Title: 3d N = 4 Gauge Theories, Mirror Symmetry and Hilbert series

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Abstract: This talk focuses on vacuum moduli spaces of N=4 supersymmetric field theories in three dimensions. A particular branch of the moduli space, known as the Coulomb branch, receives quantum corrections. We present an exact result, known as the Hilbert series, that enumerates the operators in the chiral ring of such a quantum Coulomb branch. This exact result can be applied to a large class of 3d supersymmetric field theories, with and without known Lagrangian descriptions. As an application, we present a method to compute partition functions of instantons on C^2 for any simple group, including exceptional and non-simply-laced ones.

$3d$ $\mathcal{N} = 4$ Gauge Theories, Mirror Symmetry and Hilbert series

Noppadol Mekareeya
CERN

Perimeter Institute

April 28, 2014

Based on the following works:

- [arXiv:1403.0585, 1403.2384] with S. Cremonesi, A. Hanany, A. Zaffaroni
- [arXiv:1402.0016] with A. Dey, A. Hanany, P. Koroteev
- [arXiv:1205.4741] with A. Hanany and S. Razamat
- [arXiv:1111.5624] with C. Keller, J. Song and Y. Tachikawa
- [arXiv:1110.6203] with A. Hanany
- [arXiv:1005.3026] with S. Benvenuti and A. Hanany

Features of $3d$ $\mathcal{N} = 4$ gauge theories

- Two branches of the moduli space:
  - Higgs branch: VEVs of scalars components of the hypermultiplets. Classically exact.
  - Coulomb branch: VEVs of scalars components of the vector multiplets. Receives quantum corrections.
  - Both are hyperKähler spaces.
- R-symmetry: $SO(4) \sim SU(2)_H \times SU(2)_V$.
- A quantum description of the Coulomb branch involves monopole operators.
  [Kapustin et al. from '02]
- Mirror symmetry exchanges the Higgs branch of one theory with the quantum Coulomb branch of another theory, and vice-versa. [Intriligator-Seiberg '97]
  - As a working assumption, it's very useful for studying moduli spaces of various theories.
  - E.g. $3d$ Gaiotto's type theories, whose mirrors have known Langrangians. [Benini-Tachikawa-Xie '10]

Overview of the talk

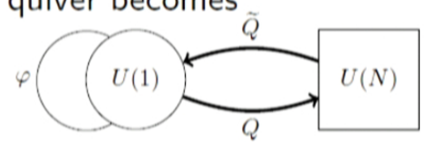
1. The Higgs branch of $3d$ $\mathcal{N}=4$ gauge theories.
2. The Coulomb branch and its quantum description.
3. Hilbert series as a generating function of the gauge invariant quantities on the moduli space.
4. $3d$ Sicilian theory as a theory of class $\mathcal{S}$ ($4d$ Gaiotto's theory) compactified on S^1.
   - Their mirror theories and the constructions.
   - The Coulomb branch of these mirror theories.
5. Connections with the moduli spaces of instantons.
   - New technology for computing instanton partition functions.
   - $5d$ instanton partition functions for k G instantons on $\mathbb{C}^2$, for any simple group G.

Higgs branch of a $3d$ $\mathcal{N} = 4$ gauge theory

- Translate the theory in $3d$ $\mathcal{N} = 2$ language: The F and D terms give rise to the moment map equations of the hyperKähler quotient.
- A suitable description is in terms of gauge invariant quantities subject to constraints from the F term relations.
- A Hilbert series is a generating function that counts these gauge invariant quantities wrt. a $U(1)$ global symmetry that is a generator of $SU(2)_H$ and wrt. the flavour symmetry.
- For the Higgs branch Hilbert series, the global $U(1)$ symmetry can be taken as a generator of the $SU(2)_H$ R-symmetry.

Example 1: $U(1)$ gauge theory with N flavours

- In $3d$ $\mathcal{N}=2$ notation, the above quiver becomes



with the superpotential $W = \widetilde{Q}_i \varphi Q^i$.

- The relevant F-term for the Higgs branch is $\partial_\varphi W = \widetilde{Q}_i Q^i = 0$.
- The Higgs branch is parametrised by the gauge invariant quantities

$$M^i{}_j = Q^i \widetilde{Q}_j \qquad \text{with} \qquad \mathrm{tr}\, M = M^i{}_i = 0 \ .$$

They transform in the adjoint rep., $\mathbf{Adj} = [1, 0, \ldots, 0, 1]$, of $SU(N)$.

- Thanks to the F-term, the square of matrix $M^2 = 0$, *i.e.* M is nilpotent:

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- Also, $\epsilon^{j_1 \ldots j_N} \epsilon_{i_1 \ldots i_N} M^{i_1}{}_{j_1} M^{i_2}{}_{j_2} = 0$.

