Title: A polynomial-time algorithm for the ground state of 1D gapped local Hamiltonians

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Abstract: <div id="Cleaner"><!--[if gte mso 9]><xml>
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# A polynomial-time algorithm for the ground state of 1D gapped local Hamiltonians

Thomas Vidick
Caltech

Joint work with Zeph Landau and Umesh Vazirani (UC Berkeley)

# Simulating quantum systems

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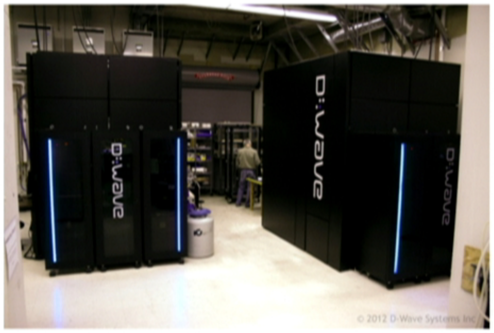

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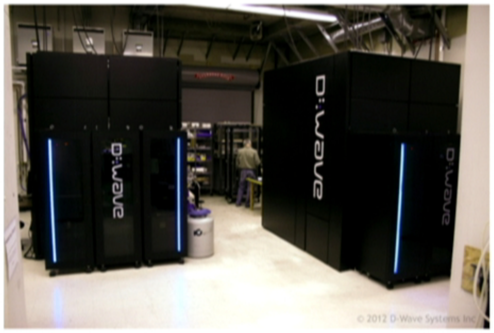


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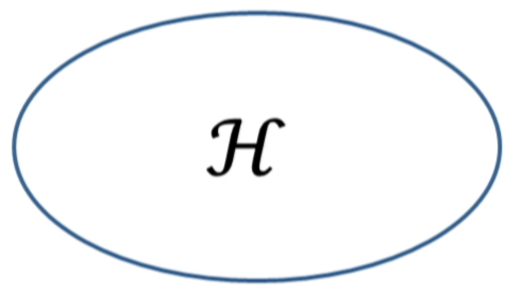

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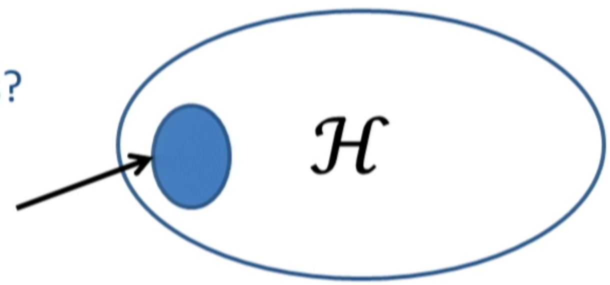
  - Goal: delineate the "physically relevant corner of Hilbert space"



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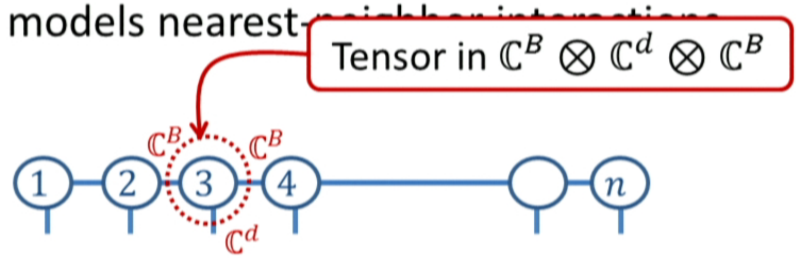

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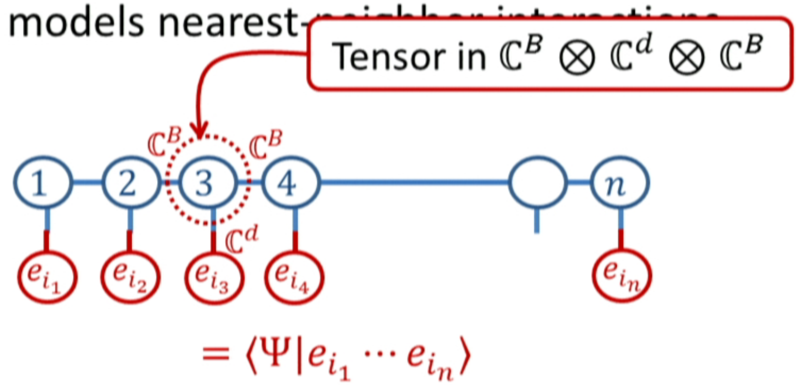

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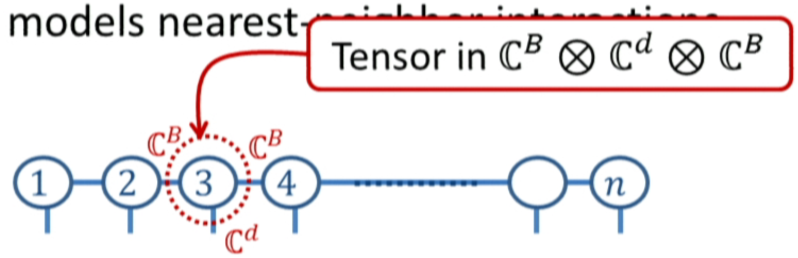


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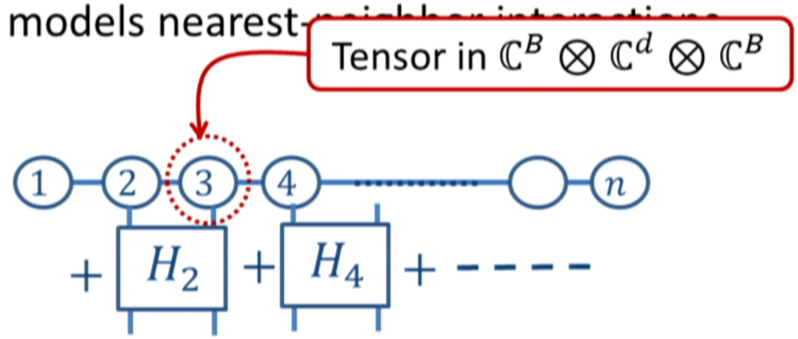


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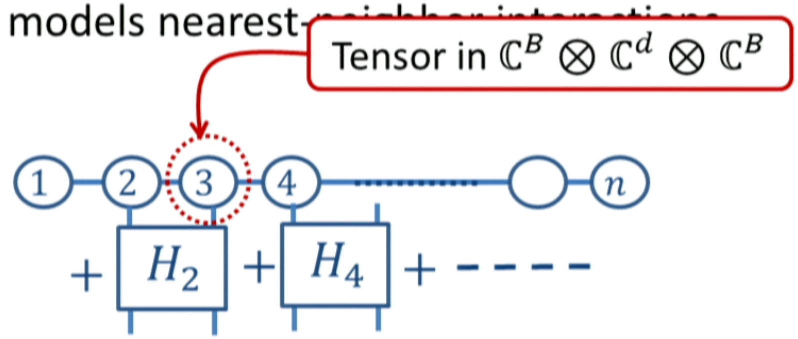


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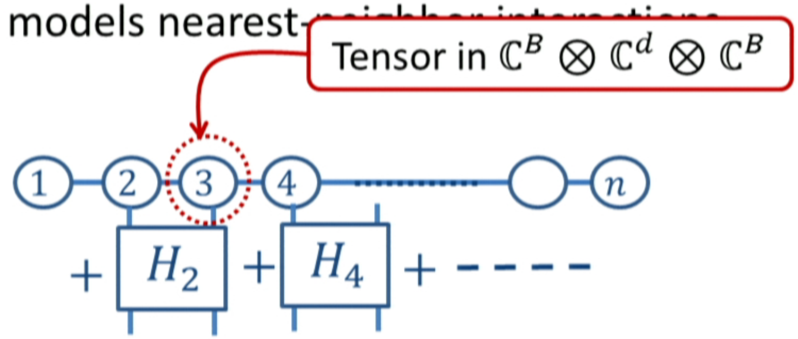


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evaluate local observables $\langle GS|O|GS\rangle$

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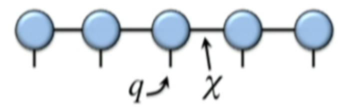
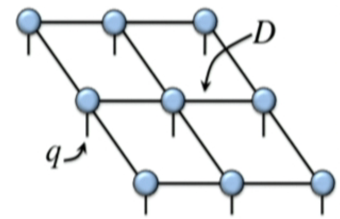
MPS




PEPS



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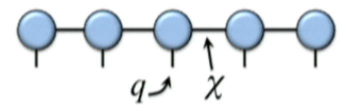


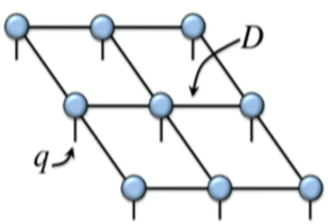


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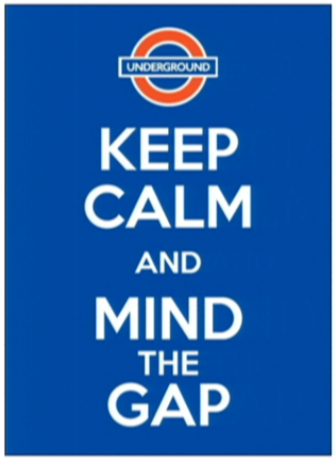
  - Good ansatz for $|GS\rangle$: MPS/MERA in 1D, PEPS in 2+D
  - DMRG algorithm [White'92] highly successful in 1D
  - No rigorous analysis; artificial "hard" examples known
- [Kit'99] Approximating $E_0$ is QMA-hard
  - Even very simple Hamiltonians: 2-local, qubits, arbitrary graphs [CM'13,CG'13]; 2D tiling problems [GI'09]; 2-local, qudits, 1D [AGIK'07,HNN'13]
  - No "efficient" classical description of $|GS\rangle$ unless QMA=NP !
- How to explain the practical successes?

