Title: Bulk-Edge Correspondence in 2+1-Dimensional Abelian Topological Phases

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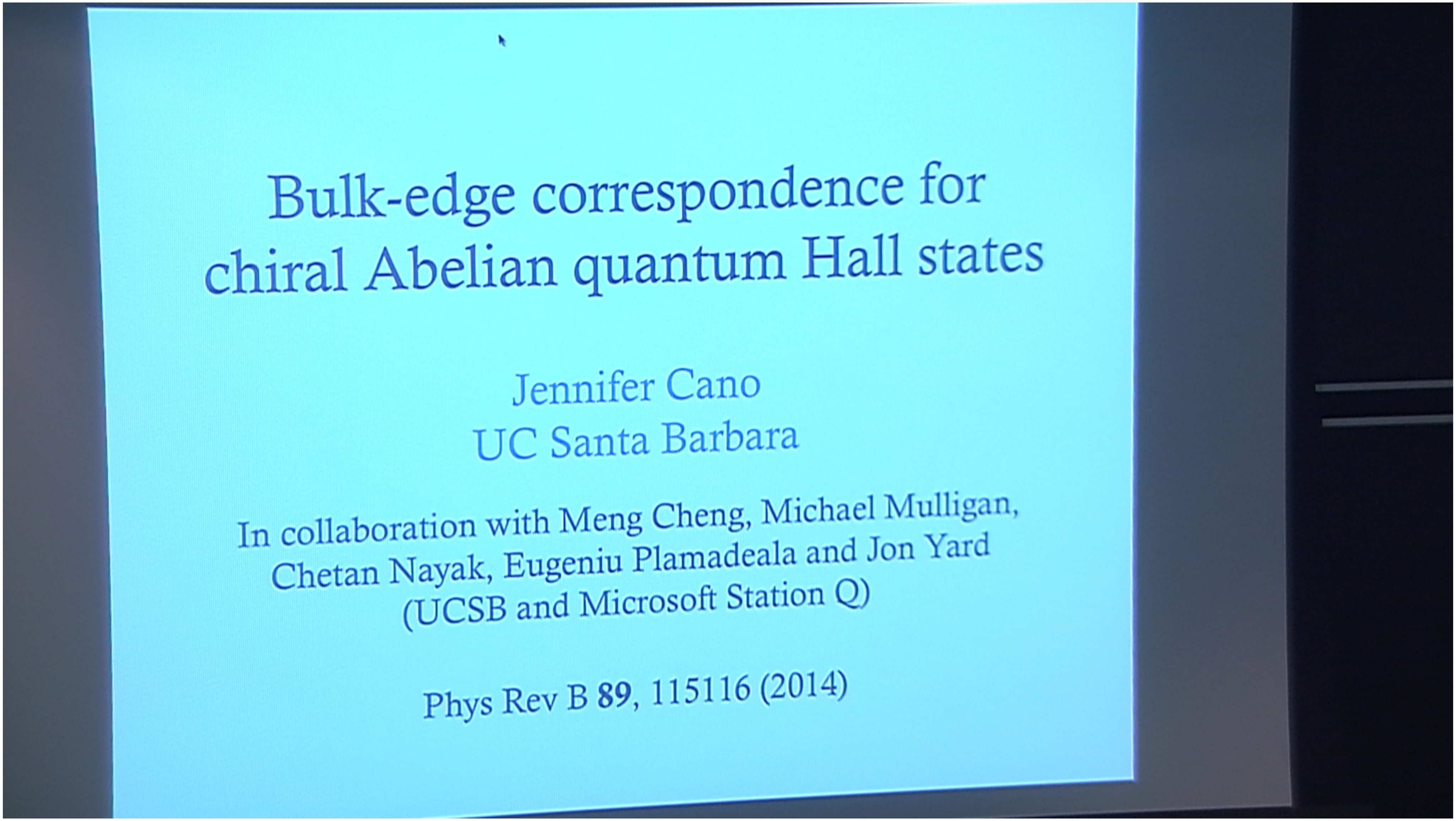
Abstract: The same bulk two-dimensional topological phase can have multiple distinct, fully-chiral edge phases. We show that this can occur in the integer quantum Hall states at fillings 8 and 12 with experimentally-testable consequences. We also show examples for Abelian fractional quantum Hall states, the simplest examples being at filling fractions 8/7, 12/11, 8/15, 16/5. For all examples, we propose experiments that can distinguish distinct edge phases. Our results are summarized by the observation that edge phases correspond to lattices while bulk phases correspond to genera of lattices. Since there are typically multiple lattices in a genus, there are usually many stable fully chiral edge phases corresponding to the same bulk. We show that fermionic systems can have edge phases with only bosonic low-energy excitations and discuss a fermionic generalization of the relation between bulk topological spins and the central charge. The latter follows from our demonstration that every fermionic topological phase can be represented as a bosonic topological phase, together with some number of filled Landau levels. Our analysis also leads to a simple demonstration that all Abelian topological phases can be represented by a Chern-Simons theory parameterized by a K-matrix.