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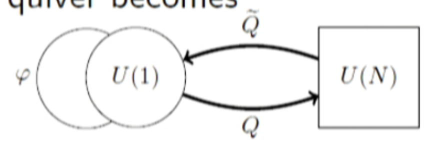

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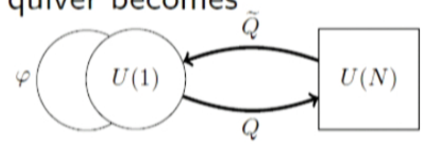

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  This space is also
  - the 'reduced' moduli space of 1 $SU(N)$ instanton on $\mathbb{C}^2$;
  - minimal nilpotent cone of $SU(N)$.
- Any gauge invariant is a product of a matrix M, which carries charge 2 under the J_3 generator of $SU(2)_H$.
  - The operators with charge p transform in $\mathrm{Sym}^p \mathbf{Adj} = \mathrm{Sym}^p [1, 0, \ldots, 0, 1]$ of $SU(N)$.
  - The (minimal) nilpotency kills all reps, except the irrep $[p, 0, \ldots, 0, p]$ in $\mathrm{Sym}^p \mathbf{Adj}$.

  [e.g. Kronheimer '90; Vinberg-Popov '72; Garfinkle '73; Gaiotto-Neitzke-Tachikawa '08, Benvenuti-Hanany-NM '10]
- The Higgs branch Hilbert series is

$$H_{\text{red. } 1\ SU(N) \text{ inst. } \mathbb{C}^2}(t; \boldsymbol{y}) = \sum_{p=0}^{\infty} \chi^{SU(N)}_{[p,0,\ldots,0,p]}(\boldsymbol{y})\ t^{2p}\ .$$

