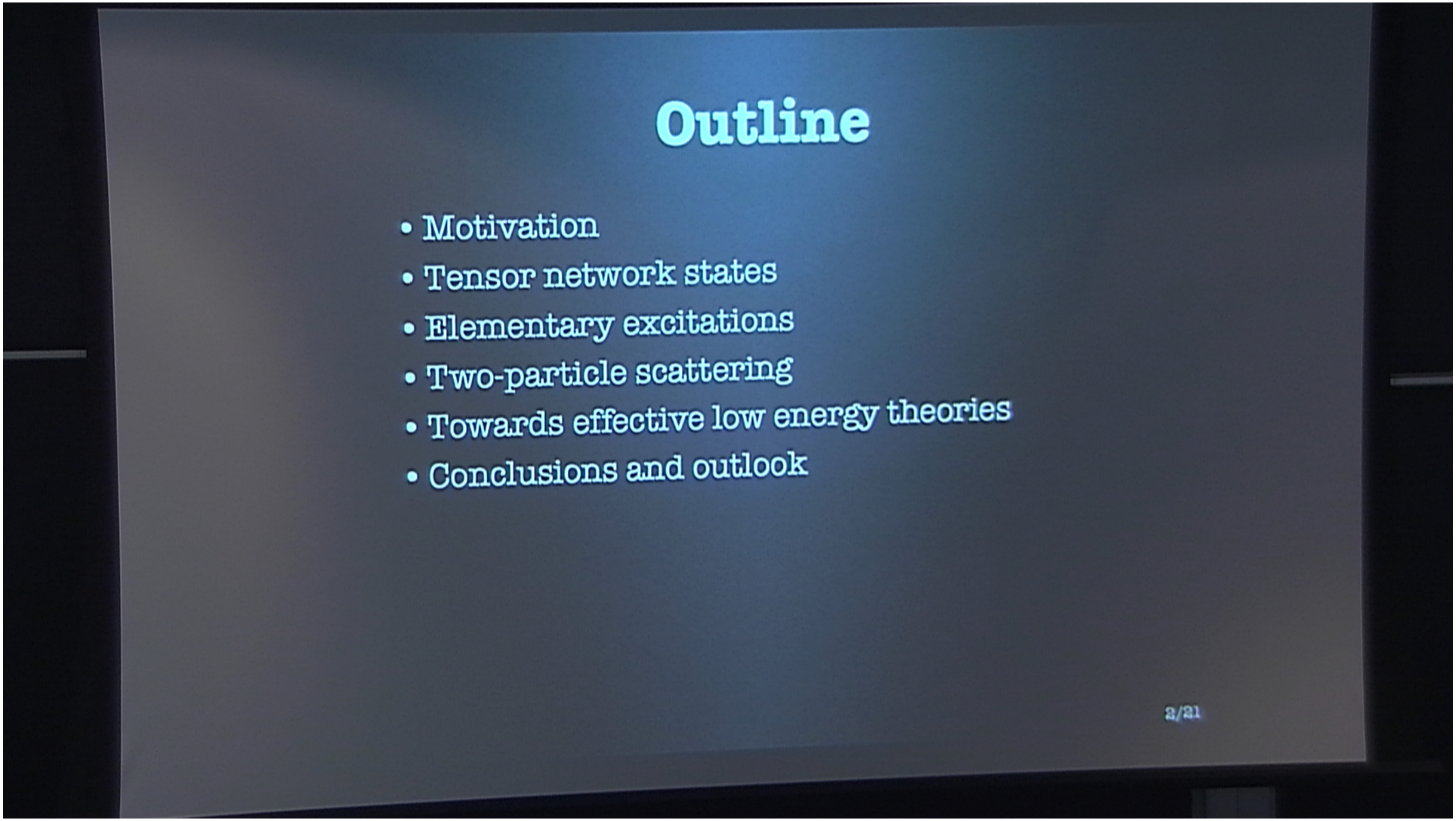
Title: Scattering of emerging excitations in Matrix Product States

Date: May 06, 2014 03:30 PM

URL: http://pirsa.org/14050010

Abstract: We review the formalism of matrix product states and one of its recent generalisations which allows to variationally determine the dispersion relation of elementary excitations in generic one-dimensional quantum spin chains. These elementary excitations dominate the low energy effective behaviour of the system. We discuss recent work where we show how we can also describe the effective interaction between these excitations â€“ as mediated by the strongly correlated ground state â€“ and how we can extract the corresponding S matrix. With these two ingredients, we can already build a highly non-trivial low-energy description of any microscopic Hamiltonian by assuming that higher order scattering processes are negligible. This allows to extract accurate information about the behaviour of the system under perturbations or at finite temperature, as we illustrate using the spin 1 Heisenberg model.


Outline
• Motivation
• Tensor network states
• Elementary excitations
• Two-particle scattering
• Towards effective low energy theories
• Conclusions and outlook
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Motivation

- Tensor network state methods have proven successful for the description of ground states of local quantum lattice systems (with gap)
  - ↪ area law of entanglement entropy
- Allows for the extraction of local expectation values, correlation functions, critical exponents, topological information (T- and S-matrices)
- Spectral information?
  - ↪ is time evolution the best we can do?

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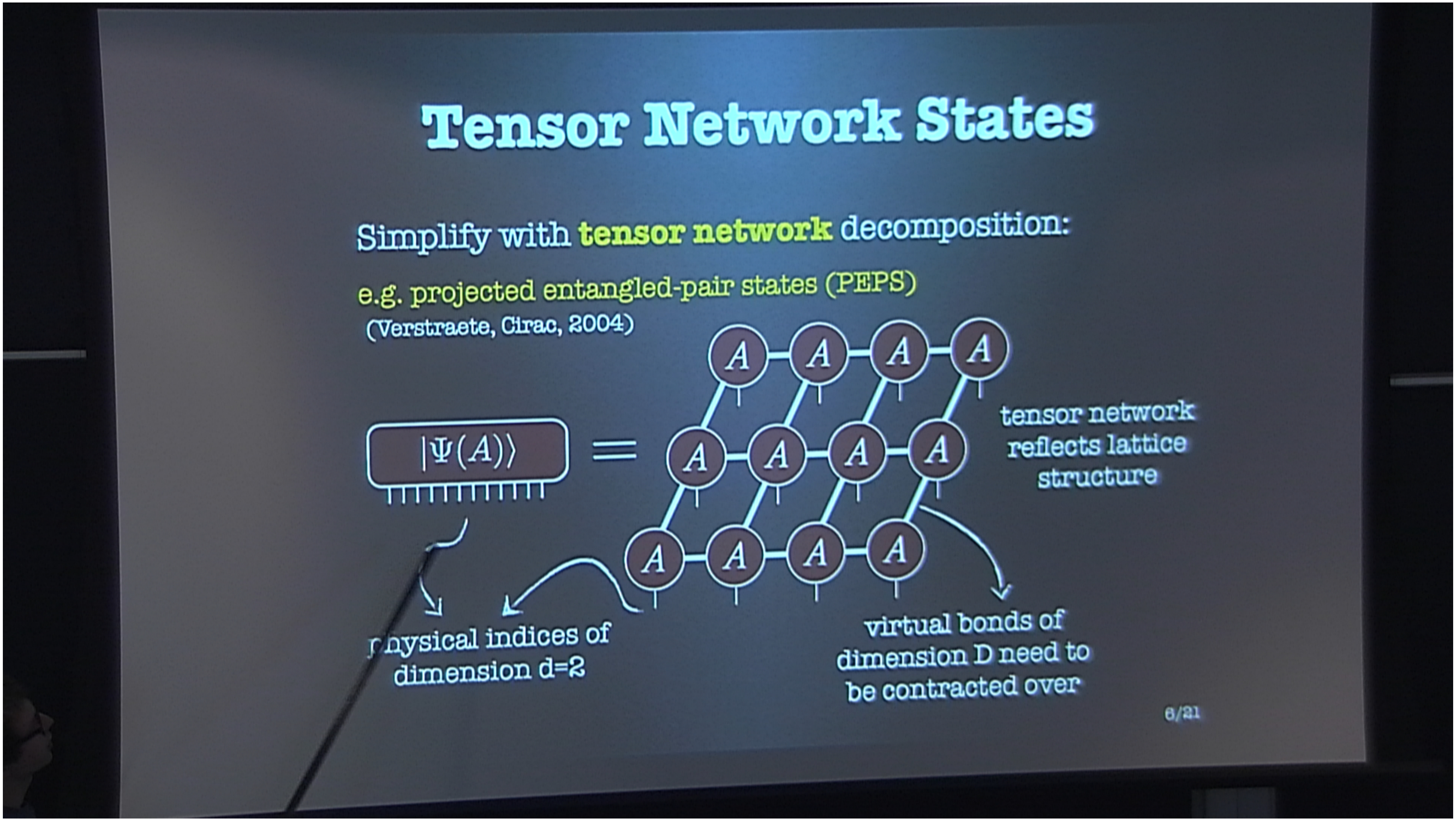
Tensor Network States
Simplify with tensor network decomposition:
e.g. projected entangled-pair states (PEPS)
(Verstraete, Cirac, 2004)
|Ψ(A)⟩
A
tensor network reflects lattice structure
physical indices of dimension d=2
virtual bonds of dimension D need to be contracted over
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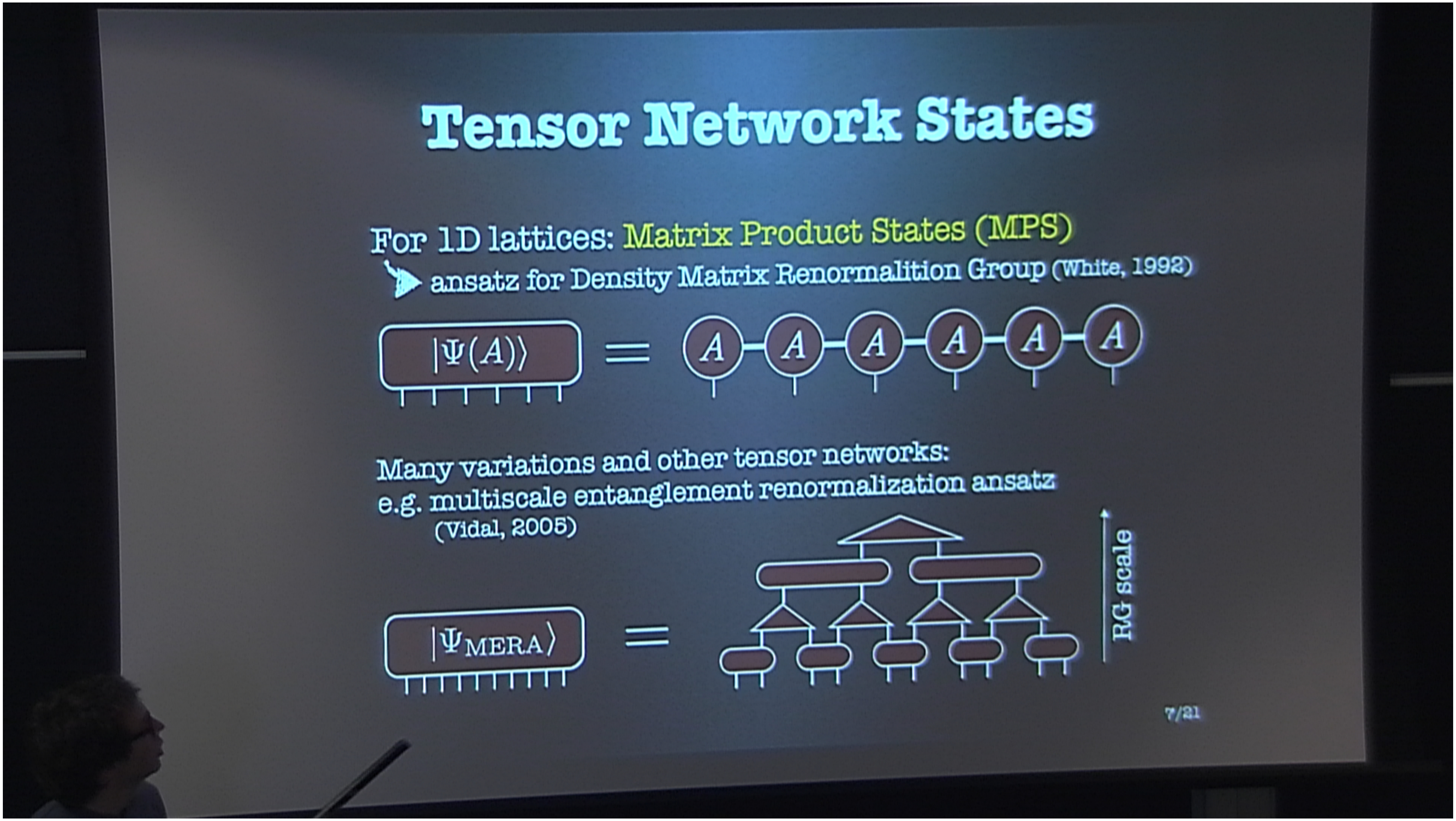
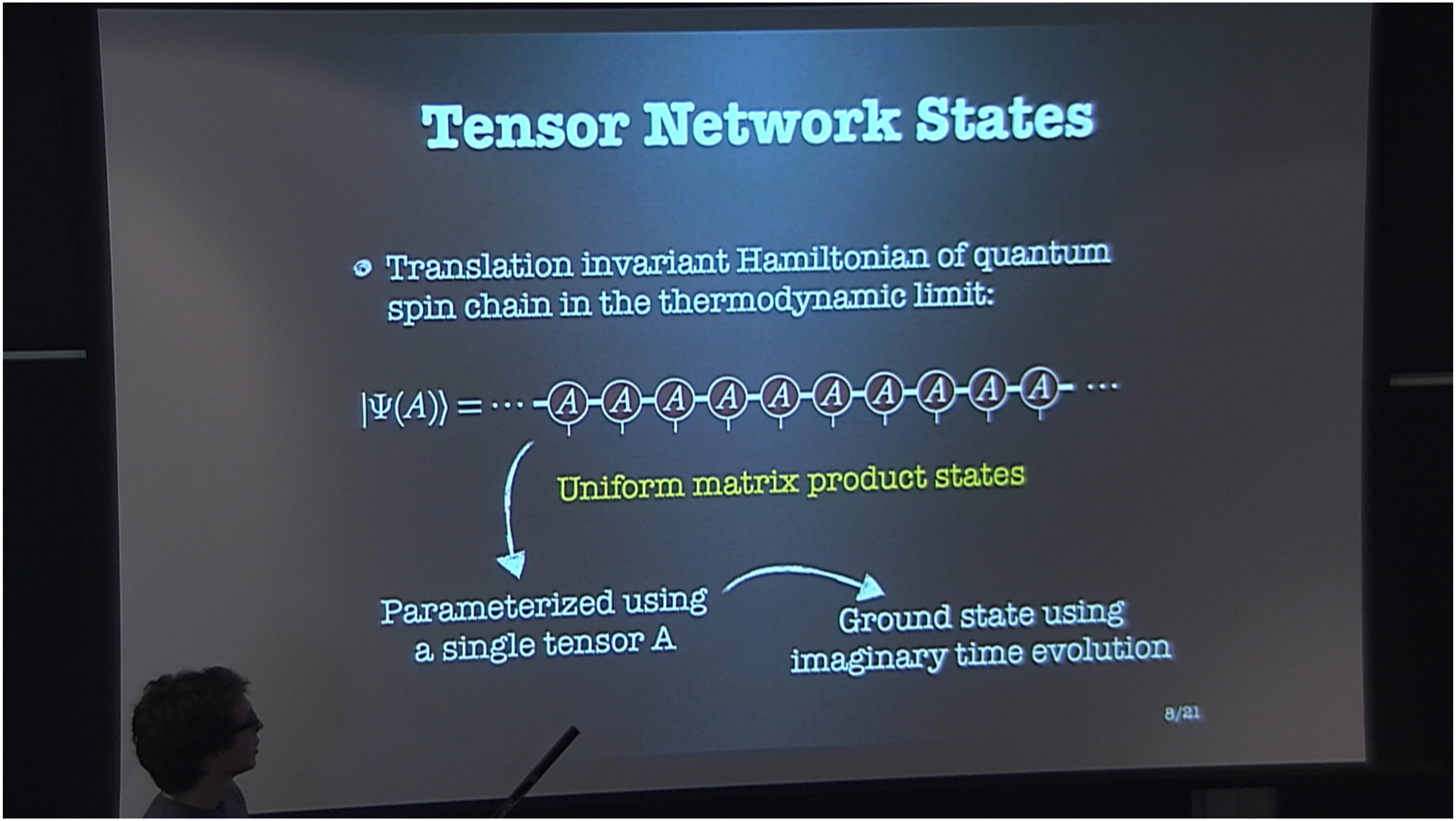
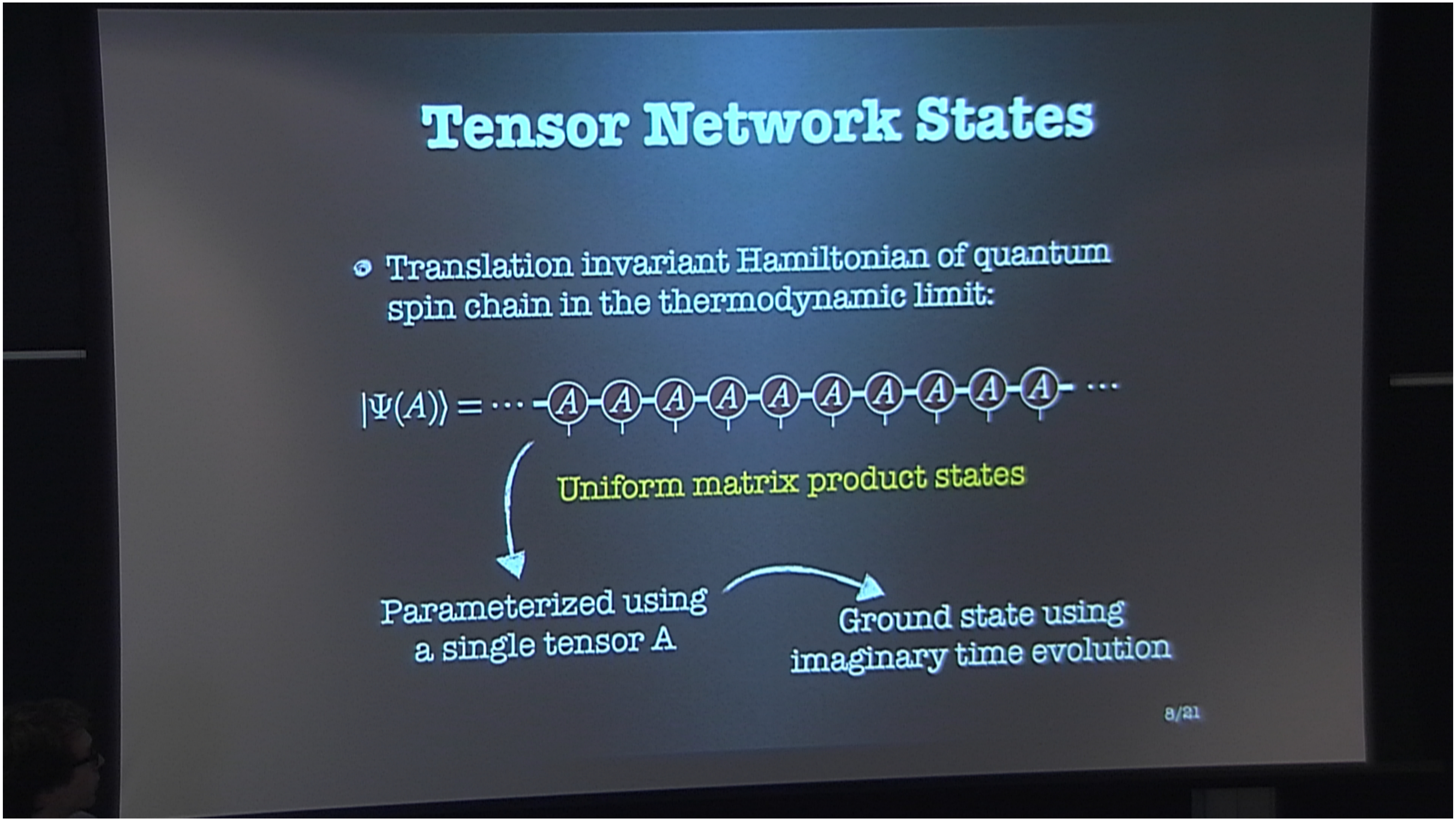

