Title: Probing renormalization group flows using entanglement entropy

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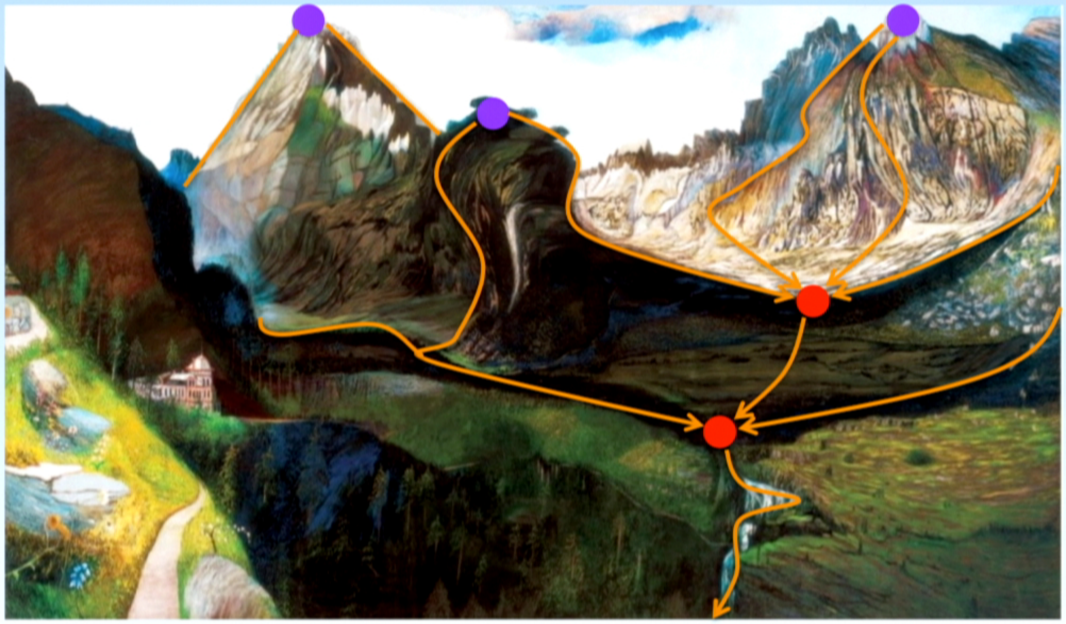
Abstract: The entanglement entropy of the vacuum of a quantum field theory contains information about physics at all scales and is UV sensitive. A simple refinement of entanglement entropy gets rid of its UV divergence, and allows us to extract entanglement per scale. In two and three spacetime dimensions this quantity can be used as a proxy for the number of degrees of freedom, as it decreases under RG flow. We investigate its behavior around fixed points, and reveal its interesting analytic structure in the space of couplings.

Probing RG flows using entanglement entropy



Mark Mezei (MIT)

Collaborator: Hong Liu [arXiv:1202.2070, 1309.6935]

Perimeter Institute
04/30/2014

Outline

Introduction
- Counting degrees of freedom
- Improved understanding of RG

Renormalized entanglement entropy
- UV finiteness
- Entanglement at a scale
- Monotonicity

Behavior near fixed points
- UV fixed points
- IR behavior from matched series expansions

Conclusions
- Entanglement entropy probes RG sensitively
- Challenges

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Wilsonian renormalization group

Strongly interacting systems:

Micro description (UV) —Guesswork→ Long-wavelength description (IR)

- QCD: Quarks, gluons —Chiral SB→ Pions
- Intuition: number of degrees of freedom decreases under RG
  Can we formalize this intuition?

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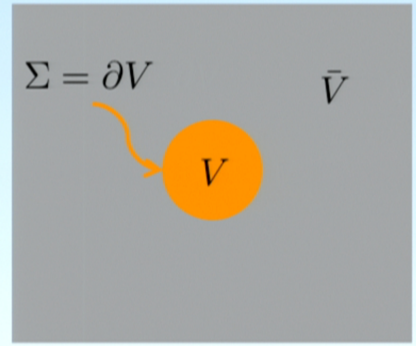
Entanglement entropy and the number of degrees of freedom (3d example)

Understanding entanglement leads to an improved picture of RG



- In a local theory: $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$
- Reduced density matrix: $\rho_r = \mathrm{Tr}_{\bar{V}} \, |0\rangle \langle 0|$
- Entanglement entropy: $S_V \equiv -\mathrm{Tr}_V \, \rho_r \log \rho_r$
- Area and universal term: $S_V = \# \frac{A_\Sigma}{\delta} - F_V$
- For a disk in a CFT: $F_{\text{disk}} = -\log Z_{S^3}$ [Casini, Huerta, Myers]

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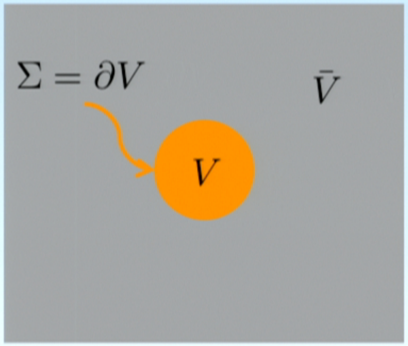