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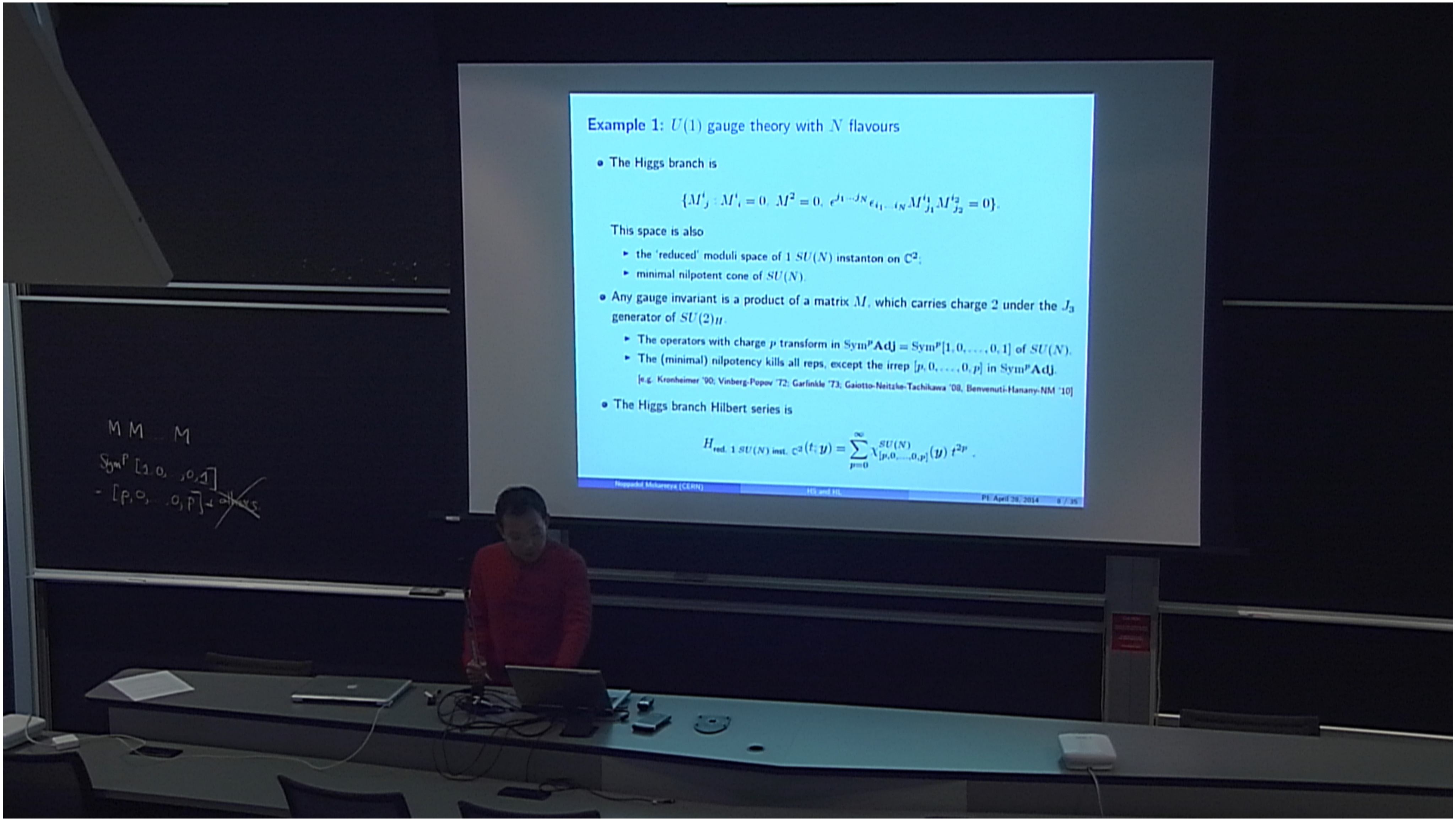
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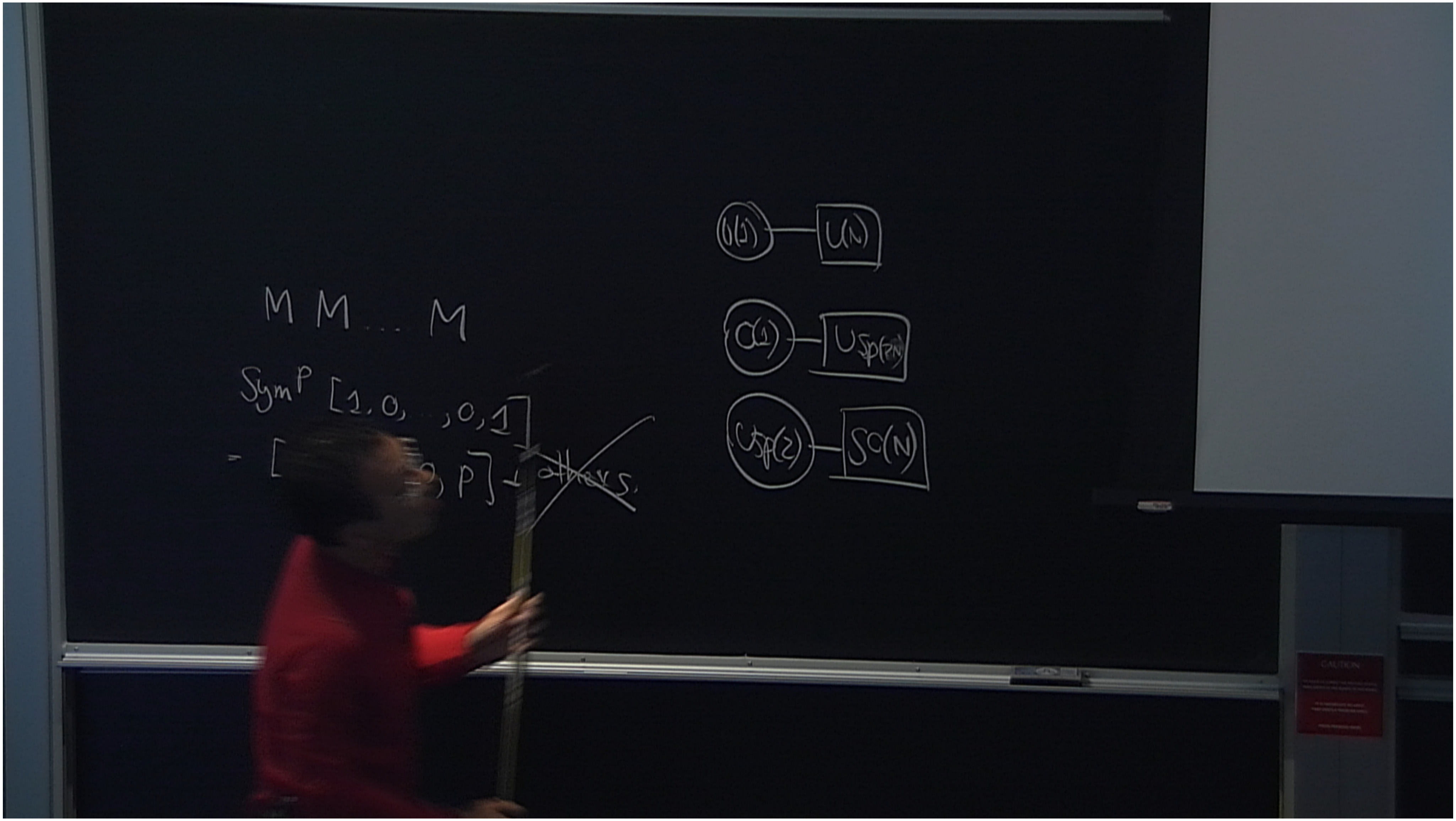
April 28, 2014