Tensor Network States
For 1D lattices: Matrix Product States (MPS)
ansatz for Density Matrix Renormalition Group (White, 1992)
|Ψ(A)⟩ =
A A A A A A
Many variations and other tensor networks:
e.g. multiscale entanglement renormalization ansatz
(Vidal, 2005)
|Ψ_MERA⟩ =
RG scale
7/21


Tensor Network States
Translation invariant Hamiltonian of quantum spin chain in the thermodynamic limit:
$|\Psi(A)\rangle = \cdots -A-A-A-A-A-A-A-A-A-A- \cdots$
Uniform matrix product states
Parameterized using a single tensor A
Ground state using imaginary time evolution
8/21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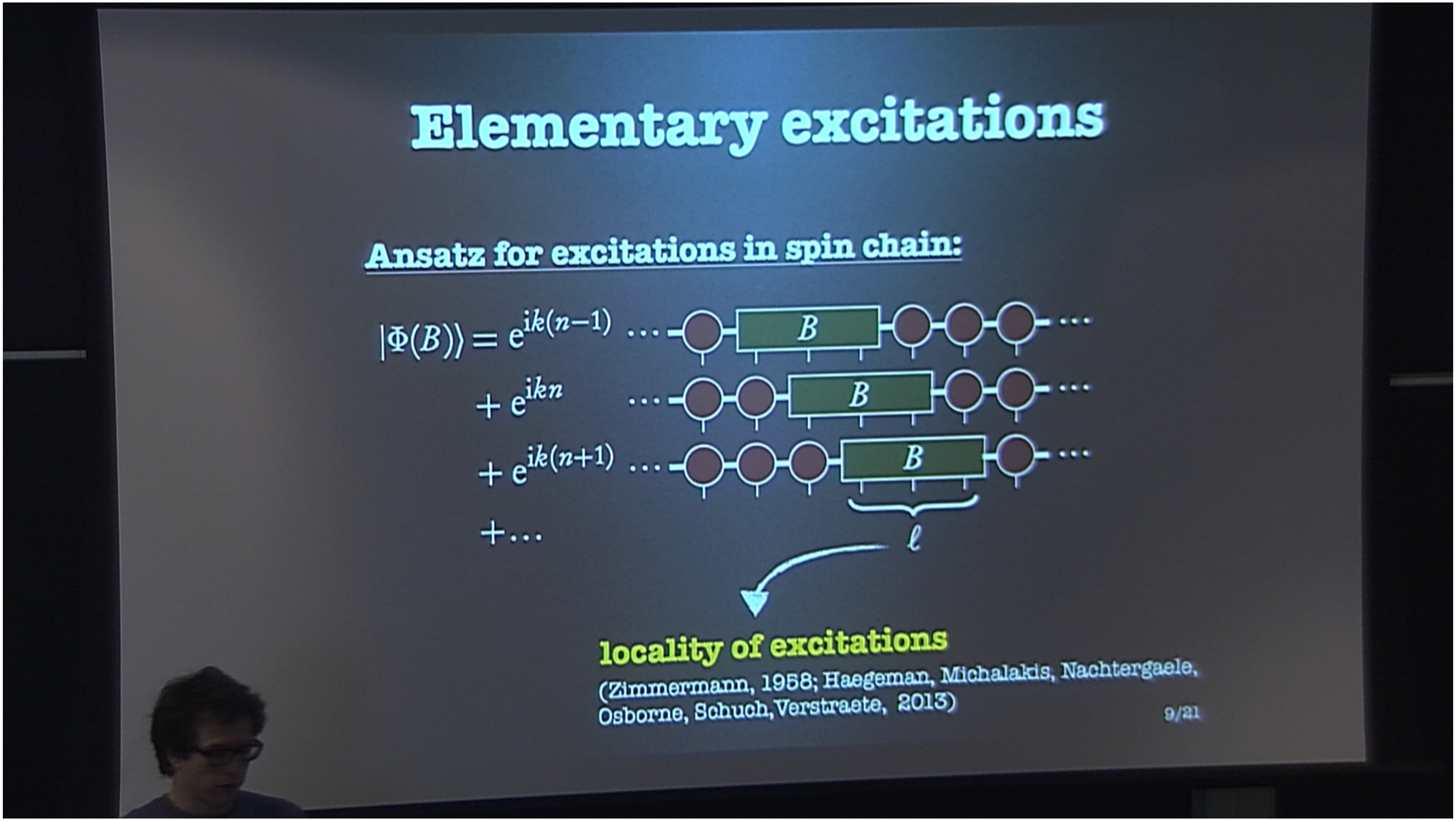
Elementary excitations
Ansatz for excitations in spin chain:
$|\Phi(B)\rangle = e^{ik(n-1)} \cdots$
$+ e^{ikn} \cdots$
$+ e^{ik(n+1)} \cdots$
$+ \ldots$
B
B
B
ℓ
locality of excitations
(Zimmermann, 1958; Haegeman, Michalakis, Nachtergaele, Osborne, Schuch, Verstraete, 2013)
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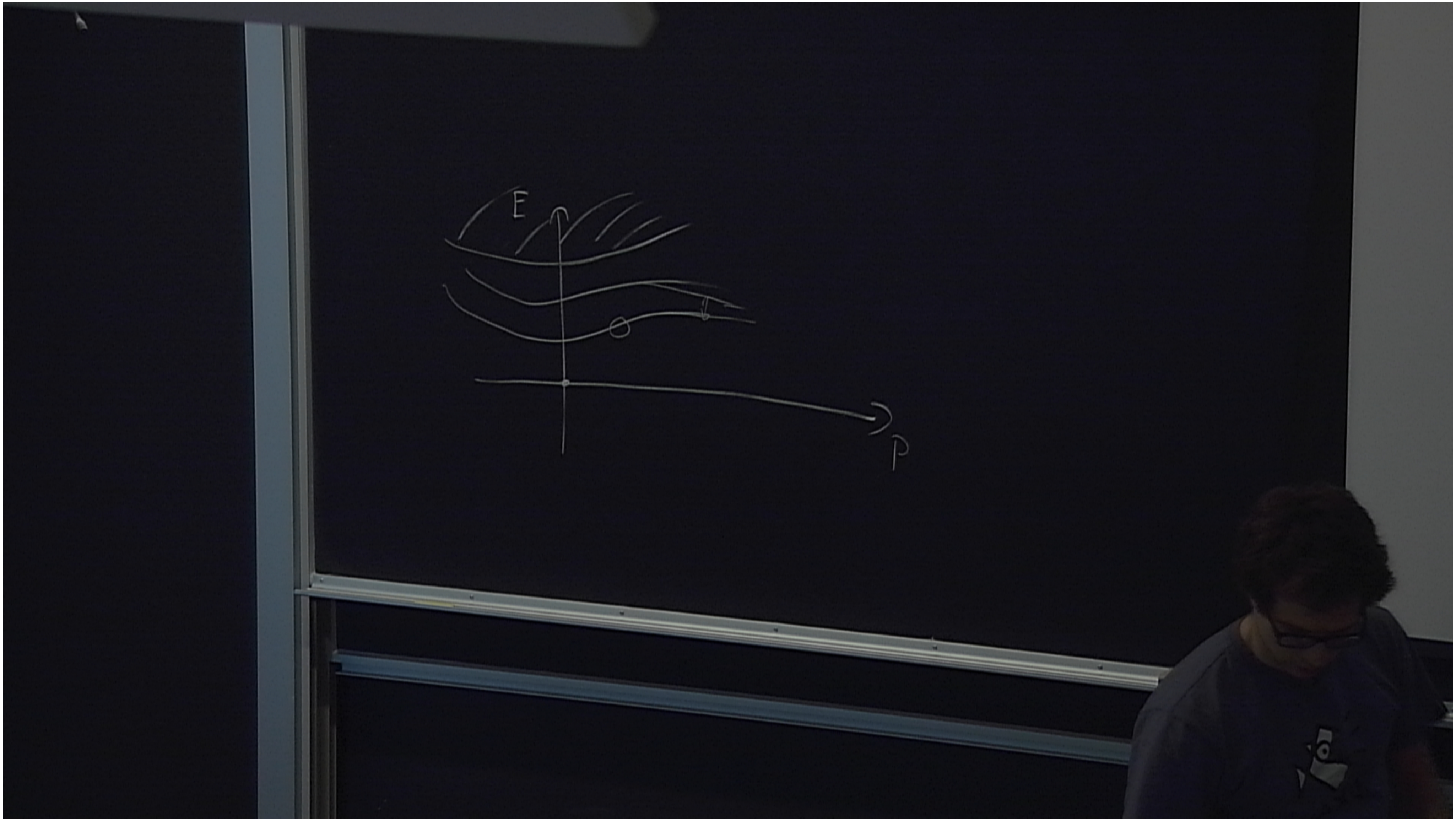
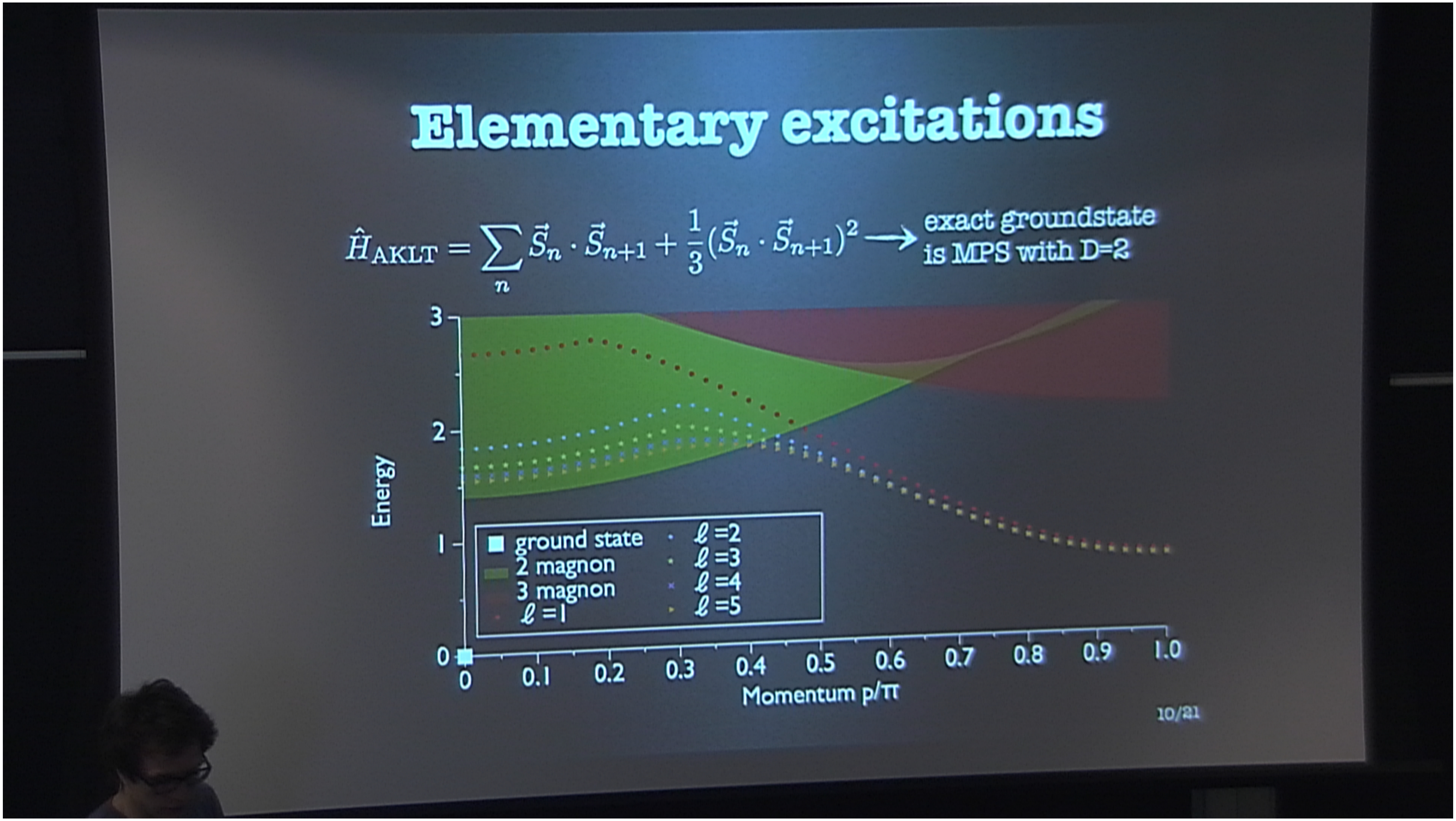
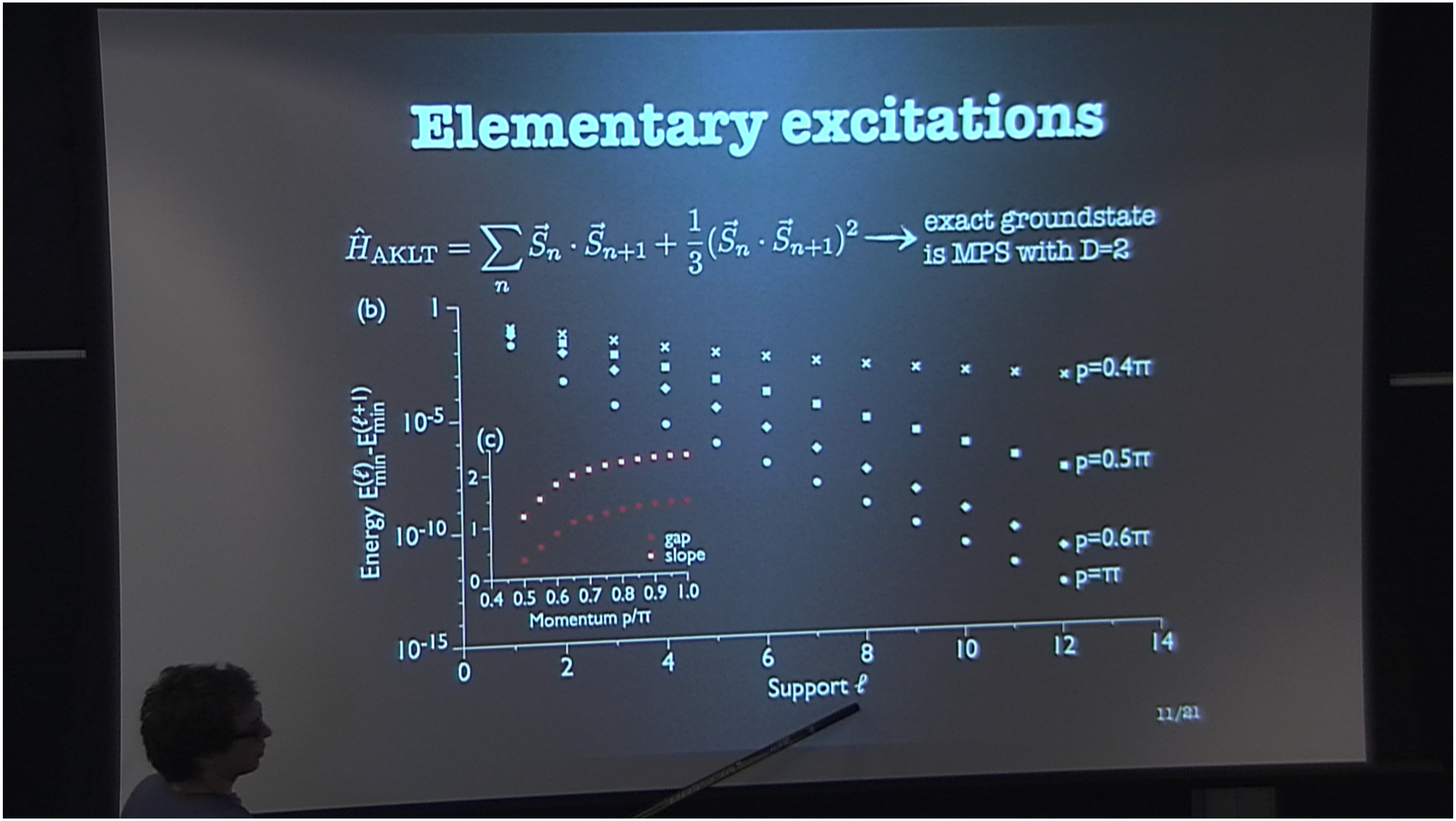

E
P


Elementary excitations
$\hat{H}_{\mathrm{AKLT}} = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + \frac{1}{3}(\vec{S}_n \cdot \vec{S}_{n+1})^2 \longrightarrow$ exact groundstate is MPS with D=2
Energy
Momentum p/π
ground state
2 magnon
3 magnon
ℓ =1
ℓ =2
ℓ =3
ℓ =4
ℓ =5
0
1
2
3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
10/21