  Can only explore tiny fraction of Hilbert space: why is $|GS\rangle$ found?

# Gapped Hamiltonians



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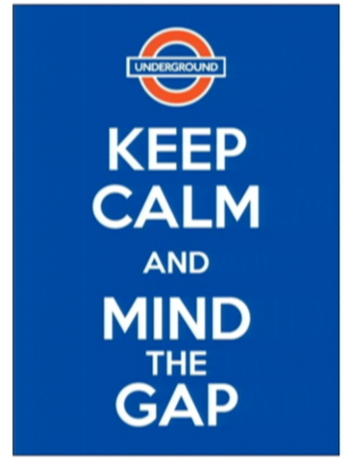

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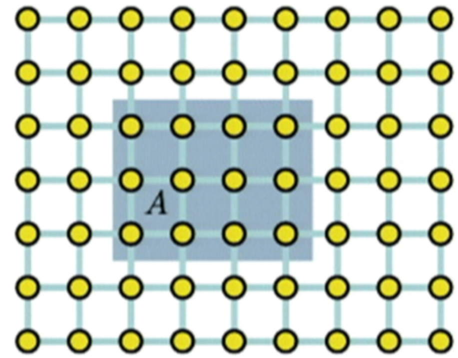


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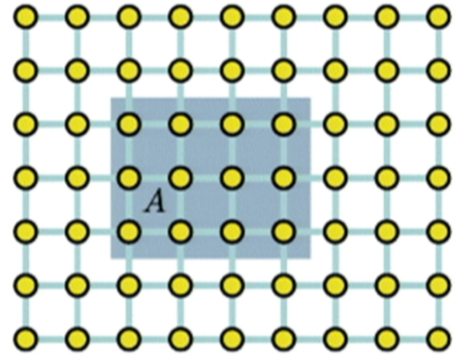


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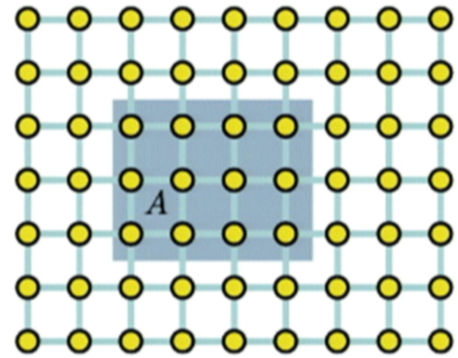


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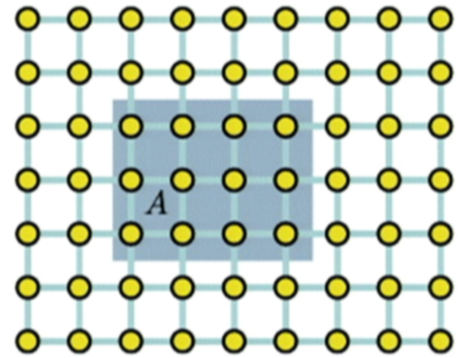


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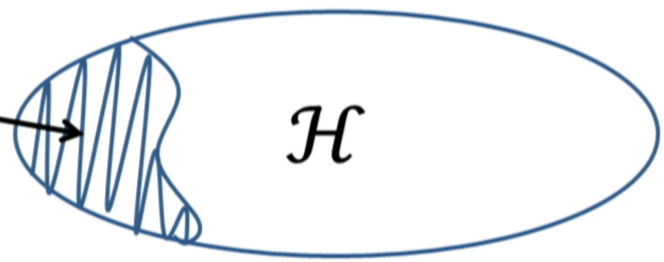

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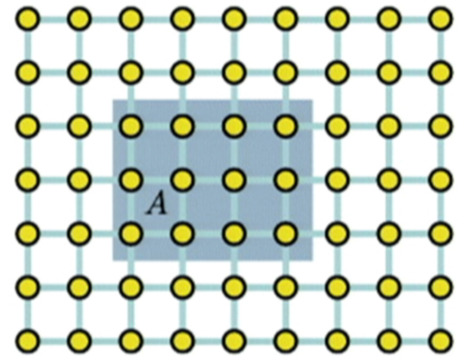


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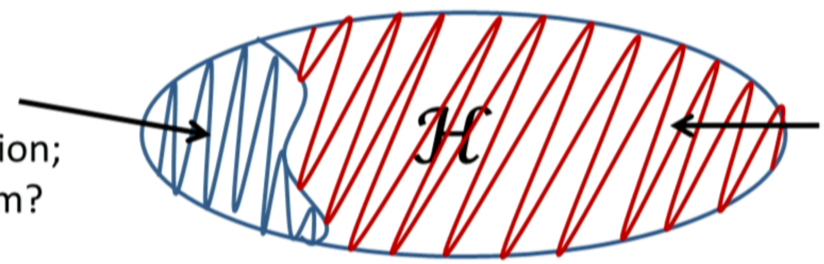


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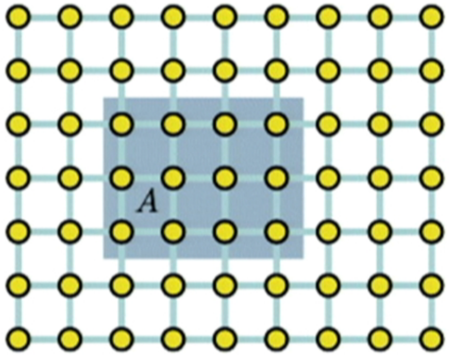


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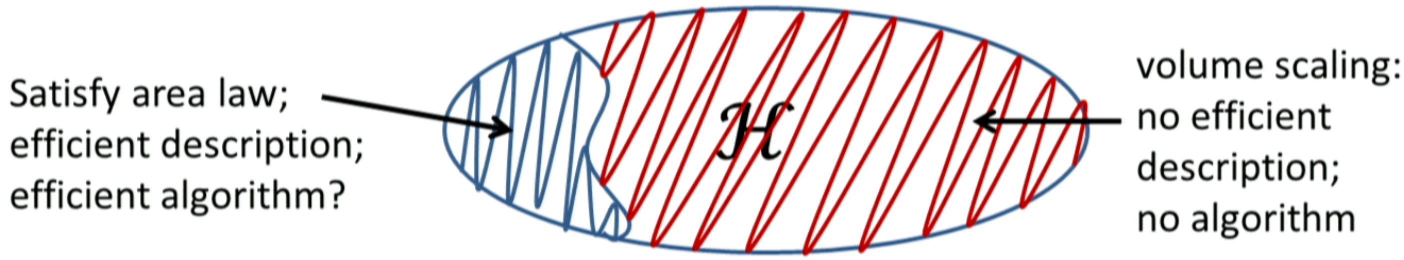


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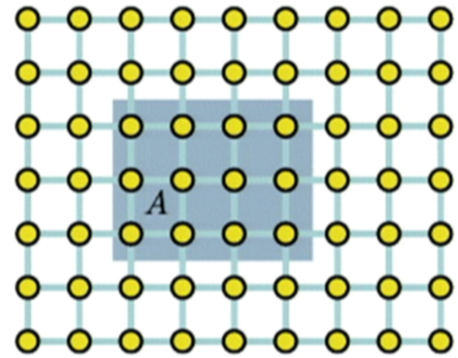


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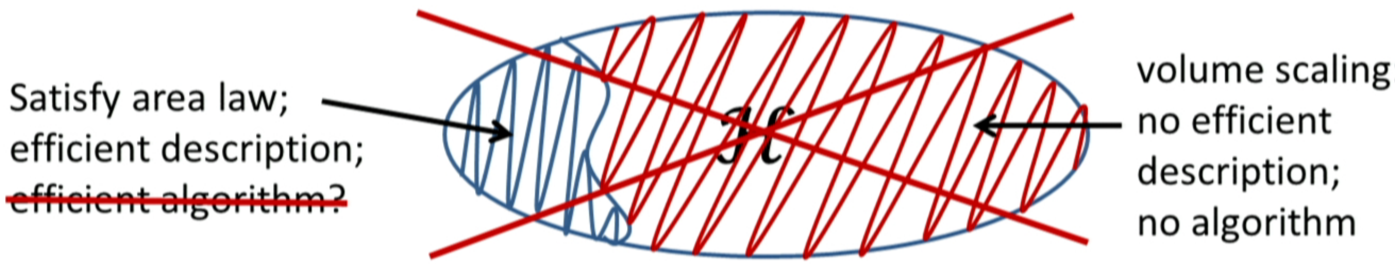


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$\Rightarrow$

$$\min \mathrm{Tr}((H_1 + \cdots + H_i)\rho) \quad \rho \in S_i \subseteq \mathrm{Dens}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$$

$S_i$ has poly-size description

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Errors in navigation at each step can add up to throw the algorithm totally off course on the exponential lake.