Bulk-edge correspondence for
chiral Abelian quantum Hall states
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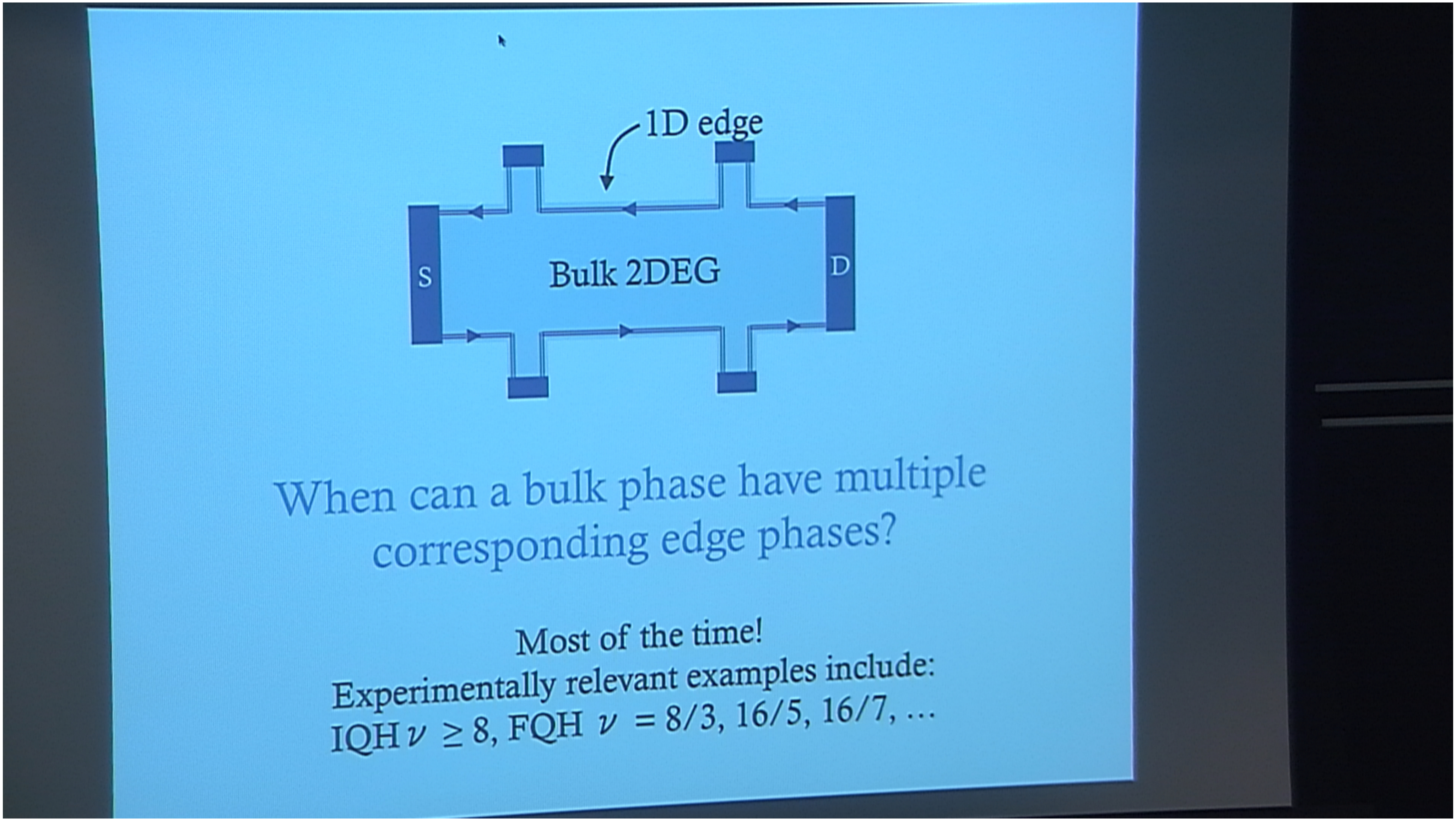
In collaboration with Meng Cheng, Michael Mulligan,
Chetan Nayak, Eugeniu Plamadeala and Jon Yard
(UCSB and Microsoft Station Q)
Phys Rev B 89, 115116 (2014)


1D edge
S
Bulk 2DEG
D
When can a bulk phase have multiple corresponding edge phases?
Most of the time!
Experimentally relevant examples include:
IQH $\nu \geq 8$, FQH $\nu = 8/3, 16/5, 16/7, \ldots$

What defines a bulk quantum Hall state?

