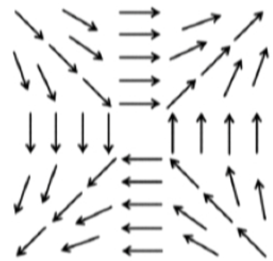
$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$J > 0$ - coupling constant
$\langle \cdots \rangle$ - nearest neighbours only
θ_i - angle variable on site i

Interesting spin excitations

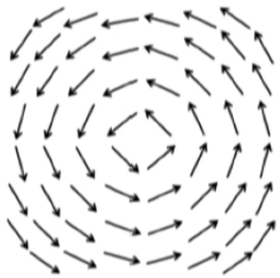
XY model

 - positive vortex

 - negative vortex

Regular $2d$ systems: Berezinskii-Kosterlitz-Thouless (BKT) transition

XY model



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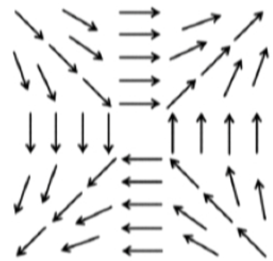

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$\Rightarrow$ universal jump in helicity modulus $\gamma(T_c)/T_c = 2/\pi$ (Nelson & Kosterlitz, 1977)

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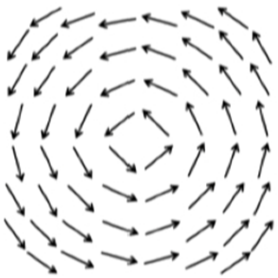

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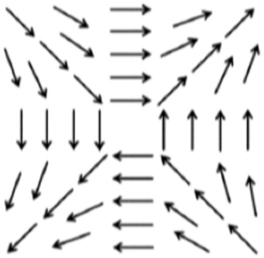

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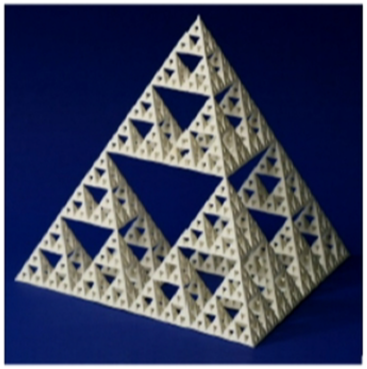
Regular $3d$ systems: continuous phase transition

Fractal lattices

	Sierpiński pyramid (SP)	**Sierpiński carpet** (SC)	**Menger sponge** (MS)
			