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- $(n, \delta, \mathrm{poly})$-viable set contains good approximation to $|GS\rangle$.
  How do we get there?

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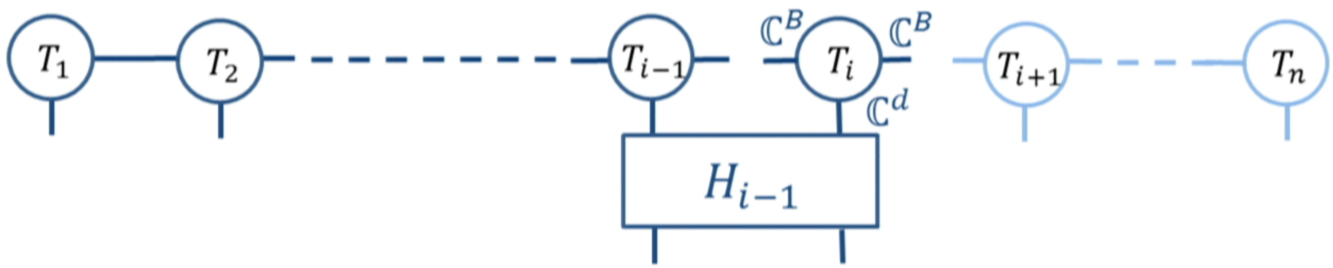


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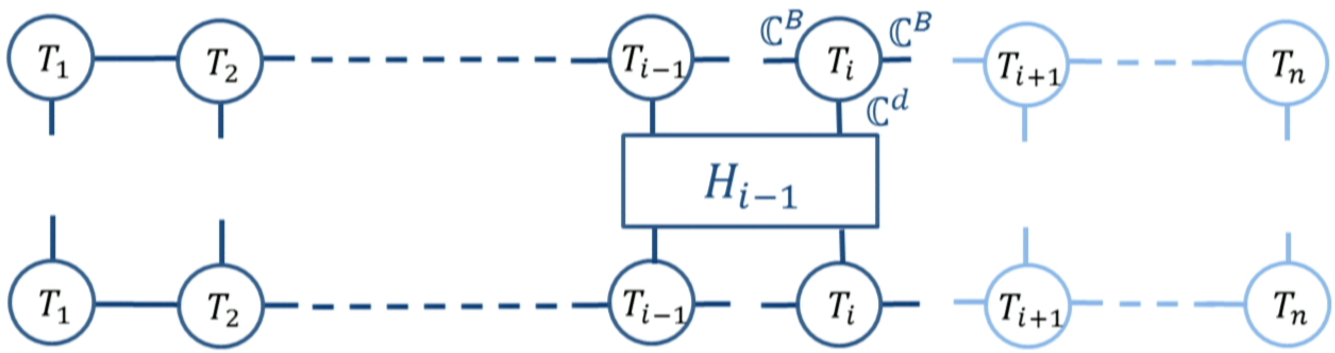


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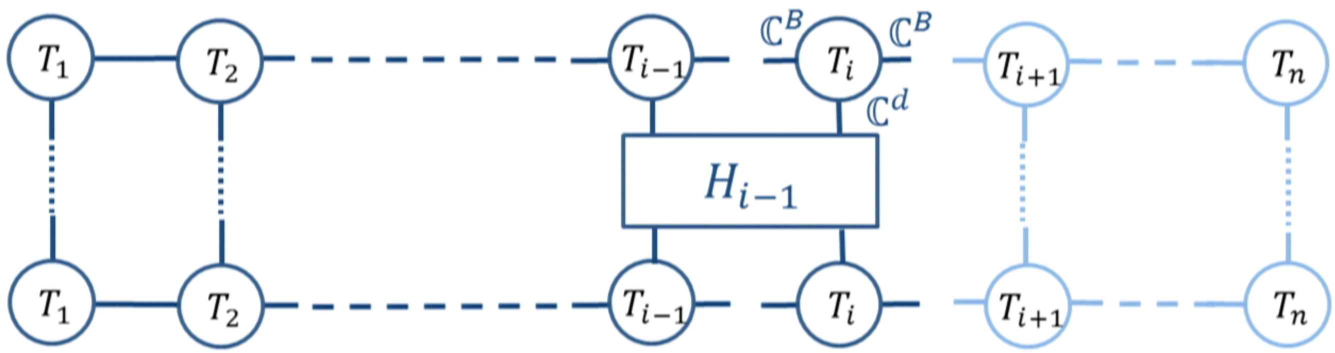


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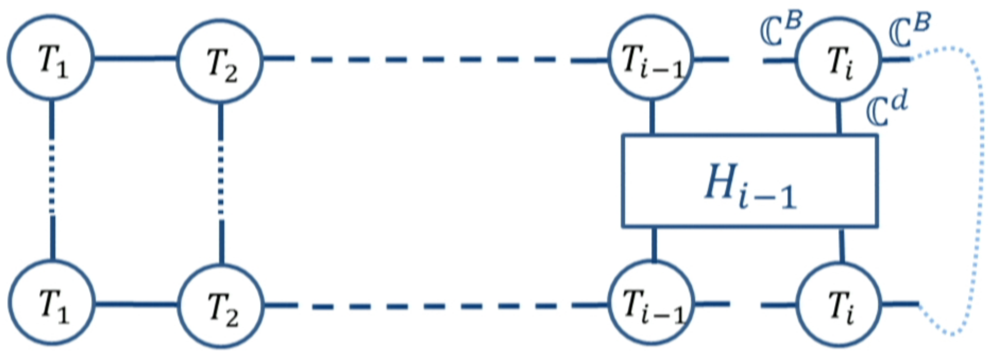


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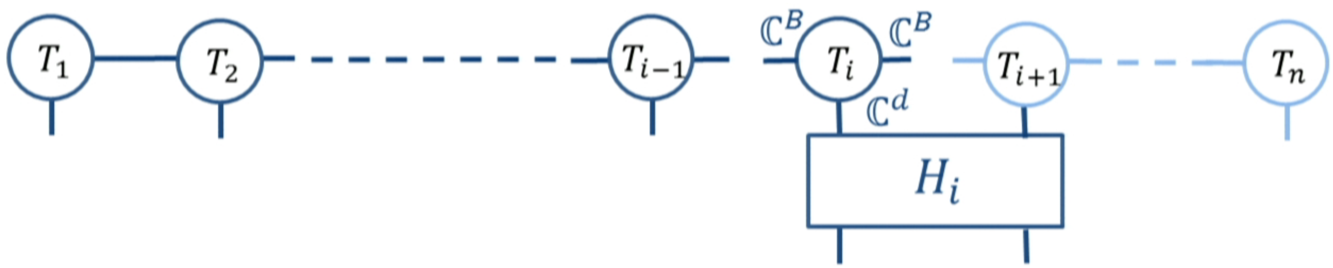


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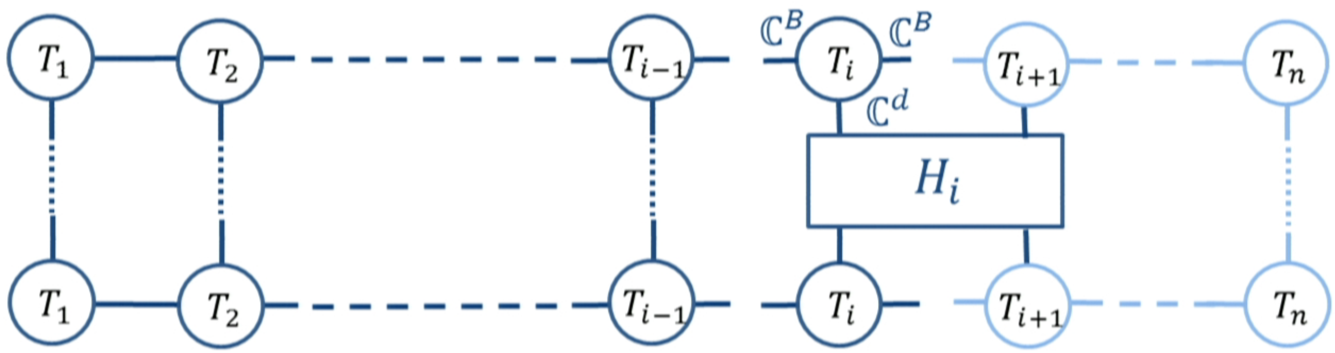