M M ... M
SymP [1, 0, ..., 0, 1]
P] + others
U(1)
U(N)
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USp(2N)
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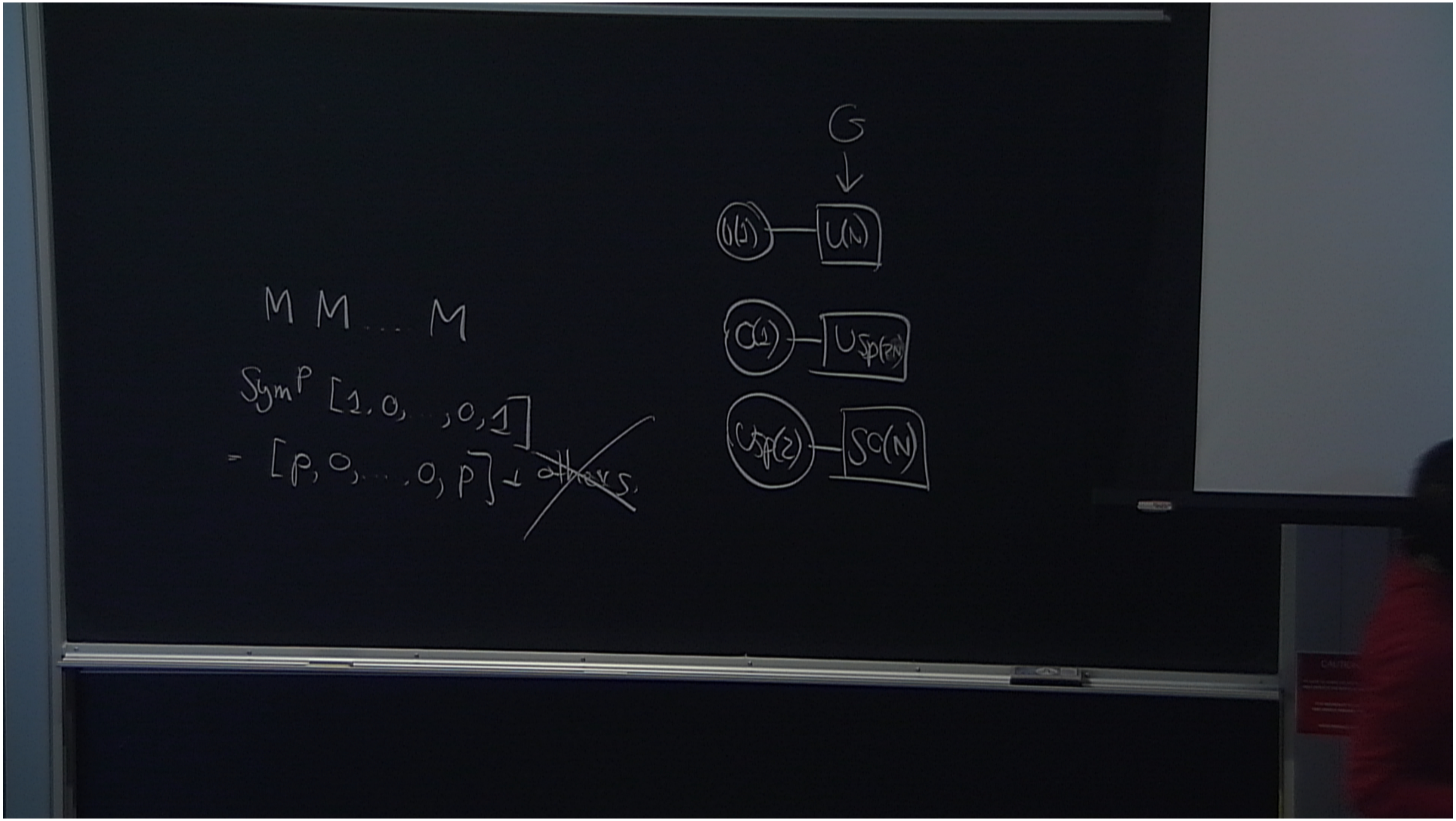
G
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Coulomb branch of a $3d$ $\mathcal{N} = 4$ gauge theory

- The magnetic flux $\boldsymbol{m}$ in $U(1)^{r_G} \subset G$ is labelled by a weight of the GNO dual group $G^\vee$, modulo the Weyl transformations $W_{G^\vee}$.
- Turning on the monopole flux $\boldsymbol{m}$ breaks G to a residual gauge symmetry $H_{\boldsymbol{m}}$.
  - **Example:** $G = U(2)$, $\boldsymbol{m} = (m_1, m_2)$.
    Up to a Weyl transformation, we can take $m_1 \geq m_2 > -\infty$.
    $H_{\boldsymbol{m}} = U(2)$ if $m_1 = m_2$, and $H_{\boldsymbol{m}} = U(1)^2$ if $m_1 \neq m_2$.
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Coulomb branch Hilbert series

