Title: Physics, Logic and Mathematics of Time

Date: May 02, 2014 11:30 AM

URL: http://pirsa.org/14050032

Abstract: <span>Consider discrete physics with a minimal time step taken to be
 tau. A time series of positions q,q',q'', ... has two classical
 observables: position (q) and velocity (q'-q)/tau. They do not commute,
 for observing position does not force the clock to tick, but observing
 velocity does force the clock to tick. Thus if VQ denotes first observe
 position, then observe velocity and QV denotes first observe velocity,
 then observe position, we have
 VQ: (q'-q)q/tau
 QV: q'(q'-q)/tau
 (since after one tick the position has moved from q to q').
 Thus [Q,V]= QV - VQ = (q'-q)^2/tau. If we consider the equation
 [Q,V] = k (a constant), then k = (q'-q))^2/tau and this is recognizably
 the diffusion constant that arises in a process of Brownian motion.
 Thus, starting with the simplest assumptions for discrete physics, we are
 lead to recognizable physics. We take this point of view and follow it
 in both physical and mathematical directions. A first mathematical
 direction is to see how i, the square root of negative unity, is related
 to the simplest time series: ..., -1,+1,-1,+1,... and making the
 above analysis of time series more algebraic leads to the following
 interpetation for i. Let e=[-1,+1] and e'=[+1,-1] denote, as ordered
 pairs, two phase-shifted versions of the alternating series above.
 Define an operator b such that eb = be' and b^2 = 1. Regard b as a time
 shifting operator. The operator b shifts the alternating series by one
 half its period. Regard e' = -e and ee' = [-1.-1] = -1 (combining term by
 term). Then let i = eb. We have ii = (eb)(eb) = ebeb = ee'bb = -1. Thus ii = -1
 through the definition of i as eb, a temporally sensitive entity that
 shifts it phase in the course of interacting with (a copy of) itself.
 By going to i as a discrete dynamical system, we can come back to the
 general features of discrete dynamical systems and look in a new way at
 the role of i in quantum mechanics. Note that the i we have constructed is
 already part of a simple Clifford algebra generated by e and b with
 ee = bb = 1 and eb + be = 0. We will discuss other mathematical physical
 structures such as the Schrodinger equation, the Dirac equation and the
 relationship of a simple logical operator (generalizing negation) with
 Majorana Fermions.

Physics, Logic and Mathematics of Time

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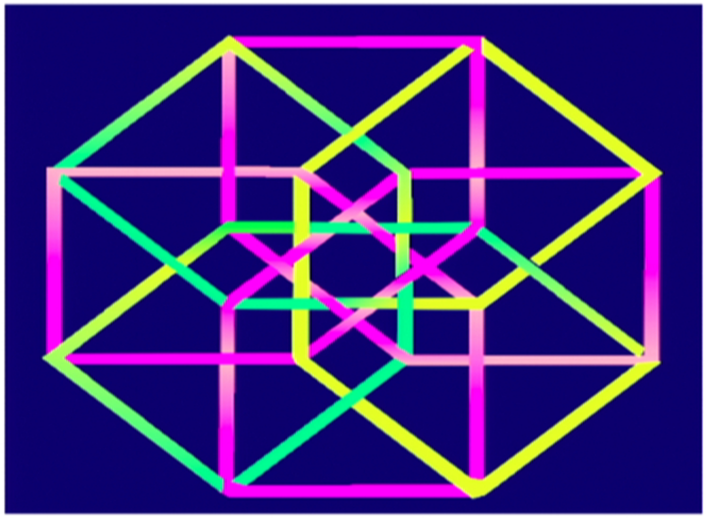


God Does Not Play Dice! Here is a little story about the square root of minus one and quantum mechanics. God said - I would really like to be able to base the universe on the Diffusion Equation

$$\partial \psi / \partial t = \kappa \partial^2 \psi / \partial x^2.$$

But I need to have some possibility for interference and waveforms. And it should be simple. So I will just put a "plus or minus" ambiguity into this equation, like so:

$$\pm \partial \psi / \partial t = \kappa \partial^2 \psi / \partial x^2.$$

This is good, but it is really terrible. I do not play dice. The $\pm$ coefficient will have to be lawful, not random. Nothing is random. What to do? Aha! I shall take $\pm$ to mean the alternating sequence

$$\pm = \cdots + - + - + - + - \cdots$$

and time will become discrete. Then the equation will become a time series

$$\psi_{t+1} - \psi_t = (-1)^t \kappa(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx))$$

where

$$\partial_x^2 \psi_t = \psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx).$$

This will do it, but I have to consider the continuum limit, and there, there is no meaning to

$$(-1)^t$$

in the realm of continuous time. What do do? Ah! In the discrete world my wave function (not a bad name for it!) divides into ψ_e and ψ_o where the time is either even or odd. So I can write

$$\partial_t \psi_e = \kappa \partial_x^2 \psi o$$

and

$$\partial_t \psi_o = -\kappa \partial_x^2 \psi_e.$$

I will take the continuum limit of ψ_e and ψ_o *separately!*

Finally! A use for that so called imaginary number that Merlin has been bothering me with (You might wonder how Merlin could do this when I have not created him yet, but after all I am that am.). This i has the property that $i^2 = -1$ so that

$$i(A + iB) = iA - B.$$

$$i = -1/i,$$

and so you see that if $i = 1$ then $i = -1$, and if $i = -1$ then $i = 1$. So i just spends its time oscillating between $+1$ and -1, but it does it lawfully and so I can regard it as a definition that

$$i = \pm 1.$$

(In fact, I can see now what Merlin what getting at. When I multiply $ii = (\pm 1)(\pm 1)$, I get -1 because the i takes a little time to oscillate and so by the time this second term multiplies the first term, they are just our of phase and so we get either $(+1)(-1) = -1$ or $(-1)(+1) = -1$. Either way, $ii = -1$ and we have the perfect ambiguity.) Heh. People will say that I am playing dice, but it is just not so. Now ± 1 behaves quite lawfully and I can write the true equation of the World

$$\psi = \psi_e + i\psi o,$$

$$i\partial\psi/\partial t = \kappa\partial^2\psi/\partial x^2.$$

I shall call this the Schroedinger equation. Now I can rest on this seventh day before the real creation. This is the imaginary creation.