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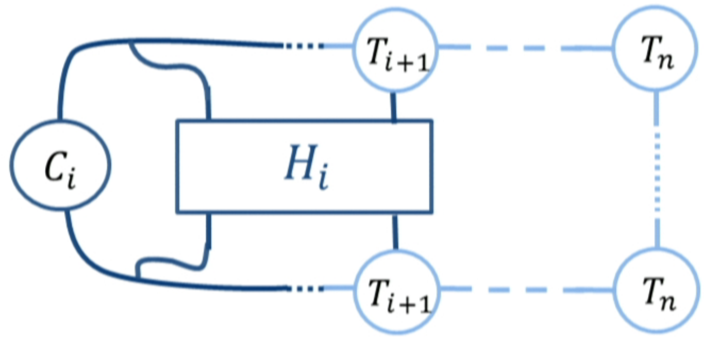


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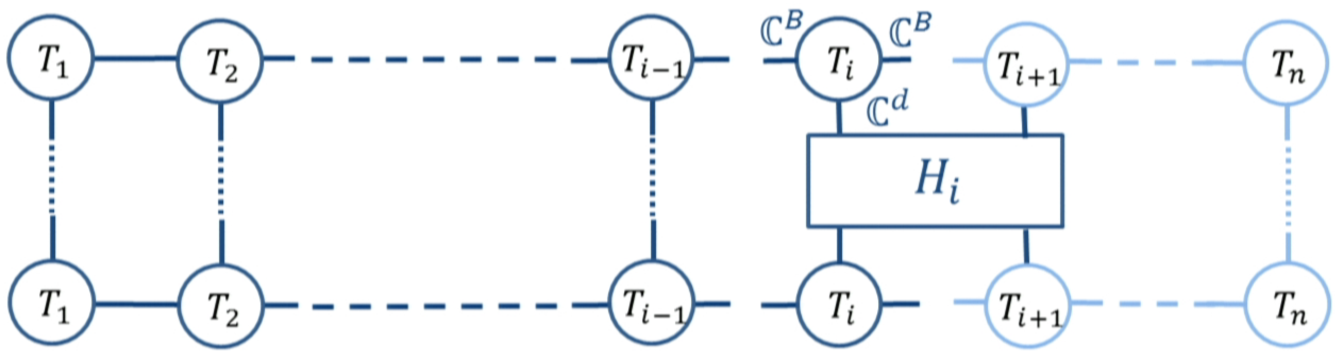


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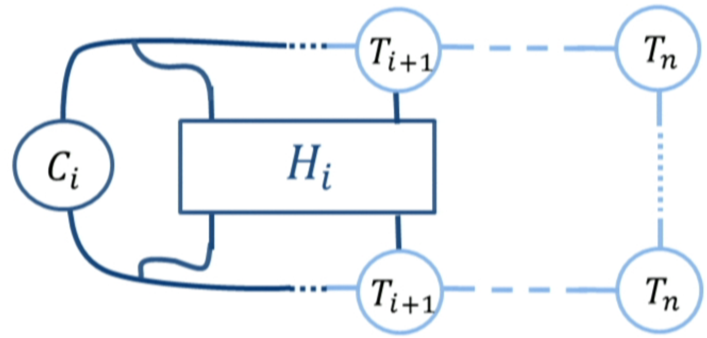


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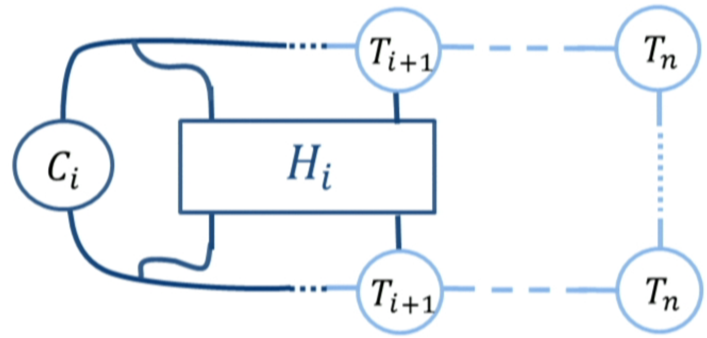


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Only depends on "right half" + contraction

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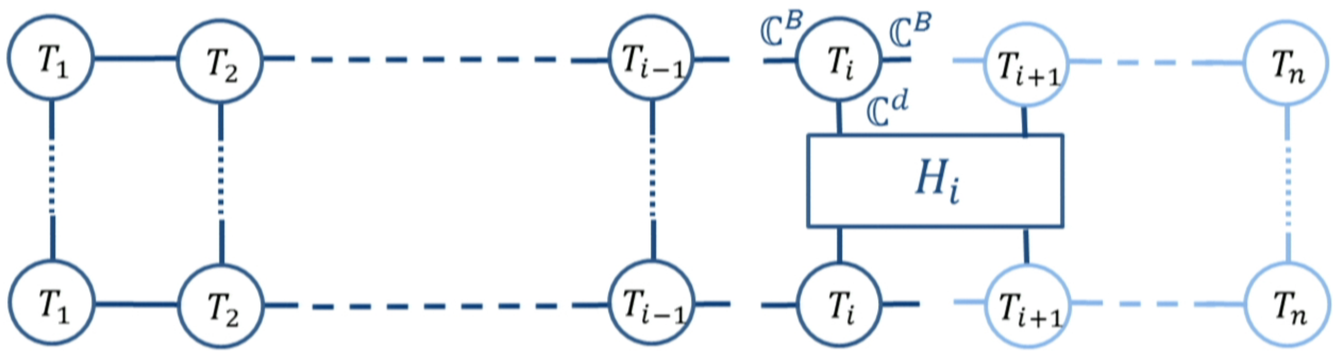


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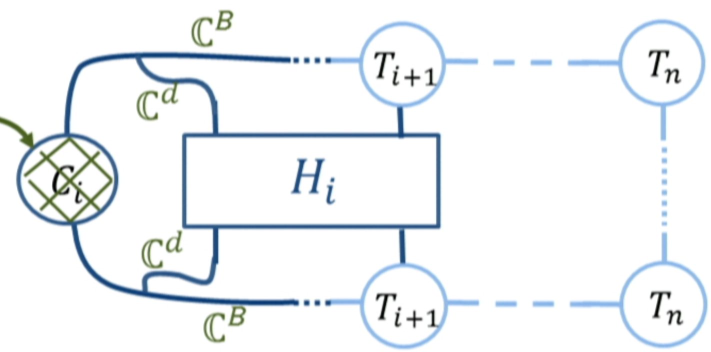
Discretize to precision $\eta$
$\rightarrow$ net of size $\approx (1/\eta)^{d^2 B^2}$

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$$= f_L(T_1, \ldots, T_i) + f_R(C_i, T_i, \ldots, T_n)$$

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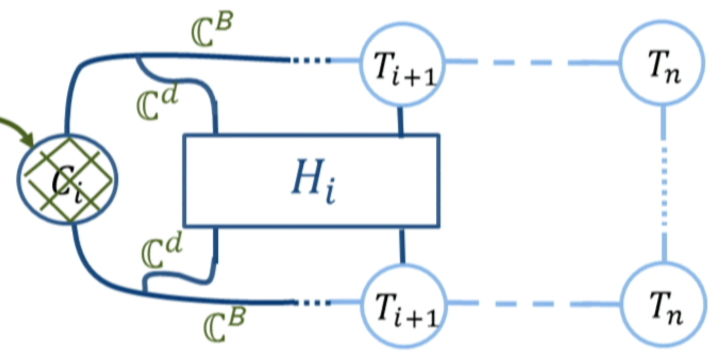


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$$= \quad f_L(T_1, \ldots, T_i) \quad + \quad f_R(C_i, T_i, \ldots, T_n)$$

- *Boundary contraction* $C_i$ is sufficient shared data to allow decoupling
- Matching "left halves" with equal energy are equally good