D	$\frac{\ln 4}{\ln 2} = 2$		
R_{min}	4		

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	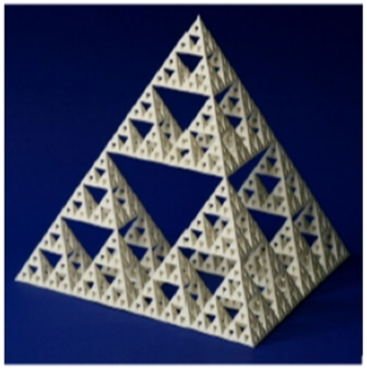	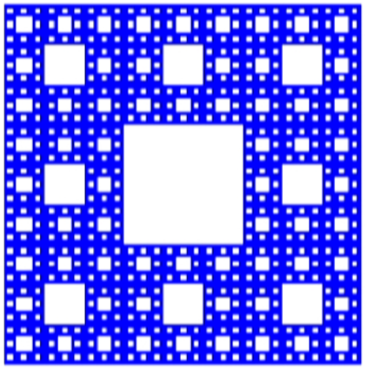	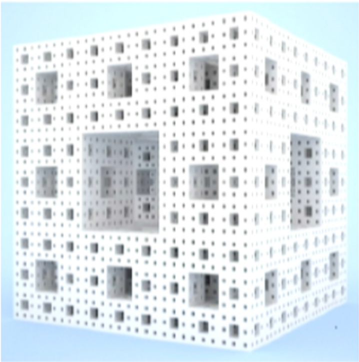
D	$\frac{\ln 4}{\ln 2} = 2$	$\frac{\ln 8}{\ln 3} = 1.8928$	$\frac{\ln 20}{\ln 3} = 2.7268$
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	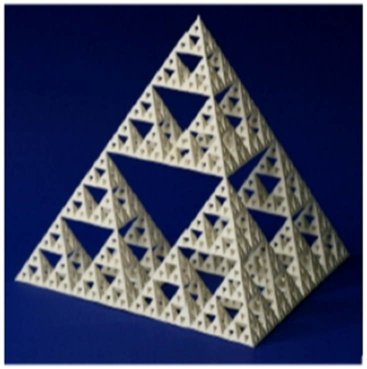	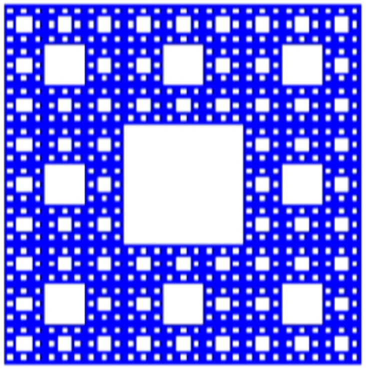	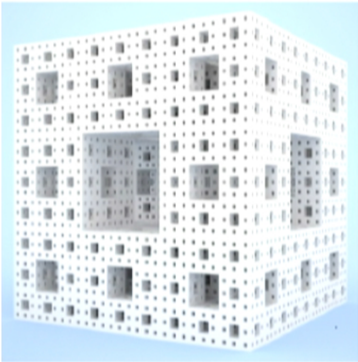
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MC simulations

SP: Metropolis algorithm (Metropolis *et al.*, 1953)

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⇒ local update scheme

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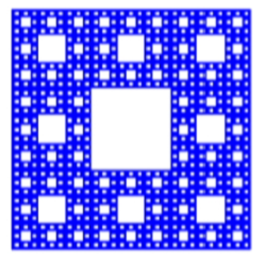
Distribution of states $\{\Omega_i\}$ with $\rho(\Omega_i) \propto e^{-E(\Omega_i)/k_B T}$

$\Rightarrow$ Thermal average: $\langle A \rangle = \frac{1}{M} \sum_{i=1}^{M} A_i$

Open and closed boundary conditions:



MC simulations: $H = -J\sum \cos(\theta_i - \theta_j)$

Heat capacity per site:

$$C = \frac{1}{N}\frac{\langle H^2\rangle - \langle H\rangle^2}{k_B T^2}$$

N - number of sites

Linear magnetic susceptibility per site:

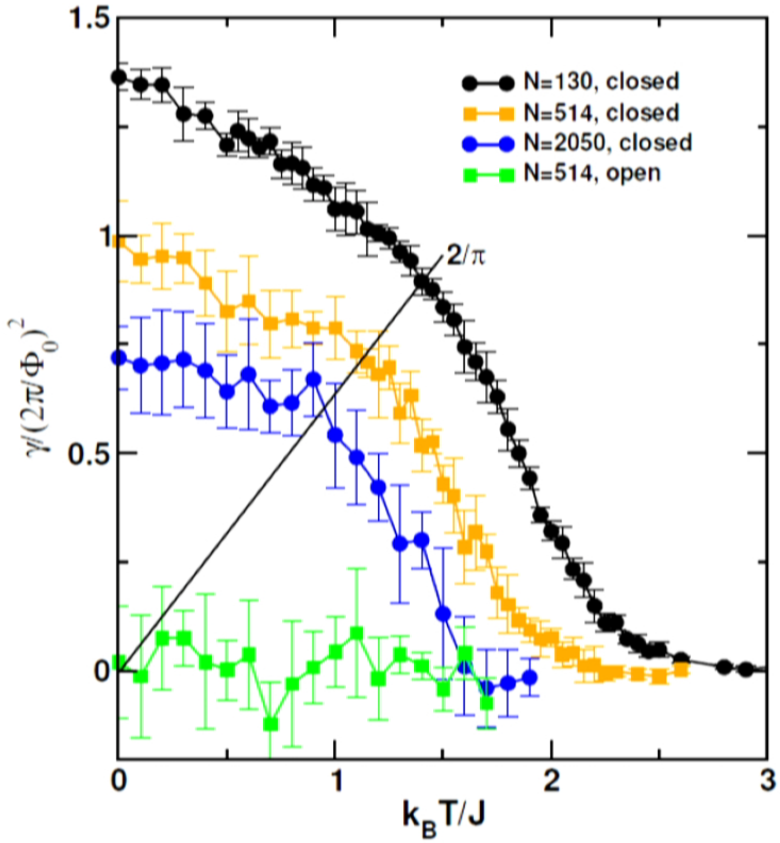
$$\chi = \frac{\langle m^2\rangle - \langle m\rangle^2}{k_B T}$$