Discrete Measurement is Intrisically Non-commutative.

Time Series: $X, \; X', \; X'', \; \ldots$

Derivative: $\dot{X} = (X' - X)/dt = dX/dt$

Here dt and dX are finite increments.

$X\dot{X}$: Observe $\dot{X}$, then observe X.

$\dot{X}X$: Observe X, then observe $\dot{X}$.

$$X\dot{X} = X'(X'-X)/dt$$

$$\dot{X}X = (X'-X)X/dt$$

$$X\dot{X} - \dot{X}X = (X'-X)(X'-X)/dt$$

$$[X,\dot{X}] = (dX)^2/dt$$

$$X\dot{X} - \dot{X}X = (X'-X)(X'-X)/dt$$

$$[X,\dot{X}] = K \text{ then } K = (dx)(dx)/dt$$

$$X' = X \pm dx$$

The discrete analog of Heisenberg's equation yields a Brownian walk with diffusion constant K.

Recalling the Diffusion Constant and the Diffusion Equation

P(x,t) = Probability that the particle is at x at time t.

$P(x,t+dt) = (1/2)[P(x+dx,t) + P(x-dx,t)]$,whence

$P(x,t+dt) - P(x,t) = (1/2)[P(x+dx,t) - P(x,t) \;\; -(P(x,t) - P(x+dx,t))]$

$dP/dt = (K/2)\, d^2 P/dx^2$ Diffusion Equation

$K = (dx)^2/dt$ Diffusion Constant

We have just seen the diffusion constant arise differently(!) in the context of discrete process commutators, with no second difference.

Discrete calculus is embedded in commutator calculus:

$\dot{X}$ is a signal to time-shift the algebra to its left.

Make algebraic by defining new operator J with
$$XJ = JX'.$$

Redefine
$$\dot{X} = J(X' - X)/dt.$$

Then $\dot{X} = (XJ - JX)/dt = [X, J/dt]$.

$\dot{X}$ satisfies the Leibniz rule.

Embed Discrete Calculus in Non-Commutative Calculus.

$\tilde{f}(x) = f(x + h)$

$Df = (\tilde{f} - f)/h$ Discrete derivative

$D(fg) = D(f)g + \tilde{f}D(g)$ Pseudo Leibniz rule

$fJ = J\tilde{f}$ Introduce Shift Operator

$\nabla(f) = JD(f)$ Redefine Derivative

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g)$$

$$\nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h]$$

Leibniz Rule is restored, and new derivative is a commutator.

Emergence of the Diffusion Constant

Thus we can interpret the equation

$$[X, \dot{X}] = Jk$$

(k a constant scalar) as

$$(X' - X)^2/\tau = k.$$

This means that the process is a Brownian walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$

where k is a constant. In other words, we have

$$k = \Delta^2/\tau.$$

We have shown that a Brownian walk with spatial step size Δ and time step τ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This means that *a given commutator equation can be satisfied by walks with arbitrarily small spatial step and time step, just so long as these steps are in this fixed ratio.*

Heisenberg/Schrödinger Equation. Here is how the Heisenberg form of Schrödinger's equation fits in this context. Let the time shift operator be given by the equation $J = (1 + H\Delta t/i\hbar)$. Then the non-commutative version of the discrete time derivative is expressed by the commutator

$$\nabla\psi = [\psi, J/\Delta t],$$

and we calculate

$$\nabla\psi = \psi[(1 + H\Delta t/i\hbar)/\Delta t] - [(1 + H\Delta t/i\hbar)/\Delta t]\psi = [\psi, H]/i\hbar,$$

$$i\hbar\nabla\psi = [\psi, H].$$

This is exactly the Heisenberg version of the Schrödinger equation.

The Heisenberg Commutator

$$[x,(Dx)] = J\,(\Delta x)^2/\Delta t$$

which we will simplify to

$$[q,\ p/m] = (\Delta x)^2/\Delta t.$$

taking q for the position x and p/m for velocity, the time derivative of position.

Understanding that Δt should be replaced by $i\Delta t$, and that

$$(\Delta x)^2/\Delta t = \hbar/m, \quad \text{(at Planck length and time)}$$

we have

$$[q,\ p/m] = (\Delta x)^2/i\Delta t = -i\ \hbar/m$$

whence

$$[p,q] = i\hbar.$$

A non-commutative world of flat coordinates suitable for advanced calculus.

The flat coordinates X_i satisfy the equations below with the P_j chosen to represent differentiation with respect to X_j:

$$[X_i, X_j] = 0, \quad \text{Coordinates Commute.}$$
$$[P_i, P_j] = 0, \quad \text{Partials commute.}$$
$$[X_i, P_j] = \delta_{ij}. \quad \text{Derivative formula.}$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F/\partial X_i = [F, P_i],$$
$$\hat{\partial}_i F = \partial F/\partial P_i = [X_i, F].$$

Temporal derivative is represented by commutation with a special (Hamiltonian) element H of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take $i\hbar dA/dt = [A, H]$.)