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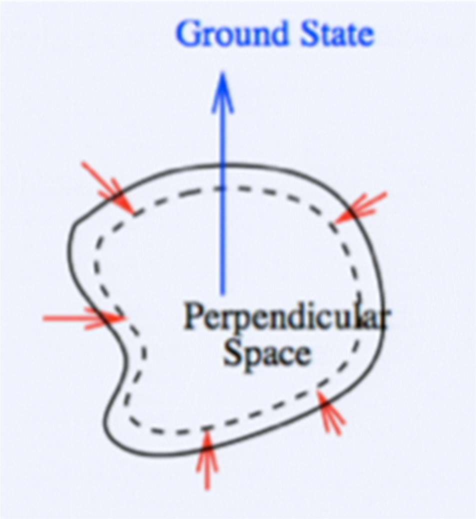

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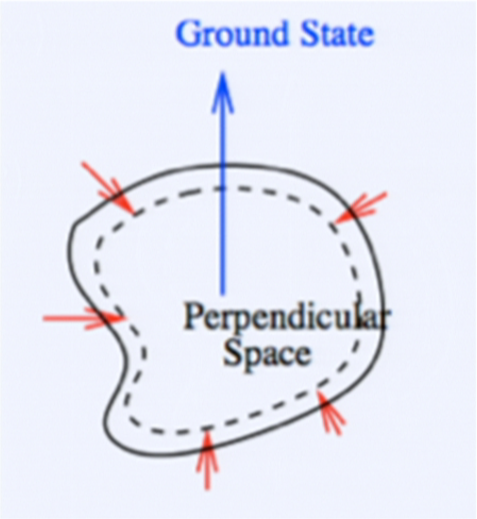


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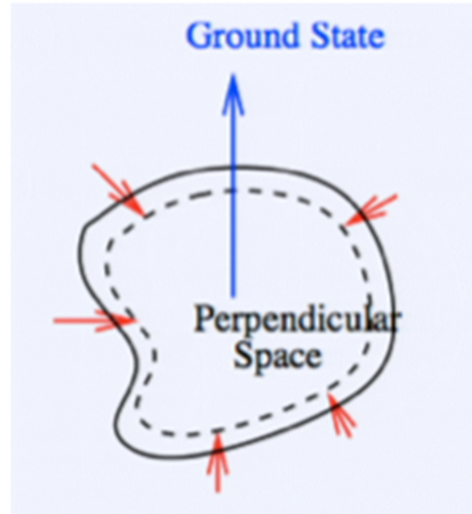


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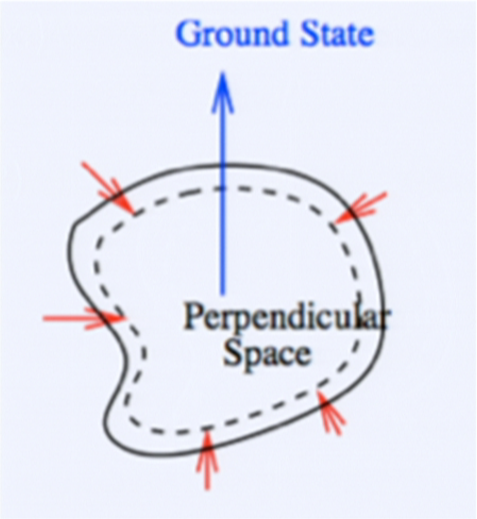


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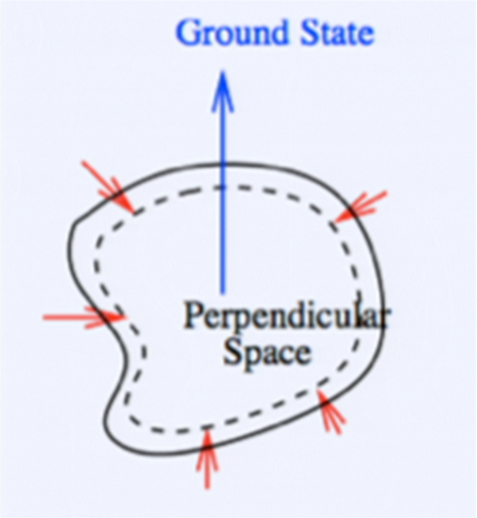


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  - Crucially relies on Hamiltonian being gapped

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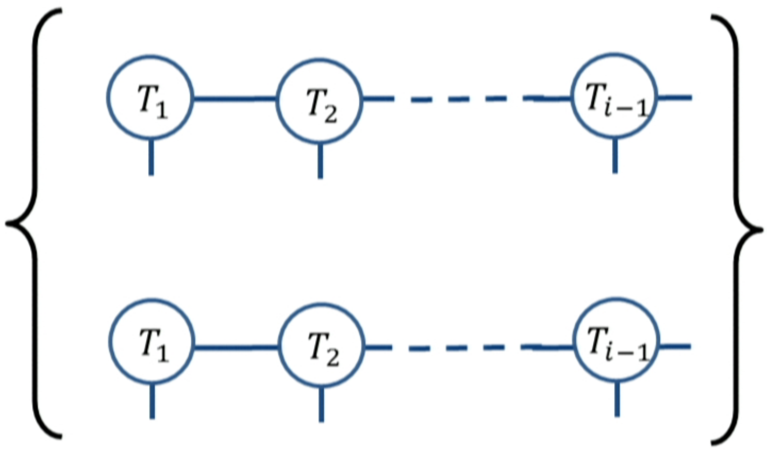

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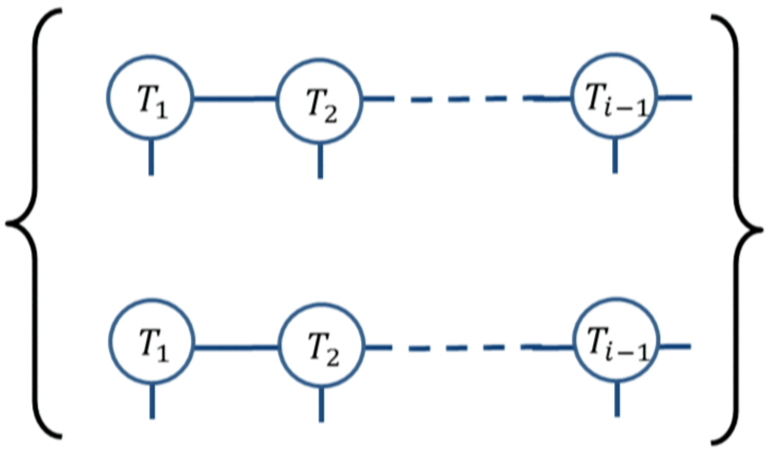

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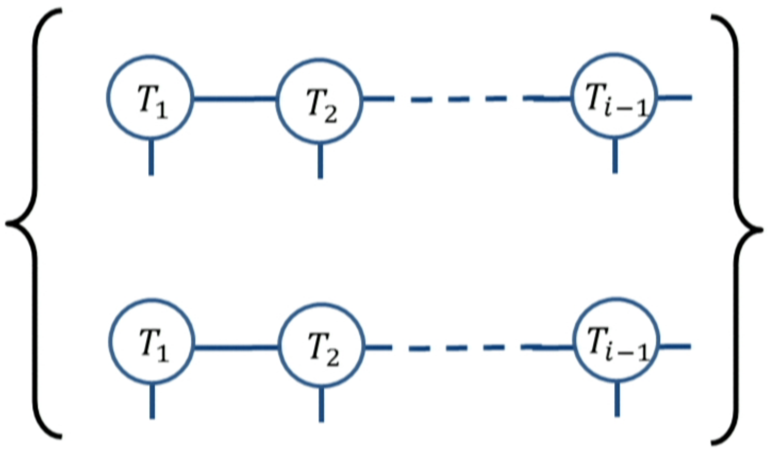

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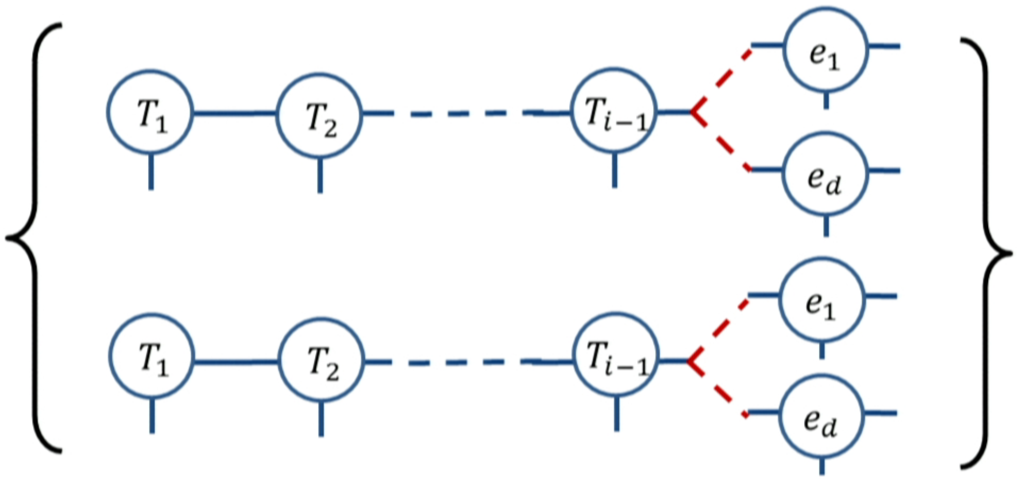