Universal

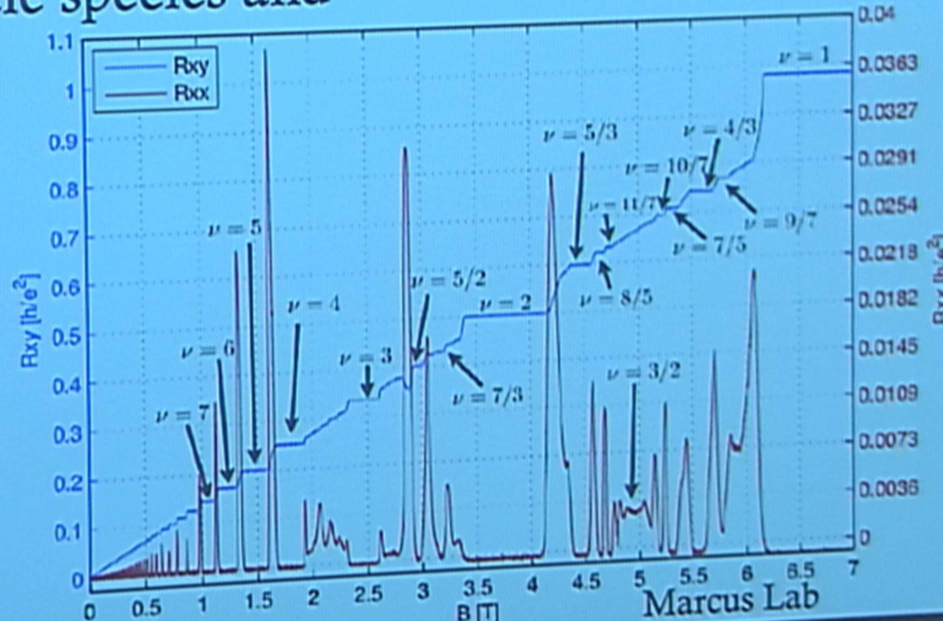
Quantized conductivities

Quasiparticle species and statistics

Non-universal

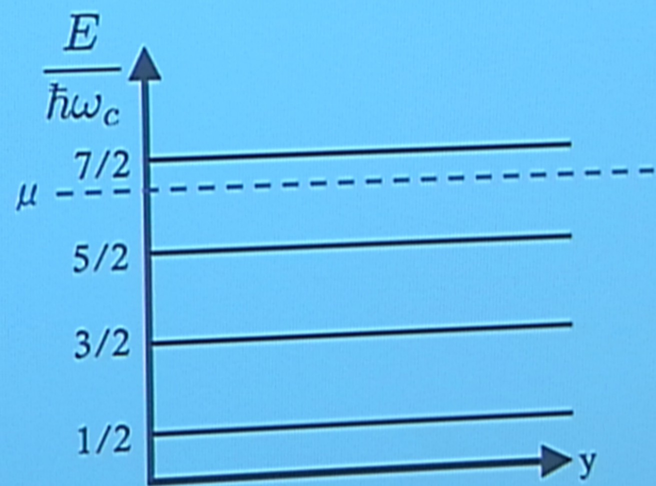
Bulk gap

Plateau

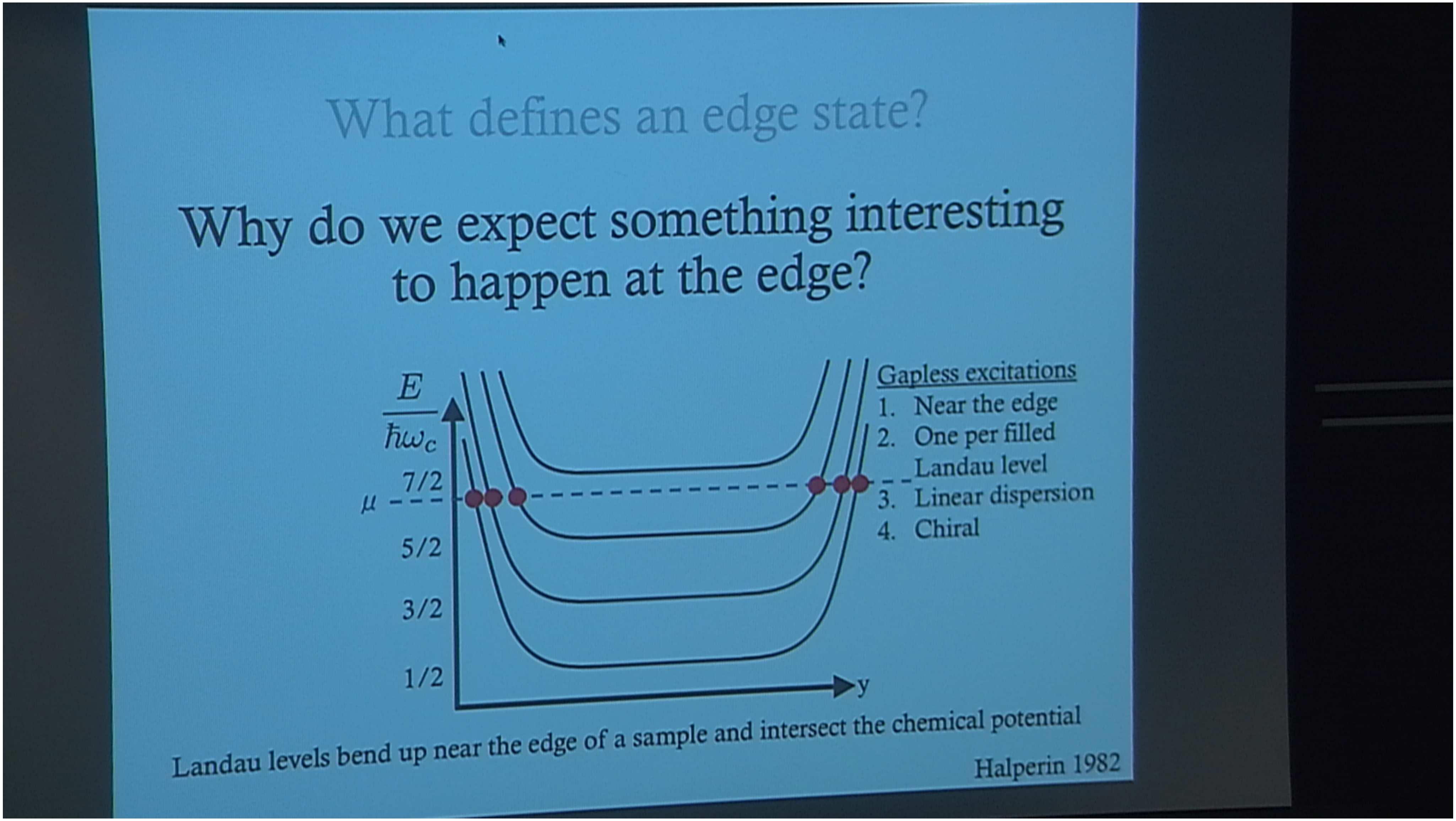


What defines an edge state?

Why do we expect something interesting to happen at the edge?



Recall: Landau levels


What defines an edge state?
Why do we expect something interesting to happen at the edge?
E
ħω_c
7/2
μ
5/2
3/2
1/2
y
Gapless excitations
1. Near the edge
2. One per filled Landau level
3. Linear dispersion
4. Chiral
Landau levels bend up near the edge of a sample and intersect the chemical potential
Halperin 1982

Edge Lagrangian: gapless chiral bosons

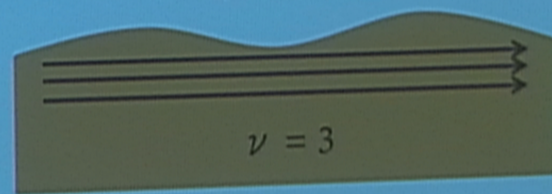
Gapless excitations

1. Near the edge
2. One per filled Landau level
3. Linear dispersion
4. Chiral

Bosonization

$$\psi_I^\dagger \sim e^{i\phi_I}$$

$$n_I = \frac{\partial_x \phi_I}{2\pi}$$



Chiral boson Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi}\int dx \sum_I \left(\partial_t + v_I \partial_x\right)\phi_I \partial_x \phi_I$$

Add density-density interactions:

$$\mathcal{L} = \frac{1}{4\pi}\int dx \sum_{IJ} \left(\delta^{IJ}\partial_t \phi_I \partial_x \phi_J + V^{IJ}\partial_x \phi_I \partial_x \phi_J\right)$$

Orthogonal transformation diagonalizes V

Wen *Int. J. Mod. Phys.* **B6** 1711 (1991)

IQH edge Lagrangian is Fermi liquid

$$\mathcal{L} = \frac{1}{4\pi} \int dx \sum_I \left(\partial_t + v\partial_x\right) \phi_I \partial_x \phi_I$$

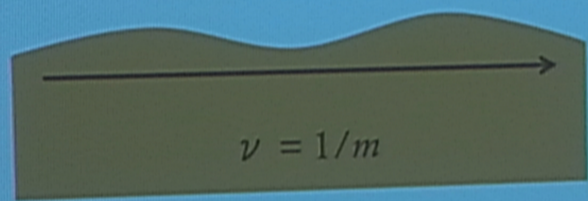
$$\langle \psi_J^\dagger(x,t)\psi_I(0,0)\rangle \sim \langle e^{i\phi_I(x,t)} e^{-i\phi_J(0,0)}\rangle \sim \frac{\delta_{IJ}}{x - vt}$$

Allowed operators: $e^{in_I\phi_I}, n_I \in \mathbb{Z}$

from boundary condition $\phi_I = \phi_I + 2\pi$

FQH Laughlin edge is Luttinger liquid

Wen's hydrodynamic theory: $\mathcal{L} = \frac{m}{4\pi} \int dx \, (\partial_t + v\partial_x) \, \phi \partial_x \phi$



$$\rho = \frac{1}{2\pi} \partial_x \phi$$

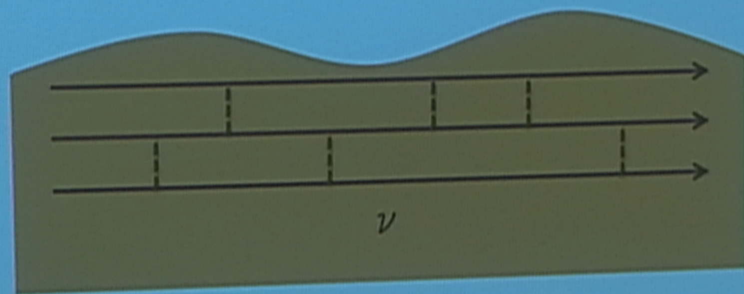
Allowed operators: $\psi_e^\dagger \sim e^{im\phi}$ $\qquad \psi_{qp}^\dagger \sim e^{i\phi}$ $\quad e^* = e/m$

Correlators: $\langle e^{im\phi(x,t)} e^{-im\phi(0,0)} \rangle \sim \frac{1}{(x - vt)^m}$

$$\langle e^{i\phi(x,t)} e^{-i\phi(0,0)} \rangle \sim \frac{1}{(x - vt)^{1/m}}$$

Wen *Int. J. Mod. Phys.* **B6** 1711 (1991)