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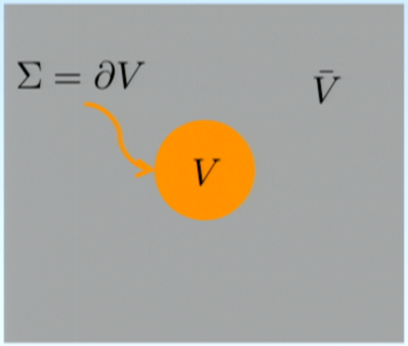

Proposal:

$$F(R) \equiv \left(R \frac{d}{dR} - 1 \right) S_{\mathrm{disk}}$$

$$\frac{dF(R)}{dR} \leq 0$$

Captures the intuition of decreasing number of degrees of freedom under RG.

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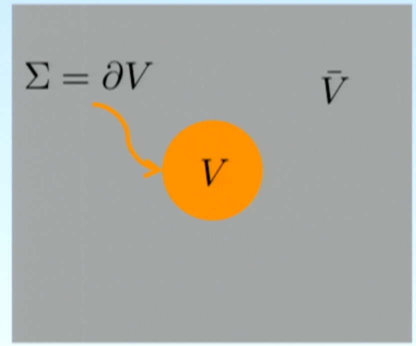

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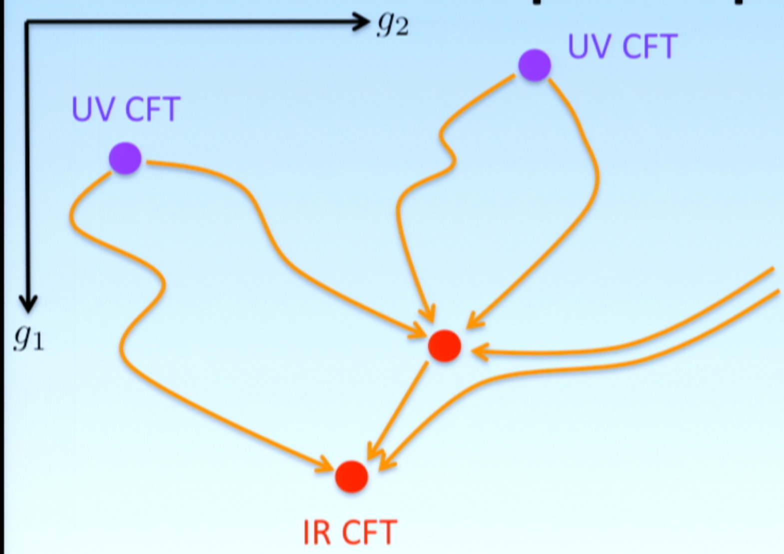

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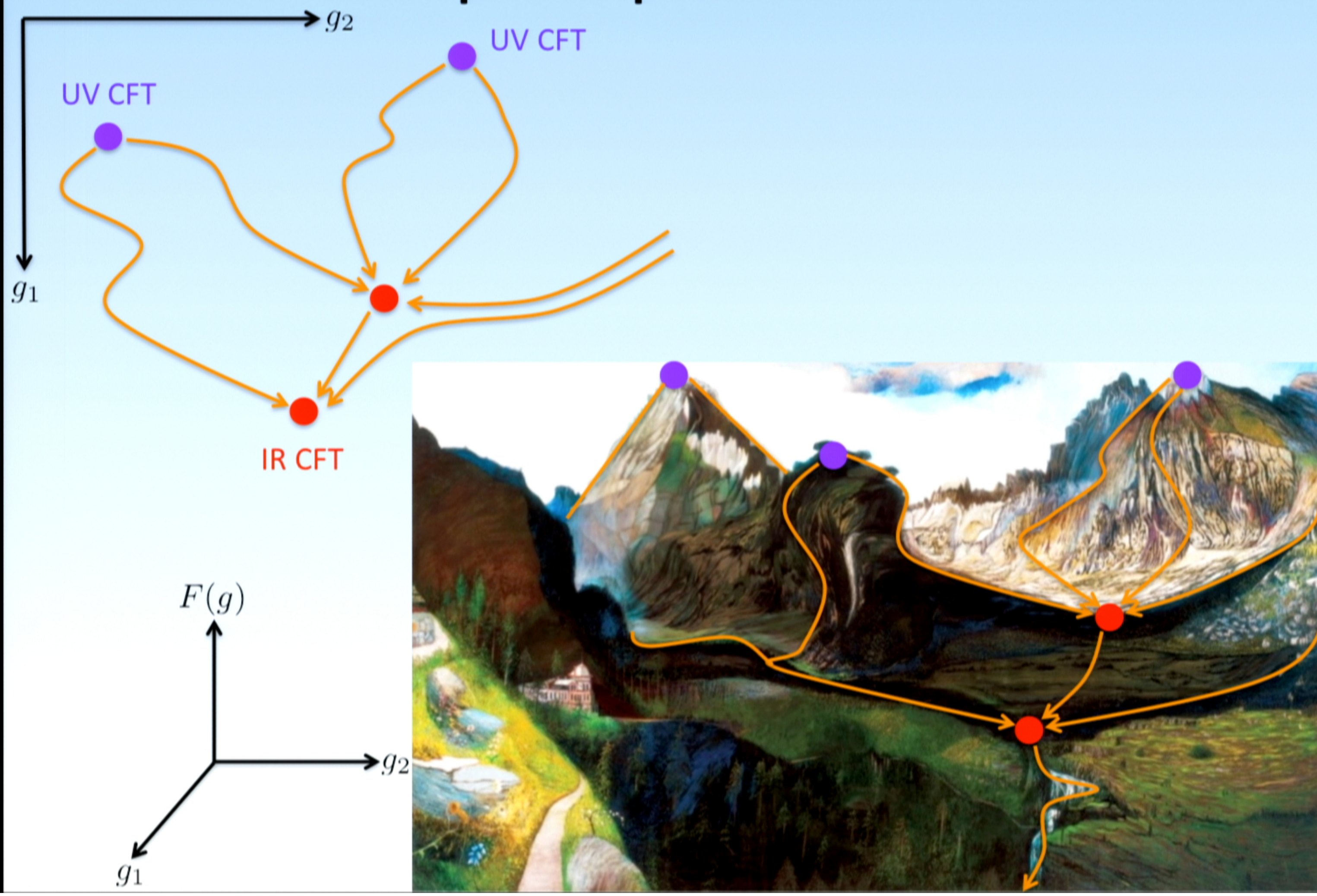