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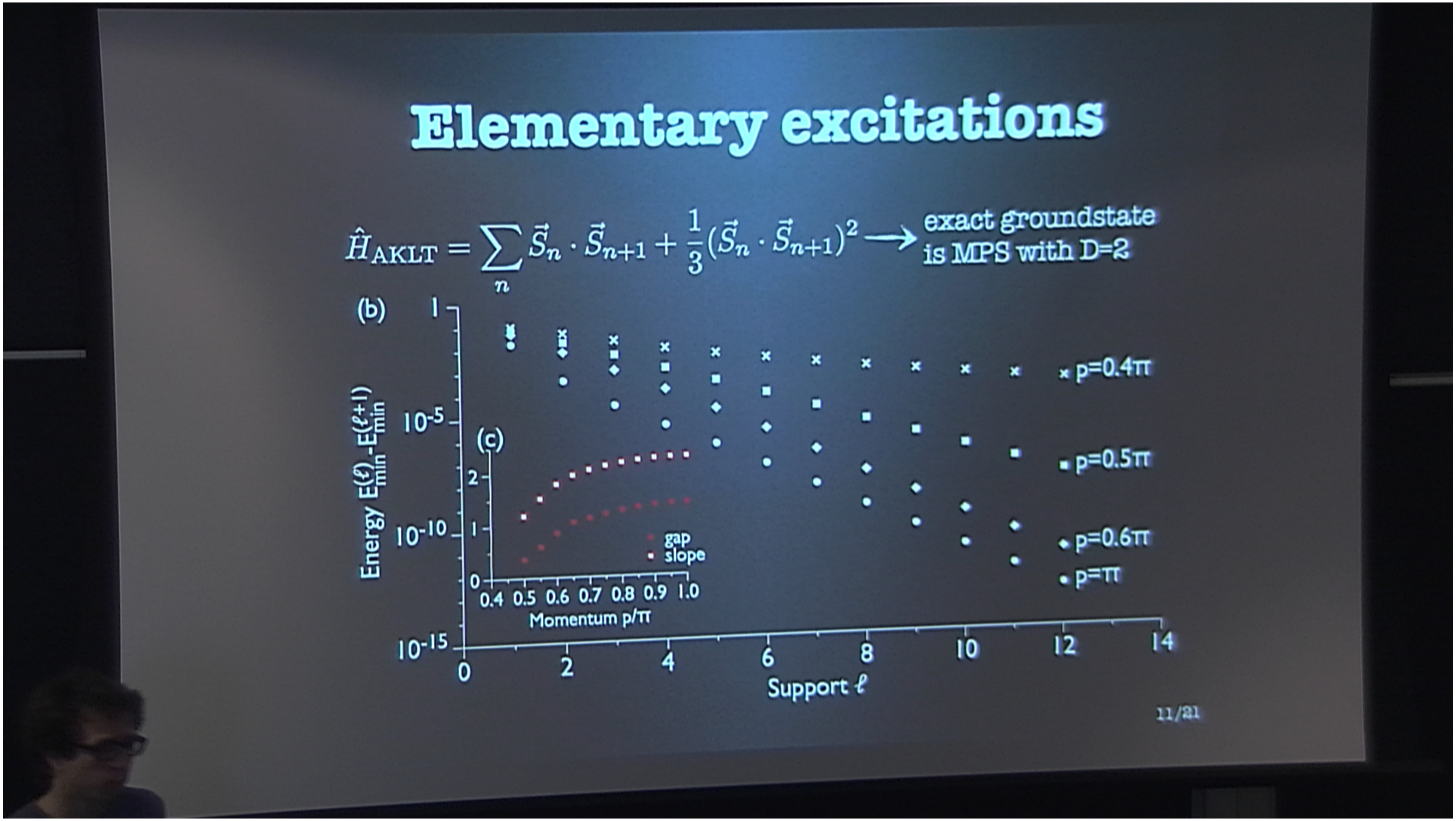
(b)
Energy $E^{(\ell)}_{min} - E^{(\ell+1)}_{min}$
Support ℓ
p=0.4π
p=0.5π
p=0.6π
p=π
(c)
gap
slope
Momentum p/π
11/21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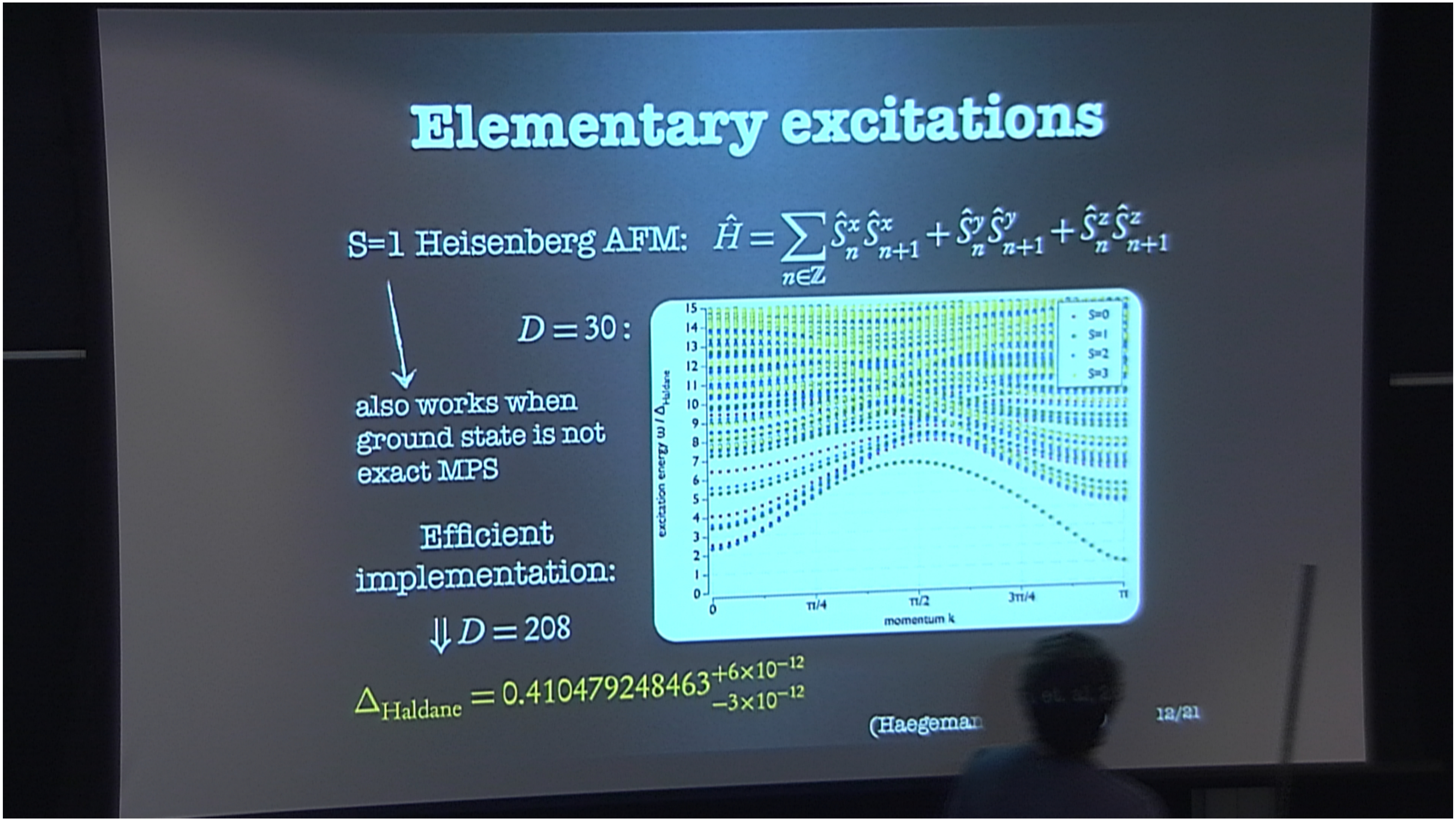
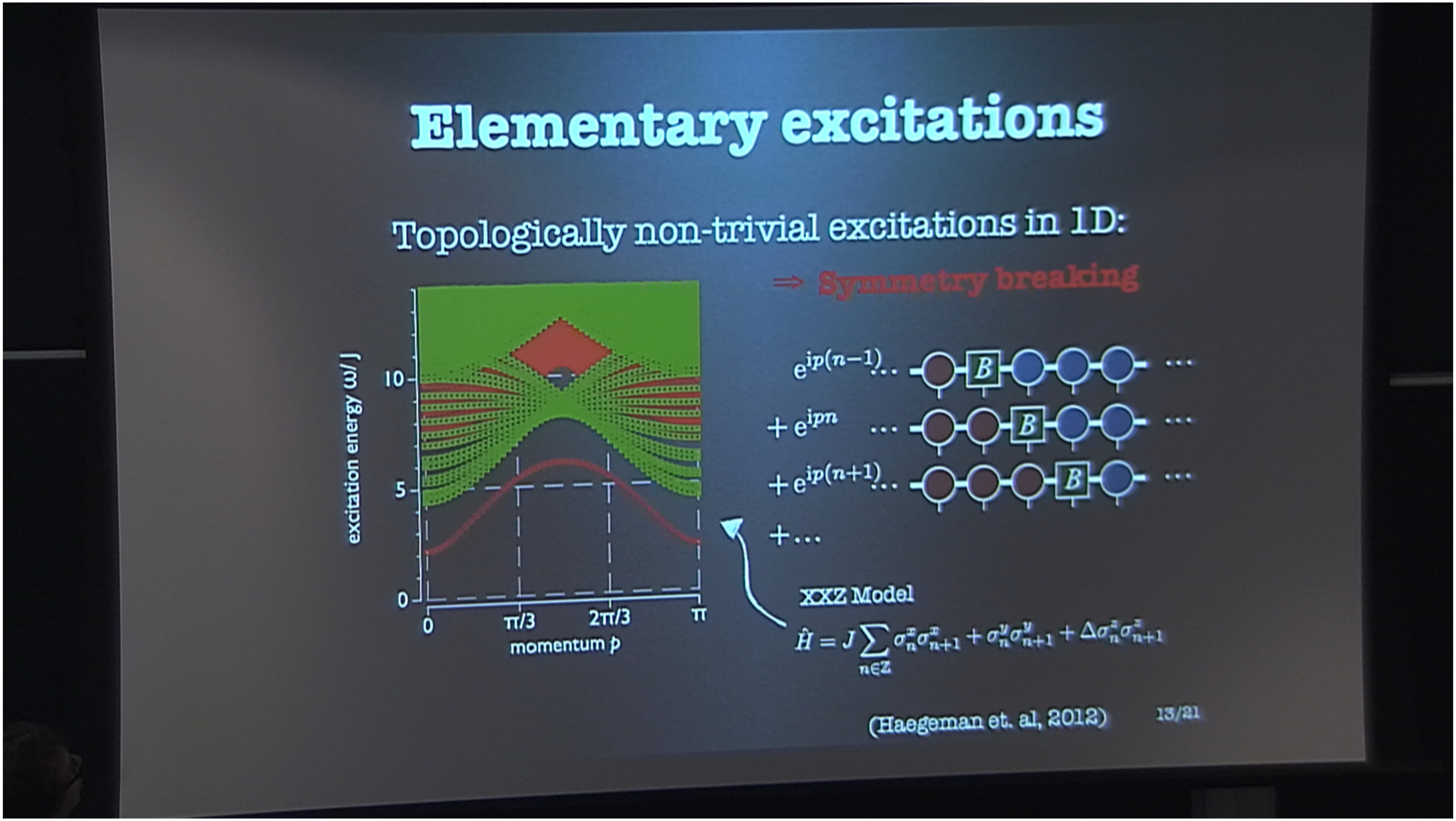
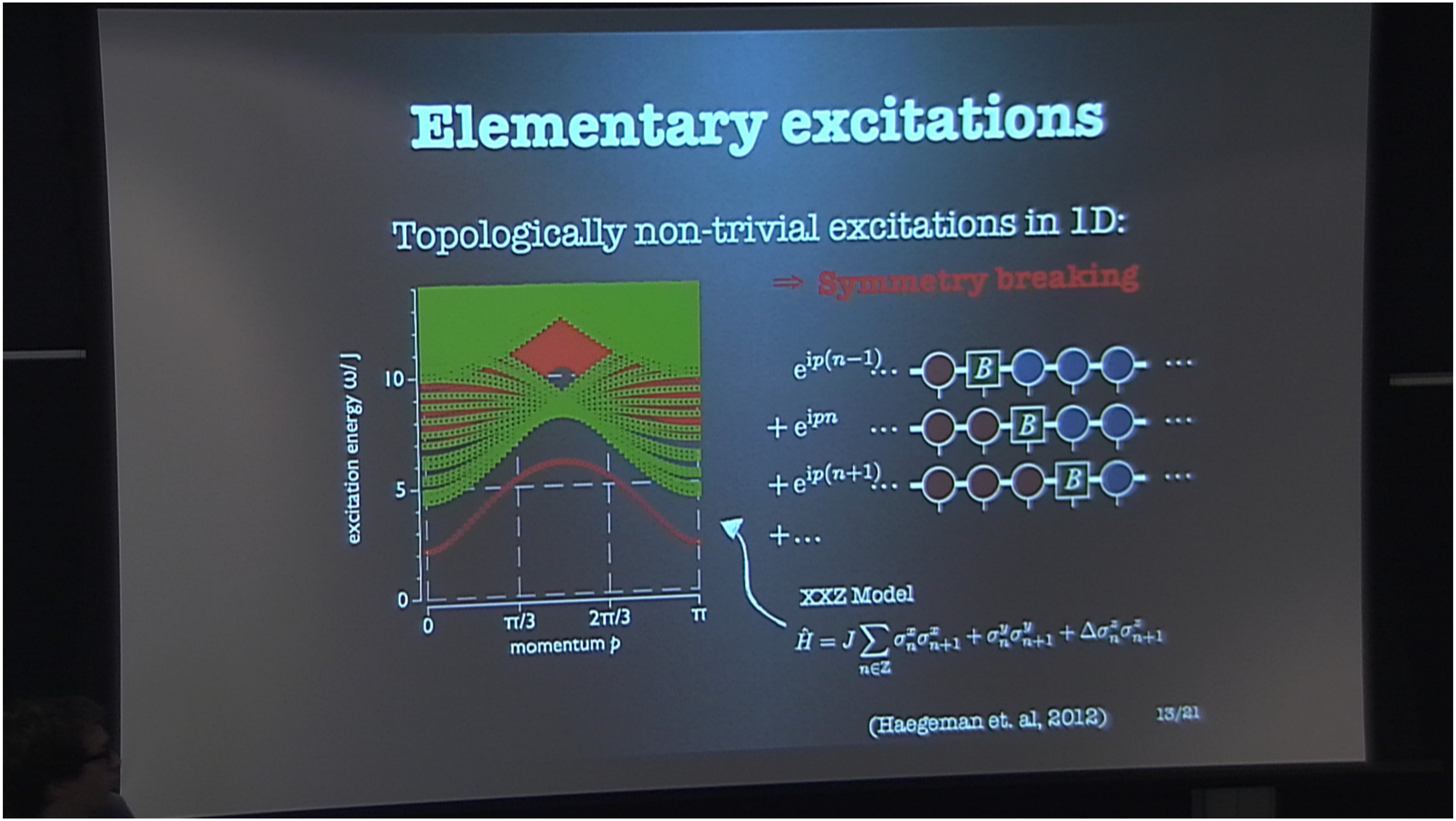
(b)
Energy $E^{(\ell)}_{\min} - E^{(\ell+1)}_{\min}$
1
10^{-5}
10^{-10}
10^{-15}
0 2 4 6 8 10 12 14
Support ℓ
p=0.4π
p=0.5π
p=0.6π
p=π
(c)
0 1 2
0.4 0.5 0.6 0.7 0.8 0.9 1.0
Momentum p/π
gap
slope
11/21


Elementary excitations
S=1 Heisenberg AFM: $\hat{H} = \sum_{n\in\mathbb{Z}} \hat{S}^x_n \hat{S}^x_{n+1} + \hat{S}^y_n \hat{S}^y_{n+1} + \hat{S}^z_n \hat{S}^z_{n+1}$
$D = 30$:
also works when ground state is not exact MPS
Efficient implementation:
$\Downarrow D = 208$
$\Delta_{\text{Haldane}} = 0.410479248463^{+6\times10^{-12}}_{-3\times10^{-12}}$
excitation energy ω / Δ_Haldane
momentum k
S=0
S=1
S=2
S=3
(Haegeman
12/21


Elementary excitations
Topologically non-trivial excitations in 1D:
⇒ Symmetry breaking
excitation energy ω/J
10
5
0
0
π/3
2π/3
π
momentum p
$e^{ip(n-1)}$ ...
$+ e^{ipn}$ ...
$+ e^{ip(n+1)}$ ...
+...
B
XXZ Model
$\hat{H} = J \sum_{n \in \mathbb{Z}} \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z$
(Haegeman et. al, 2012)
13/21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momentum p
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B
...
$+ e^{ipn}$ ...
B
...
$+ e^{ip(n+1)}$ ...
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...
+...
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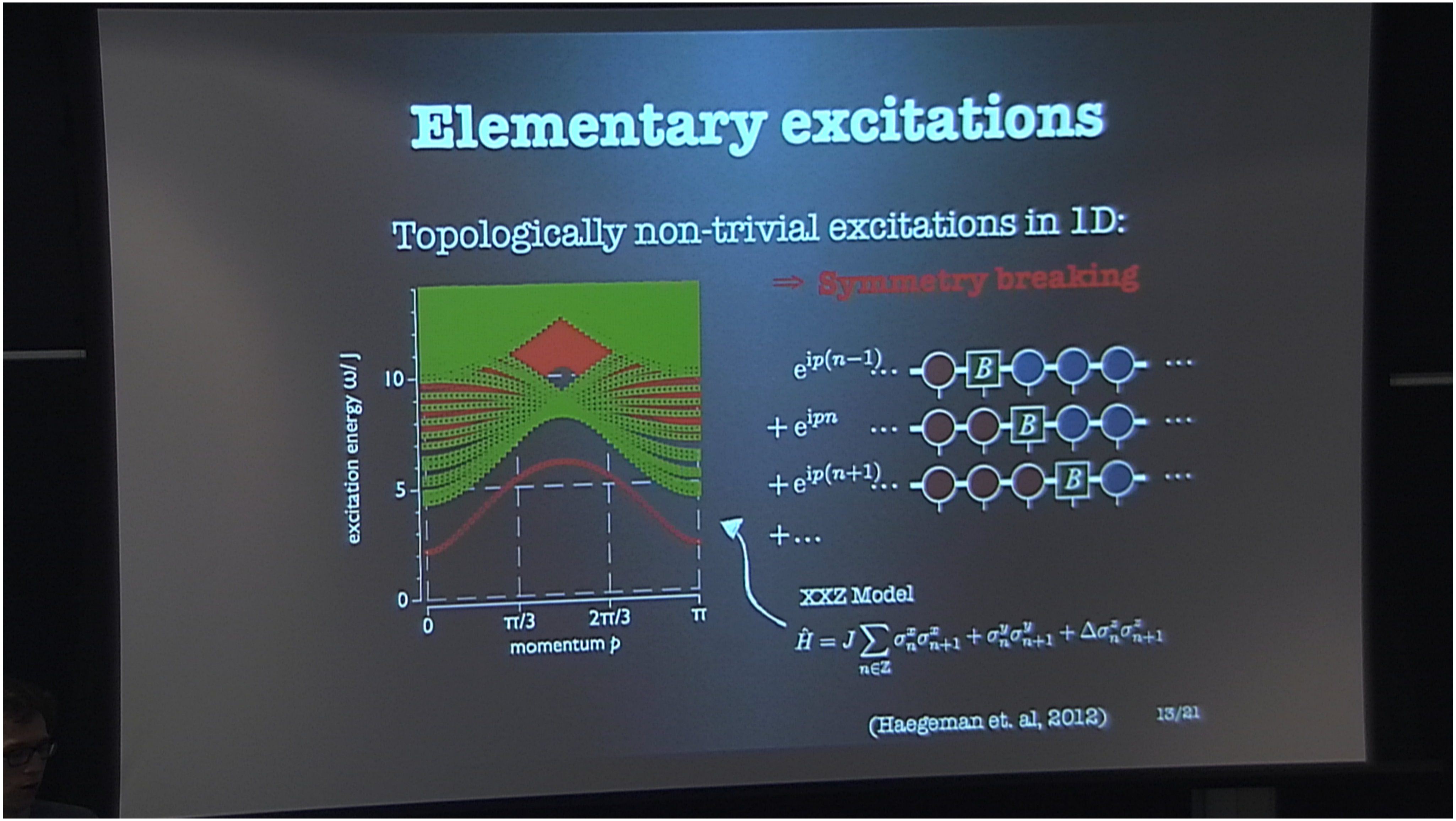