General QH edge has several channels and interactions between them



$$S_{LL} = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right)$$

<u>The K-matrix contains universal physics</u>

The V-matrix parameterizes velocities and interactions

The t-vector carries the charge of each mode

Wen *Int. J. Mod. Phys.* **B6** 1711 (1991)

The K-matrix (with charge vector) contains all the universal physics

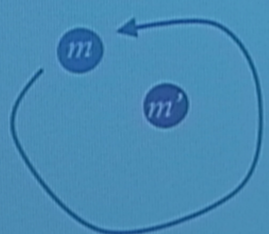
Filling fraction: $\nu = t^T K^{-1} t$

Ground state degeneracy (torus): $\det(K)$

Chiral central charge: $\mathrm{sign}(K) = r_+ - r_-$

<u>Quasiparticle charge and statistics</u>

Physical quasiparticles: $e^{im\phi} \equiv e^{im_I \phi_I} \qquad m_I \in \mathbb{Z}$



$$e^* = m^T K^{-1} t$$

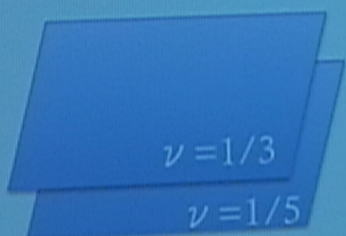
$$S_{m,m'} \propto e^{-2\pi i m^T K^{-1} m'} \longrightarrow \text{S-matrix: statistics}$$

Edge physics uniquely determines the bulk phase!

Examples of K-matrices

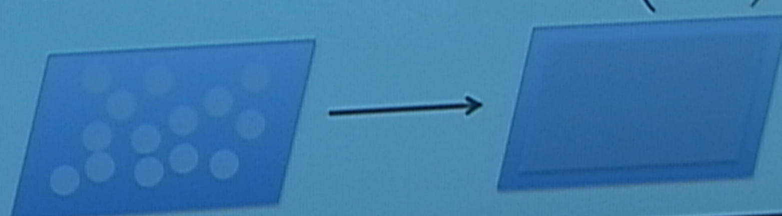
IQH: $\nu = \text{N}$ $K = \mathbb{I}_N$

Laughlin: $\nu = 1/m$ $K = (m)$



Bilayer system: $\nu = 1/3 + 1/5$ $K = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

Hierarchy: $\nu = 2/5$ $K = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$



Can two K-matrices describe the same edge physics?

$$K_1 = W^T K_2 W \qquad t_1 = t_2 W \qquad W \in GL(n, \mathbb{Z})$$

$$S_{LL} = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right)$$

Change of variables $\phi^I = W^I_J \tilde{\phi}^J$

Preserves boundary conditions $\tilde{\phi}^I = \tilde{\phi}^I + 2\pi$

One-to-one mapping between quasiparticles

Equivalence classes of K-matrices describe the same edge physics

Read, *PRL* **65** 1502 (1990); Fröhlich and Thiran *J. Stat. Phys.* **76**, 209 (1994)

Distinct classes of K-matrices can be distinguished by their scaling dimensions

If K is chiral, scaling dimensions are universal:

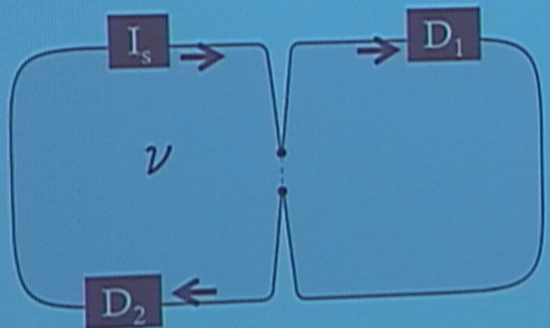
$$\langle e^{im_I \phi^I(0,t)} e^{-im_J \phi^J(0,0)} \rangle \sim \frac{1}{t^{\Delta_m}}$$

Scaling dimension: $\Delta_m = \frac{1}{2} m_I K^{-1}_{IJ} m_J$

When K is non-chiral, Δ depends on V-matrix

Scaling dimensions can be used to physically distinguish edge phases

1) Tunneling across a QPC



$$\mathcal{L}_{tun} = \sum_{m_I} v_m e^{i m_I \phi_R^I} e^{-i m_I \phi_L^I} + h.c.$$

Creates R-mover   Annihilates L-mover

Many terms: most relevant minimizes $mK^{-1}m$

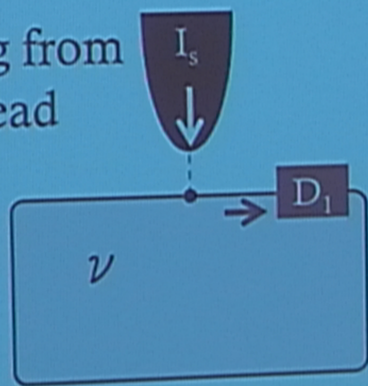
Except most relevant term to dominate backscattered current:

$$I_b \propto |v_m|^2 V^{2mK^{-1}m-1}$$

Chamon, Freed Wen (1994)
Kane and Fisher (1992)

Scaling dimensions can be used to physically distinguish edge phases

2) Tunneling from a metallic lead



Most relevant term minimizes $mK^{-1}m + n^2$

$$I_{lead} \propto |t_m|^2 V^{mK^{-1}m+n^2-1}$$

Tunnel one electron:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} e^{im_I \phi^I_R}$$

$$mK^{-1}t = 1$$

Tunnel two electrons:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} \partial \psi^\dagger_{lead} e^{im_I \phi^I_R}$$

$$mK^{-1}t = 2$$

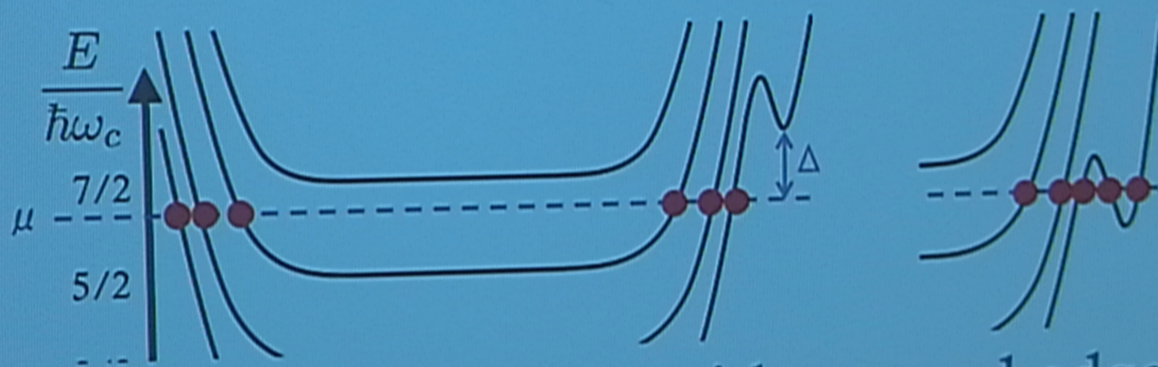
Tunnel n electrons:

$$\mathcal{L}_{lead} = t_m \underbrace{\left[\psi^\dagger_l \partial \psi^\dagger_l \partial^2 \psi^\dagger_l \dots\right]}_{n \text{ terms}} e^{im_I \phi^I_R}$$