m - magnetization per site

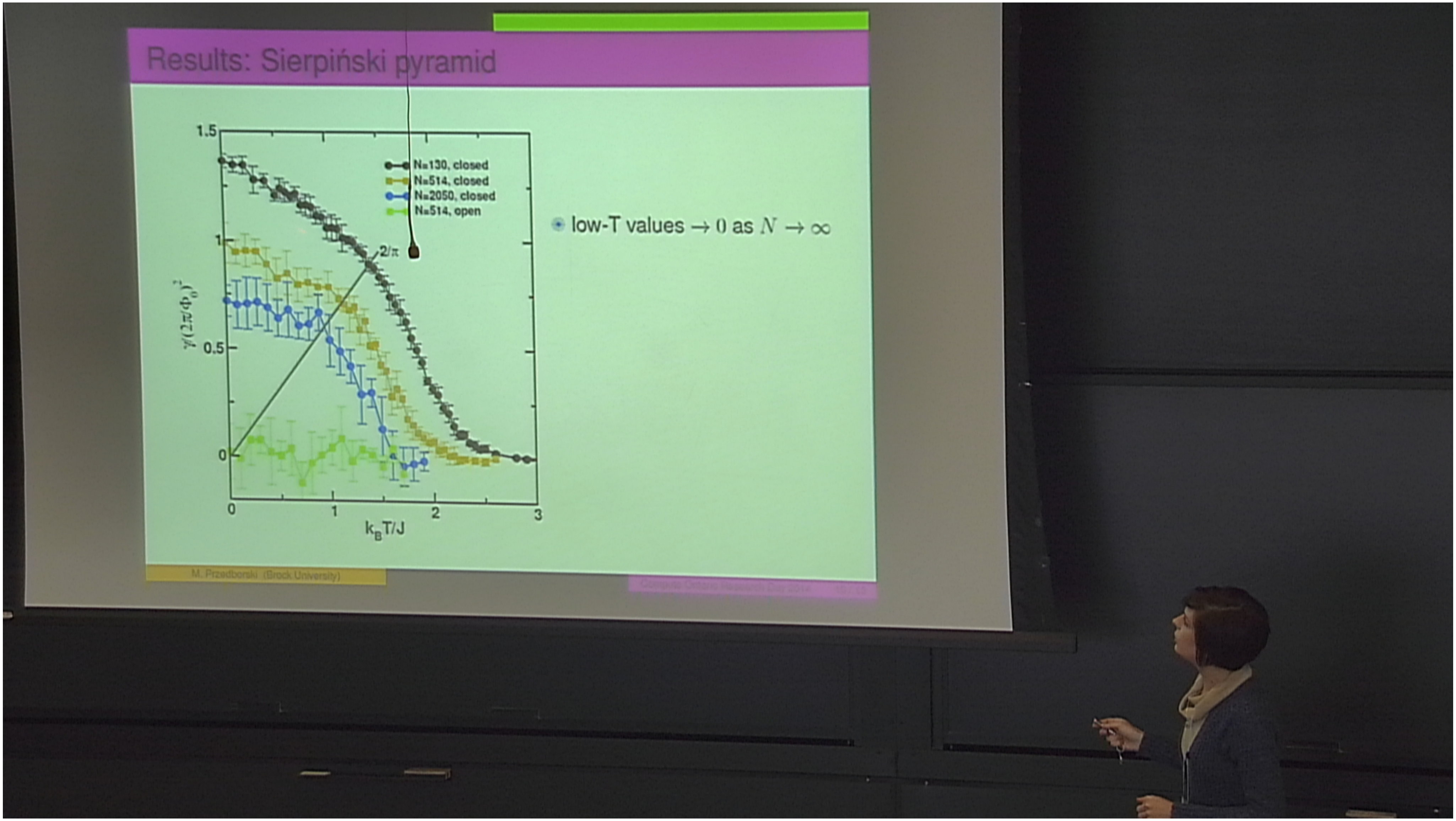
Helicity modulus per site:

$$\gamma = \left\langle \left(\frac{\partial^2 H}{\partial A^2}\right)_{A=0} \right\rangle - \frac{1}{k_B T}\left\langle \left(\frac{\partial H}{\partial A}\right)^2_{A=0} \right\rangle + \frac{1}{k_B T}\left\langle \left(\frac{\partial H}{\partial A}\right)_{A=0} \right\rangle^2$$

Results: Sierpiński pyramid



- low-T values $\to 0$ as $N \to \infty$



Results: Sierpiński pyramid
N=130, closed
N=514, closed
N=2050, closed
N=514, open
2/π
low-T values → 0 as N → ∞
k_B T/J
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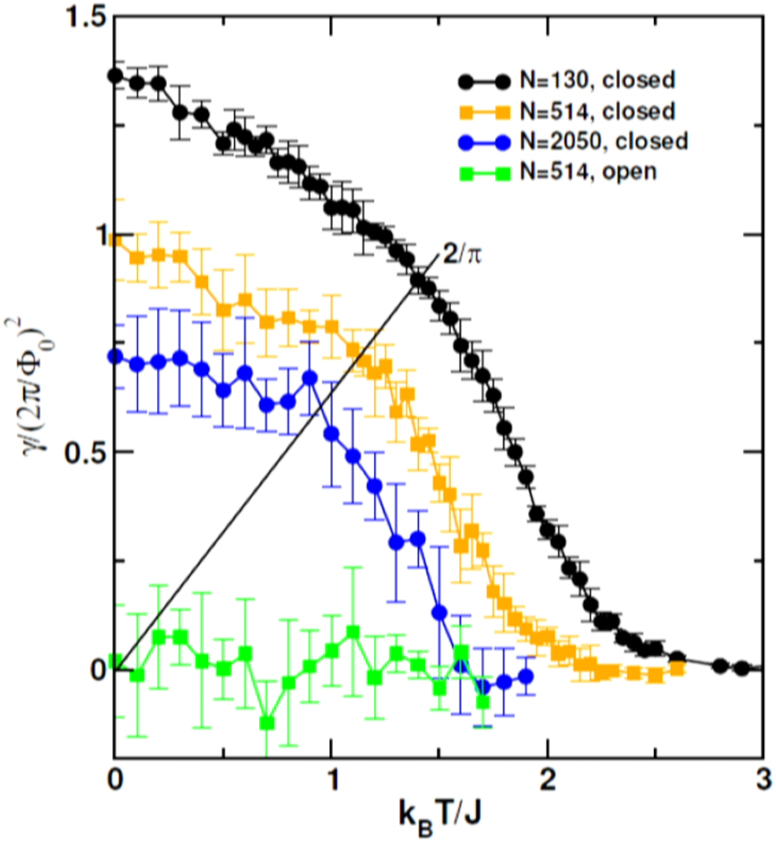
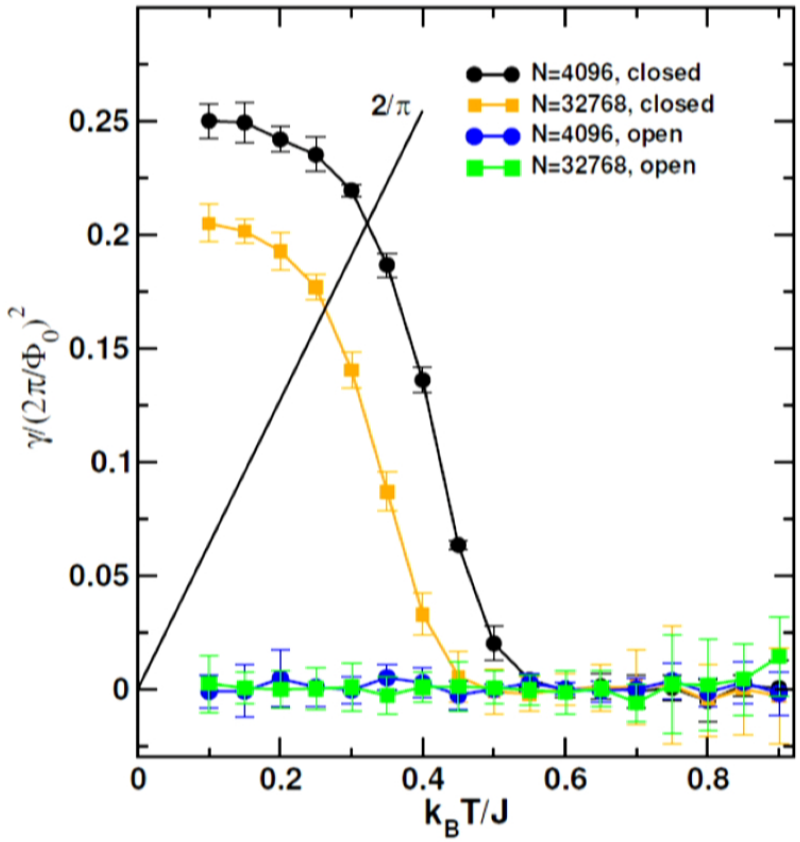
Results: Sierpiński pyramid