- Operators on the Coulomb branch: Monopole operators dressed with powers of scalars in the residual gauge group.
- Count these operators with respect to a $U(1)$ global symmetry, which is a generator of $SU(2)_V$ and wrt. the topological $U(1)_J$ charges.
- The **monopole formula** for the Coulomb branch HS: [Cremonesi, Hanany, Zaffaroni '13]

$$H(t;\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\Gamma_{G^\vee}/W_{G^\vee}} t^{\Delta(\boldsymbol{m})} \boldsymbol{z}^{J(\boldsymbol{m})} P_G(t;\boldsymbol{m}) ,$$

  - Fugacity t keeps track of the $U(1)$ global symmetry.
  - Fugacities $\boldsymbol{z} = (z_1, z_2, \ldots)$ keep track of a collection of the $U(1)_J$ charges, $J(\boldsymbol{m})$.
  - $P_G(t;\boldsymbol{m}) = \prod_i 1/(1-t^{2d_i})$, with d_i the degrees of independent Casimirs of $H_{\boldsymbol{m}}$.
  - Dimension of monopole operators:

$$\Delta(\boldsymbol{m}) = \left(\sum_{\text{all hypers}} \sum_{\substack{w:\text{ weights} \\ \text{of rep of each hyper}}} |w(\boldsymbol{m})| \right) - 2 \sum_{\alpha\in\Delta_G^+} |\alpha(\boldsymbol{m})| .$$

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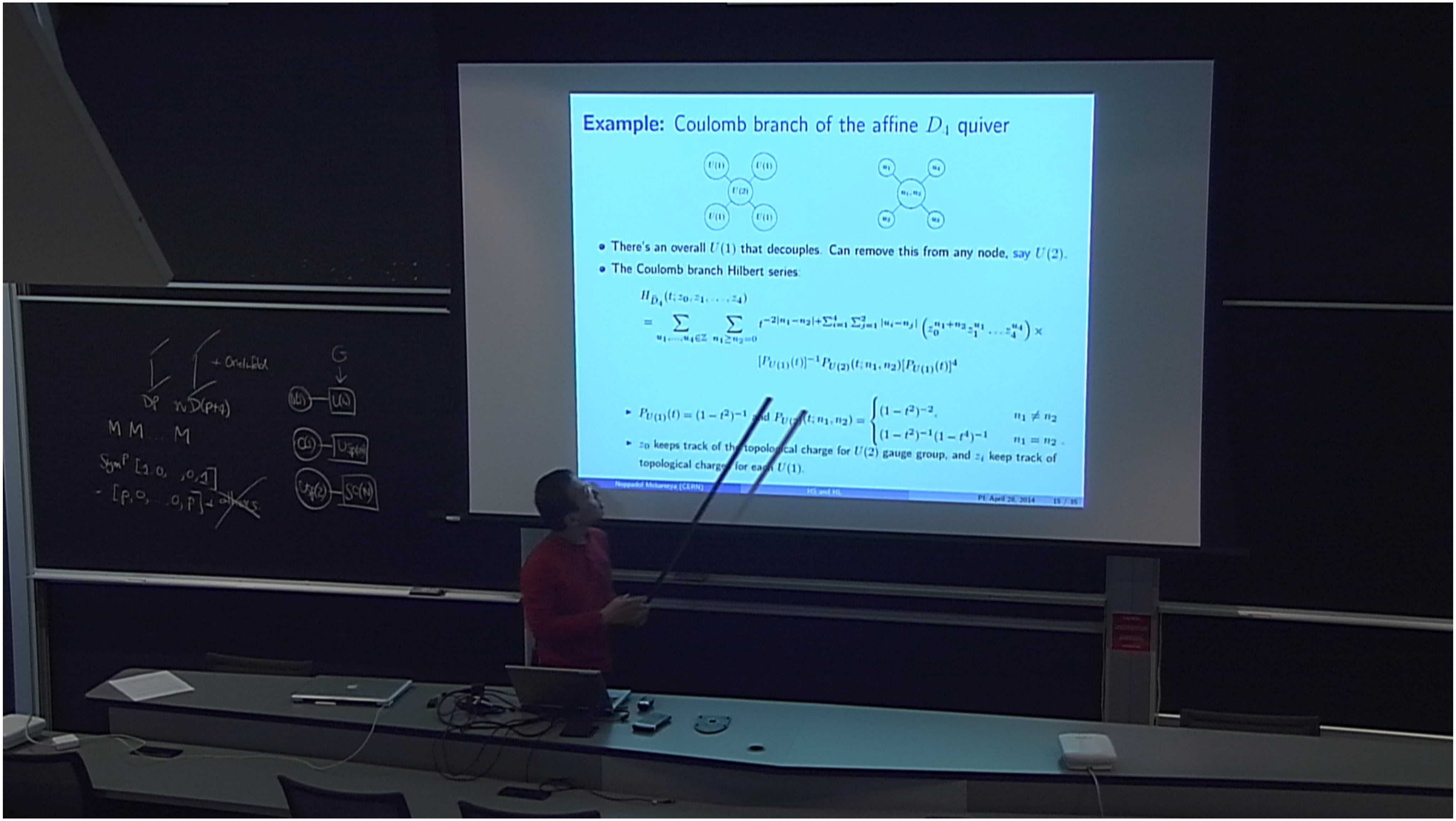