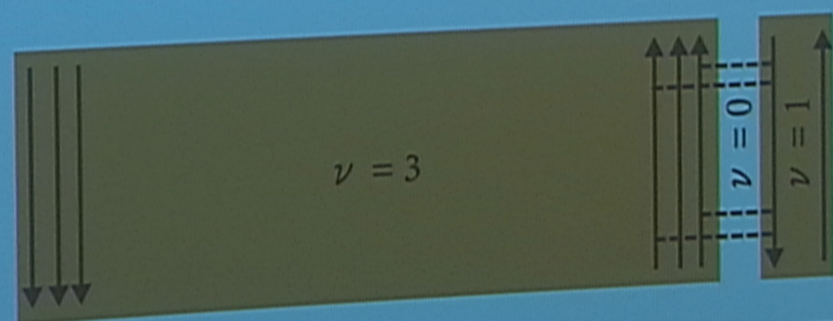
$$mK^{-1}t = n$$

Chamon, Freed Wen (1994) Kane and Fisher (1992)

Consider interactions with gapped modes



Can interactions with gapped edge modes change the phase of the edge?



Describe the original + gapped edge modes in K-matrix formalism

$$K \to K \oplus \sigma_z \qquad \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 5 & 7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \to \left(\begin{array}{cccc|cc} 1 & 2 & 3 & \cdots & 0 & 0 \\ 2 & 5 & 7 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{array}\right)$$

$$t \to (t, 1, -1)$$

$\nu = 1$ strip

<u>ν = 1 strip is topologically trivial</u>

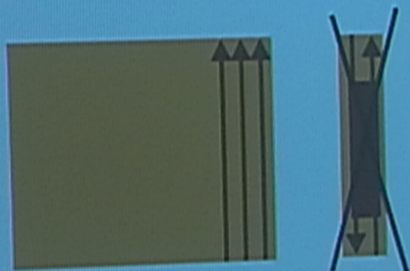
Leaves invariant:

- Filling fraction
- Chiral central charge
- Determinant
- Allowed quasiparticle spectrum

(New quasiparticles $e^{i\phi_{N+1}}, e^{i\phi_{N+2}}$ are electrons)

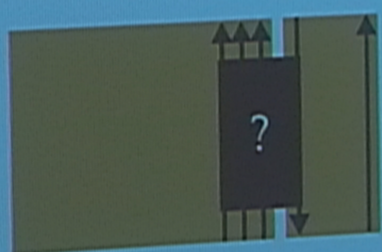
Counterpropagating mode generates backscattering terms

$$S_{LL} = \int dx\, dt \left(\frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J + \frac{1}{2\pi} t_I \epsilon_{\mu\nu} \partial_\mu \phi^I A_\nu \right)$$



Thin strip, away from other edge modes

Backscattering: $e^{i\phi_{N+1}} e^{-i\phi_{N+2}} + h.c.$

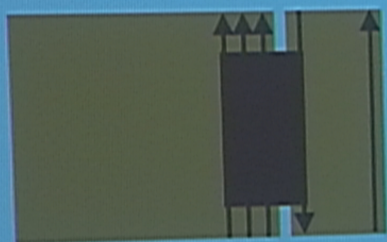


Near other edge modes

Backscattering: $e^{i n_I \phi_I} + h.c.$

Will this term open a gap?

When does scattering open a gap?



Scattering: $e^{in_I\phi_I} + h.c. \propto \cos(n\phi)$

Fixing cosine → gap

No external source of charge: $n_I K^{-1}_{IJ} t_J = 0$

Must commute with self at different positions: $n_I K^{-1}_{IJ} n_J = 0$

Not met for a chiral edge. Example:

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$n_I K^{-1} n_J = n_1^2 + n_2^2 + n_3^2 \neq 0!$

When $K \to K \oplus \sigma_z$, can gap

Haldane *PRL* 74 2090 (1995)

What is the left over theory?

Example: $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}, t_1 = (1, -1)^T$

Enlarge: $K_1 \to K_1 \oplus \sigma_z \quad t_1 \to (1, -1, 1, 1)^T$

1) 

$$S' = \int dx dt u' \cos(\phi_3 + \phi_4)$$

2) 

$$S'' = \int dx dt u'' \cos(\phi_1 - 11\phi_2 + 2\phi_3 + 4\phi_4)$$

$$n = (1, -11, 2, 4)$$

Strategic variable change $\phi = W\phi'$ ⇒

$$\begin{cases} S'' = \int dx dt u'' \cos(\phi_3' + \phi_4') \\ K \to K_2 \oplus \sigma_z \qquad K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \end{cases}$$

What is the leftover theory?

Example, cont

1) Most relevant term: $u' \cos(\phi_3 + \phi_4)$

2) Most relevant term: $u'' \cos(\phi_3' + \phi_4')$

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}, t_1 = (1, -1)^T \qquad K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, t_2 = (-1, -2)^T$$

Are these phases distinct? $\Delta = \frac{1}{2} m_I K_{IJ}^{-1} m_J$

$\Delta_{min} = 1/11$ $\qquad$ $\Delta_{min} = 3/11$

Stable equivalence

$$W^T K_1 W \neq K_2$$

K_1 and K_2 describe distinct edge phases

$$W^T (K_1 \oplus \sigma_z) W = K_2 \oplus \sigma_z$$

$K_1 \oplus \sigma_z$ and $K_2 \oplus \sigma_z$ describe same edge
→ surround same bulk

K_1 and K_2 surround the same bulk

More topologically trivial additions

Superfluid strip

$$K \to K \oplus \sigma_x$$

Several $\nu = 1$ strips

$$K \to K \oplus \sigma_z \oplus \sigma_z \oplus \cdots \oplus \sigma_z$$

Strip of anything non-chiral with trivial quasiparticles

$$K \to K \oplus L, \quad |\det(L)| = 1 \quad \mathrm{sig}(L) = (n, n)$$

Distinction between topological phases of bosons and fermions

Bosonic K-matrix has only even numbers on the diagonal

➡ σ_z is not allowed!