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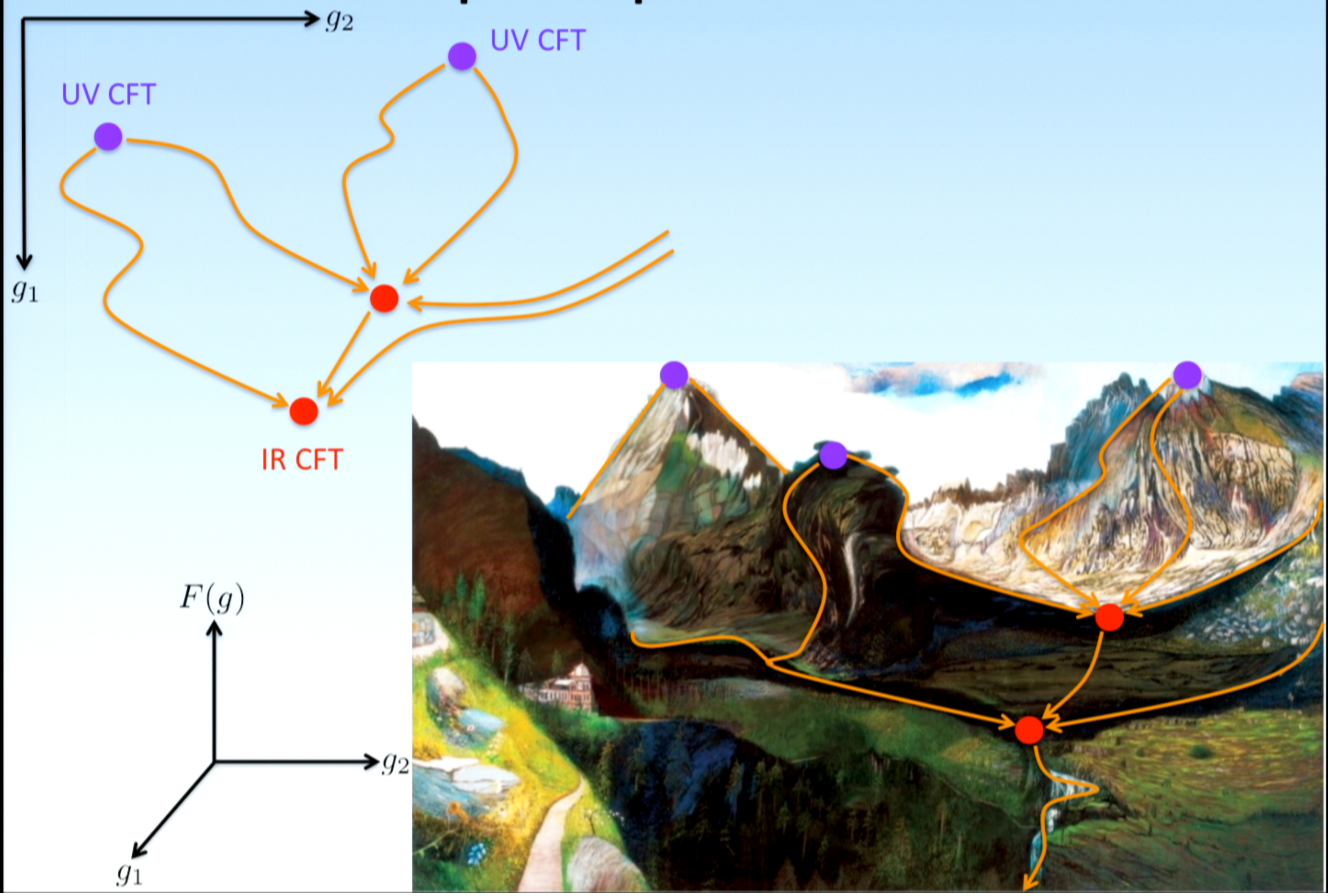