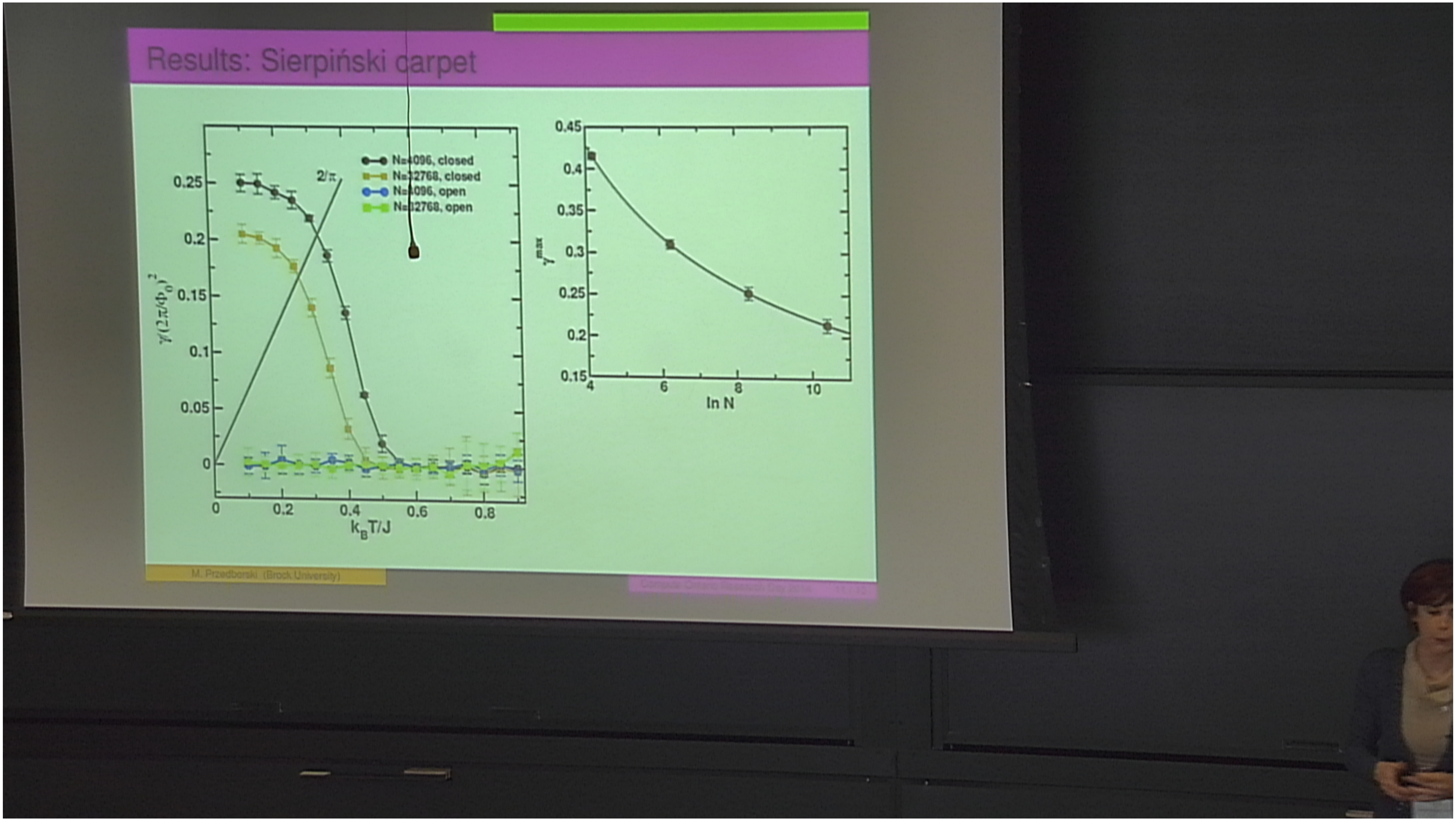


- low-T values $\rightarrow 0$ as $N \rightarrow \infty$
- $T_\gamma \rightarrow 0$ as $N \rightarrow \infty$
- for open BC, $\gamma = 0$ for all N

$\Rightarrow$ **no phase transition in SP**

Results: Sierpiński carpet