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Topologically non-trivial excitations in 1D:
⇒ Symmetry breaking
excitation energy ω/J
10
5
0
0
π/3
2π/3
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momentum p
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$+ e^{ipn} \cdots$ B $\cdots$
$+ e^{ip(n+1)} \cdots$ B $\cdots$
$+ \cdots$
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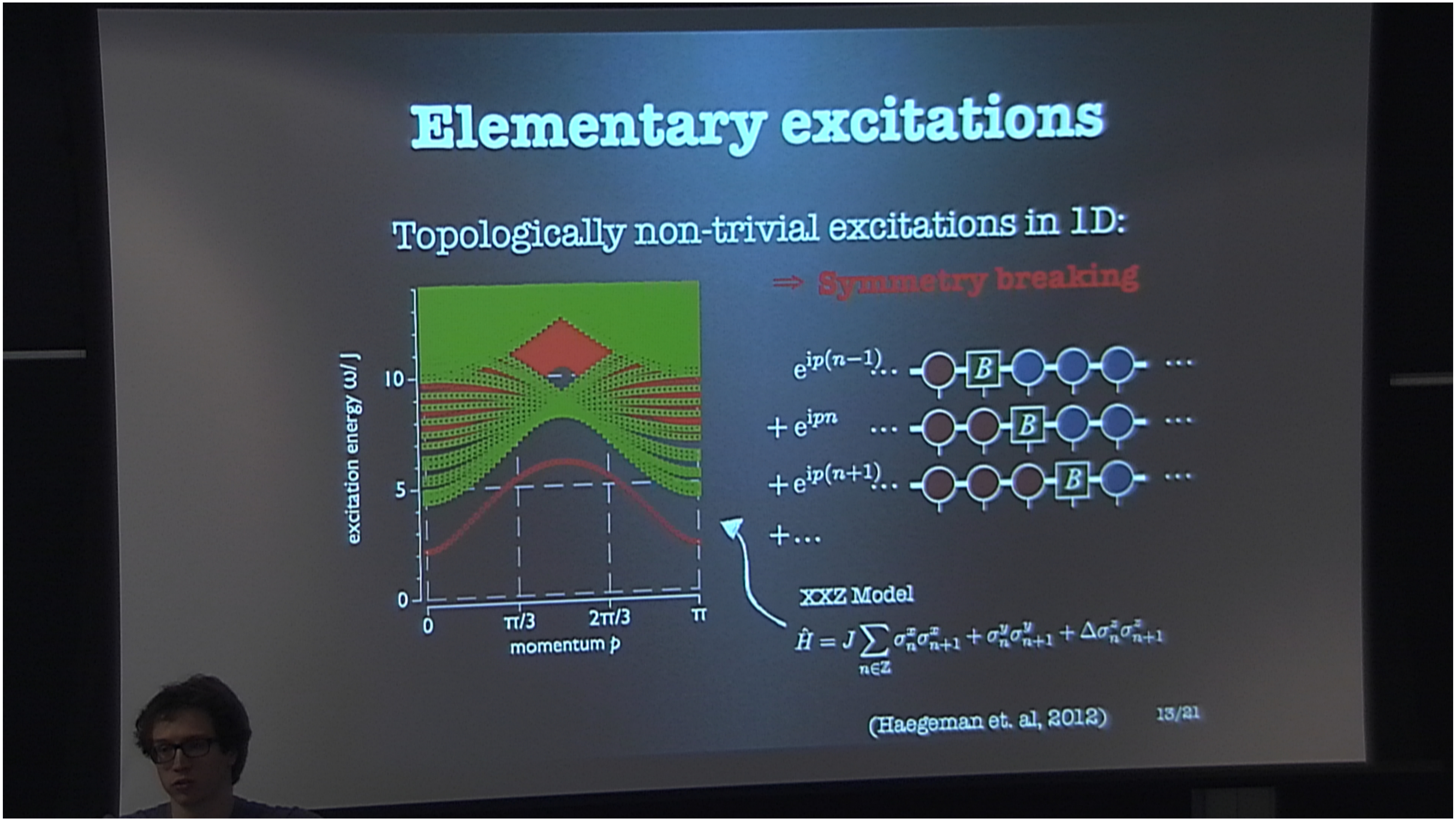
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10
5
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π
momentum p
$e^{ip(n-1)}$ ...
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$+ \dots$
B
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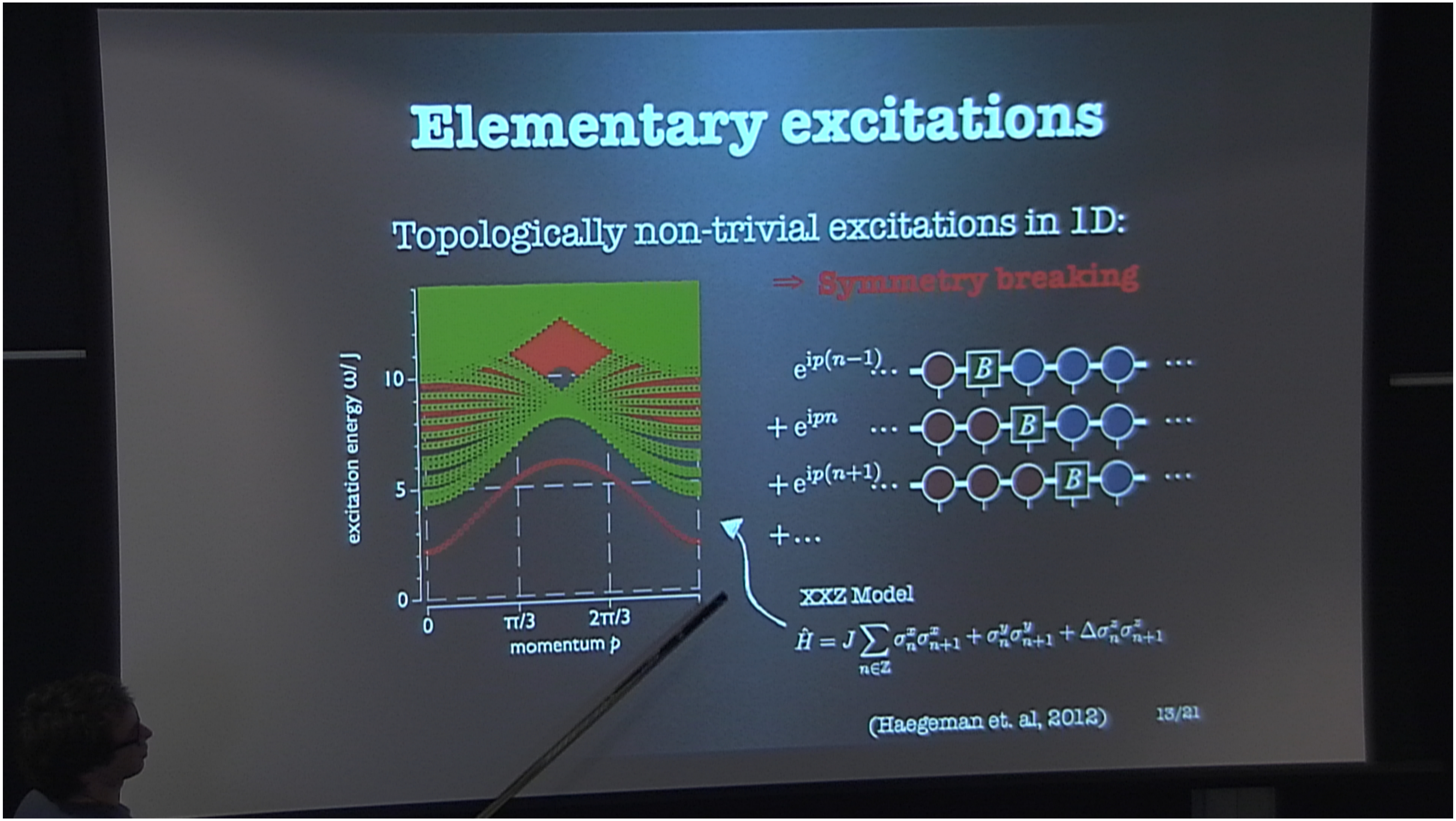
13/21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+ e^{ipn} ... B ...
+ e^{ip(n+1)} ... B ...
+...
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Elementary excitations

S=1 Heisenberg AFM: $\hat{H} = \sum_{n \in \mathbb{Z}} \hat{S}^x_n \hat{S}^x_{n+1} + \hat{S}^y_n \hat{S}^y_{n+1} + \hat{S}^z_n \hat{S}^z_{n+1}$

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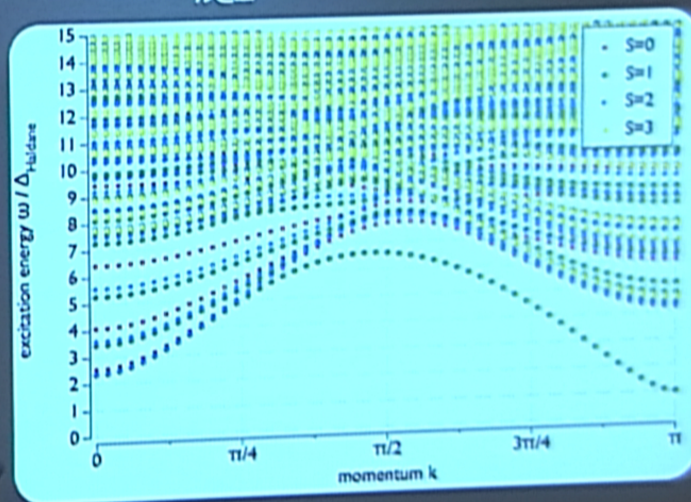

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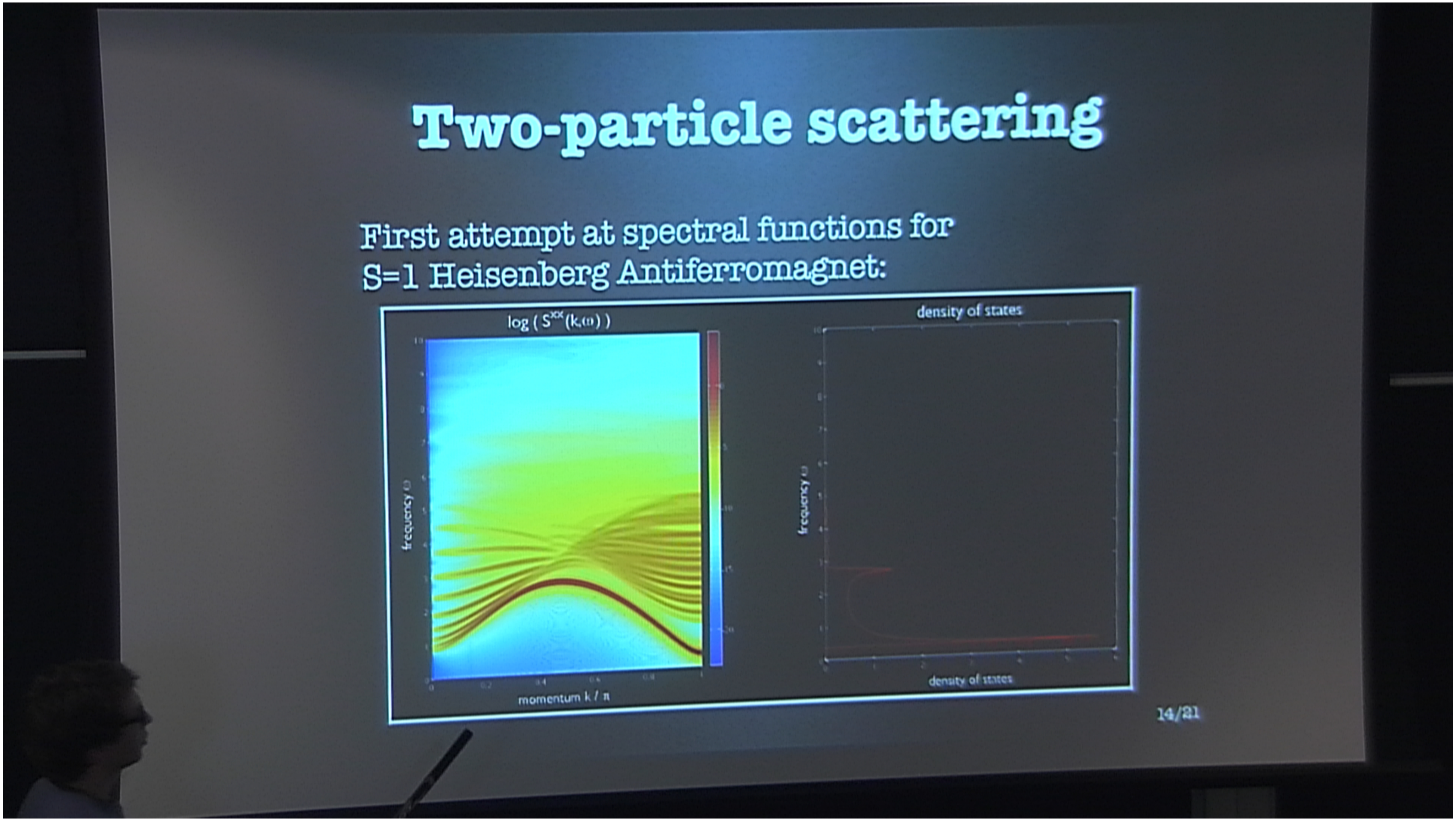
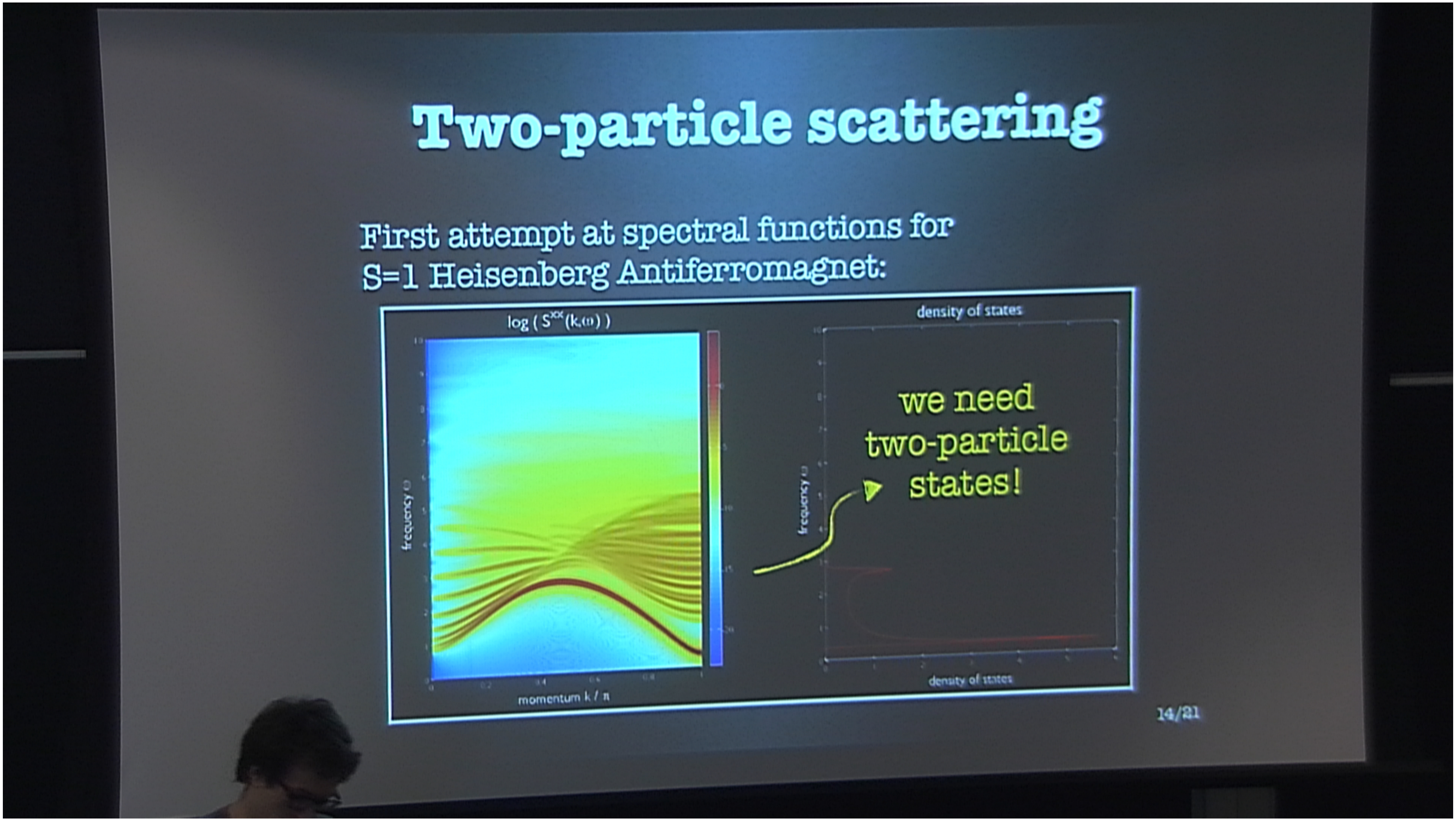