If $[Q, P] = 1$ $\quad \partial F/\partial Q = [F, P]$

$\dot{F} = [F, H]$ $\quad \partial F/\partial P = [Q, F]$

Then

$$\dot{P} = [P, H] = -[H, P] = -\partial H/\partial Q$$
$$\dot{Q} = [Q, H] = \partial H/\partial P$$

Hamilton's Equations follow from non-commutative calculus.

Hamilton's Equations express the Mathematics of a Non-Commutative Flat World.

$$dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H/\partial X_i$$

$$dX_i/dt = [X_i, H] = \partial H/\partial P_i.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

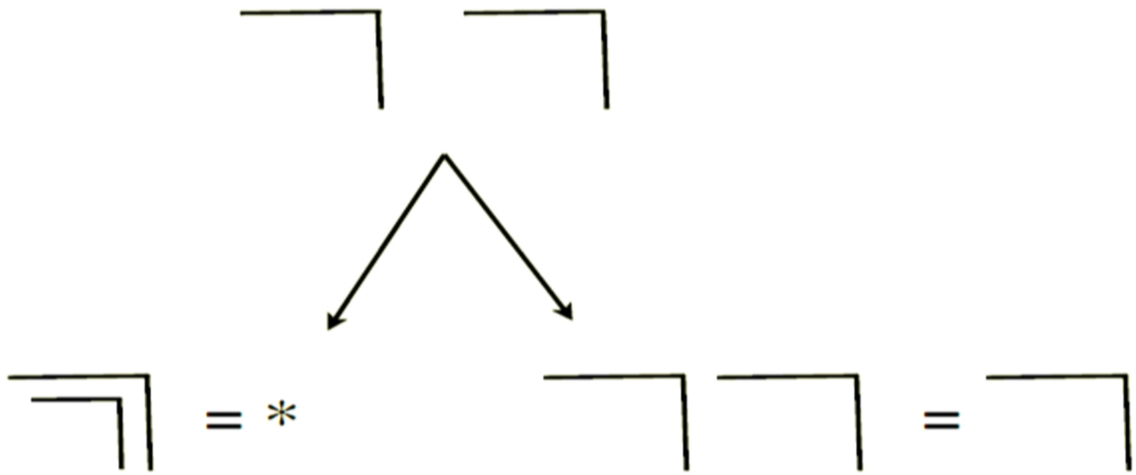
One can explore formulations of discrete physics in non-commutative context.
We will not go further with this in the present slide show.

The context of non-commutative calculus where the derivatives are represented by commutators is directly related to physics in a new way by this translation of the discrete. This also suggests opening the books again on the relationship of commutators and quantum theory.

We now examine how elements of matrix algebra can be seen as discrete dynamical systems. Thus Time and Algebra are intertwined as Hamilton knew.

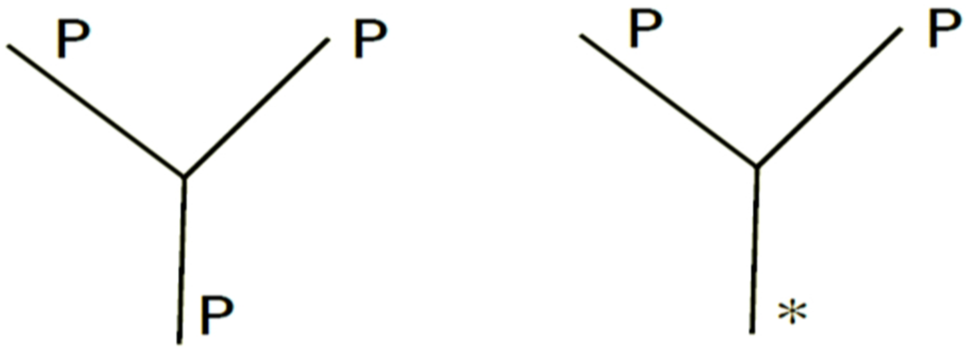
This part is motivated by G. Spencer-Brown's invention of a 'logical particle' that interacts with itself to either confirm itself or to cancel itself. This interaction, combined with recursion, leads both to matrix algebra and the very elementary mathematics of a Majorana Fermion.

In Laws of Form (G. Spencer-Brown)
Negation emerges from an operator that interacts with itself either to annihilate itself, or to produce itself.