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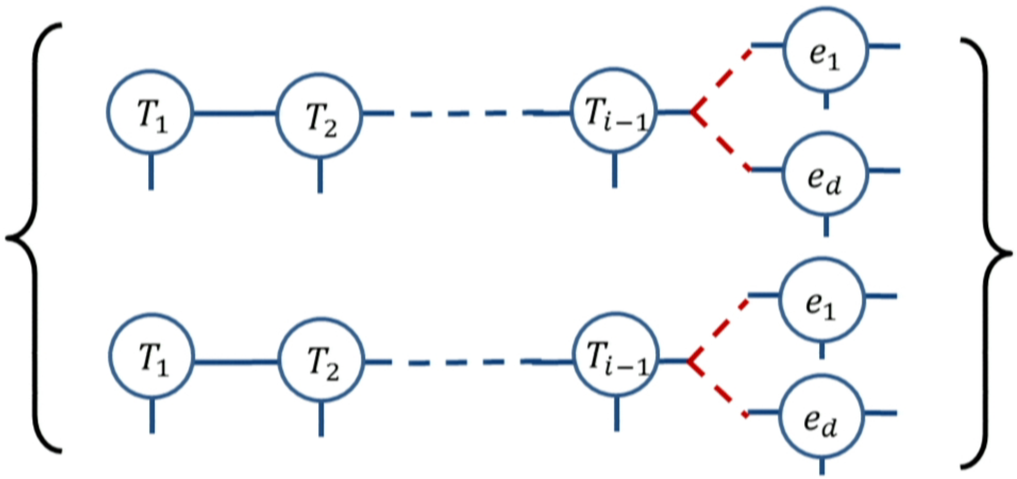

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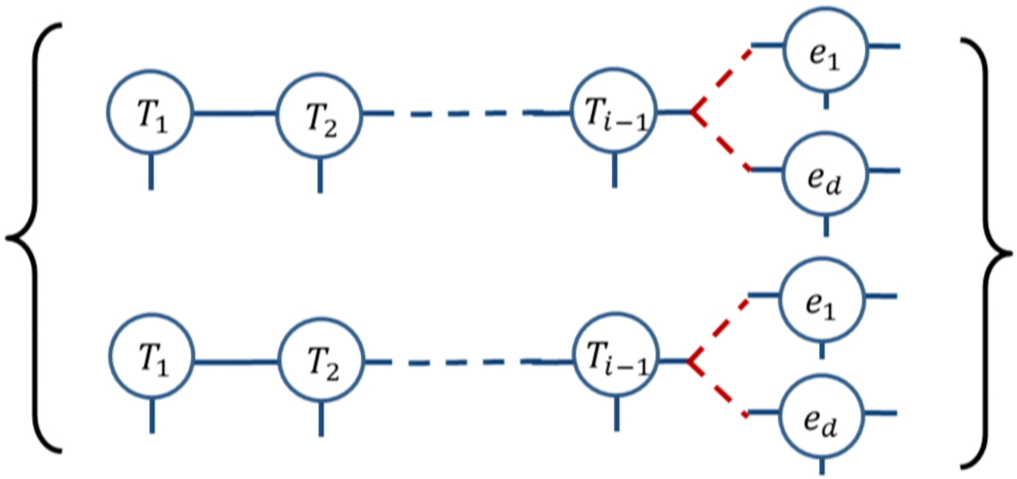

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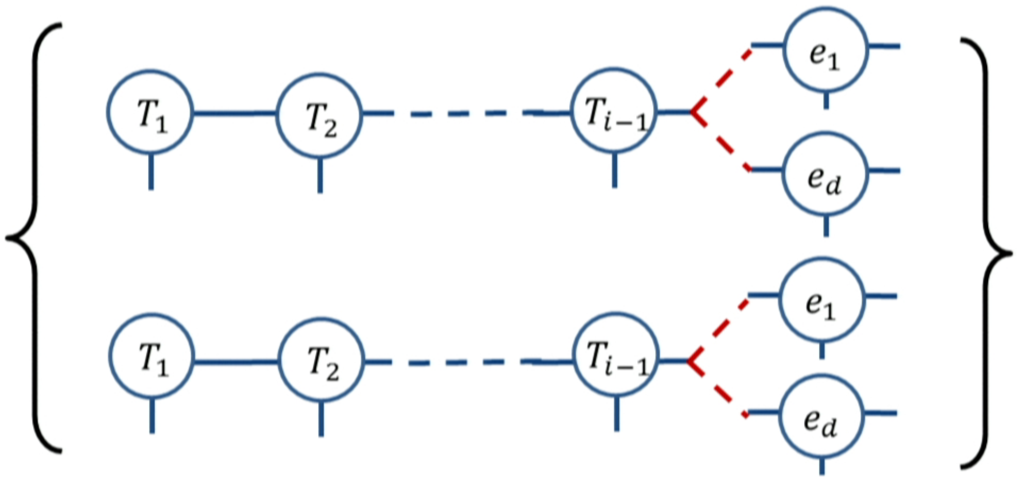
Decoupling

For each candidate $C$, keep minimal-energy “left half”



# A poly-time algorithm

$(i, \delta, s)$-viable set $S$: there exists a $\delta$-approximation to $|\mathrm{GS}\rangle$ whose reduced density on qudits $1, \ldots, i$ is supported on Span$(S)$.

- Initialization: $\{1\}$ is trivially $(0, \delta, 1)$-viable
- Invariant: $(i-1, \delta \approx 1/n^2, s)$- viable set $S_i$

1. Extend: $(i-1, \delta, s) \rightarrow (i, \delta, sd)$ (size blow-up)
2. Reduce: $(i, \delta, sd) \rightarrow (i, \frac{1}{10}, p(n))$ (error blow-up)

Easy: $S \rightarrow S \otimes \{e_1, \ldots e_d\}$

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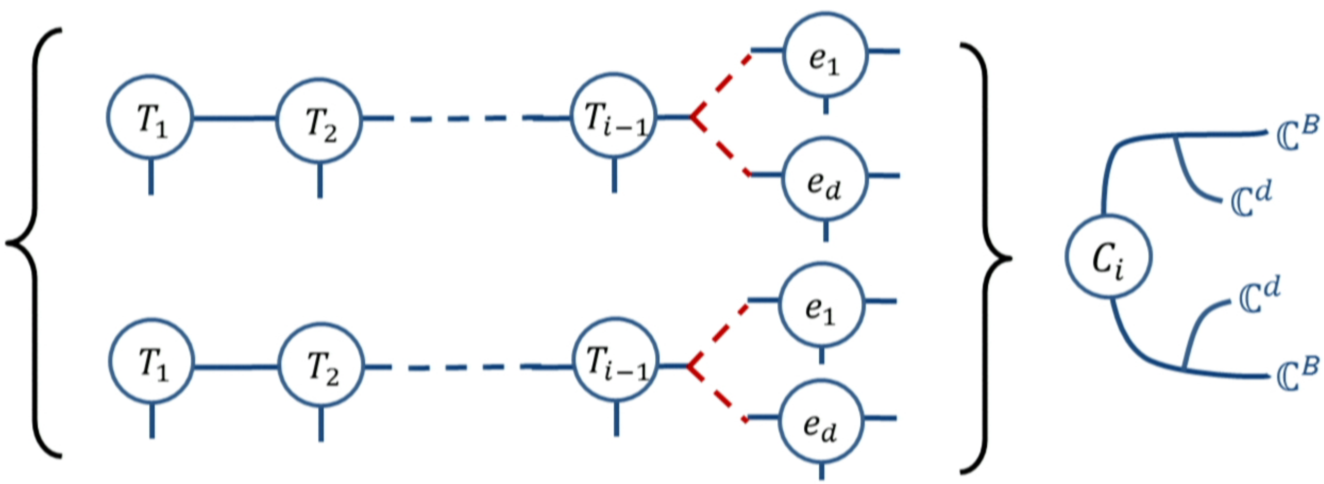