Two-particle scattering
First attempt at spectral functions for
S=1 Heisenberg Antiferromagnet:
log (S^xx(k,ω))
density of states
frequency ω
momentum k / π
frequency ω
density of states
14/21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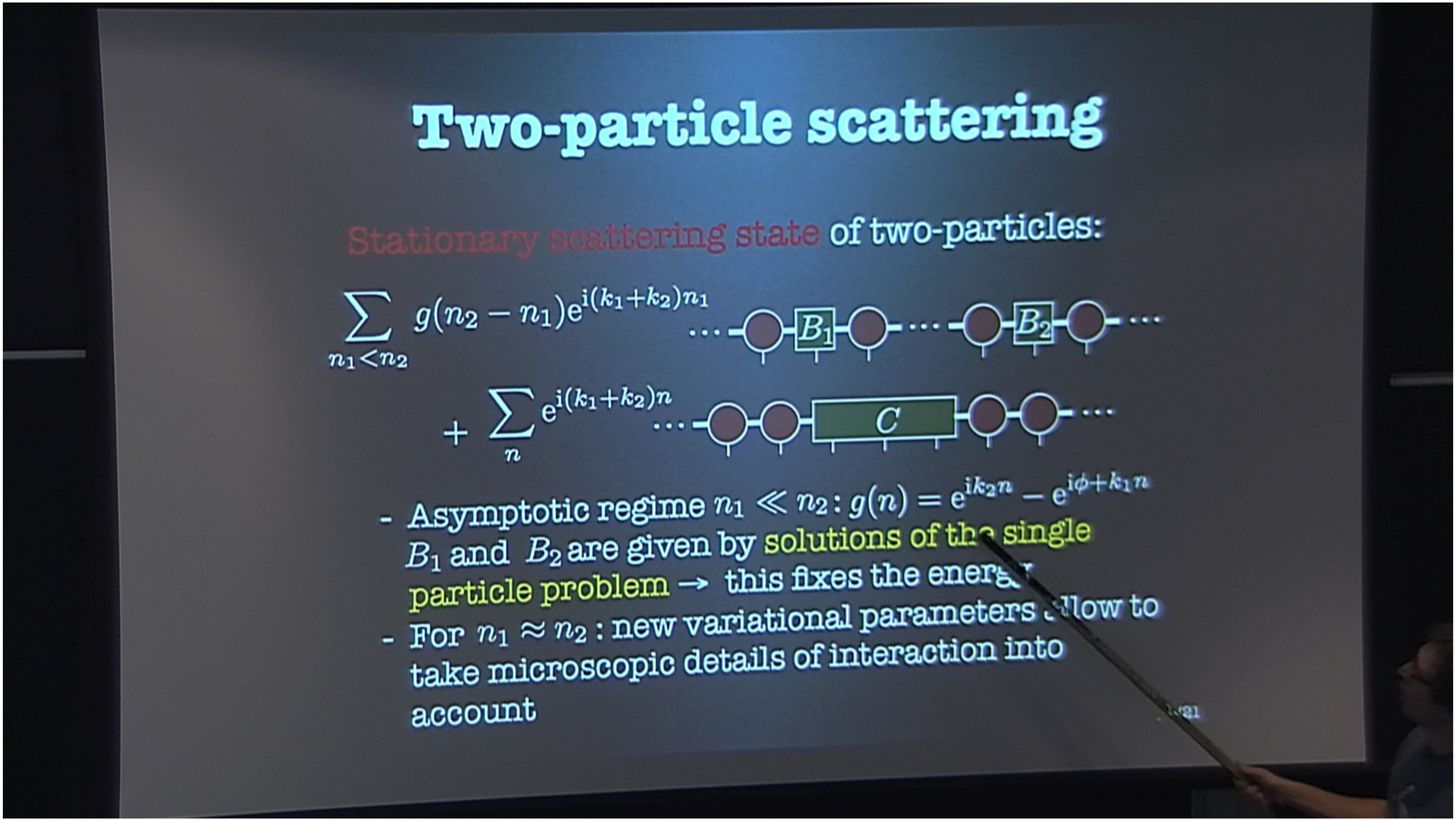
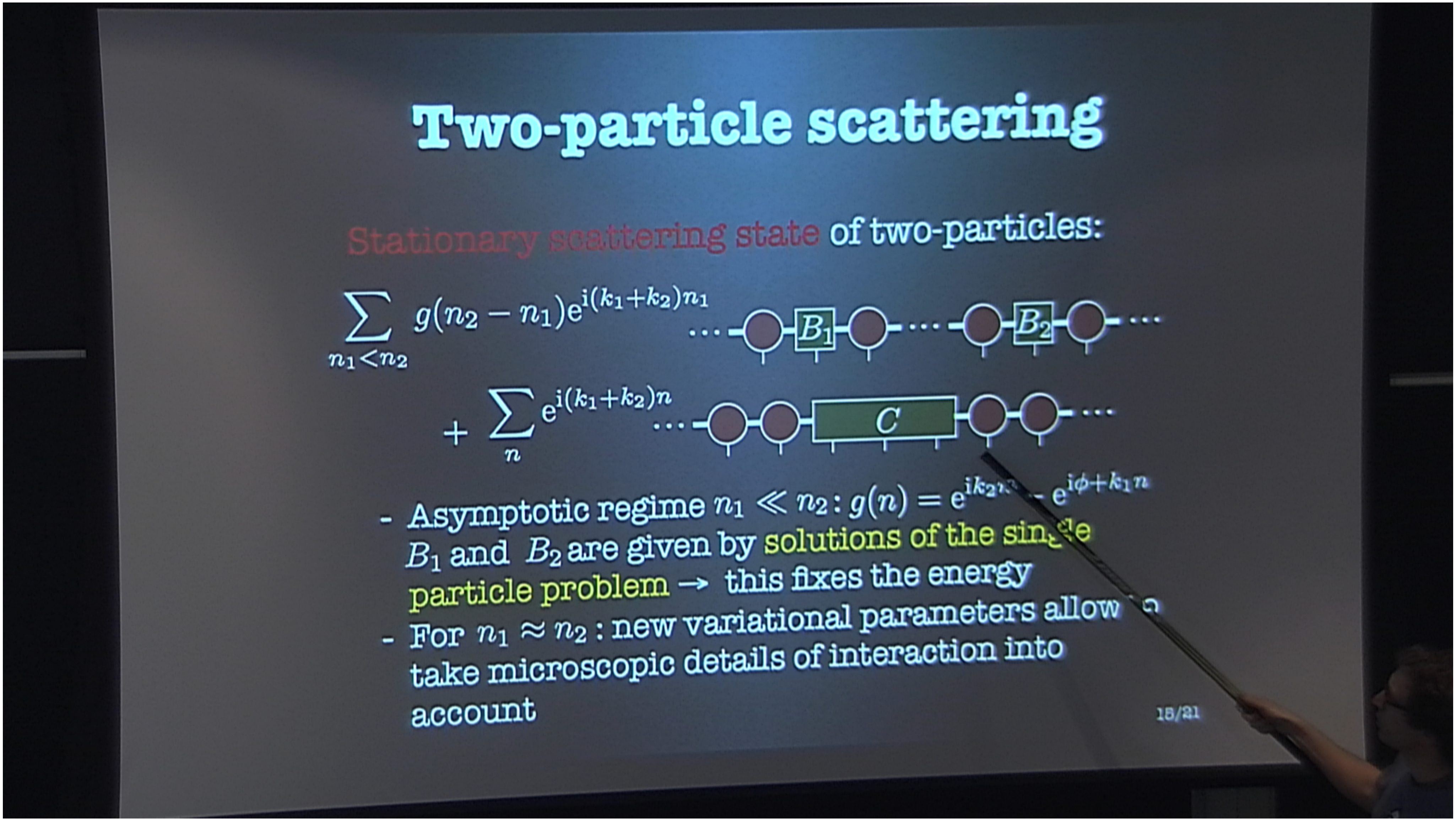
frequency ω
momentum k / π
density of states
frequency ω
density of states
we need
two-particle
states!
14/21


Two-particle scattering
Stationary scattering state of two-particles:
$\sum_{n_1<n_2} g(n_2-n_1)\mathrm{e}^{\mathrm{i}(k_1+k_2)n_1}$
B_1
B_2
$+\sum_n \mathrm{e}^{\mathrm{i}(k_1+k_2)n}$
C
- Asymptotic regime $n_1 \ll n_2$: $g(n) = \mathrm{e}^{\mathrm{i}k_2 n} - \mathrm{e}^{\mathrm{i}\phi + k_1 n}$
B_1 and B_2 are given by solutions of the single particle problem → this fixes the energy
- For $n_1 \approx n_2$: new variational parameters allow to take microscopic details of interaction into account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15/21

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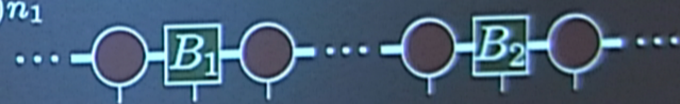
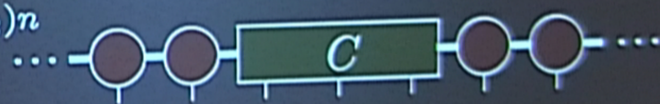


$$+\sum_{n} \mathrm{e}^{\mathrm{i}(k_1+k_2)n}$$



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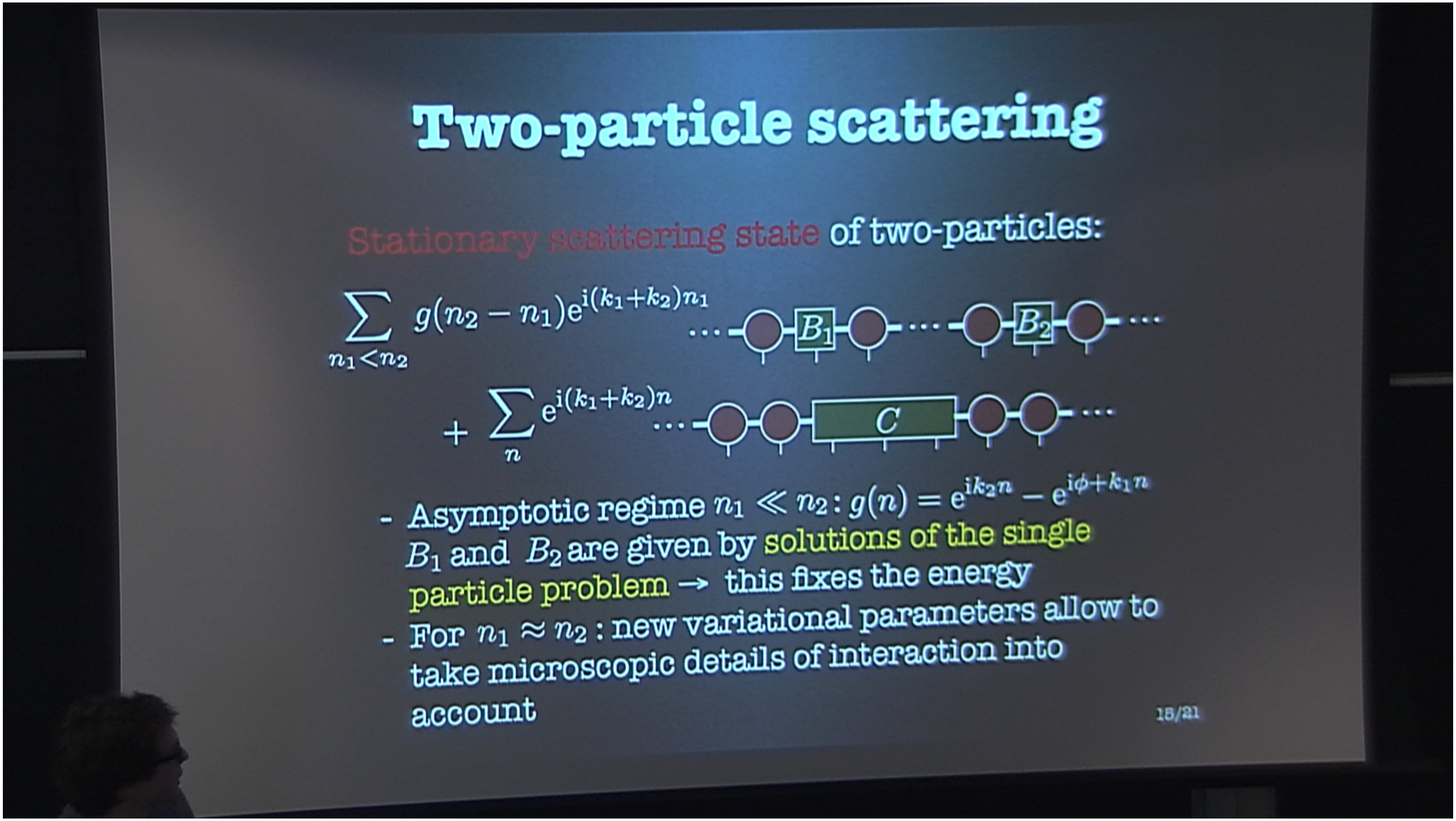
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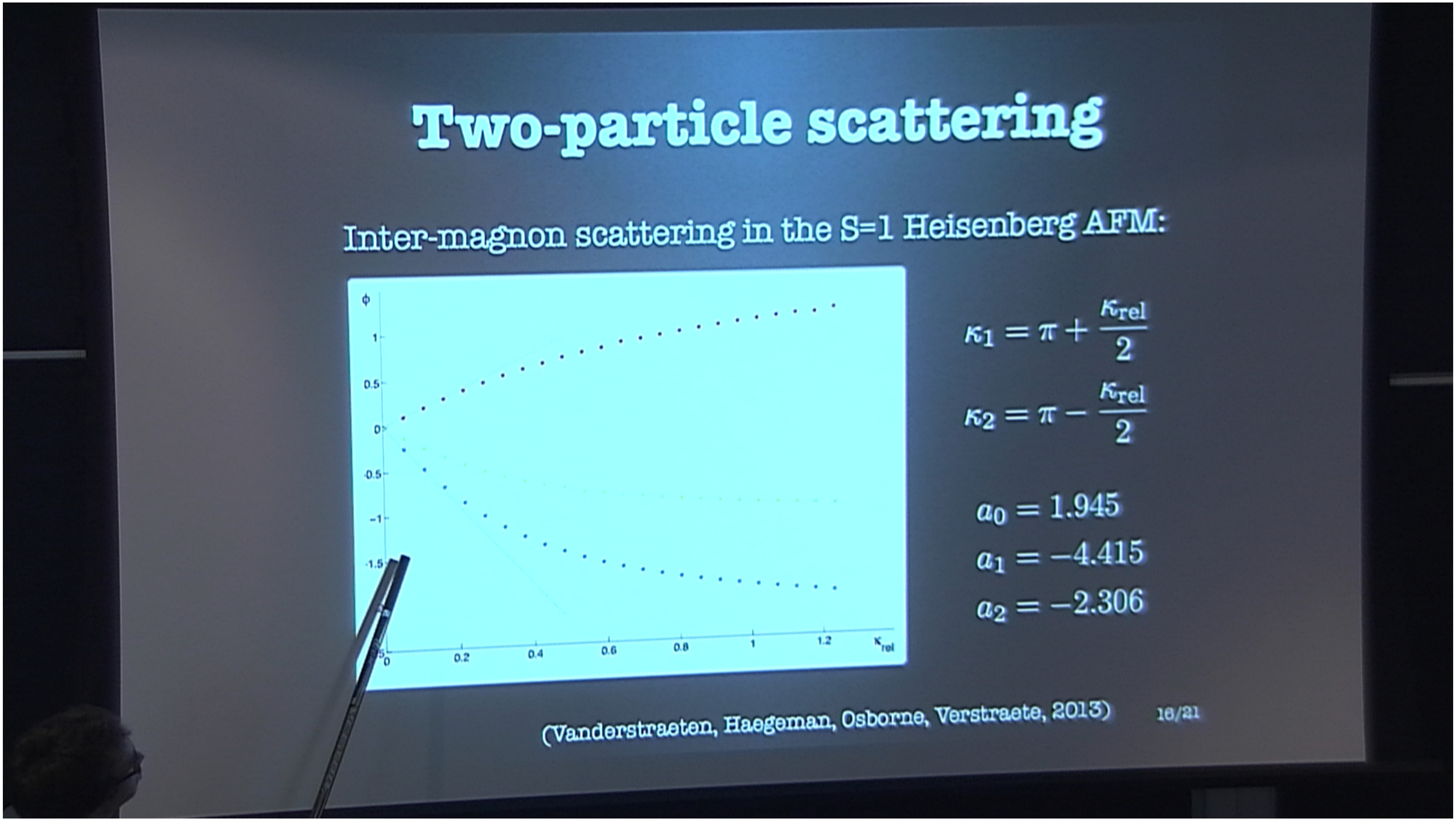
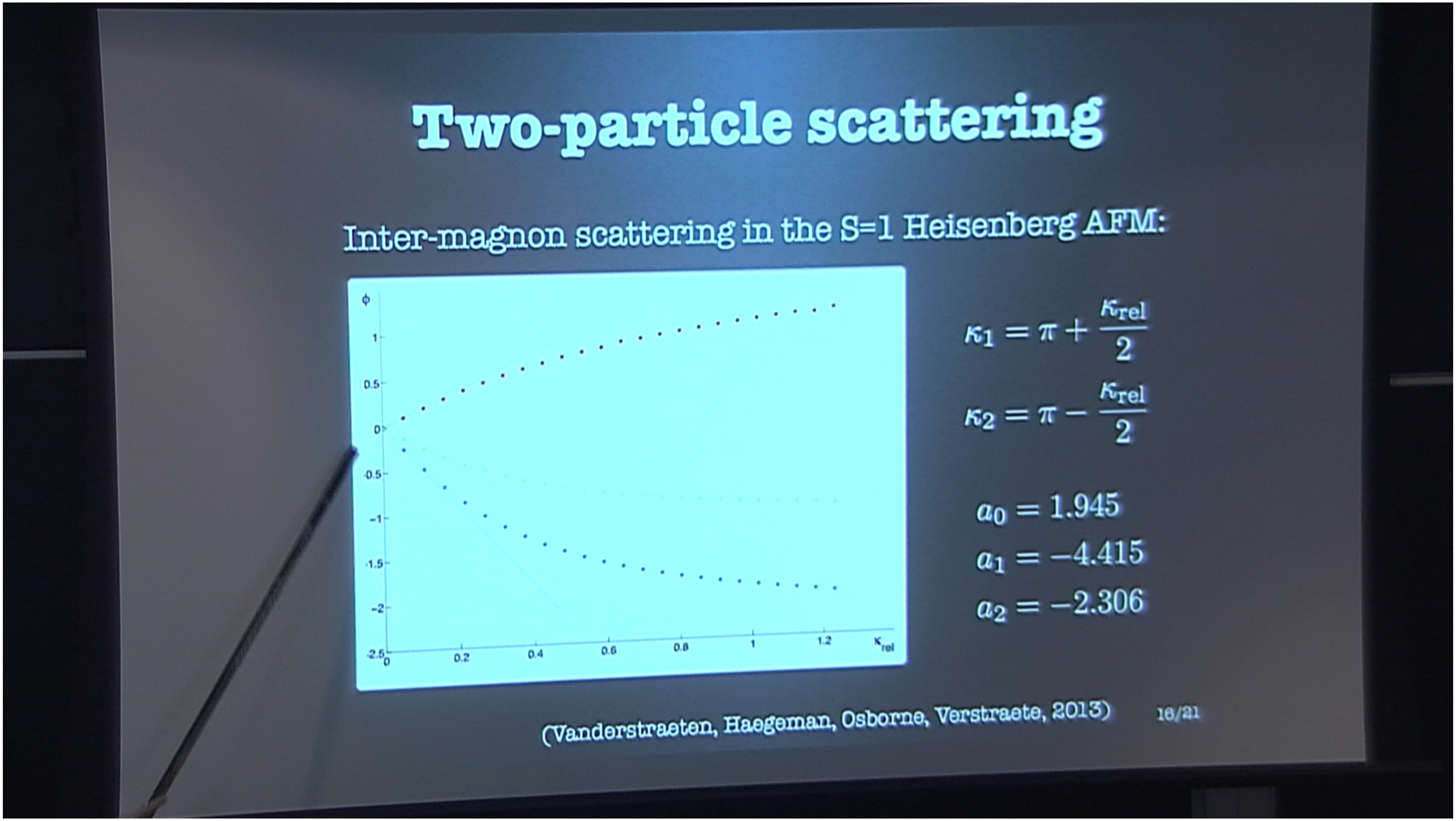