Fermionic K-matrix has no restriction

Is there a difference: $K \to K \oplus \sigma_z$

$K \to K \oplus \sigma_x$

No: $W^T(K_{\text{odd}} \oplus \sigma_z)W = (K_{\text{odd}} \oplus \sigma_x)$

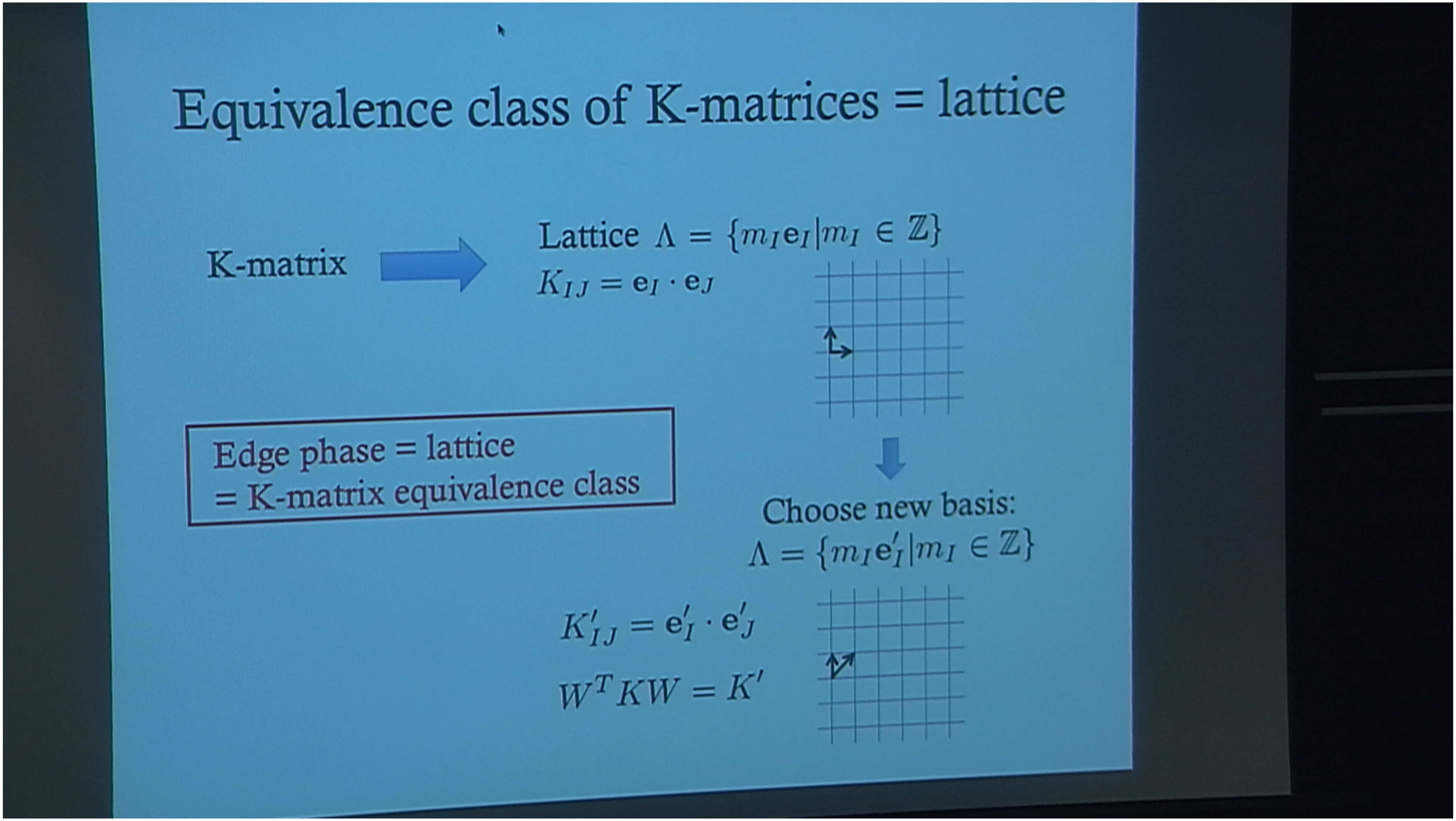
Questions

When do distinct edge phases exist for a given bulk?

Does anything different happen when multiple copies of σ_x, or other topologically trivial matrices, are appended to the K-matrix?

When there are two distinct edge phases, is one preferred because its gapping term is more relevant?

MATHEMATICAL FORMULATION


Equivalence class of K-matrices = lattice
K-matrix
Lattice $\Lambda = \{m_I \mathbf{e}_I | m_I \in \mathbb{Z}\}$
$K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$
Edge phase = lattice
= K-matrix equivalence class
Choose new basis:
$\Lambda = \{m_I \mathbf{e}'_I | m_I \in \mathbb{Z}\}$
$K'_{IJ} = \mathbf{e}'_I \cdot \mathbf{e}'_J$
$W^T K W = K'$

Lattice yields physical quantities:

Minimum length of vector = minimum scaling dimension Δ_{min}

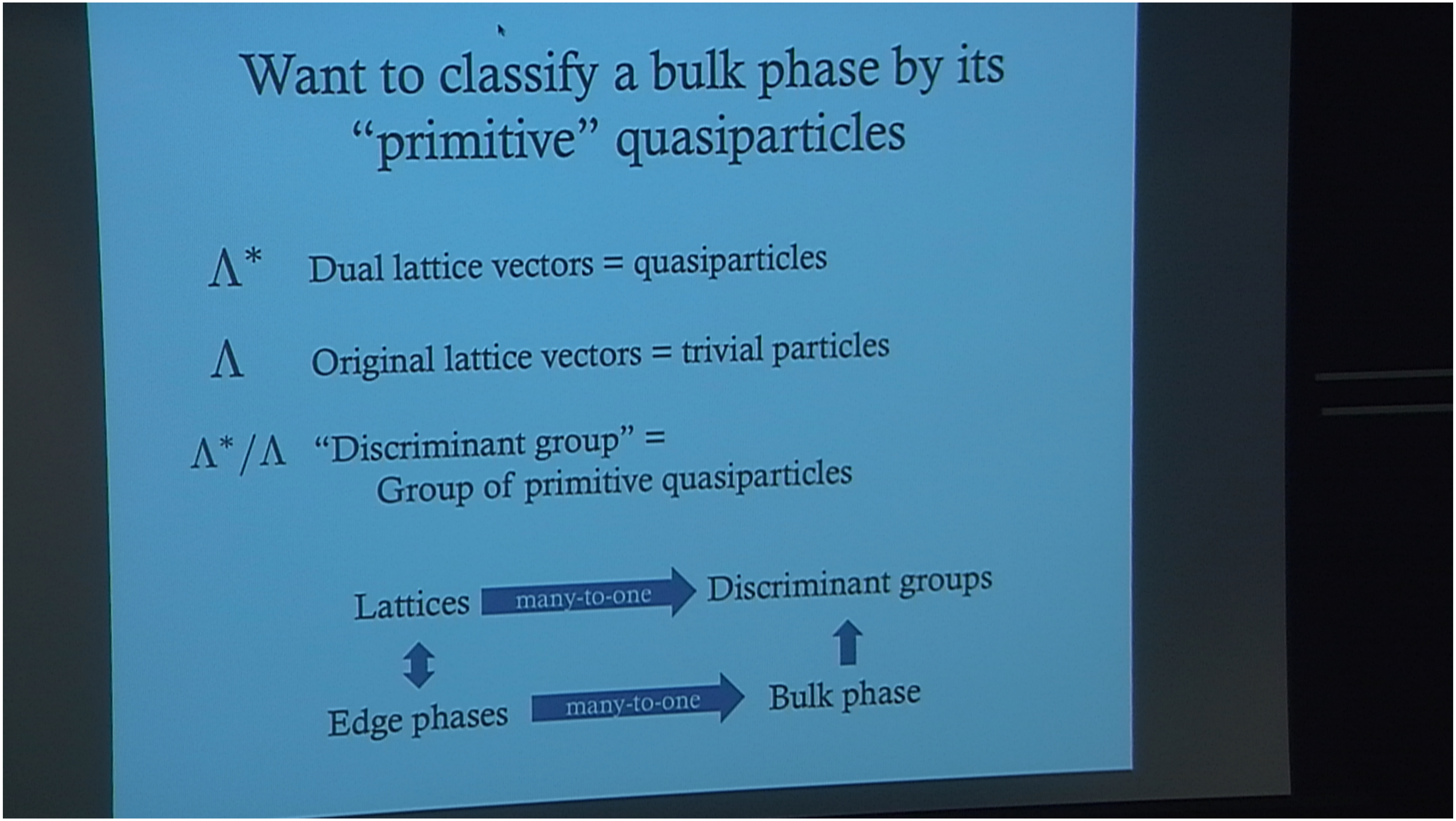
Dual lattice vectors = quasiparticles

$$\Lambda^* = \{m_I \mathbf{f}_I | m_I \in \mathbb{Z}\} \qquad f_a^I = \left(K^{-1}\right)^{IL} e_{La}$$

Inner product = statistics

$$\mathbf{f}_I \cdot \mathbf{f}_J = K^{-1}_{IJ}$$

$$S_{m,m'} \propto e^{-2\pi i m^T K^{-1} m'}$$


Want to classify a bulk phase by its "primitive" quasiparticles
Λ* Dual lattice vectors = quasiparticles
Λ Original lattice vectors = trivial particles
Λ*/Λ "Discriminant group" = Group of primitive quasiparticles
Lattices
many-to-one
Discriminant groups
Edge phases
many-to-one
Bulk phase

Lattices are classified by genus

A genus is defined by its:

- Discriminant group A
- Inner product
- Signature (r_+, r_-)
- Parity

A bulk phase is defined by its:

- Quasiparticle statistics
- Chiral central charge $r_+ - r_-$ (mod 24)
- Constituent particles

Genus determines a bulk phase

For chiral bulk ($r_- = 0$), bulk determines a family of genera

Nikulin: genus of lattices = stable equivalence of inner product
for any type of stable equivalence

Nikulin *Math. USSR Izv.* **14**, 103 (1980)

In principle, can count the lattices in a genus by the Smith-Siegel-Minkowski mass formula

Sum over lattices in a genus = sum over edge phases

$$\sum_{\Lambda \in g} \frac{1}{|\mathrm{Aut}(\Lambda)|} = m(K)$$

Conway and Sloane 1988

↑ # of automorphisms of lattice counts symmetries of K matrix

- Chiral Abelian quantum Hall states with at least 10 edge modes have multiple distinct chiral edge phases[1]
- Else, if chiral central charge ≥2, a finite set of bulk states have only one edge phase; all others have multiple[2]

→Multiple edge phases are the norm, not the exception

[1]G. Watson, Proc. London Math Soc. 12 57787 (1962)
[2]D. Lorch and M. Kirschmer, LMS Journal of Computation and Mathematics **16**, 172 (2013)

Stable equivalence between even and odd matrices

$$K_{\text{odd}} \oplus \sigma_z = W^T (K_{\text{even}} \oplus \sigma_z) W$$

Example

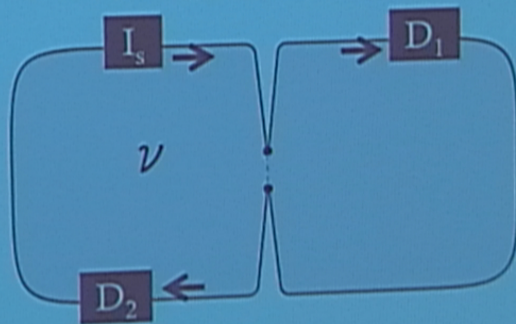
IQH $\nu = 8$: $K = \mathbb{I}_8$ $\qquad W_8^T\,(K_{E_8} \oplus \sigma_z)\,W_8 = \mathbb{I}_8 \oplus \sigma_z$

$$\mathbb{I}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Even lattice!

Can distinguish experimentally between candidate $\nu = 8$ phases

1) Tunneling across a QPC



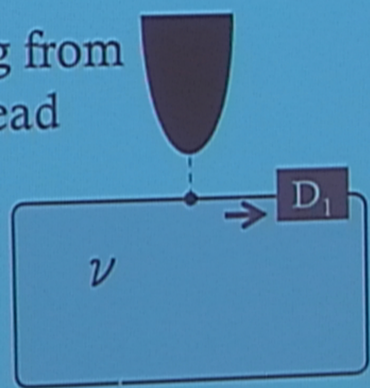
$$I_b \propto V^{2\Delta_{min}-1}$$

$\mathbb{I}_8 \to I_b \propto V$ (electron tunneling)

$K_{E_8} \to I_b \propto V^3$ (charge $2e$ tunneling of composite particle)

Can distinguish experimentally between candidate $\nu = 8$ phases

2) Tunneling from a metallic lead



In I_8 state tunnel one electron:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} e^{i\phi_I}$$

$$I_{lead} \propto V$$

E_8 state does not have a charge e operator!

➡ Tunnel two electrons from lead:

Spin-polarized: $\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} \partial \psi^\dagger_{lead} e^{i m_I \phi^I_R}$ $\quad I_{lead} \propto V^5$

Not spin-polarized: $\mathcal{L}_{lead} = t_m \psi^\dagger_{lead,\uparrow} \psi^\dagger_{lead,\downarrow} e^{i m_I \phi^I_R}$ $\quad I_{lead} \propto V^3$

Formalize the odd-even correspondence

Utilize another theorem from Nikulin:

For every fermionic bulk phase, there is a corresponding* edge phase which yields the same** bulk quasiparticles and statistics and has no gapless charge e operator

Caveats
*might have different number of edge modes
**mod e

Nikulin *Math. USSR Izv.* **14**, 103 (1980)