Results: Sierpiński carpet
N=4096, closed
N=32768, closed
N=4096, open
N=32768, open
2/π
γ(2π/Φ0)^2
k_B T/J
γ^max
ln N
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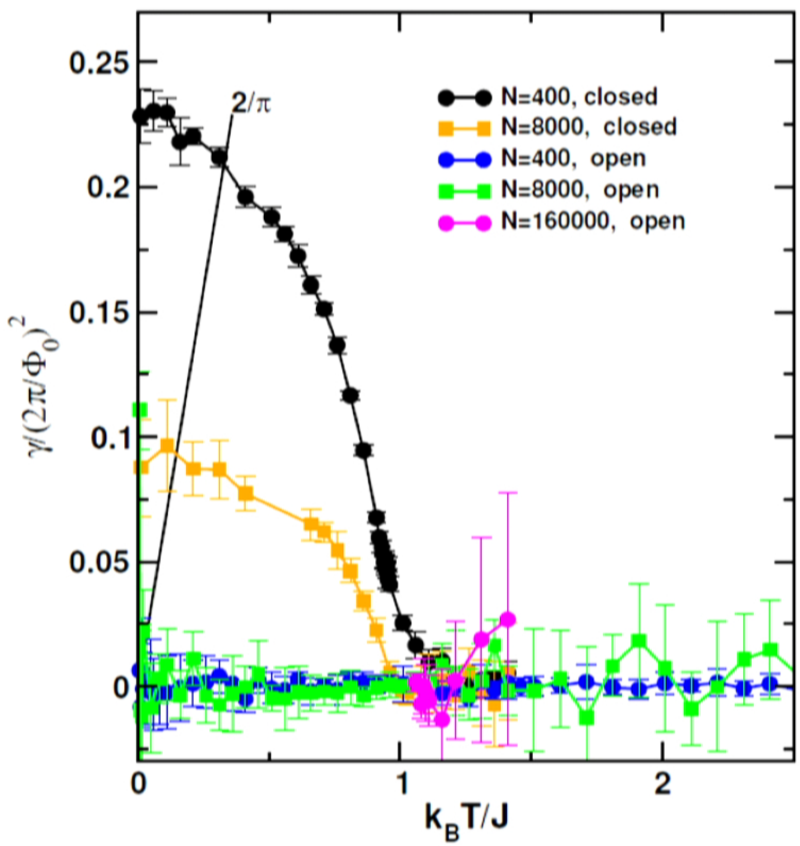
Results: Menger sponge