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- The F-theorem provides fundamental constraints:

  $F(\bullet) \geq F(\bullet)$

- Confinement in 3d gauge theories
  - QED_3 with $N_f \geq 12$ fermion flavors deconfines
  - Detailed analysis of the dynamics gives agreement [Dyer, MM, Pufu]

- Phase transitions [Grover]

  - ● cannot be the critical O(3) model

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Entanglement entropy

Entanglement entropy (EE) depends on physics at all length scales

- Dominated by short distance correlations

$$S_V = \# \frac{A_\Sigma}{\delta^{d-2}} + \dots$$

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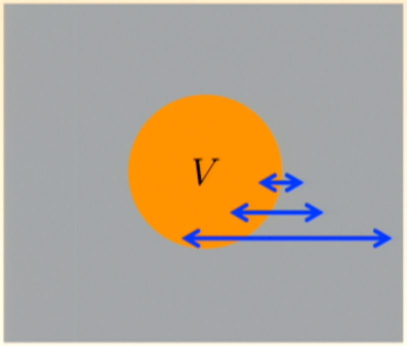
- Common practice: subtract UV divergent parts
- After subtraction could still depend on d.o.f. at much smaller scales than size of V



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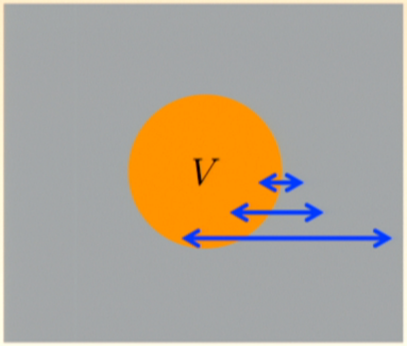

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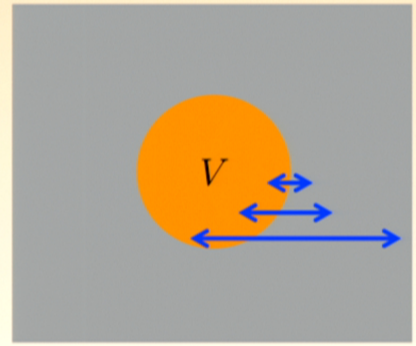


d=3 scalar:

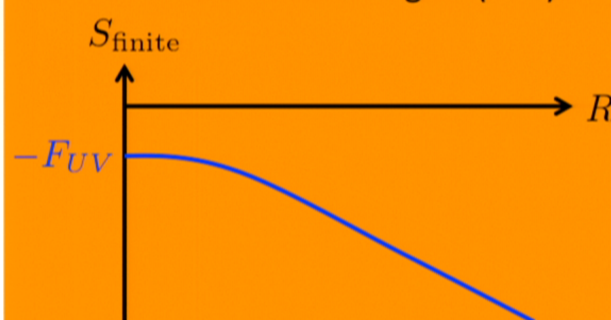
$$S_{\text{scalar}}(mR) = \# \frac{R}{\delta} - \frac{\pi}{6} mR - \frac{\pi}{240} \frac{1}{mR} + \cdots$$

- At long distances this is an empty theory, we would have liked EE to go to zero.
- Redefining the cutoff by

$$\delta \to \delta(1 + m\delta + \dots)$$

would change the result.

A CFT to CFT flow would give (d=3):



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Renormalized entanglement entropy

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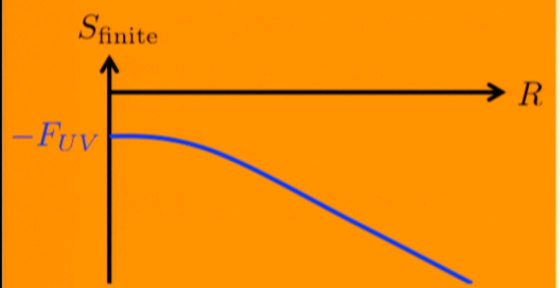

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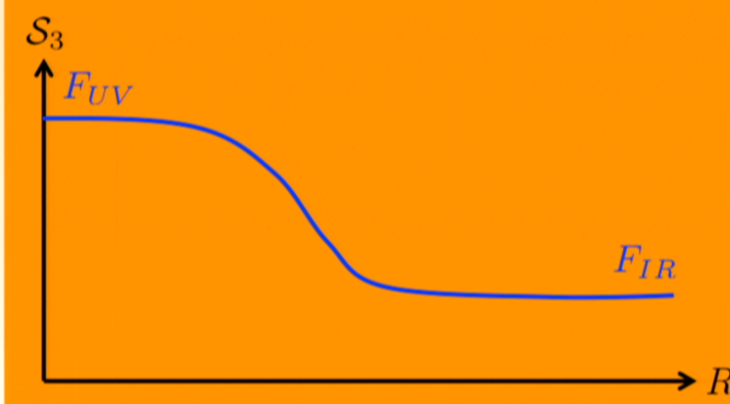
Convert by taking:

$$\left(R\frac{d}{dR}-1\right)S$$



Renormalized entanglement entropy

REE has the following attractive properties:

1. It is UV finite in the continuum limit.
   - The divergence is coming from local, cutoff scale physics around Σ.

$$S^{(\Sigma)}_{\rm div} = \int_\Sigma d^{d-2}\sigma \sqrt{h} F(K_{ab}, h_{ab})$$

   - From $S_V = S_{\bar{V}}$ valid for pure states: $S^{(\Sigma)}_{\rm div} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$
   - For a CFT fixed point

$$S^{(\Sigma)} = \begin{cases} \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R}{\delta_0} + (-1)^{\frac{d-1}{2}} s_d^{(\Sigma)} + \frac{\delta_0}{R} + \cdots & \text{odd d} \\ \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R^2}{\delta_0^2} + (-1)^{\frac{d-2}{2}} s_d^{(\Sigma)} \log \frac{R}{\delta_0} + \text{const} + \frac{\delta_0^2}{R^2} + \cdots & \text{even d} \end{cases}$$

   - For a renormalizable flow:

$$a_1 = \frac{1}{\delta_0^{d-2}} h_1(\mu\delta_0) \ , \qquad h_1(\mu\delta_0) = c_0 + c_2(\mu\delta_0)^{2(d-\Delta)} + \dots$$

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2. For a CFT it is given by an R-independent constant, $s_d^{(\Sigma)}$.
3. For a renormalizable quantum field theory, it interpolates between the $s_d^{(\Sigma)}$ values of the UV and IR fixed points as R is increased from zero to infinity.
4. It is most sensitive to degrees of freedom at scale R.

Renormalized entanglement entropy

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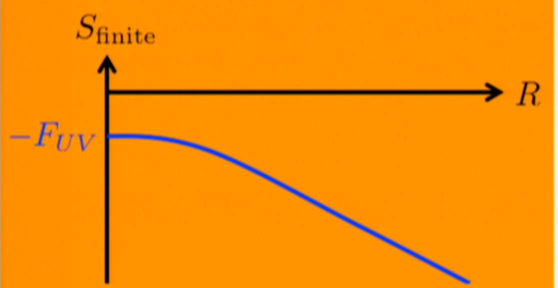

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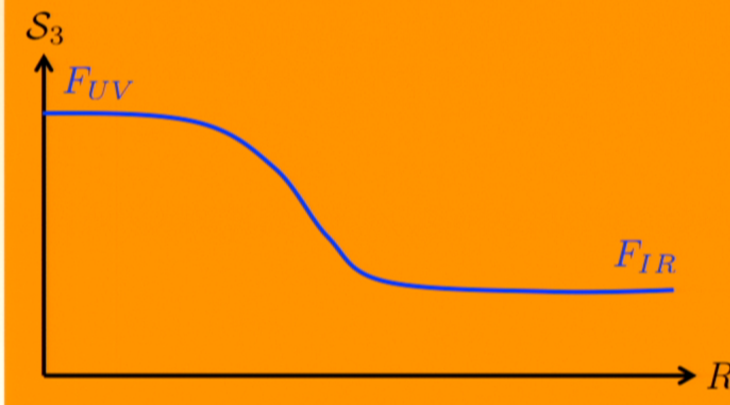

Introduce renormalized entanglement entropy (REE):

- d=2,3: $\mathcal{S}_2(R) = R\frac{dS(R)}{dR}$, $\mathcal{S}_3^{(\Sigma)}(R) = \left(R\frac{d}{dR}-1\right)S^{(\Sigma)}(R)$.
- In general:

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases}$$

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The number of degrees of freedom

RG intuitively leads to the loss of degrees of freedom:

- A field contributes to the count at scales below its Compton wavelenght.
- Its effects only manifest in corrections to parameters at distances longer than its wavelenght, and the field doesn't contribute to the count.

In d=2 the Zamolodchikov c-theorem quantifies the RG intuition:

- c can be isolated from many quantities, e.g. from EE [Holzhey, Larsen, Wilczek]

$$S_{\rm CFT} = \frac{c}{3} \log \frac{R}{\delta_0}$$

- The c-theorem can also be proven using the strong subadditivity of EE [Casini, Huerta]

In d>2 universal terms in EE across S^{d-2} have been identified as satisfying: $c_{UV} > c_{IR}$ [Myers, Sinha]

