Title: Geometric inequalities for black holes

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Abstract: A geometric inequality in General Relativity relates quantities that have both a physical interpretation and a geometrical definition. It is well known that the parameters that characterize the Kerr-Newman black hole (angular momentum, charge, mass and horizon area) satisfy several important geometric inequalities. Remarkably enough, some of these inequalities also hold for dynamical black holes. This kind of inequalities, which are valid in the dynamical and strong field regime, play an important role in the characterization of the gravitational collapse. They are closed related with the cosmic censorship conjecture. Also, variants of these inequalities are valid for ordinary bodies. In this talk I will review recent results in this subject.

Geometric inequalities for black holes

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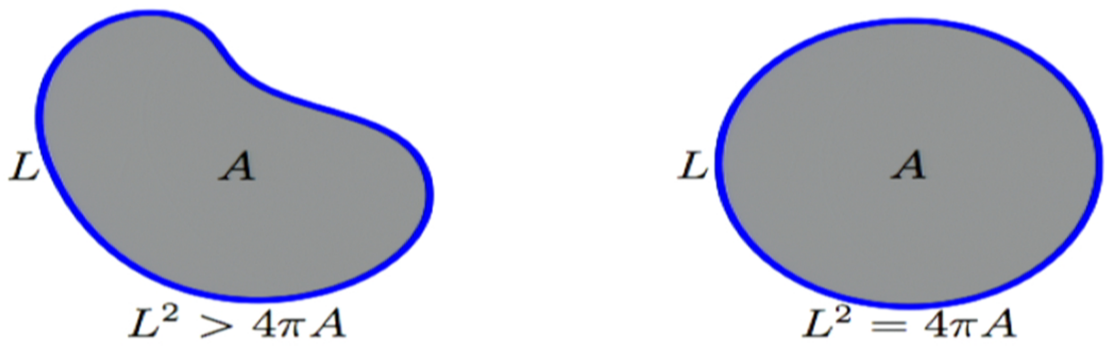
Geometric inequalities

Geometric inequalities have an ancient history in Mathematics. A classical example is the **isoperimetric inequality** for closed plane curves given by

$$L^2 \geq 4\pi A \quad (= \text{ circle})$$

where A is the **area** enclosed by a curve C of **length** L, and where **equality** holds if and only if C is a **circle**.

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L
A
L² > 4πA
L
A
L² = 4πA

- The inequality $L^2 \geq 4\pi A$ applies to complicated geometric objects (i.e. arbitrary closed planar curves).
- The equality $L^2 = 4\pi A$ is achieved only for an object of "optimal shape" (i.e. the circle). This object has a variational characterization: the circle is uniquely characterized by the property that among all simple closed plane curves of given length L, the circle of circumference L encloses the maximum area.

Geometrical inequalities in General Relativity

- General Relativity is a **geometric theory**, hence it is not surprising that geometric inequalities appear naturally in it. Many of these inequalities are similar in spirit as the isoperimetric inequality.
- However, General Relativity as a **physical theory** provides an important extra ingredient. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggests geometric inequalities which can be highly non-trivial from the mathematical point of view.
- The **interplay between physics and geometry** gives to geometric inequalities in General Relativity their distinguished character.

Plan of the talk

- **Part I**: Physical picture.
- **Part II**: Theorems.
- **Part III**: Open problems and recent results on bodies.

Positive mass theorem

Let m be the total ADM mass on an asymptotically flat complete initial data such that the dominant energy condition is satisfied, then we have:

$$0 \leq m \quad (= \text{ Minkowski}).$$

- The mass of the spacetime measures the total amount of energy and hence it should be positive from the physical point of view.
- The mass m in General Relativity is represented by a geometrical quantity on a Riemannian manifold.

From the geometrical mass definition, without the physical picture, it would be very hard to conjecture that this quantity should be positive. In fact the proof turn out to be very subtle (**Schoen-Yau 79**, **Witten 81**).

- A key assumption in the positive mass theorem is that the matter fields should satisfy an **energy condition**. This condition is expected to hold for all physically realistic matter.
- This kind of general properties which do not depend very much on the details of the model are not easy to find for a **macroscopic object**. And hence it is difficult to obtain simple and general geometric inequalities among the parameters that characterize ordinary macroscopic objects.
- Black holes represent a unique class of **very simple** macroscopic objects. They are natural candidates for geometrical inequalities.