Example: Coulomb branch of the affine D_4 quiver
U(1) U(1) U(2) U(1) U(1)
There's an overall $U(1)$ that decouples. Can remove this from any node, say $U(2)$.
The Coulomb branch Hilbert series:
$H_{\hat{D}_4}(t; z_0, z_1, \ldots, z_4)$
$= \sum_{u_1,\ldots,u_4 \in \mathbb{Z}} \sum_{n_1 \geq n_2 = 0} t^{-2|n_1-n_2| + \sum_{i=1}^4 \sum_{j=1}^2 |u_i - n_j|} \left(z_0^{n_1+n_2} z_1^{u_1} \ldots z_4^{u_4} \right) \times$
$[P_{U(1)}(t)]^{-1} P_{U(2)}(t; n_1, n_2) [P_{U(1)}(t)]^4$
$P_{U(1)}(t) = (1-t^2)^{-1}$ and $P_{U(2)}(t; n_1, n_2) = \begin{cases} (1-t^2)^{-2}, & n_1 \neq n_2 \\ (1-t^2)^{-1}(1-t^4)^{-1} & n_1 = n_2 \end{cases}$
z_0 keeps track of the topological charge for $U(2)$ gauge group, and z_i keep track of topological charge for each $U(1)$.
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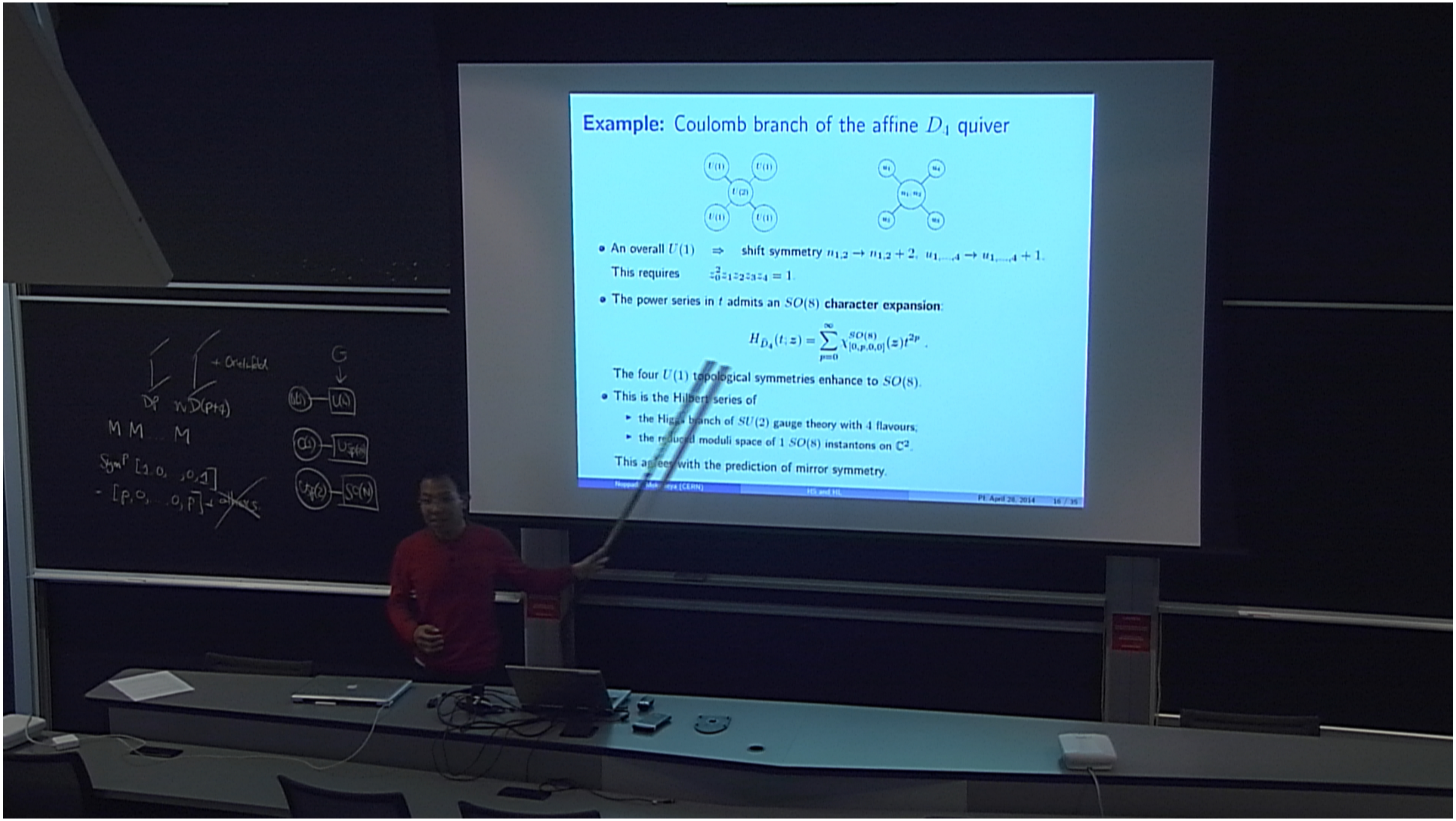