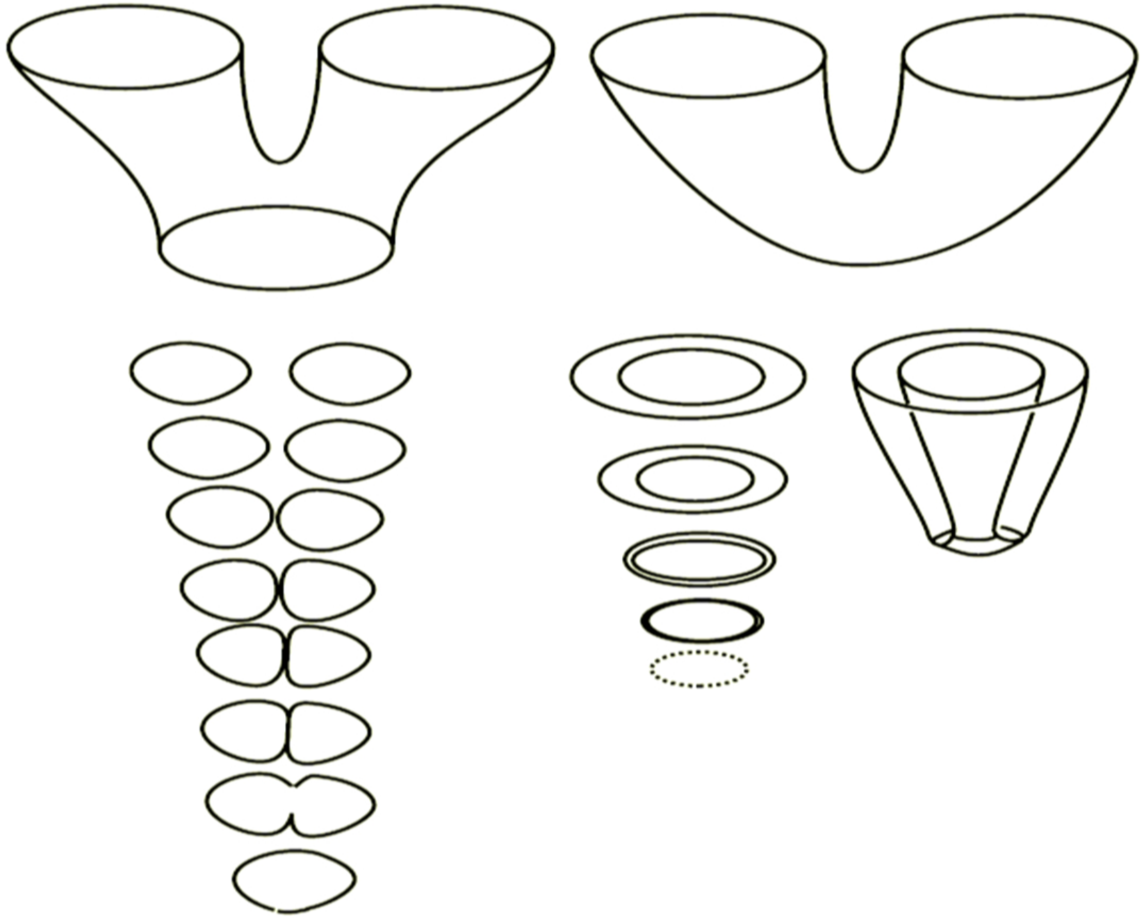


The Mark is a "logical particle" for a level of logic deeper than Boolean Logic.

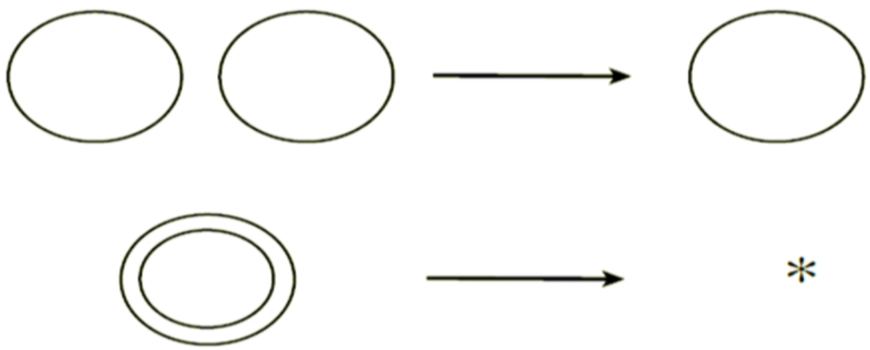
A Very Elementary Particle - Fusion Rules for a Majorana Fermion



The “particle” P interacts with P
to produce either P or *.
The particle * is neutral.



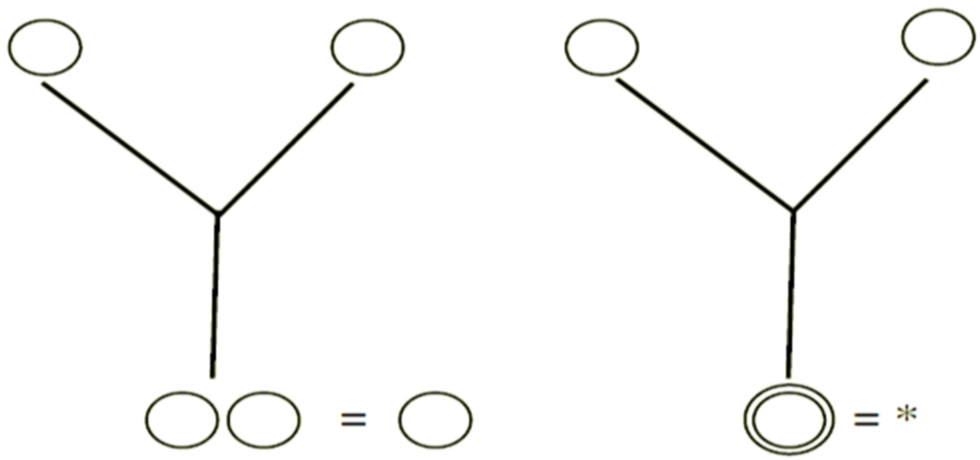
Formally, we can distinguish the two interactions via adjacency and concentricity.



PP = * + P

And From Logic Alone?

Note that we have shown how the formalism of the mark, as logical particle is coherent with its interpretation as a Majorana Fermion.



Ludwig Wittgenstein

Tractatus Logico-Philosophicus

4.0621 That, however, the signs "p" and "$\sim p$" *can* say the same thing is important, for it shows that the sign "$\sim$" corresponds to nothing in reality.