Two-particle scattering
Inter-magnon scattering in the S=1 Heisenberg AFM:
$\kappa_1 = \pi + \frac{\kappa_{rel}}{2}$
$\kappa_2 = \pi - \frac{\kappa_{rel}}{2}$
$a_0 = 1.945$
$a_1 = -4.415$
$a_2 = -2.306$
(Vanderstraeten, Haegeman, Osborne, Verstraete, 2013)
16/21



Two-particle scattering
Inter-magnon scattering in the S=1 Heisenberg AFM:
ϕ
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.2
0.4
0.6
0.8
1
1.2
κ_{rel}
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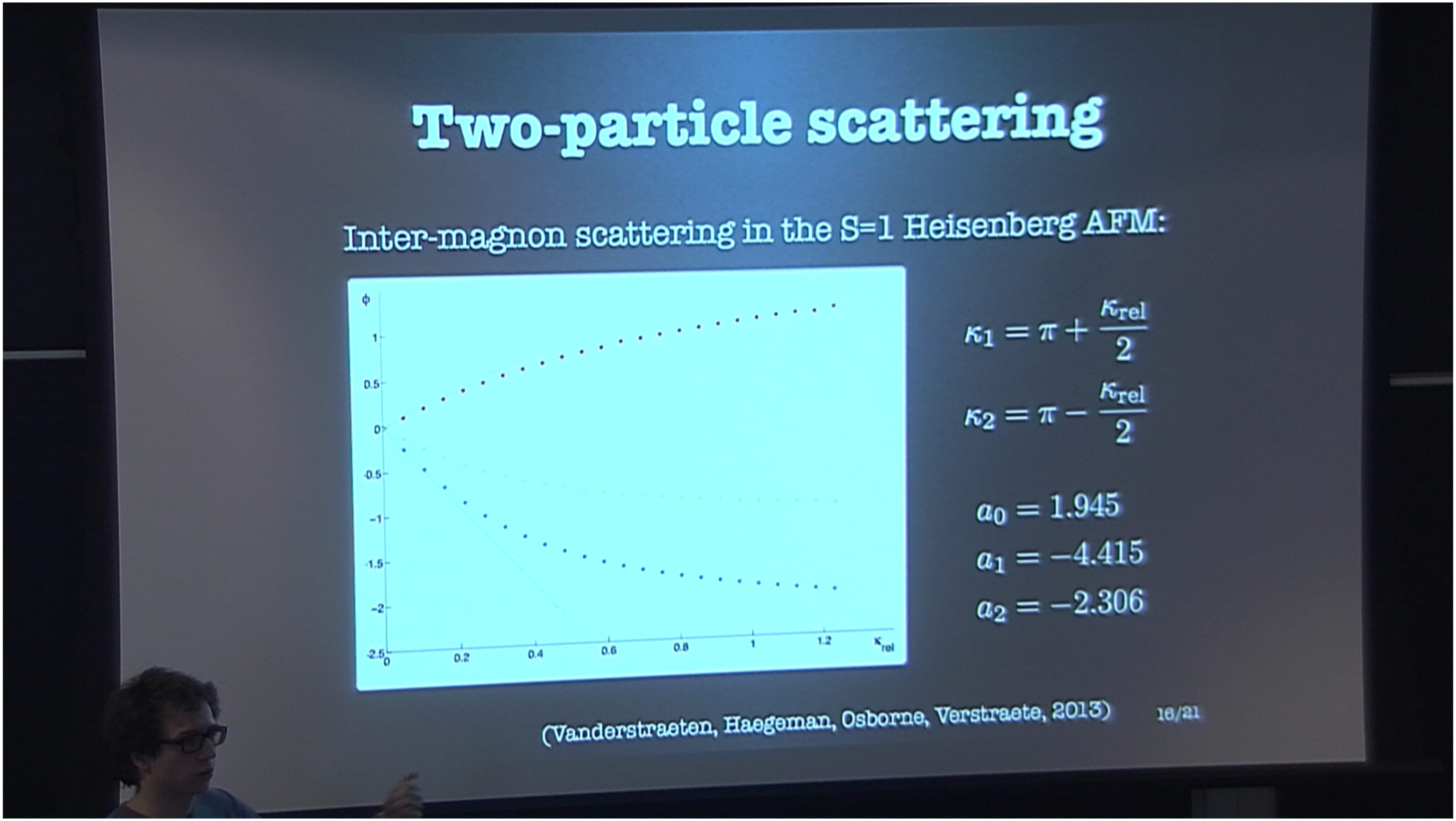
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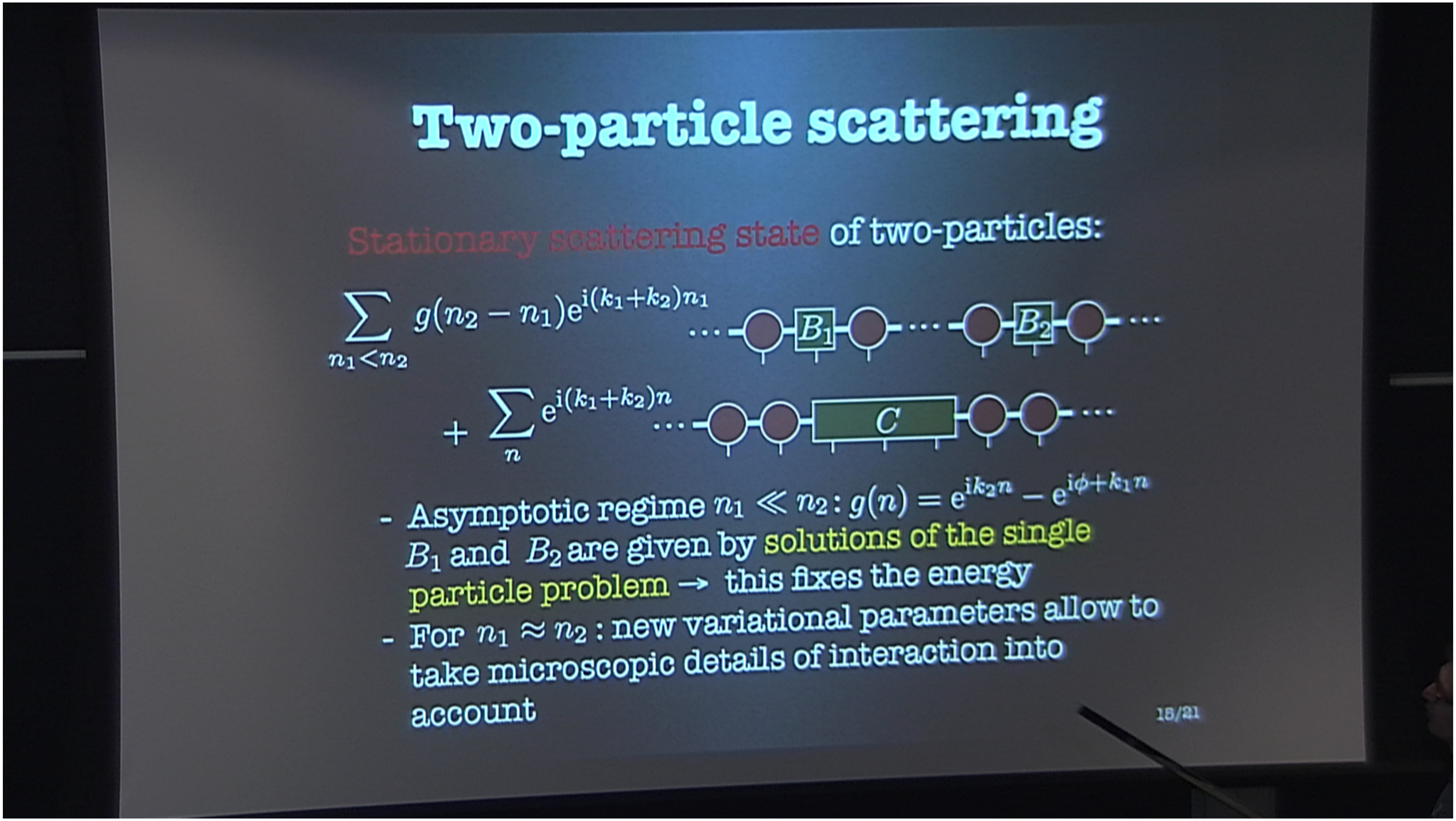
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B_1
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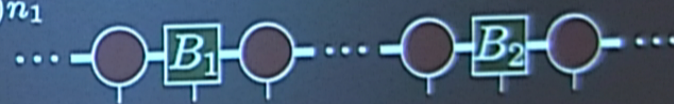
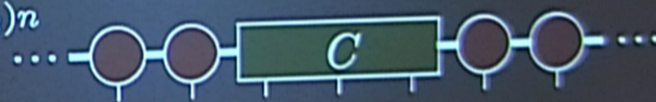


$$+\sum_{n} \mathrm{e}^{\mathrm{i}(k_1+k_2)n}$$



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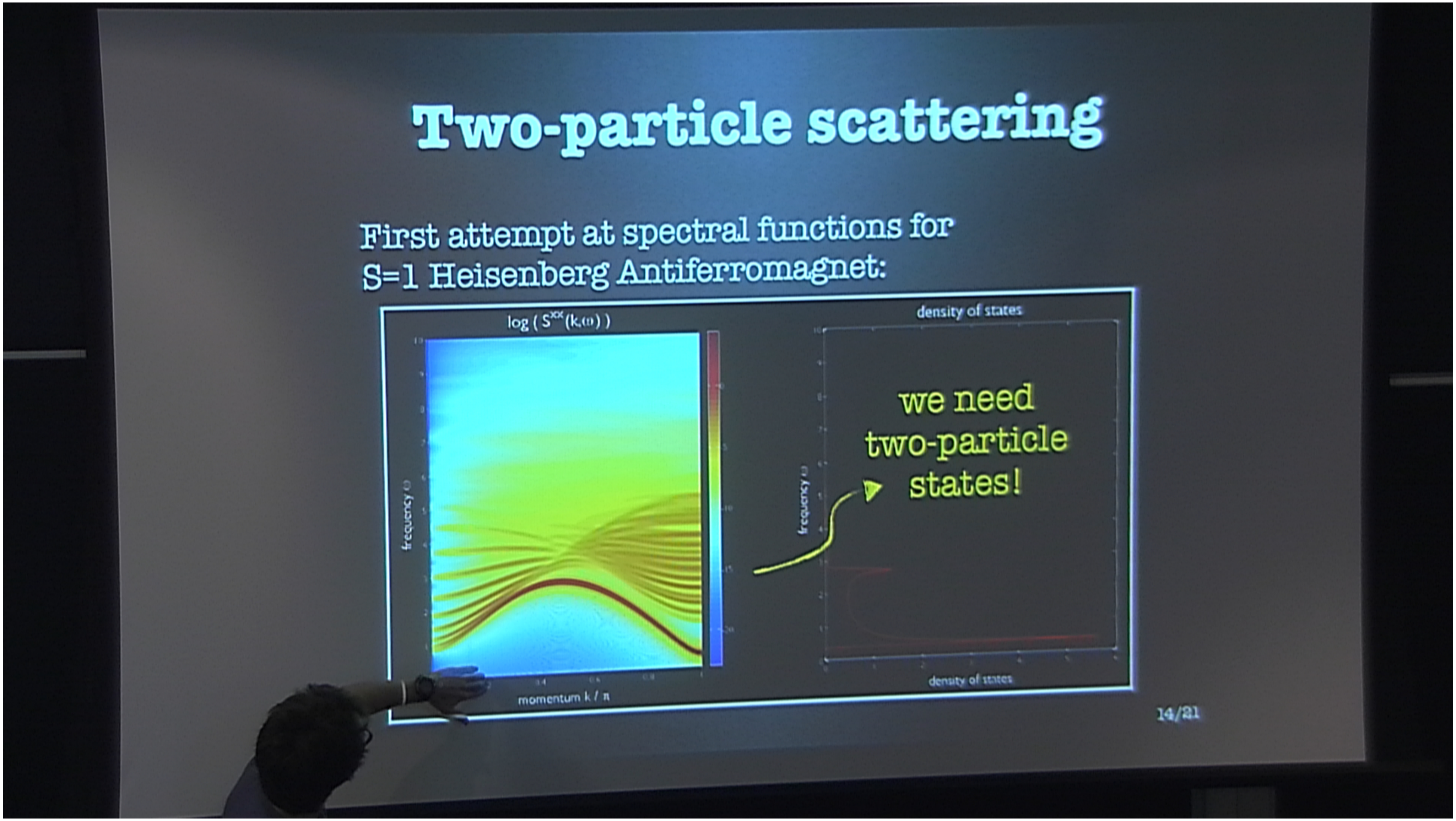

Two-particle scattering
First attempt at spectral functions for
S=1 Heisenberg Antiferromagnet:
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frequency ω
momentum k / π
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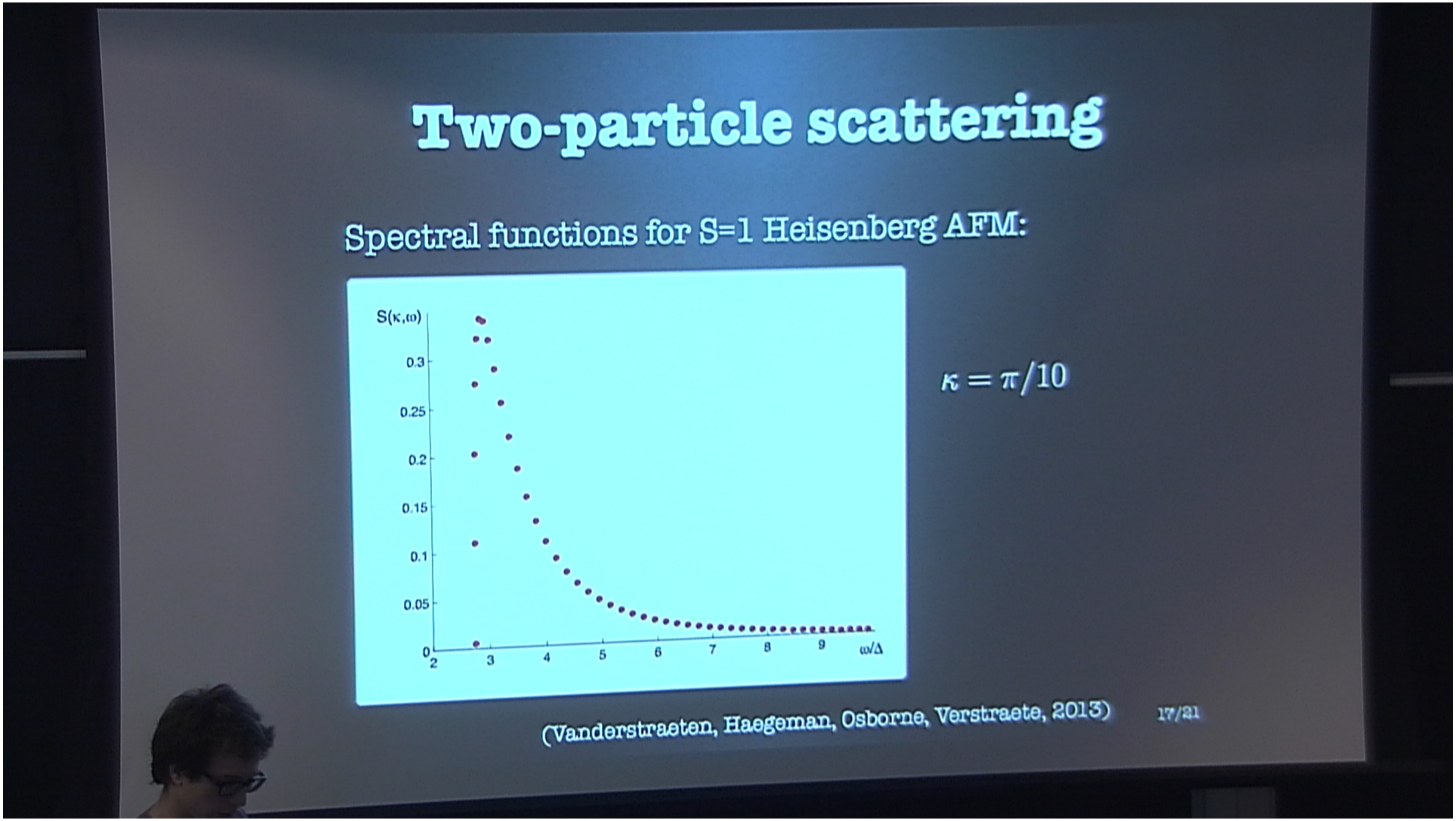
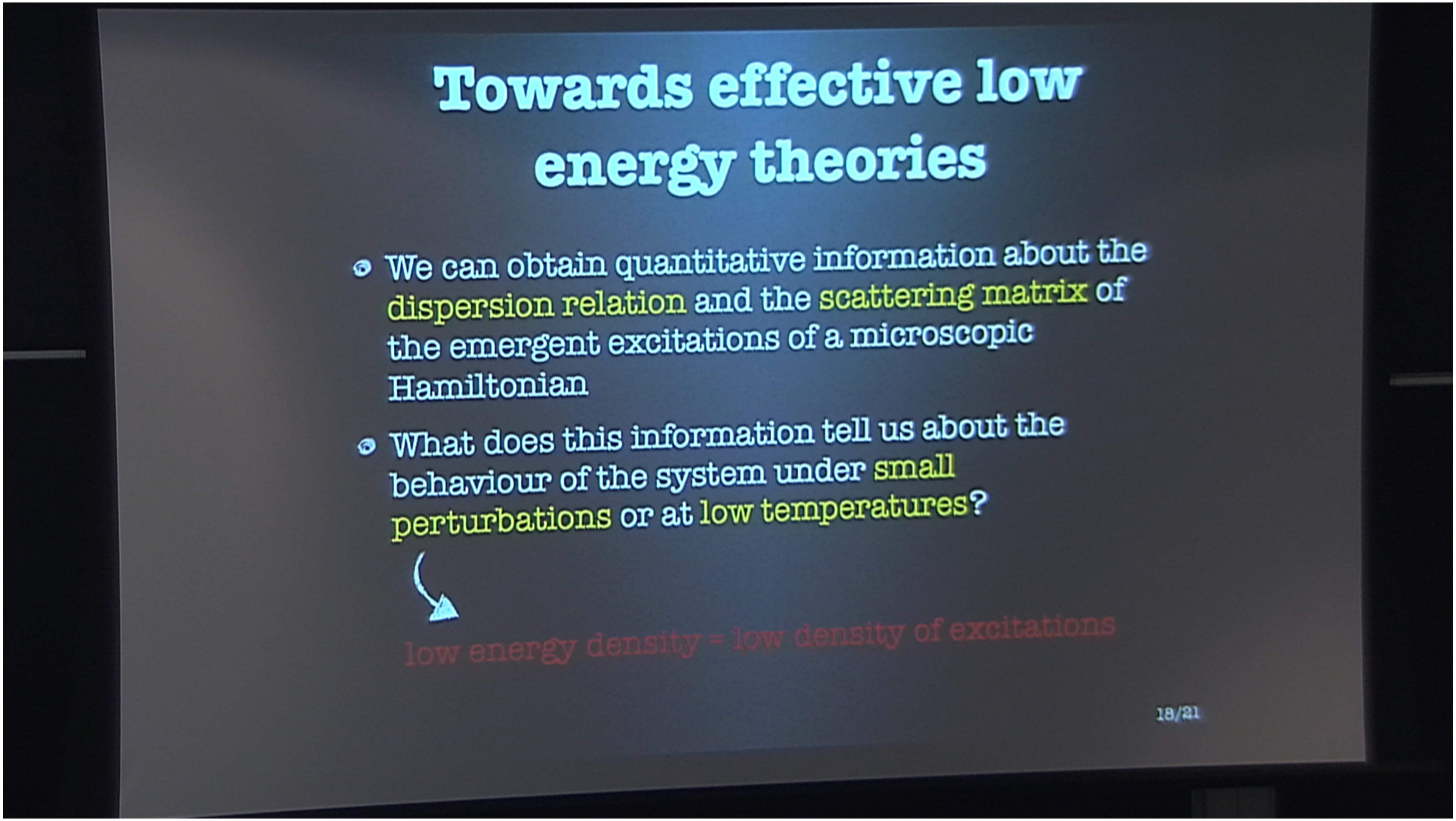
density of states
we need
two-particle
states!
14/21