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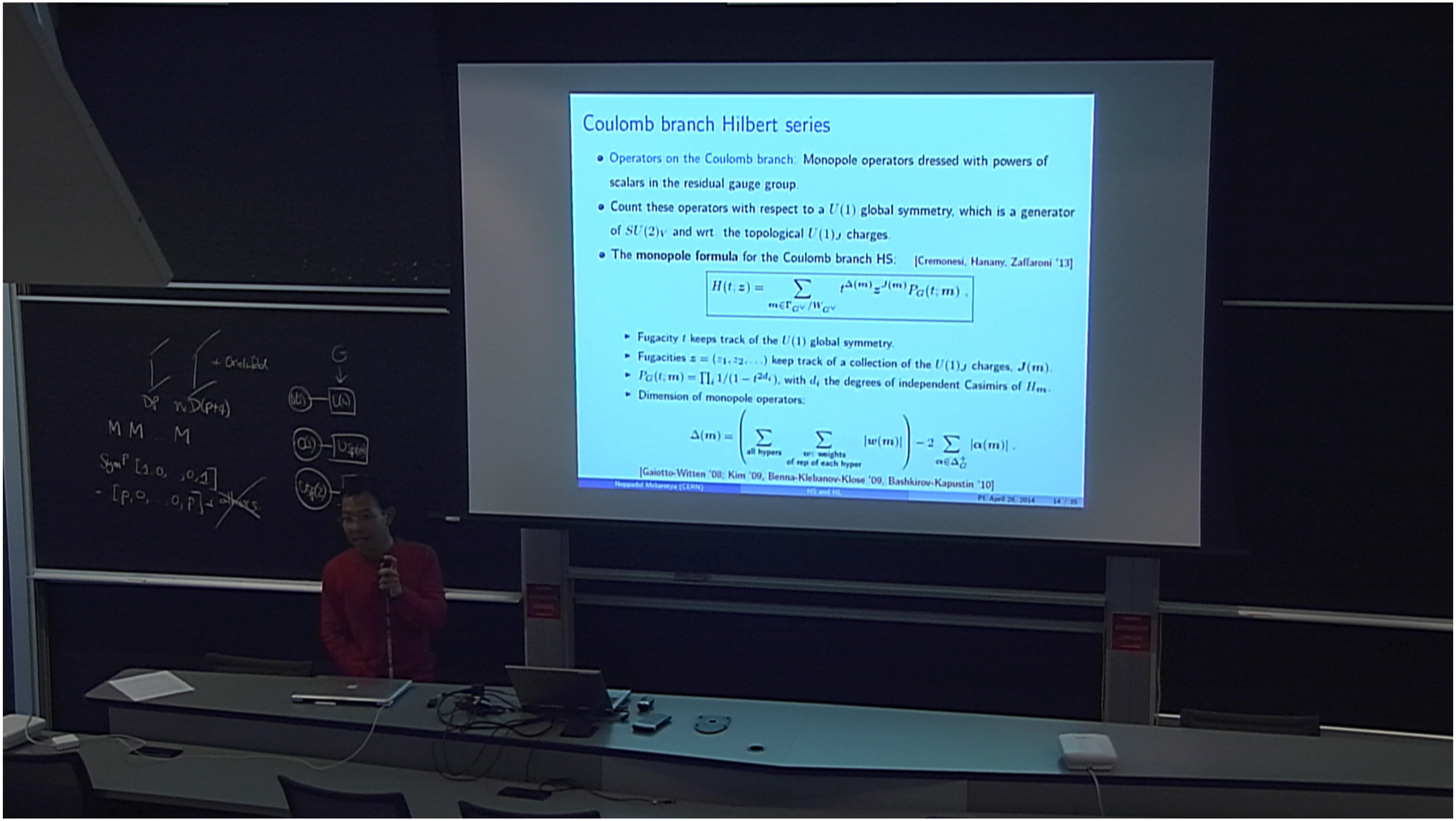
An overall $U(1)$ ⇒ shift symmetry $n_{1,2} \to n_{1,2} + 2$, $u_{1,\dots,4} \to u_{1,\dots,4} + 1$.
This requires $z_0^2 z_1 z_2 z_3 z_4 = 1$.
The power series in t admits an $SO(8)$ character expansion:
$H_{\hat{D}_4}(t; z) = \sum_{p=0}^{\infty} \chi^{SO(8)}_{[0,p,0,0]}(z) t^{2p}$.
The four $U(1)$ topological symmetries enhance to $SO(8)$.
This is the Hilbert series of
the Higgs branch of $SU(2)$ gauge theory with 4 flavours;
the reduced moduli space of 1 $SO(8)$ instantons on $\mathbb{C}^2$.
This agrees with the prediction of mirror symmetry.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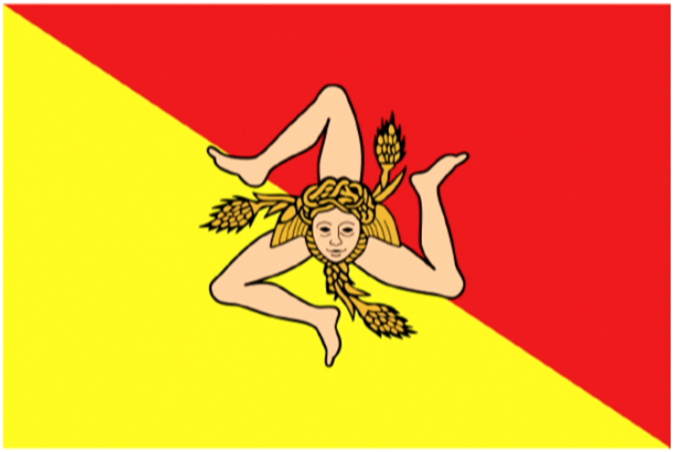
Part III: $3d$ Sicilian theories and beyond!