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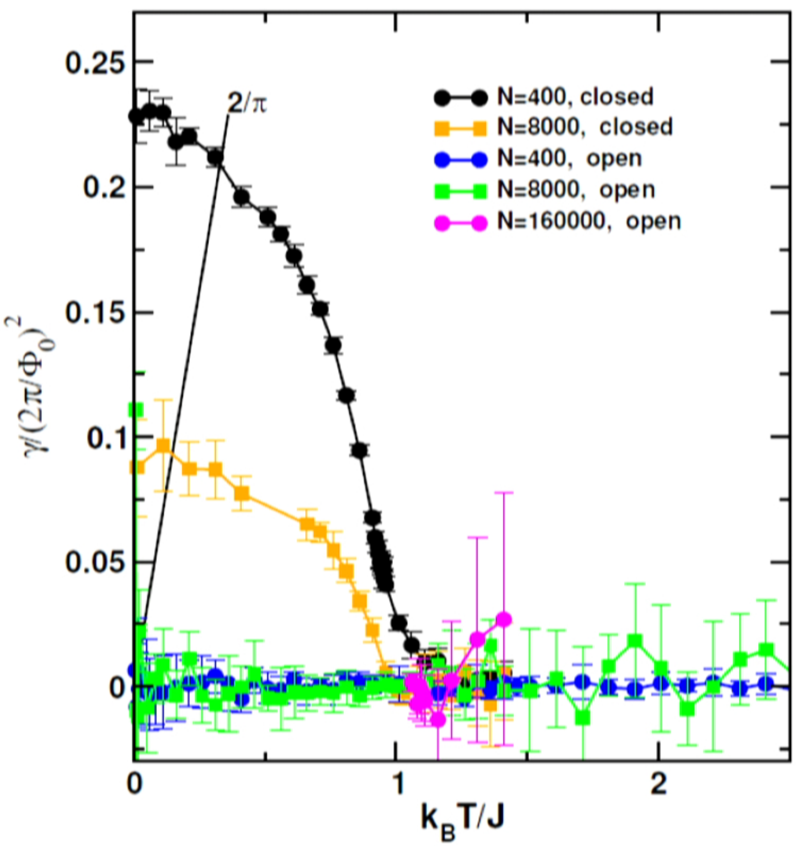
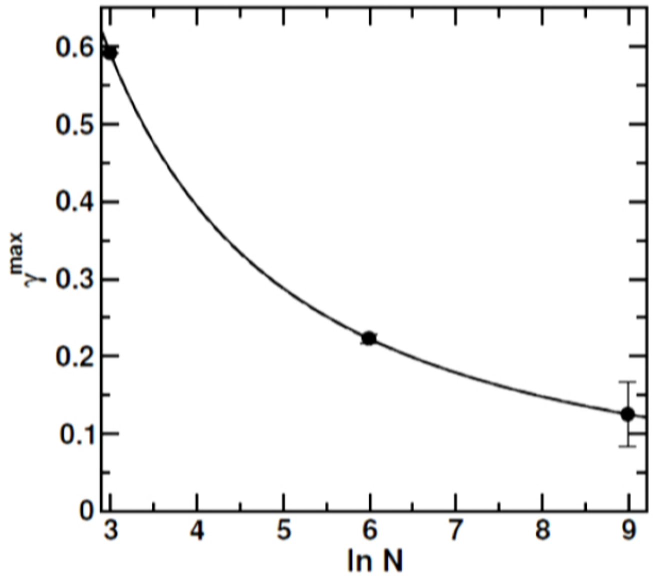
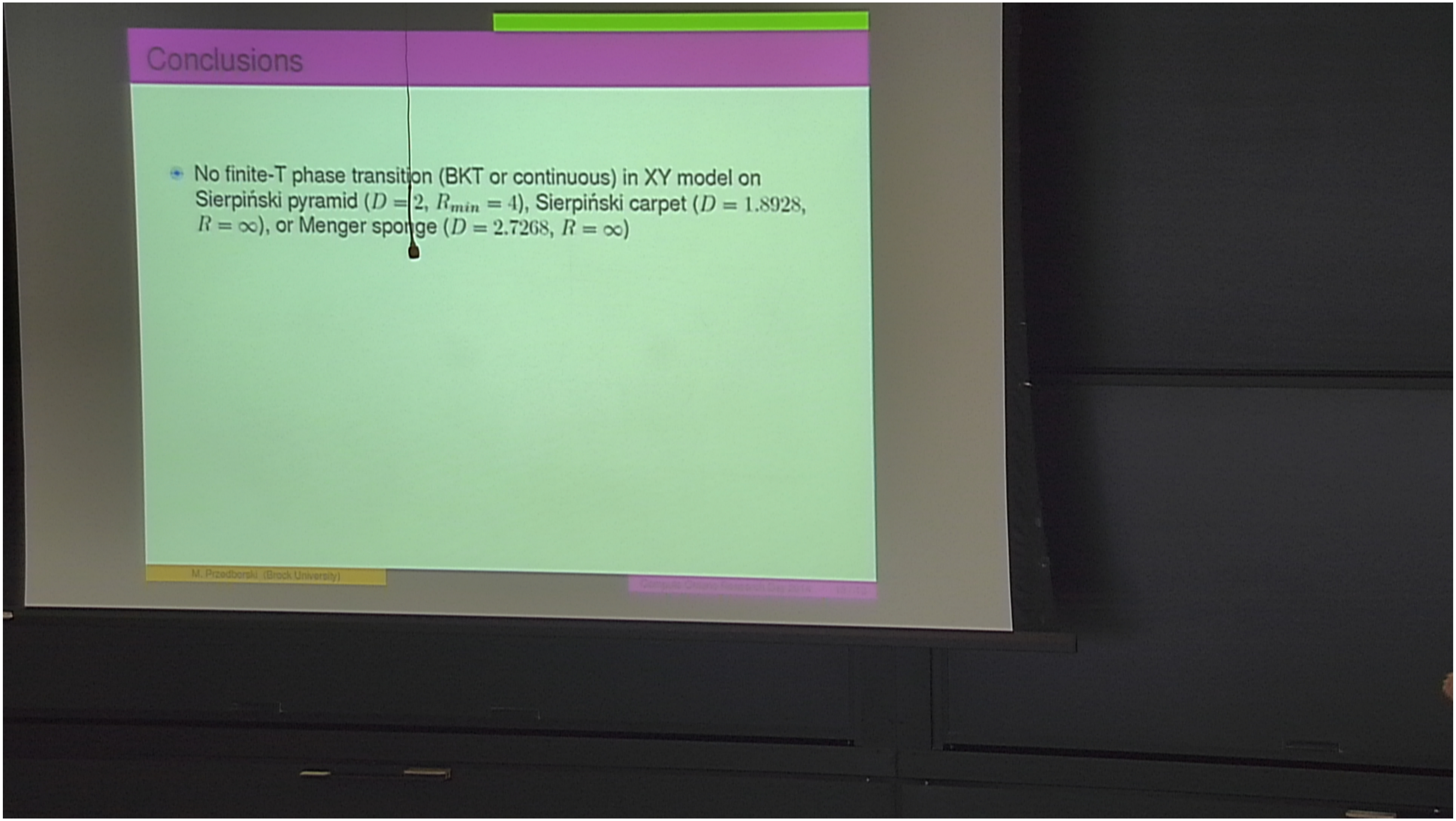




$$\gamma^{\max} = \frac{2.7844}{(\ln N)^{1.41103}}$$


Conclusions
No finite-T phase transition (BKT or continuous) in XY model on Sierpiński pyramid ($D = 2$, $R_{min} = 4$), Sierpiński carpet ($D = 1.8928$, $R = \infty$), or Menger sponge ($D = 2.7268$, $R = \infty$)
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Conclusions

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- Fractal dimension is not the deciding factor for whether the transition takes place
- Other topological properties (lacunarity) may influence critical behavior of fractal lattices