Example: even-odd equivalence by adding edge modes

$$\nu = 1/5$$

$$K = (5)$$

No non-trivial stable equivalence

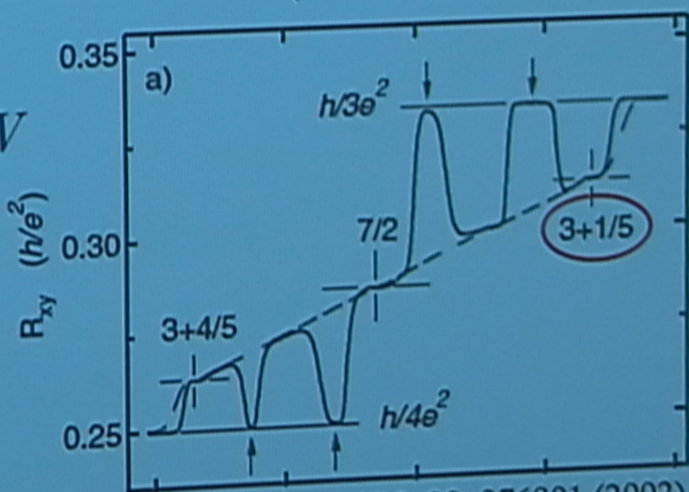
$$\nu = 3 + 1/5$$

$$K_1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

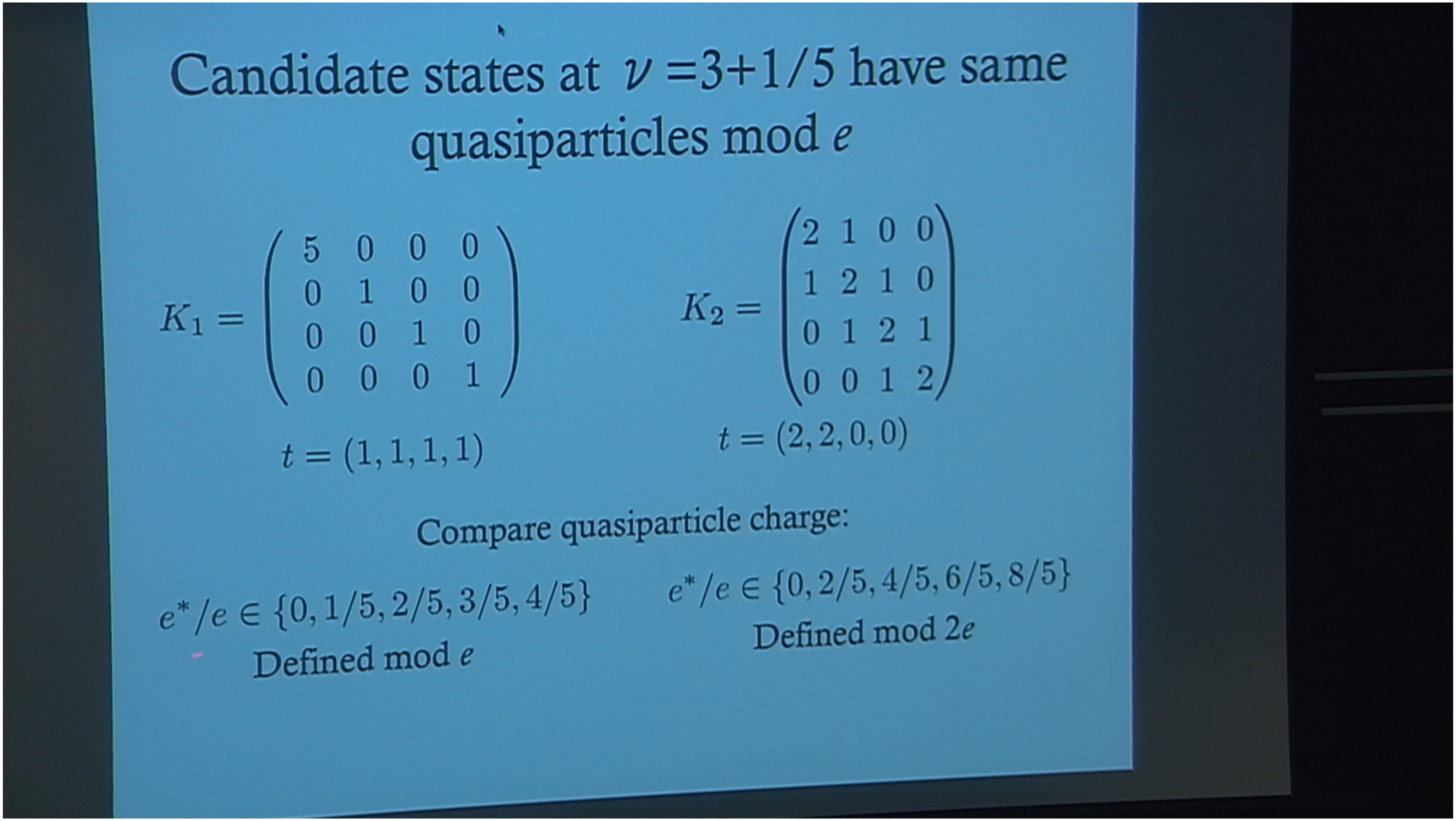
$$K_1 \oplus \sigma_z = W^T (K_2 \oplus \sigma_z) W$$

$$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Even lattice



Eisenstein, et al, PRL **88**, 076801 (2002)


Candidate states at ν =3+1/5 have same quasiparticles mod e
$K_1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$t = (1,1,1,1)$
$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$
$t = (2,2,0,0)$
Compare quasiparticle charge:
$e^*/e \in \{0, 1/5, 2/5, 3/5, 4/5\}$
Defined mod e
$e^*/e \in \{0, 2/5, 4/5, 6/5, 8/5\}$
Defined mod $2e$

Which gapping vector is most relevant?

Adding counter propagating modes

↓

Scaling dimension determined by V-matrix

↓

With control over V, can make any gapping term arbitrarily relevant

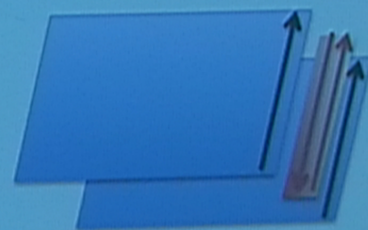
Truth: experimentalists don't have precise control
Theorists don't know what they would like to have controlled

Conclusions

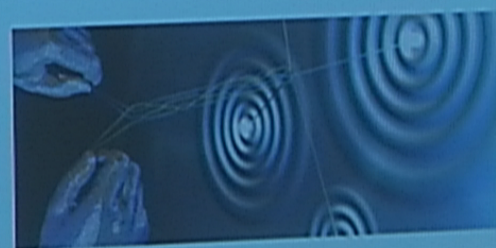
- A bulk Abelian topological phase generically has multiple edge phases
- Edge phases correspond to lattices while bulk theories correspond to genera of lattices
- Every fermionic topological phase is stably equivalent to a bosonic phase after adding a certain number of filled Landau levels
- Distinct edge phases can be physically distinguished by tunneling experiments

Further directions?

- Physical implementation:
  bilayers?



- Generalize: non-chiral edge phases? non-Abelian?



- Interesting (theoretical) application: entanglement entropy?