5.511 How can the all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, to the great mirror.

5.6 *The limits of my language* mean the limits of my world.

5.632 The subject does not belong to the world but it is a limit of the world.

The Bare Bones of a Majorana Fermion from Logic Alone?

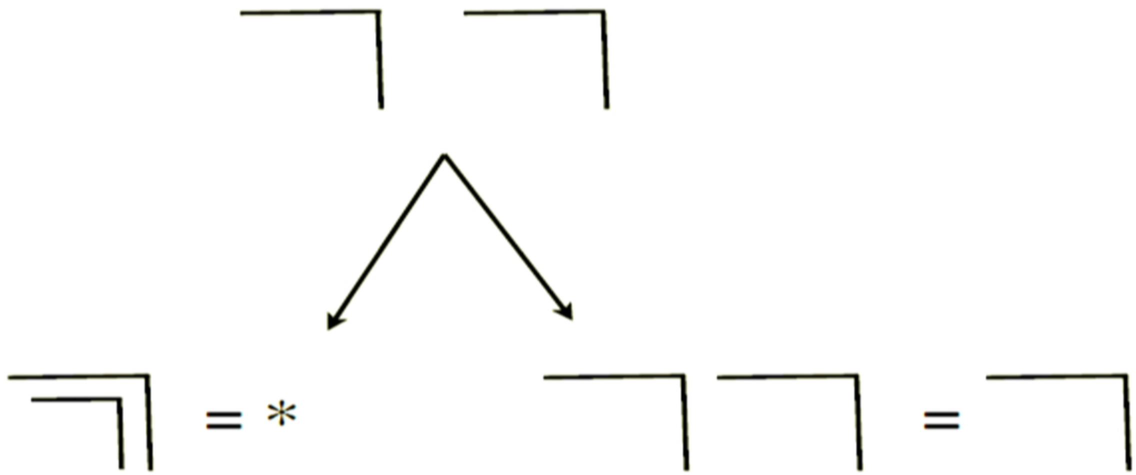
$\sim\sim Q = Q$ in Boolean logic.

Can we write
$\sim\sim = *$?

Can negation interact
with itself to produce
Nothing (as above)?

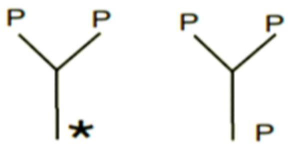
Can negation interact
with itself to produce itself?

In Laws of Form (G. Spencer-Brown)
Negation emerges from an operator that interacts with itself either to annihilate itself, or to produce itself.



The Fibonacci particle is a "logical particle" for a level of logic deeper than Boolean Logic.

Fibonacci Model



$$\delta = -A^2 - A^{-2}$$

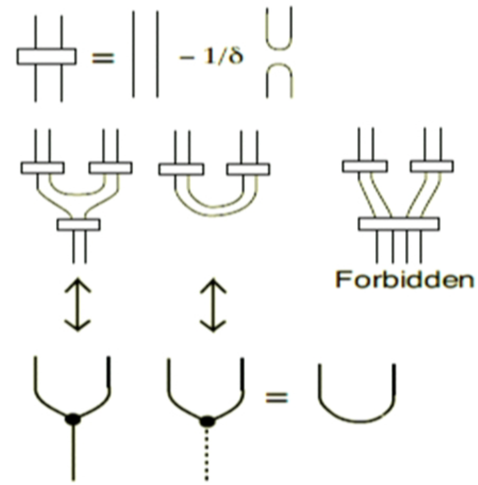
$$\Delta = \delta = (1 + \sqrt{5})/2.$$

$$F = \begin{pmatrix} 1/\Delta & 1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & -1/\Delta \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

$$R = \begin{pmatrix} -A^4 & 0 \\ 0 & A^8 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}.$$

Braid Representations Dense in Unitary Groups

$$A = e^{3\pi i/5}.$$



Temperley Lieb Representation of Fibonacci Model

The Fibonacci Model yields a braid group representation that is universal for quantum computation. It is a braid group representation that is dense in the unitary groups.

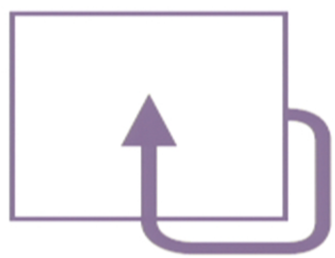
The structure of this representation is also theoretically realized in the present models of the quantum Hall effect.

THE SQUARE ROOT OF MINUS ONE IS A CLOCK.

From $G = \overline{\;G\;}|$ to $i = -1/i$.

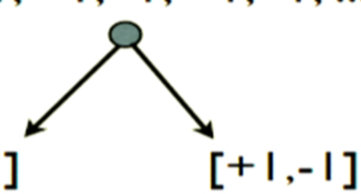
i as an imaginary value,
defined in terms of itself.

$i = -1/i$ $\qquad\qquad$ $ii = -1$



The square root
of minus one
"is"
a discrete oscillation.

... +1, -1, +1, -1, +1, -1, ...