Stationary black holes

- The black hole uniqueness theorem ensures that **stationary** black holes in **vacuum** (for simplicity we will not consider the electromagnetic field) are characterized by the Kerr exact solution of Einstein equations (important aspects of black hole uniqueness remain open see **Chruściel – Lopes Costa – Heusler, Living Review 12**).
- The Kerr metric depends on two parameters: the **mass** m and the **angular momentum** J.
- The Kerr metric is well defined for any choice of the parameters. However, it represents a black hole if and only if the following inequality holds

$$\sqrt{|J|} \leq m.$$

Physical interpretation

- $\sqrt{\frac{A}{16\pi}} \leq m$

  The difference $m - \sqrt{\frac{A}{16\pi}}$ is the rotational energy of the Kerr black hole. This is the maximum amount of energy that can be extracted by the Penrose process (**Christodoulou**).
- $\sqrt{|J|} \leq m$

  From Newtonian considerations, we can interpret this inequality as follows (**Wald**): in a collapse the gravitational attraction ($\approx m^2/r^2$) at the horizon ($r \approx m$) dominates over the centrifugal repulsive forces ($\approx J^2/mr^3$).
- $8\pi|J| \leq A$

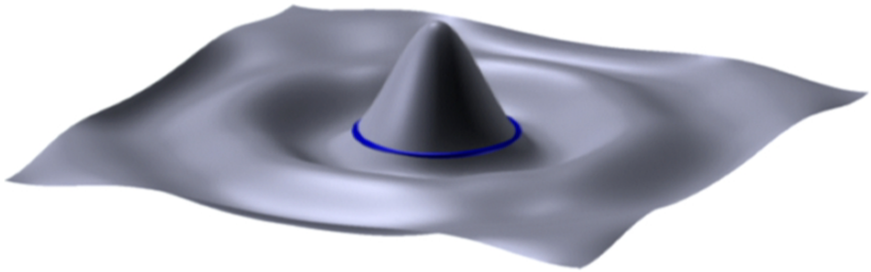
  The black hole **temperature**

$$\kappa = \frac{1}{4m}\left(1 - \frac{(8\pi J)^2}{A^2}\right),$$

  is positive $\kappa \geq 0$ and it is zero if and only if the black hole is extreme.

Geometric inequalities for dynamical black holes

- **Black holes are not stationary in general:**



- Astrophysical phenomena like the formation of a black hole by gravitational collapse or a binary black hole collision are highly dynamical.
- **Dynamical black holes can not be characterized by few parameters as in the stationary case.**
- Remarkably, these inequalities extend to the fully dynamical regime.
- The inequalities are deeply connected with global properties of the gravitational collapse: cosmic censorship conjecture.

Kerr black hole: geometrical inequalities

The area of the horizon is given by the important formula

$$A = 8\pi \left(m^2 + \sqrt{m^4 - J^2} \right).$$

This formula implies the following three geometric inequalities between the three relevant parameters (A, m, J):

$$\sqrt{\frac{A}{16\pi}} \leq m \qquad (= \text{Schwarzschild})$$

$$\sqrt{|J|} \leq m \qquad (= \text{Extreme Kerr})$$

$$8\pi|J| \leq A \qquad (= \text{Extreme Kerr})$$

The inequality $8\pi|J| \leq A$ for dynamical black holes

Consider a dynamical black hole. Physical quantities that are well defined for this spacetime are:

- The total ADM mass m of the spacetime: the sum of the black hole mass and the mass of the gravitational waves surrounding it. In the stationary case, the mass of the black hole is equal to the total mass of the spacetime. The mass m is a global quantity: it carries information on the whole spacetime.
- The area A of the horizon. The area A is a quasi-local quantity: it carries information on a bounded region of the spacetime.

What are the quasi-local mass and quasi-local angular momentum of a dynamical black hole? In general, it is difficult to find physically relevant quasi-local quantities like mass and angular momentum (see **Szabados, Living Review, 09**).