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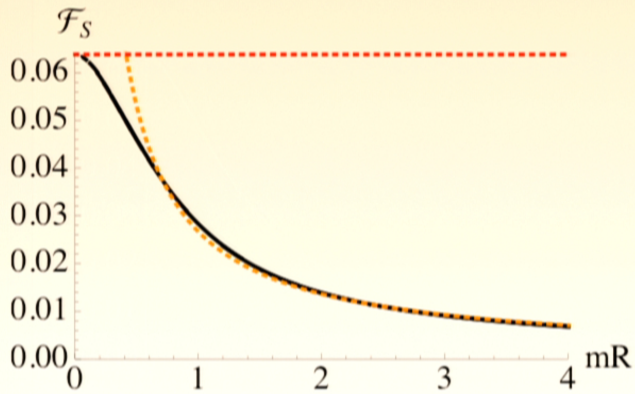
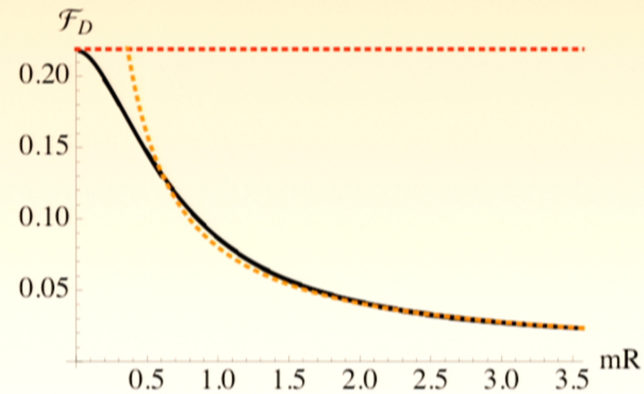
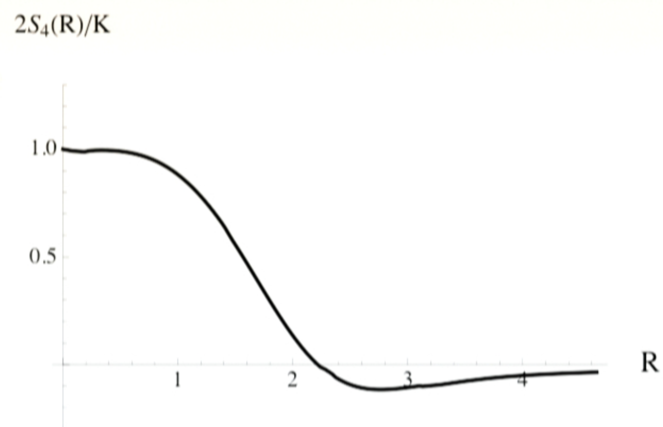
Testing monotonicity (d=3):

- Free theories [Liu, MM; Safdi]





- Holographic examples
  - Variety of examples
  - For closely separated fixed points can prove monotonicity for all d
- Casini and Huerta proved monotonicity subsequently
- REE is neither positive nor monotonic in higher dimensions



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- UV fixed points
- IR behavior from matched series expansions

Conclusions
- Entanglement entropy probes RG sensitively
- Challenges

Behavior in the vicinity of fixed points

Survey of results

- Naïve field theory expectation:

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  We will see these terms in holographic computations.

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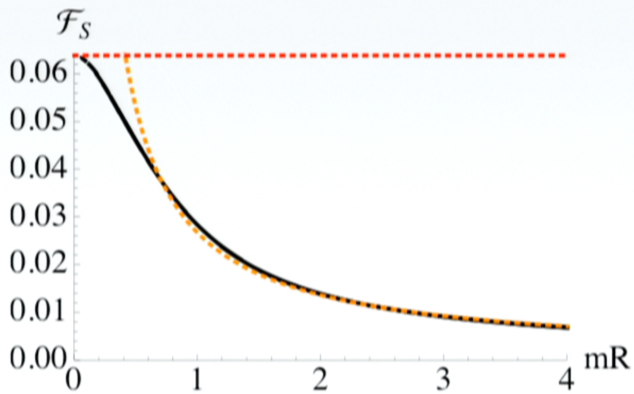

- Not stationary at UV fixed point. [Klebanov, Nishioka, Pufu, Safdi]
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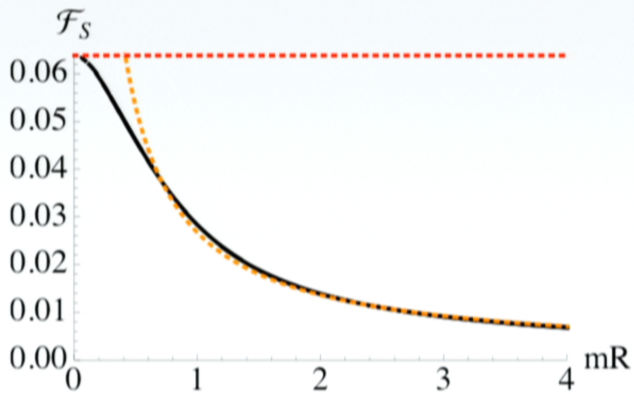