We introduce a *temporal shift operator* η such that

$$[a, b]\eta = \eta[b, a]$$

and

$$\eta\eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots ababab ab \cdots .$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define i by the equation

$$i = [1, -1]\eta.$$

This makes i both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

$$e = [1, -1].$$

$$e^2 = [1, 1] = 1$$

$$e\eta = [1, -1]\eta = [-1, 1]\eta = -e\eta.$$

$$e^2 = 1,$$

$$\eta^2 = 1,$$

$$e\eta = -\eta e.$$

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

$$[a,b] + [c,d]\eta \longleftrightarrow \boxed{\begin{matrix} a & c \\ d & b \end{matrix}}$$

Let $A = [a,b]$ and $B = [c,d]$ and let $C = [r,s]$, $D = [t,u]$. With $A' = [b,a]$, we have

$$(A + B\eta)(C + D\eta) = (AC + BD') + (AD + BC')\eta.$$

This writes 2 x 2 matrix algebra in the form of a hypercomplex number system. From the point of view of iterants, the sum $[a,b] + [b,c]\eta$ can be regarded as a superposition of two types of observation of the iterants $I\{a,b\}$ and $I\{c,d\}$. The operator-view $[c,d]\eta$ includes the shift that will move the viewpoint from $[c,d]$ to $[d,c]$, while $[a,b]$ does not contain this shift. Thus a shift

The Creation/Annihilation algebra for a Majorana Fermion is very simple.
Just an element a with $aa = 1$.
If there are two Majorana Fermions, we have
a, b
with $aa = 1$, $bb = 1$ and
$ab + ba = 0$.

Majorana Fermions are their own antiparticles.

Mathematically an Electron's creation and annihilation operators are combinations of Majorana Fermion operators:

$$U = a + ib \quad \text{and} \quad U^* = a - ib$$

where $ab+ba = 0$ and $aa = bb = 1$.

Note $UU = (a+ib)(a+ib) = aa - bb + i(ab + ba) = 0$ and $U^*U^* = 0$.

$$UU^* + U^*U = (U + U^*)(U+U^*) = 4aa = 4$$

This is (unnormalized) creation/annihilation algebra for an electron.

A row of n electrons can be regarded as a row of 2n Majorana Fermions.

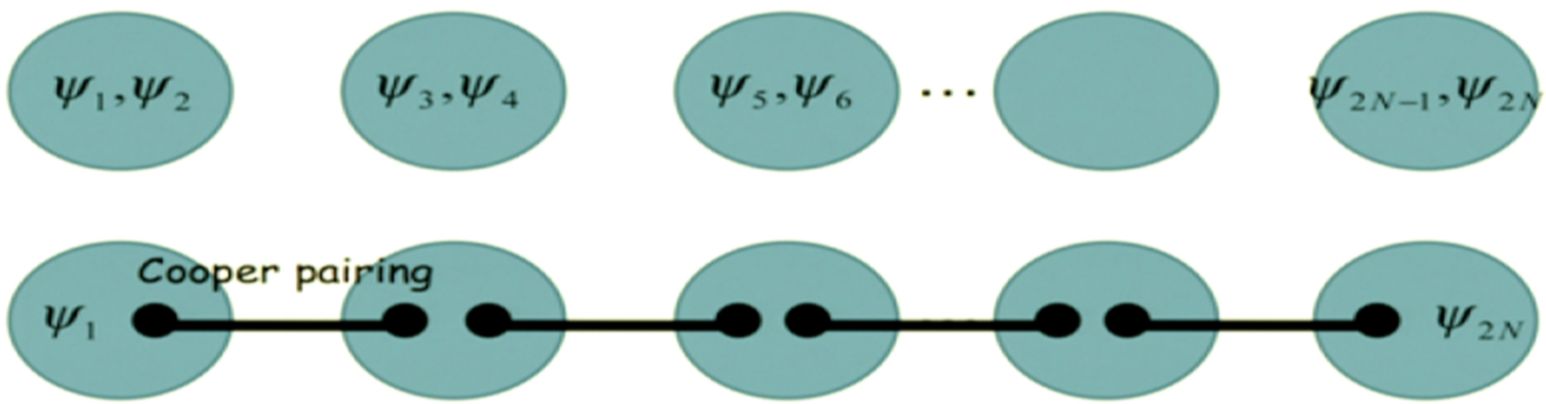
Recent work suggests that Majorana Fermions can be detected in nanowires.

Majorana (real) Fermions

f^+, f Usual (complex) fermions

$\psi = (f^+ + f)/\sqrt{2}$ ➡ $\psi = \psi^+$ $\psi^2 = 1$

$f = (\psi_1 + i\psi_2)/\sqrt{2}$ "half" of the usual (complex) fermion
"real" fermion



ψ_1, ψ_{2N} Single fermion → 1 q-bit

It is worth noting that a triple of Majorana fermions say a, b, c gives rise to a representation of the quaternion group. This is a generalization of the well-known association of Pauli matrices and quaternions. We have $a^2 = b^2 = c^2 = 1$ and they anticommute. Let $I = ba$, $J = cb$, $K = ac$. Then

$$I^2 = J^2 = K^2 = IJK = -1,$$

giving the quaternions. The operators

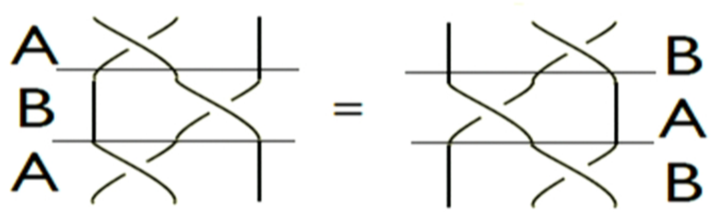
$$A = (1/\sqrt{2})(1 + I)$$

$$B = (1/\sqrt{2})(1 + J)$$

$$C = (1/\sqrt{2})(1 + K)$$

braid one another:

$$ABA = BAB, BCB = CBC, ACA = CAC.$$



Majoranas are related to standard fermions as follows: The algebra for Majoranas is $c = c^\dagger$ and $cc' = -c'c$ if c and c' are distinct Majorana fermions with $c^2 = 1$ and $c'^2 = 1$. One can make a standard fermion from two Majoranas via

$$\psi = (c + ic')/2,$$

$$\psi^\dagger = (c - ic')/2.$$

Similarly one can mathematically make two Majoranas from any single fermion. Now if you take a set of Majoranas

$$\{c_1, c_2, c_3, \cdots, c_n\}$$

then there are natural braiding operators that act on the vector space with these c_k as the basis. The operators are mediated by algebra elements

$$\tau_k = (1 + c_{k+1}c_k)/\sqrt{2},$$

$$\tau_k^{-1} = (1 - c_{k+1}c_k)/\sqrt{2}.$$

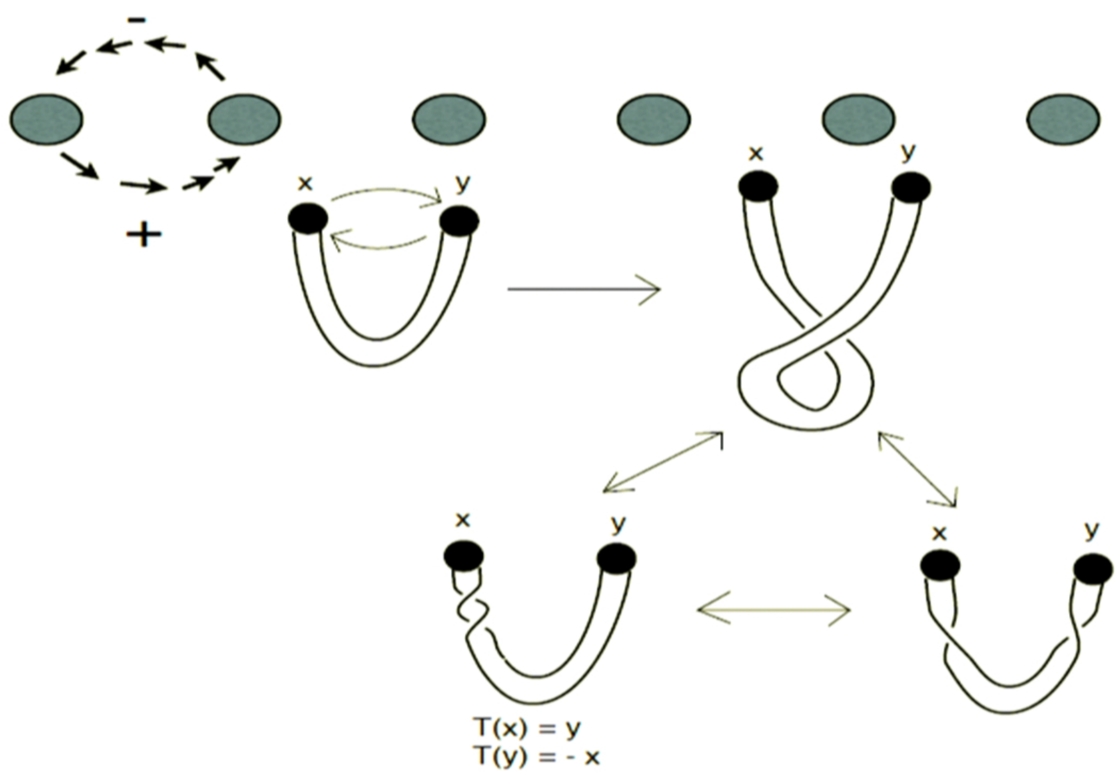
Then the braiding operators are

$$T_k : Span\{c_1, c_2, \cdots, , c_n\} \longrightarrow Span\{c_1, c_2, \cdots, , c_n\}$$

via

$$T_k(x) = \tau_k x \tau_k^{-1}.$$

Braiding Majorana Fermions



Now we show how the Majorana Fermion algebra is at the base of the Dirac equation, and how nilpotent operators (representing Fermions) arise naturally in relation to plane wave solutions to the Dirac equation.

5.2 Relativity and the Dirac Equation

Starting with the algebra structure of e and η and adding a commuting square root of -1, i, we have constructed fermion algebra and quaternion algebra. We can now go further and construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy E, momentum p and mass m are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$

Recapitulation and One More

We start with $\psi = e^{i(px-Et)}$ and the operators

$$\hat{E} = i\partial/\partial_t$$

and

$$\hat{p} = -i\partial/\partial_x$$

so that

$$\hat{E}\psi = E\psi$$

and

$$\hat{p}\psi = p\psi.$$

The Dirac operator is

$$\mathcal{O} = \hat{E} - \alpha\hat{p} - \beta m$$

and the modified Dirac operator is

$$\mathcal{D} = \mathcal{O}\beta\alpha = \beta\alpha\hat{E} + \beta\hat{p} - \alpha m,$$

so that

$$\mathcal{D}\psi = (\beta\alpha E + \beta p - \alpha m)\psi = U\psi.$$

If we let

$$\tilde{\psi} = e^{i(px+Et)}$$

(reversing time), then we have

$$\mathcal{D}\tilde{\psi} = (-\beta\alpha E + \beta p - \alpha m)\psi = U^{\dagger}\tilde{\psi},$$

giving a definition of $U^{\dagger}$ corresponding to the anti-particle for $U\psi$.

We have that

$$U^2 = U^{\dagger 2} = 0$$

and

$$UU^{\dagger} + U^{\dagger}U = 4E^2.$$

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation.