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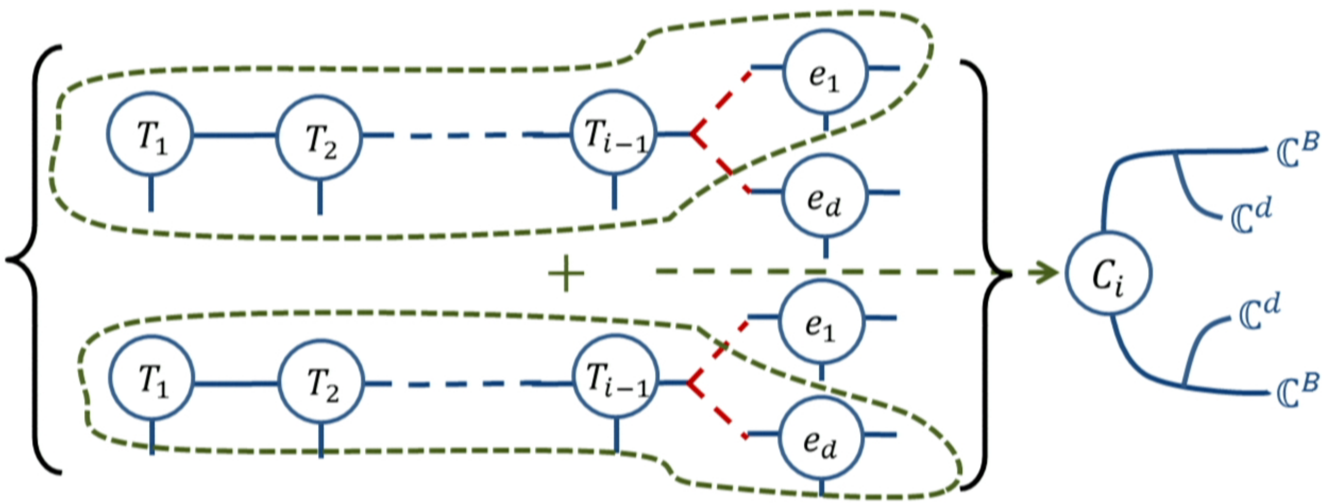

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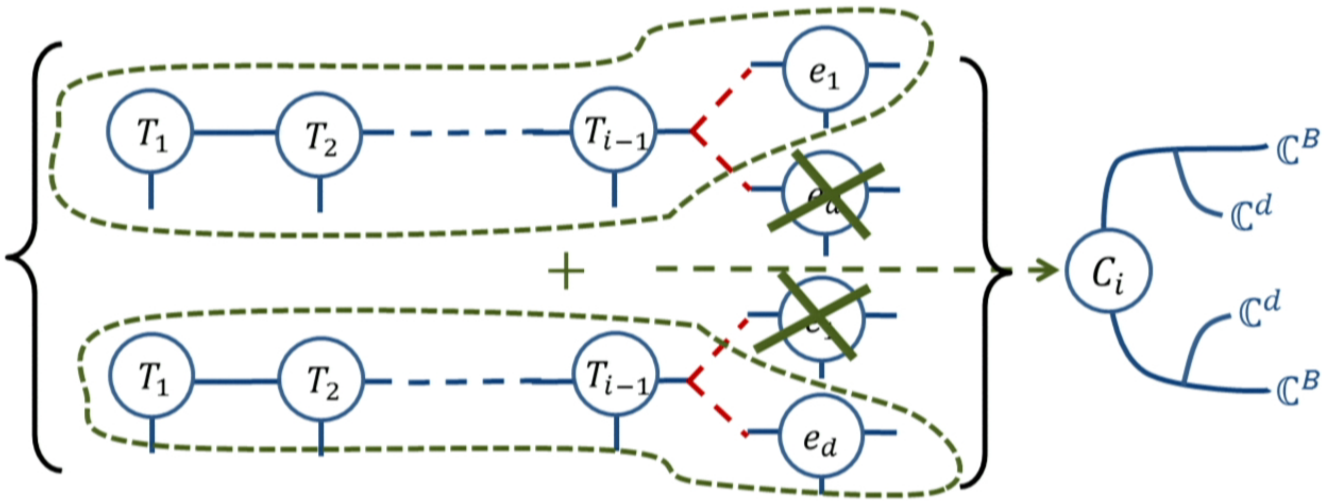

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3. Refine: $\left(i, \frac{1}{10}, p(n)\right) \rightarrow (i, \delta, s)$ (size blow-up)

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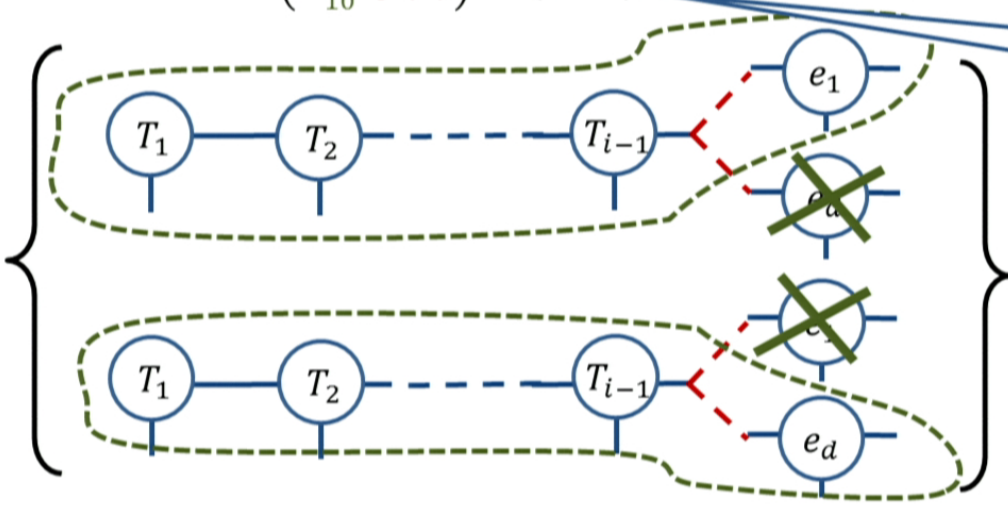
**Decoupling**
For each candidate $C$, keep minimal-energy "left half"

**AGSP**
"error reduction" procedure



# Step 2 (decoupling): reducing the cardinality of $S_i$

$(i, \delta, s)$-viable set $S_i$: there exists a $\delta$-approximation to $|\mathrm{GS}\rangle$ whose reduced density on qudits $1, \ldots, i$ is supported on $\mathrm{Span}(S_i)$.

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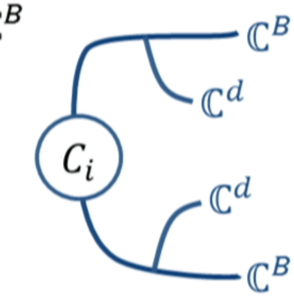

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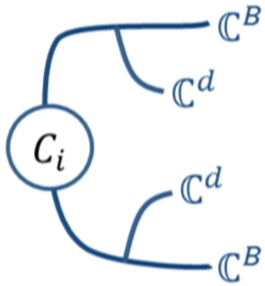

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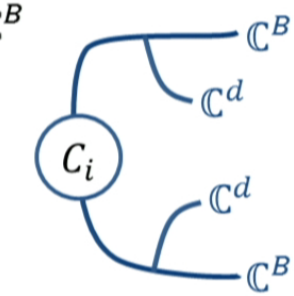


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$$\min \mathrm{Tr}((H_1 + \cdots + H_{i-1})\rho)$$
$$\rho \in \mathrm{Dens}(S_i), \quad \mathrm{Tr}_{1 \cdots i-1}\, \rho \approx \mathrm{C_i}$$

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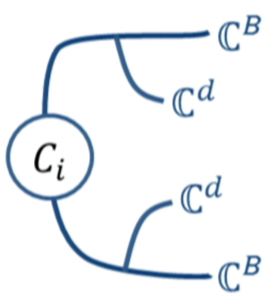


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- Convex program can be solved in time $\mathrm{poly}(n, s)$

# Step 3: improving the approximation quality

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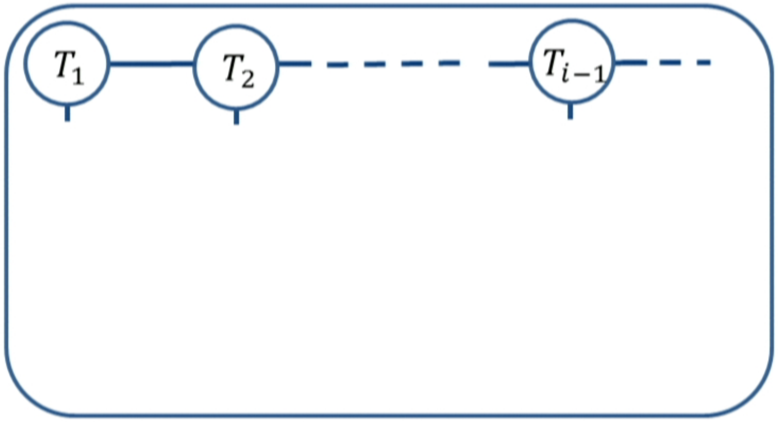