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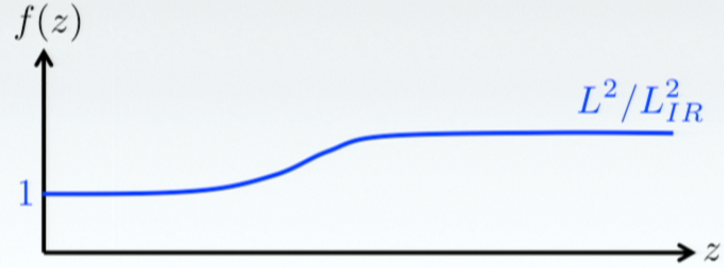
Holographic calculations

Holographically: CFT to CFT RG flow = AdS to AdS domain wall

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2 + \frac{dz^2}{f(z)}\right)$$



- Closely separated fixed point:

  $f(z) = 1 + \epsilon g(z) + \dots$

- The Ryu-Takayanagi formula can be evaluated perturbatively, giving a simple formula for REE:

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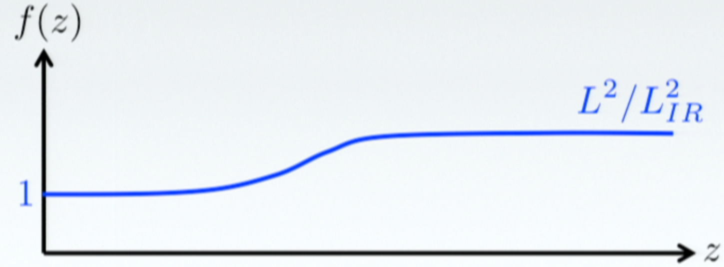

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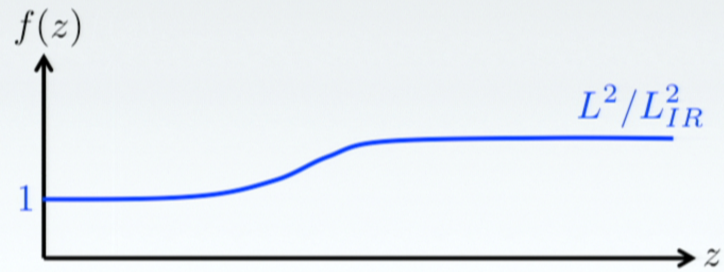

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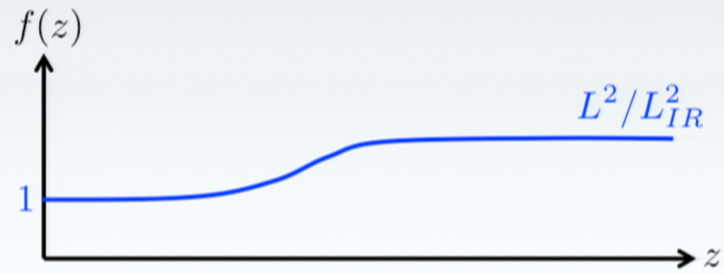

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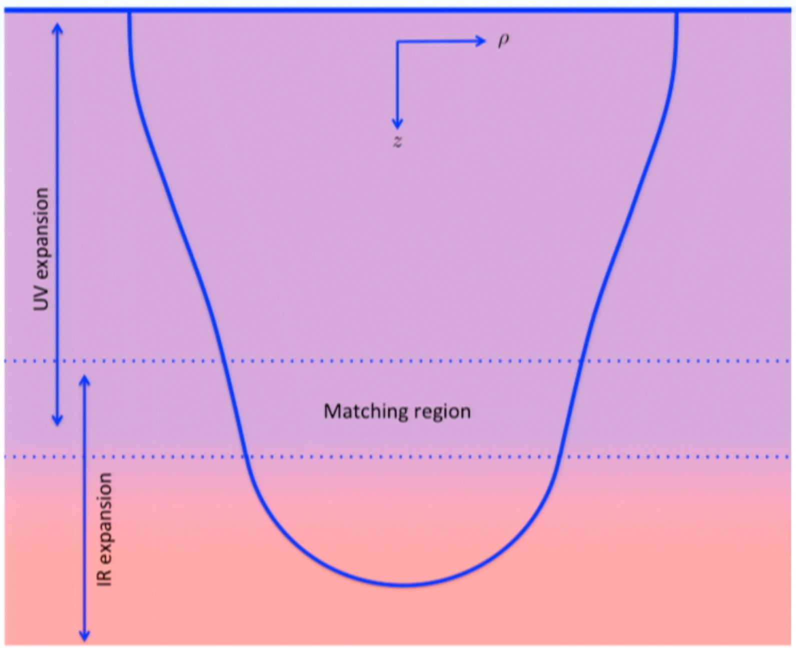