Two-particle scattering
Spectral functions for S=1 Heisenberg AFM:
S(κ,ω)
0.3
0.25
0.2
0.15
0.1
0.05
0
2
3
4
5
6
7
8
9
ω/Δ
$\kappa = \pi/10$
(Vanderstraeten, Haegeman, Osborne, Verstraete, 2013)
17/21


Towards effective low energy theories
We can obtain quantitative information about the dispersion relation and the scattering matrix of the emergent excitations of a microscopic Hamiltonian
What does this information tell us about the behaviour of the system under small perturbations or at low temperatures?
low energy density = low density of excitations
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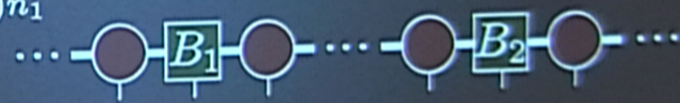
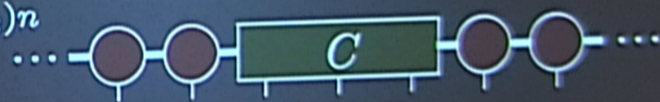
Two-particle scattering

Stationary scattering state of two-particles:

$$\sum_{n_1<n_2} g(n_2-n_1)\mathrm{e}^{\mathrm{i}(k_1+k_2)n_1}$$



$$+\sum_{n} \mathrm{e}^{\mathrm{i}(k_1+k_2)n}$$



- Asymptotic regime $n_1 \ll n_2$: $g(n) = \mathrm{e}^{\mathrm{i}k_2 n} - \mathrm{e}^{\mathrm{i}\phi + k_1 n}$
  B_1 and B_2 are given by solutions of the single particle problem → this fixes the energy
- For $n_1 \approx n_2$: new variational parameters allow to take microscopic details of interaction into account

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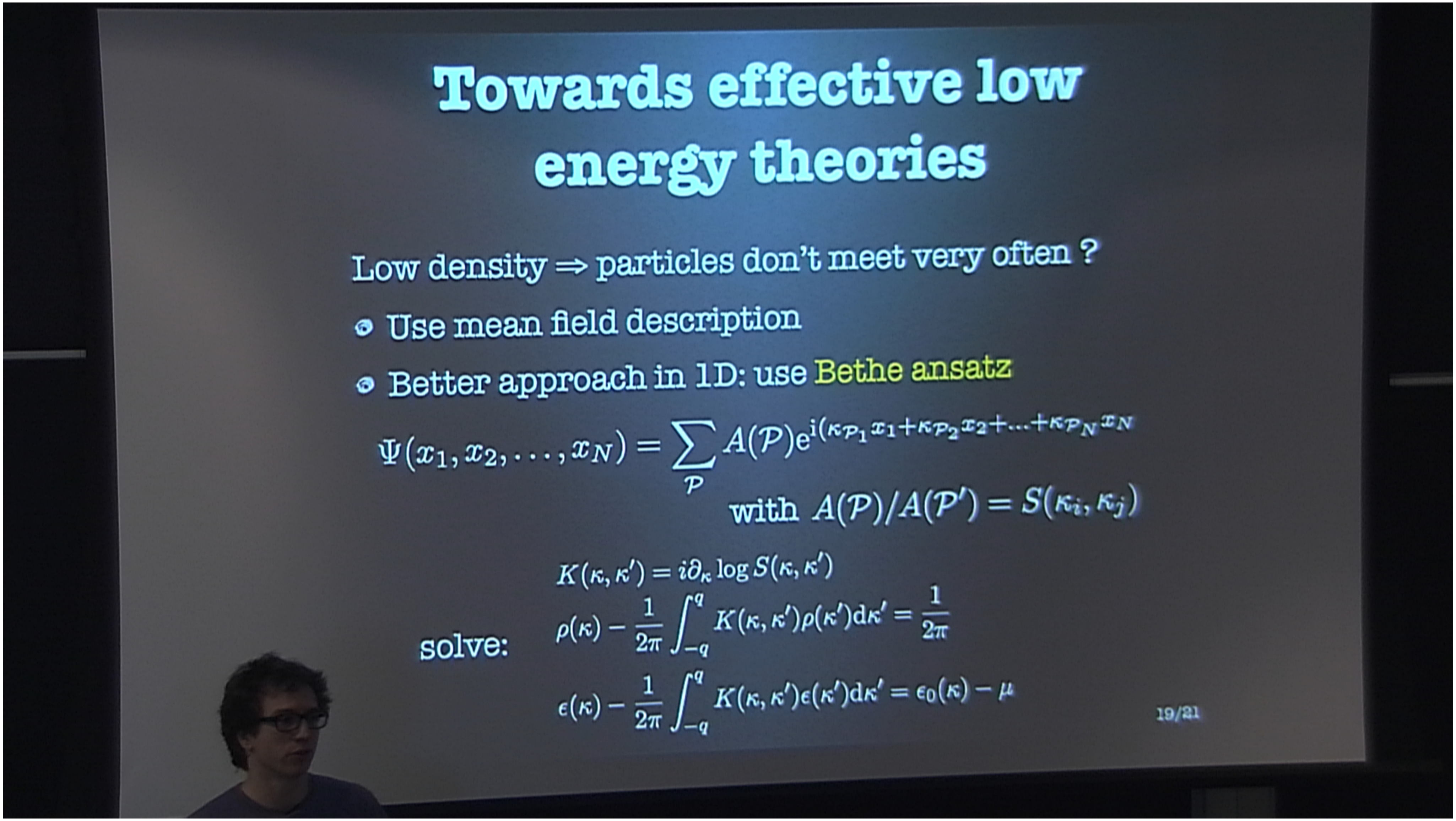
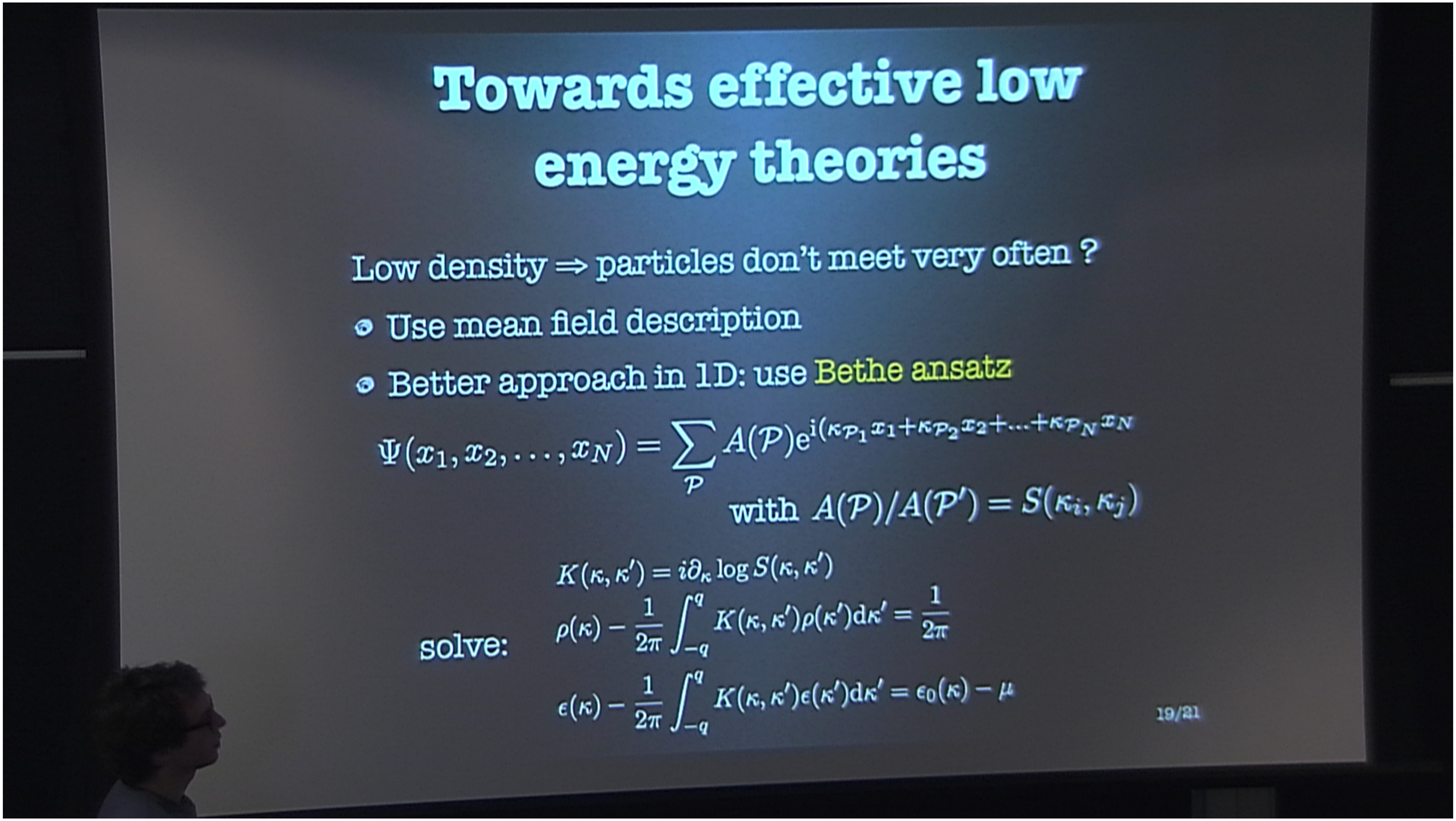

Towards effective low energy theories
Low density ⇒ particles don't meet very often ?
Use mean field description
Better approach in 1D: use Bethe ansatz
$\Psi(x_1, x_2, \ldots, x_N) = \sum_{\mathcal{P}} A(\mathcal{P}) e^{i(\kappa_{\mathcal{P}_1} x_1 + \kappa_{\mathcal{P}_2} x_2 + \ldots + \kappa_{\mathcal{P}_N} x_N}$
with $A(\mathcal{P})/A(\mathcal{P}') = S(\kappa_i, \kappa_j)$
$K(\kappa, \kappa') = i\partial_\kappa \log S(\kappa, \kappa')$
solve:
$\rho(\kappa) - \frac{1}{2\pi} \int_{-q}^{q} K(\kappa, \kappa') \rho(\kappa') d\kappa' = \frac{1}{2\pi}$
$\epsilon(\kappa) - \frac{1}{2\pi} \int_{-q}^{q} K(\kappa, \kappa') \epsilon(\kappa') d\kappa' = \epsilon_0(\kappa) - \mu$
19/21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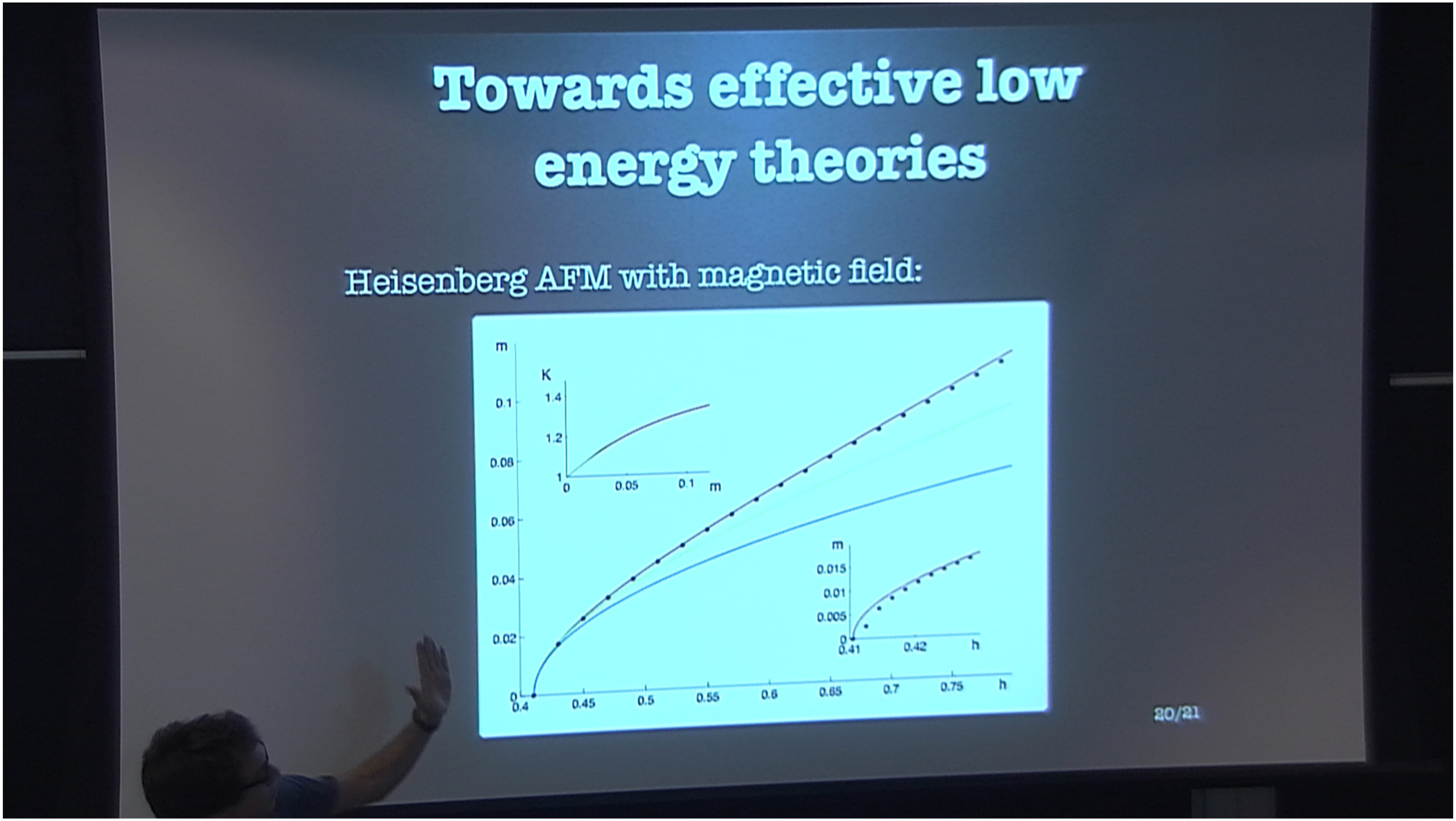
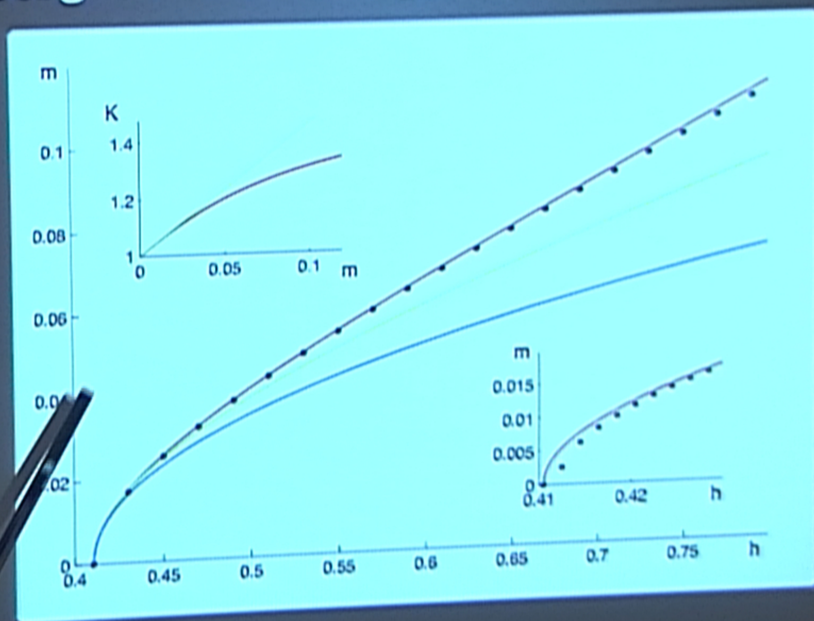
19/21


Towards effective low energy theories
Heisenberg AFM with magnetic field:
m
K
1.4
1.2
1
0
0.05
0.1
m
0.1
0.08
0.06
0.04
0.02
0
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
h
m
0.015
0.01
0.005
0
0.41
0.42
h
20/21

Towards effective low energy theories

Heisenberg AFM with magnetic field:



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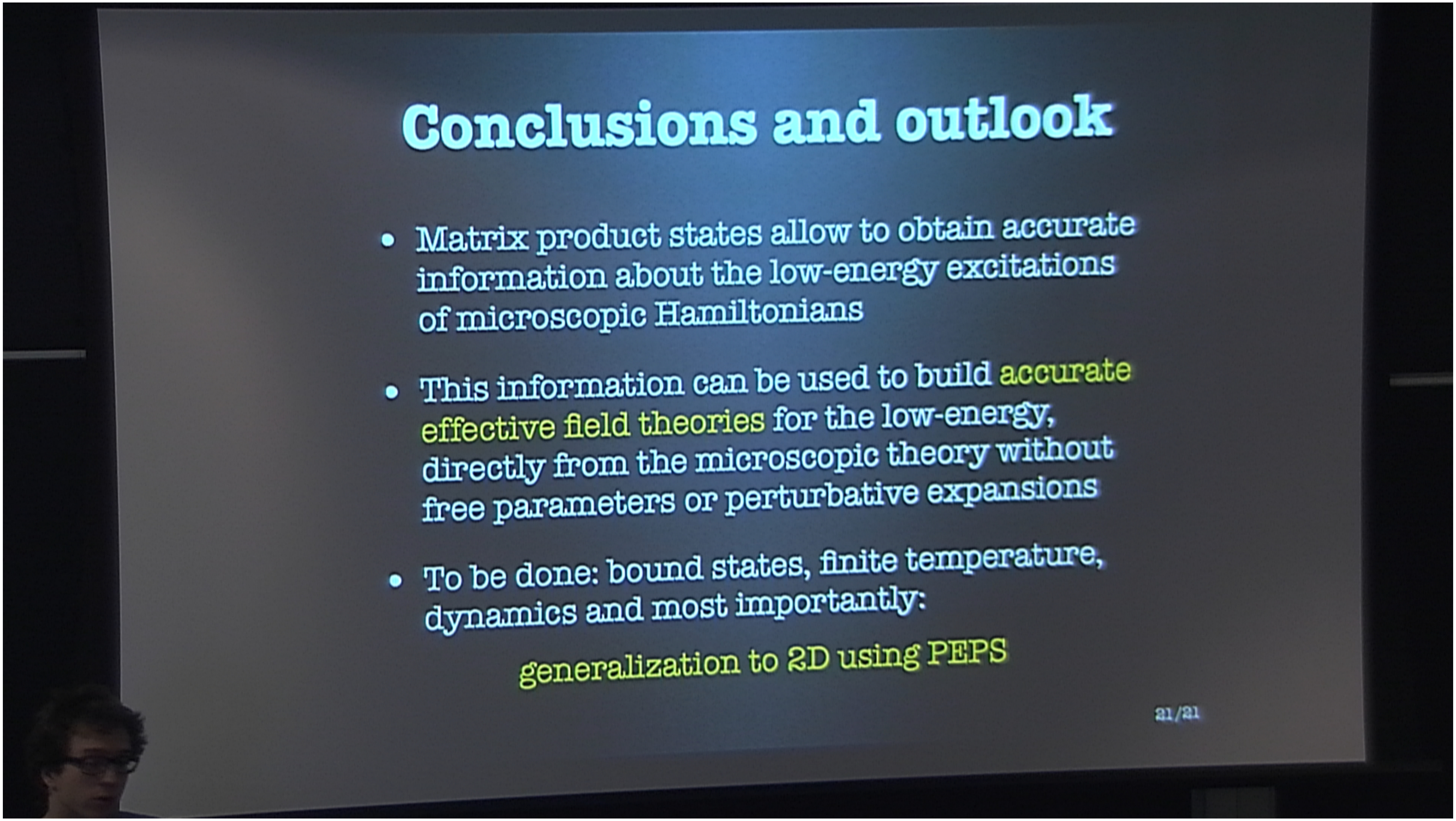
Conclusions and outlook
• Matrix product states allow to obtain accurate information about the low-energy excitations of microscopic Hamiltonians
• This information can be used to build accurate effective field theories for the low-energy, directly from the microscopic theory without free parameters or perturbative expansions
• To be done: bound states, finite temperature, dynamics and most importantly:
generalization to 2D using PEPS
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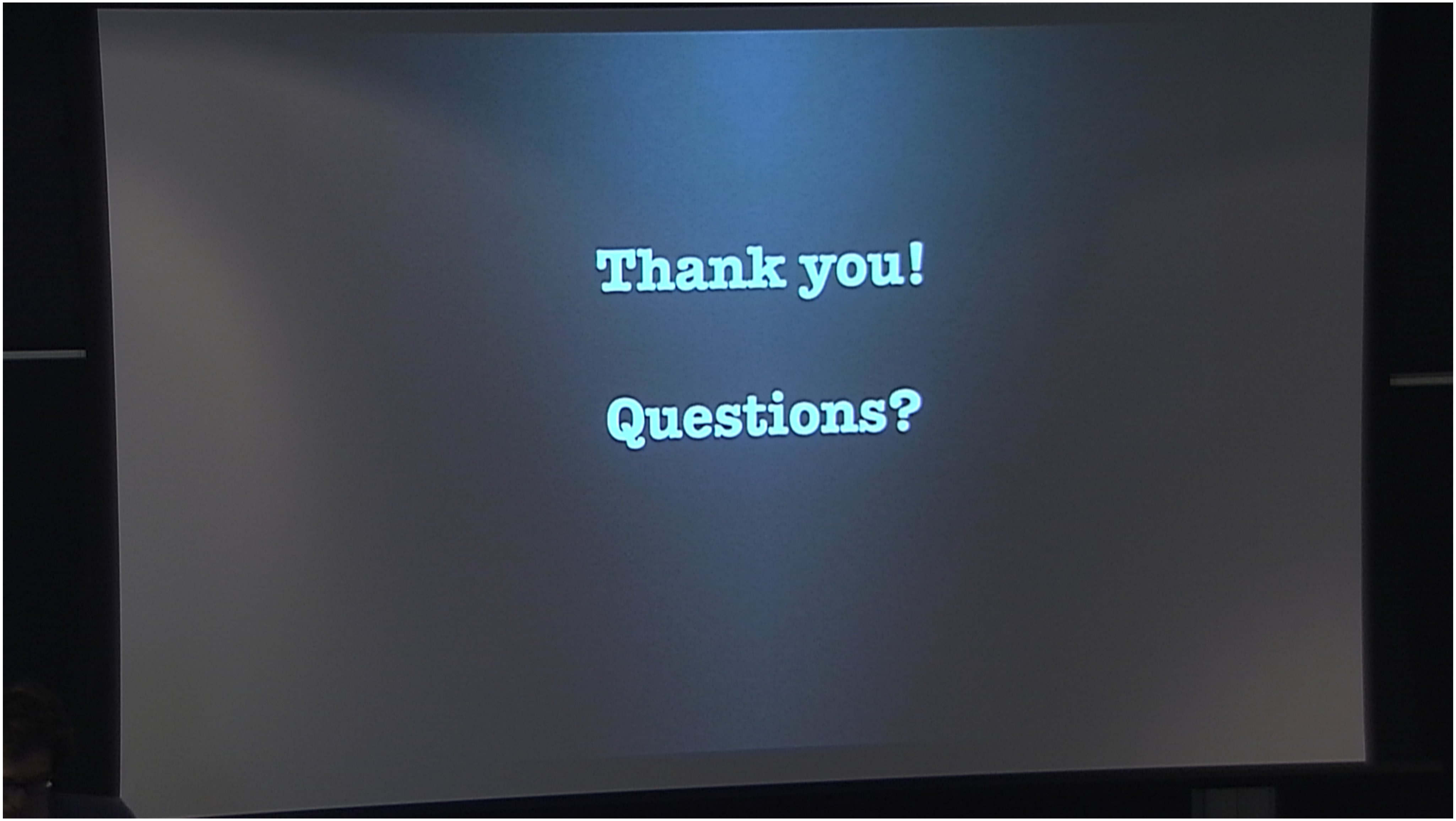
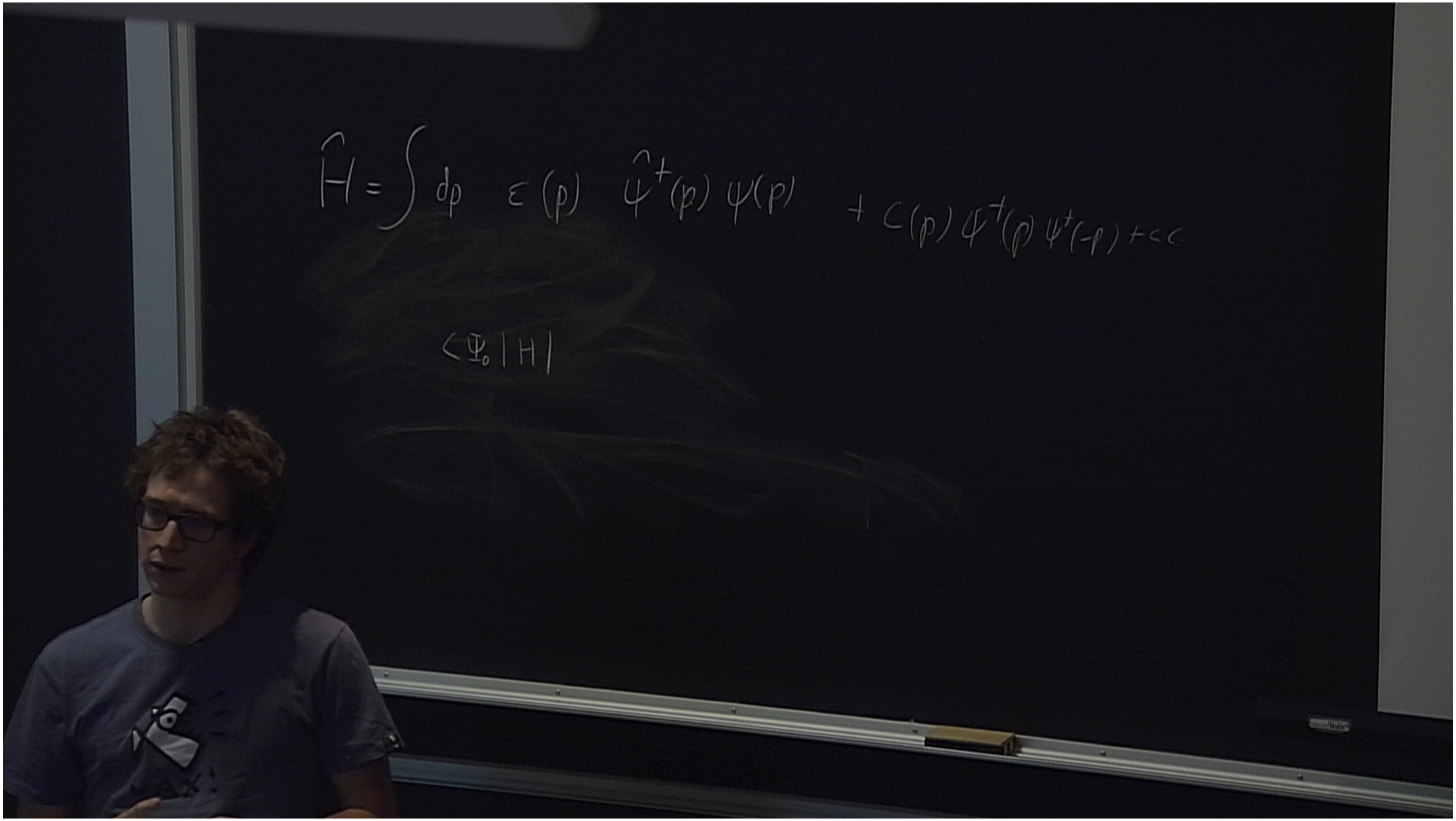
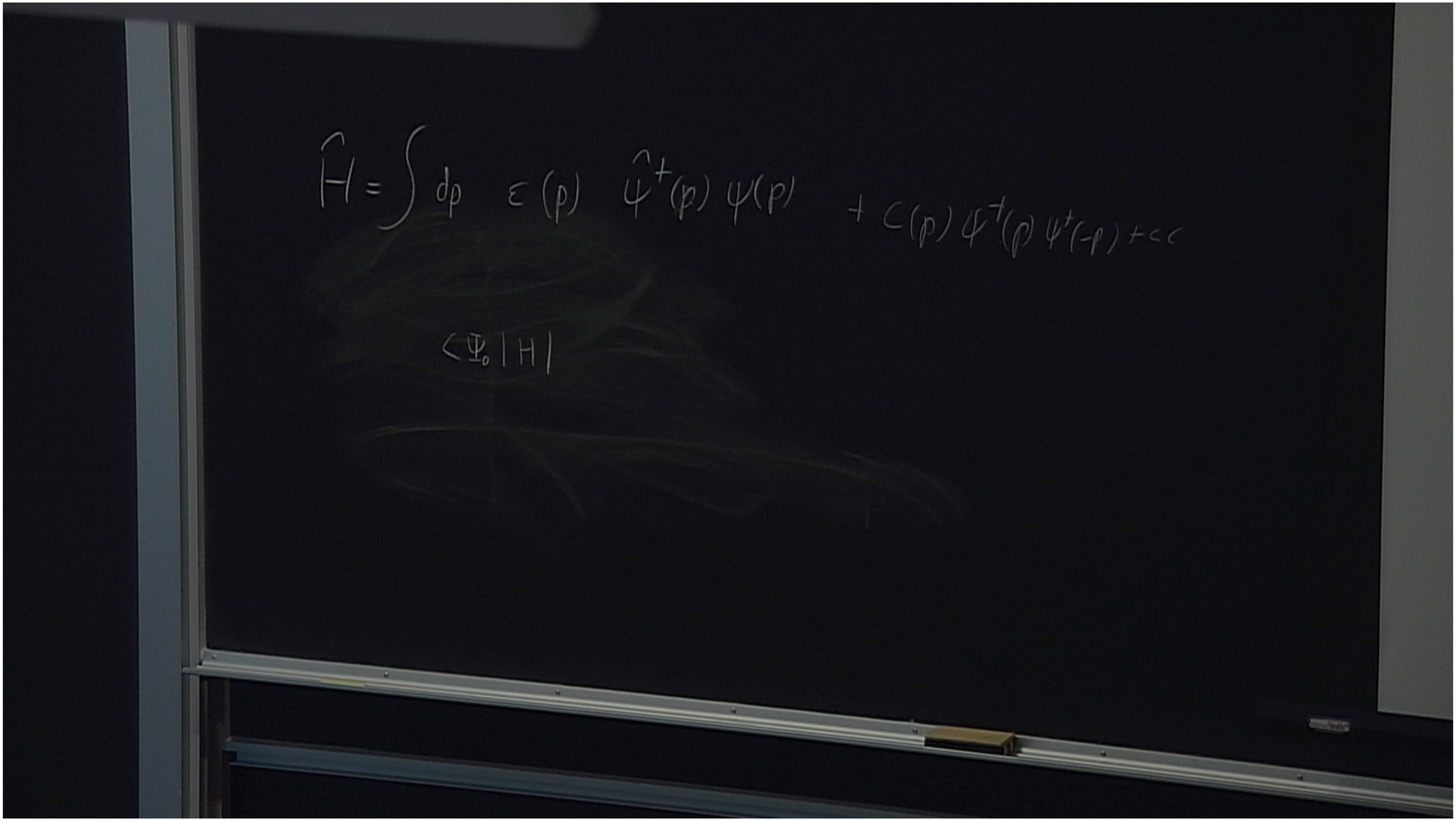
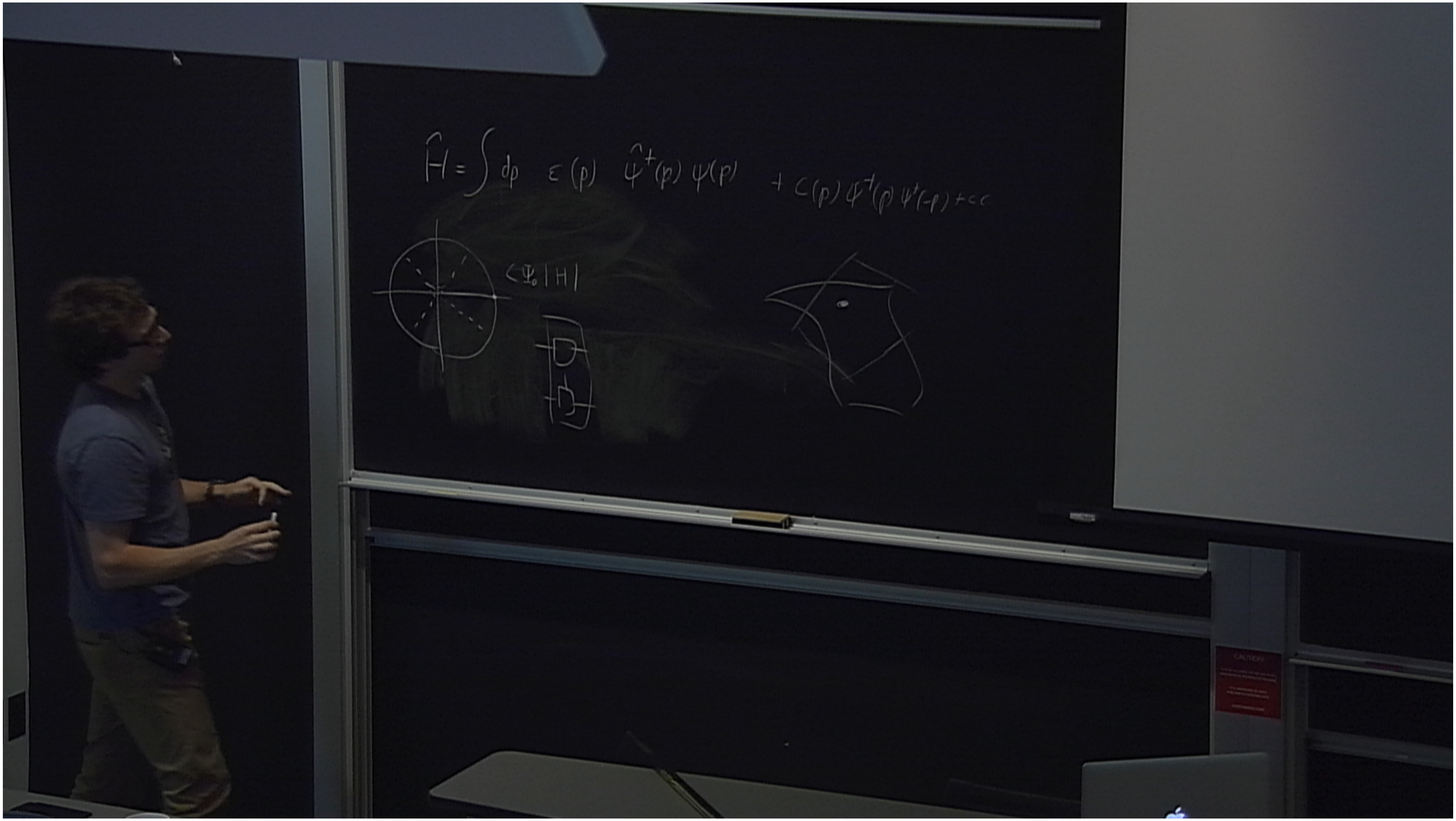

Thank you!
Questions?


Ĥ = ∫ dp ε(p) ψ̂†(p) ψ(p) + c(p) ψ†(p) ψ†(-p) + cc
<Ψ0 | H |


$\hat{H} = \int dp \;\; \varepsilon(p) \;\; \hat{\psi}^{+}(p)\, \psi(p) \;\; + C(p)\, \psi^{+}(p)\, \psi^{+}(-p) + c.c$
$\langle \Psi_0 | H |$


$\hat{H} = \int dp \; \varepsilon(p) \; \hat{\psi}^{+}(p)\, \psi(p) \; + C(p)\, \psi^{+}(p)\, \psi^{+}(-p) + c.c$
$\langle \Psi_0 | H |$