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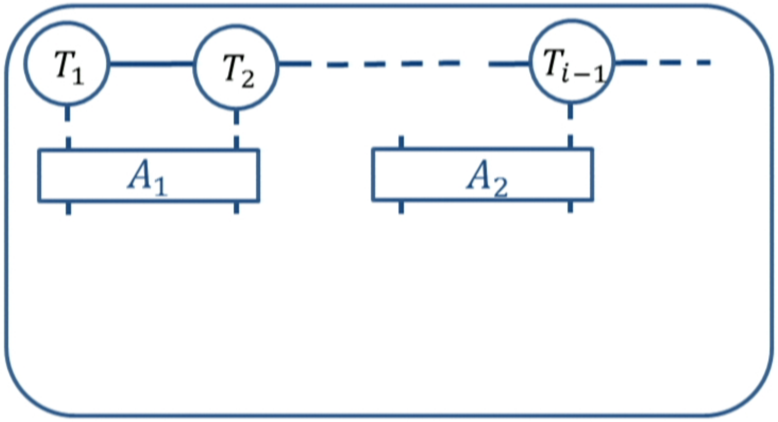


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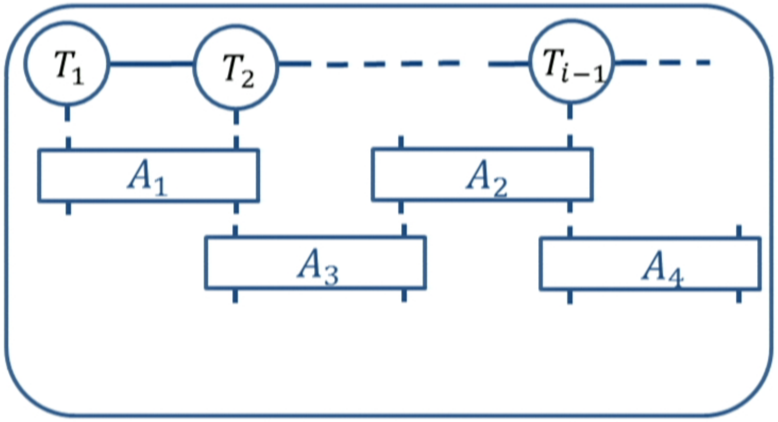


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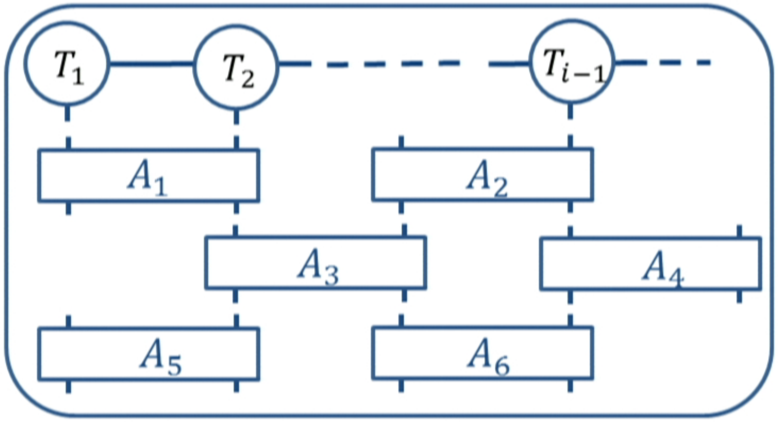


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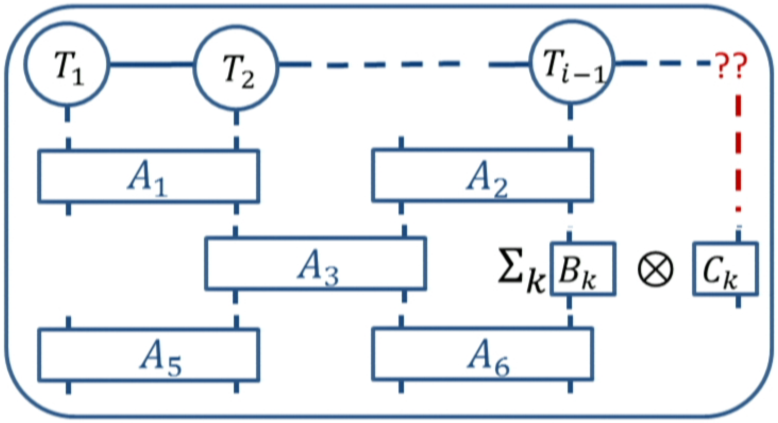


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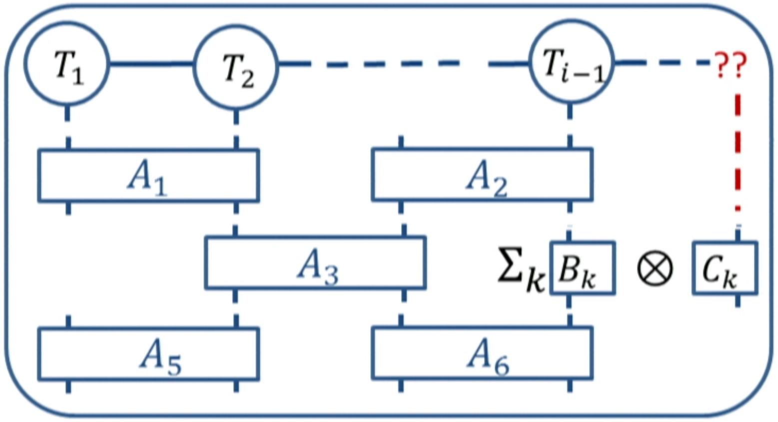


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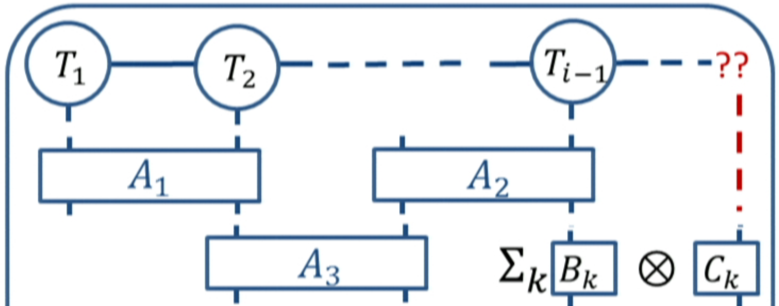


Need to ensure vectors in $S_i$ keep poly-size MPS representation
→ Additional technical "trimming" steps to keep bond dimensions poly-size

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# An efficient AGSP

Approximate Groud State Projection:

1. Local operator $A = \prod A_j$
2. Leaves GS invariant: $A|GS\rangle = |GS\rangle$
3. Reduces norm of any vector $\perp$ to GS: $\||A|GS^{\perp}\rangle\|| \leq (1-\Delta)\|||GS^{\perp}\rangle\||$

- Frustration-free case: $H|\mathrm{GS}\rangle = 0,$ $\langle \mathrm{GS}^{\perp}|H|\mathrm{GS}^{\perp}\rangle \geq \Delta$

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  → 2. and 3. are ok. What about locality?
- Expand: $\tilde{A} = \frac{1}{n^n}\big((1-H_1) + \cdots + (1-H_n)\big) = \sum_{i_1,\dots,i_n}(1-H_{i_1})\cdots(1-H_{i_n})$

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  → 2. and 3. are ok. What about locality?
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# An efficient AGSP

Approximate Groud State Projection:

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- Matrix-valued Chernoff bound: $\tilde{A} \approx \sum_{i_1,\ldots,i_n \in T} (1-H_{i_1})\cdots(1-H_{i_n})$
  for random subset $T \subseteq \{1, \ldots, n\}^n$ of $|\mathrm{T}| = \mathrm{poly}(\mathrm{n}, \log \eta^{-1})$ many terms

# Algorithm summary

$(i, \delta, s)$-viable set $S$: there exists a $\delta$-approximation to $|\mathrm{GS}\rangle$ whose reduced density on qudits $1, \ldots, i$ is supported on Span($S$).

- Initialization: $\{1\}$ is trivially $(0, \delta)$-viable
- Invariant: $(i-1, \delta, s)$-viable set $S_{i-1}$. Three steps:
  1. Extend: $(i-1, \delta, s) \to (i, \delta, sd)$ (size blow-up)
  2. Reduce: $(i, \delta, sd) \to (i, 1/10, p(n))$ (error blow-up)
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- Gapped 1D systems admit efficient descriptions and can be found efficiently
- Algorithm more complex than heuristics (non-local), but uses different ideas. Can they help improve the practical algorithms?
- Going beyond 1D remains a challenge
  - Is the gap sufficient? Area law not known in 2D
  - What is the appropriate notion of AGSP?

# Thank you!