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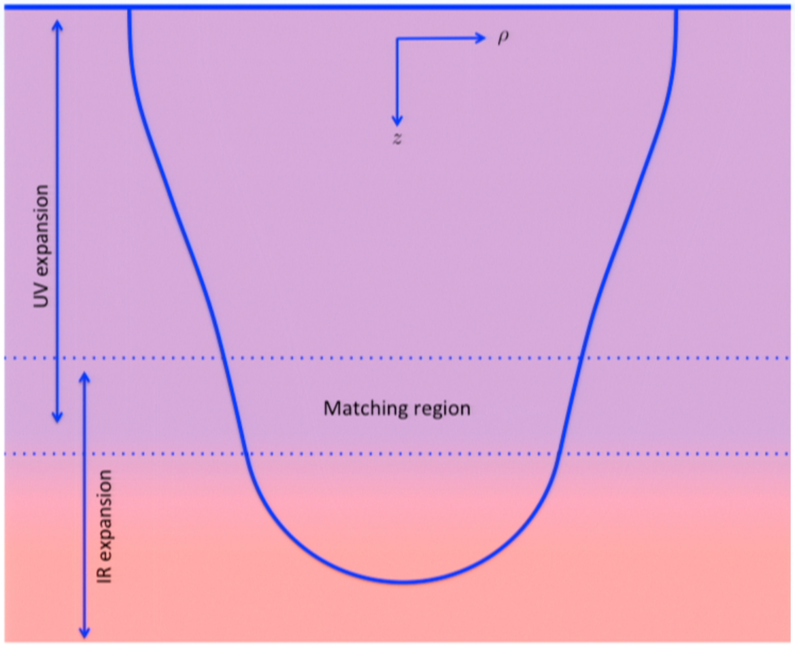

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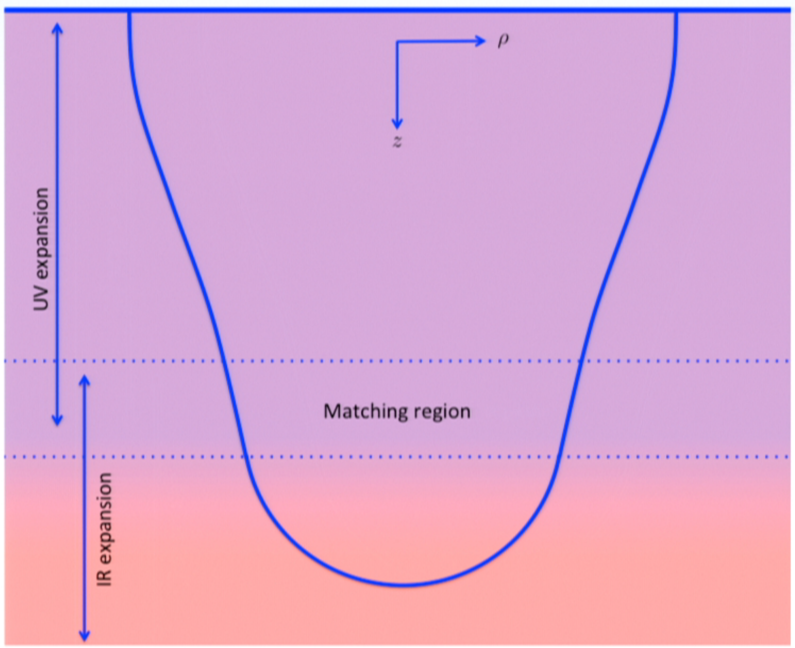
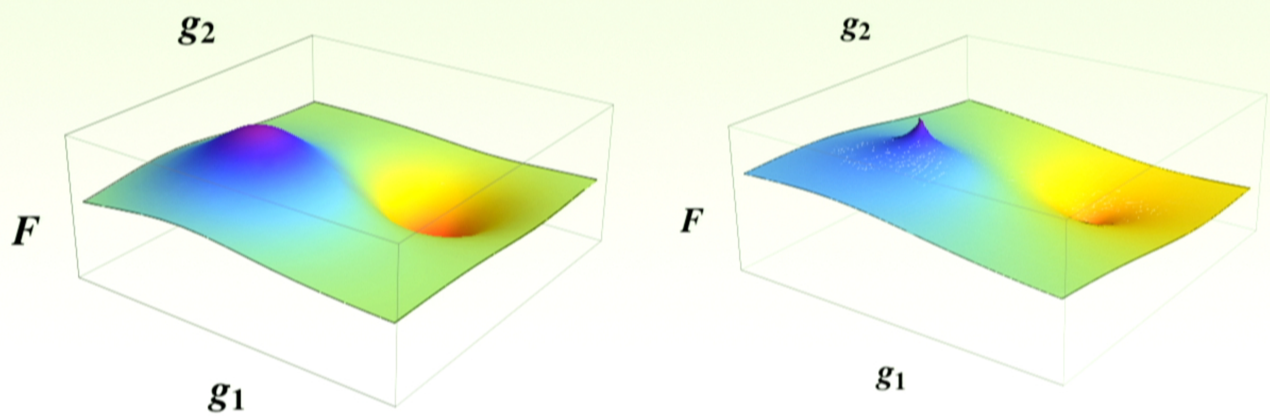


Conclusions and outlook

REE is a sensitive probe of RG flows

- Height function on coupling space in d=2,3
- Stationarity near an IR fixed point is decided by the dimension of the leading irrelevant operator



Field theory understanding is in its infancy

- Why do free massive scalars give non-stationary REE?
- What gives the 1/R term near an IR fixed point?
- Continuum definition of REE?