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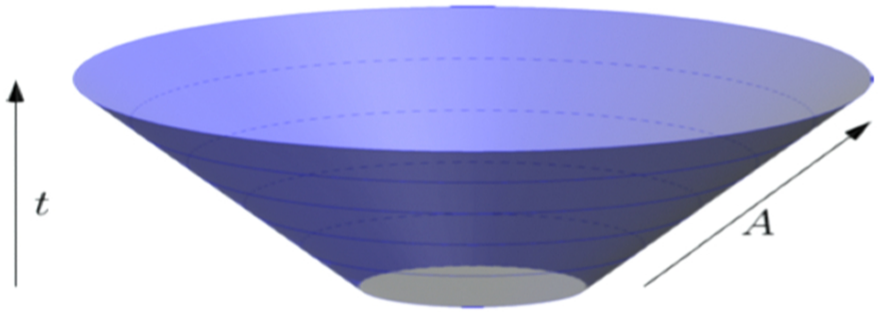
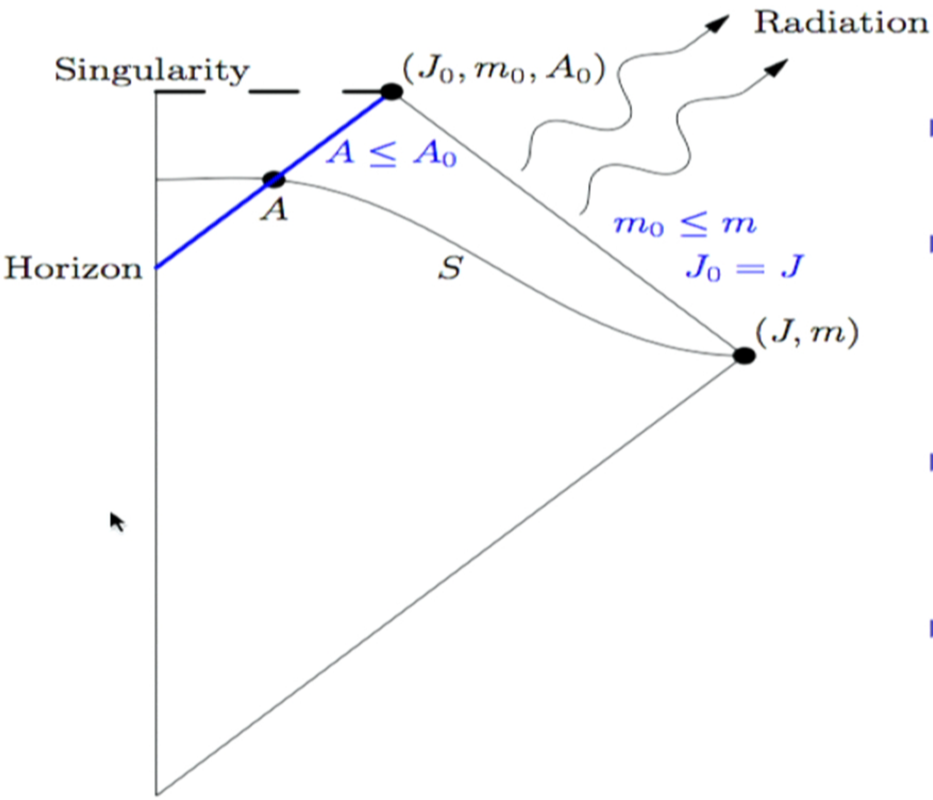
Evolution of the quasi-local mass m_{bh}

- By the area theorem, we know that the horizon area A increase with time.



- **In axial symmetry there is not transfer of angular momentum by gravitational waves.** Then, the quasi-local mass of the black hole should increase with the area, since there is no mechanism at the classical level to extract energy from the black hole (**no Penrose process in axial symmetry**).
- **Then, both the area A and the quasi-local mass m_{bh} should monotonically increase with time.**



- S: Cauchy surface such that the collapse has already occurred.
- **Gravitational waves carry positive energy:** $m \geq m_0$.
- A: area of the intersection of the event horizon with S. By the black hole **area theorem** we have $A_0 \geq A$.
- Angular momemtum is **conserved** (axial symmetry): $J_0 = J$.
- Since we have assumed that $8\pi|J| \leq A$ then m_{bh} also increase with time, then: $m_{bh} \leq m_0 \leq m$.

Physical arguments

The arguments in support of this inequality are based in the following three physical principles:

(i) The speed of light c is the maximum speed.

(ii) For bodies which are not contained in a black hole the following inequality holds (**hoop conjecture**)

$$\mathcal{R}(\Omega) \gtrsim \frac{G}{c^2} m(\Omega),$$

where $m(\Omega)$ is the mass of the body and $\mathcal{R}(\Omega)$ is some measure of size of Ω.

(iii) The conjectured inequality for bodies holds for black holes. The inequality

$$8\pi |J| \frac{G}{c^3} \leq A$$

can be interpreted as a version of this inequality for black holes.