And: Light Cone Coordinate with m = 1.

$$i\,\frac{\partial\psi}{\partial t} = \left(-\alpha i\,\frac{\partial}{\partial x} + \beta\right)\psi\ .$$

$$RI: \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}: \quad i\,\frac{\partial\psi}{\partial t} = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)\psi$$

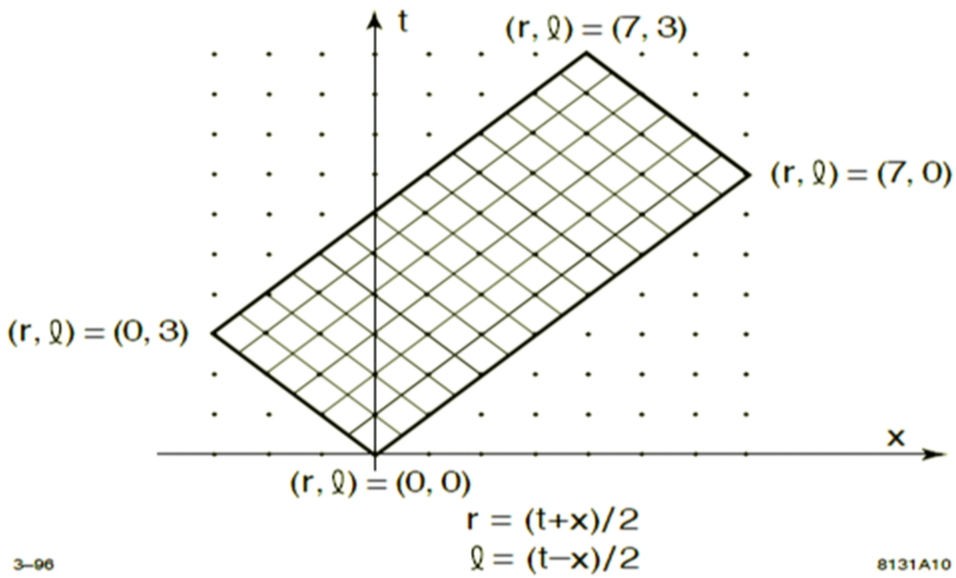
$$RII: \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \quad \frac{\partial\psi}{\partial t} = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\psi\ .$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}: \quad \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial t} - \frac{\partial\psi_1}{\partial x} \\ \frac{\partial\psi_2}{\partial t} + \frac{\partial\psi_2}{\partial x} \end{pmatrix}\ . \quad \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial \ell} \\ \frac{\partial\psi_2}{\partial r} \end{pmatrix}$$

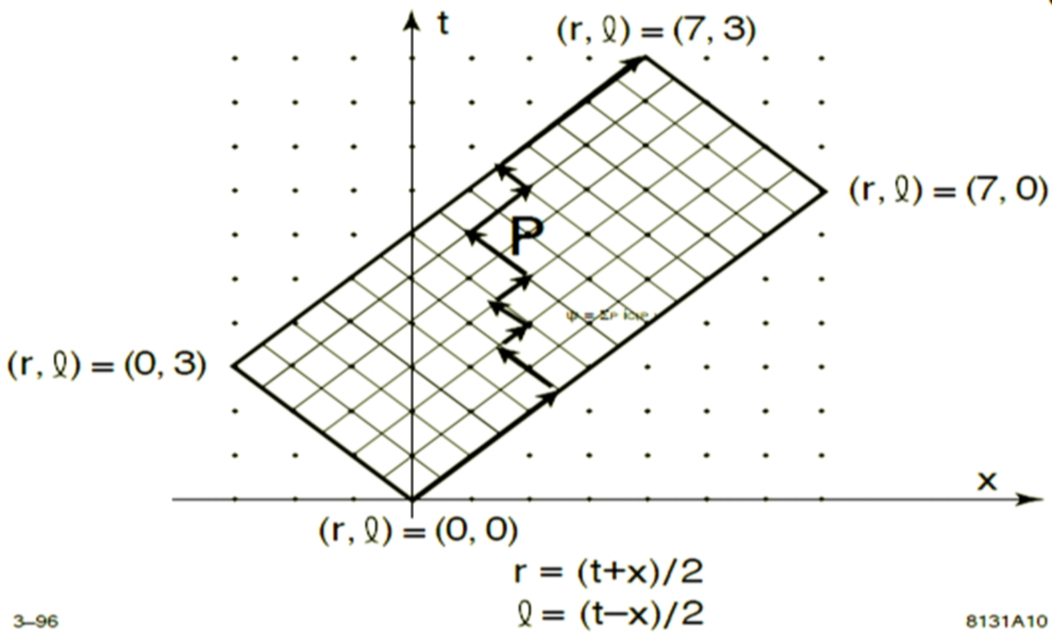
RI: Real Solutions -- 1+1 Majorana Fermions

Feynman Checkerboard

$$\text{RII: } \begin{pmatrix} -i\psi_2 \\ -i\psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \psi_1}{\partial l} \\ \frac{\partial \psi_2}{\partial r} \end{pmatrix}$$



The Feynman Checkerboard



$$\psi = \Sigma_P i^{C(P)}$$

Dirac Amplitude

C(P) =
number
of corners in
path P

In the RII, Majorana Fermion case we have

$$\partial\psi_2/\partial r = \psi_2$$

$$\partial\psi_1/\partial l = -\psi_1$$

Thus the Checkerboard works with plus/minus cornering
and we are returned to the
viewpoint from the beginning
of the talk.