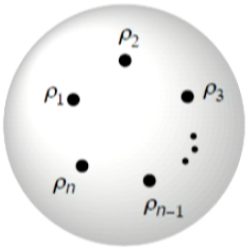
Picture taken from http://en.wikipedia.org/wiki/Flag_of_Sicily

Credit for the name: [Benini, Tachikawa, Wecht '09; Benini, Tachikawa, Xie '09]

$3d$ Sicilian theories

- Upon the compactification, one may introduce a twist. This gives rise to other types of embedding and hence other types of partitions, e.g. B and C partitions.
  [Chacaltana, Distler, Tachikawa '12, Chacaltana, Distler, Trimm '13]
- Given a puncture ρ associated with group H, there is a global symmetry G_ρ associated with it. Let r_k be the number of times that k appears in the partition ρ.

$$G_\rho = \begin{cases} S\left(\prod_k U(r_k)\right) & H = U(N)\ , \\ \prod_{k \text{ odd}} SO(r_k) \times \prod_{k \text{ even}} USp(r_k) & H = SO(2N+1) \text{ or } SO(2N)\ , \\ \prod_{k \text{ odd}} USp(r_k) \times \prod_{k \text{ even}} SO(r_k) & H = USp(2N)\ . \end{cases}$$



- For a Riemann surface with a collection of punctures $\{\rho_1, \rho_2, \ldots\}$, the global symmetry is $\prod_i G_{\rho_i}$. This may enhance to a larger group.

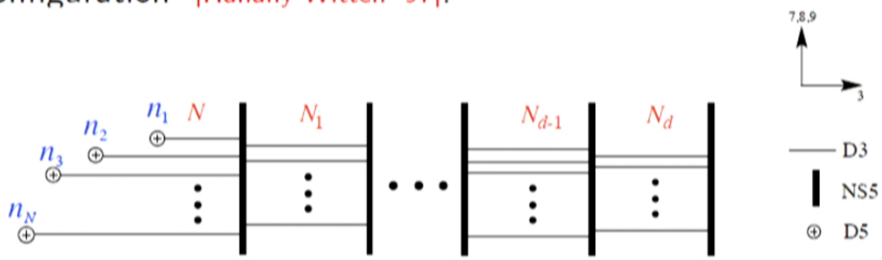
The $T_\rho(G)$ theory [Gaiotto-Witten ' 08]

- $T_\rho(G)$ is constructed as a boundary theory of $4d$ $\mathcal{N} = 4$ SYM on a half-space, with the half-BPS boundary condition specified by $\boldsymbol{\rho} : su(2) \to \mathrm{Lie}(G^\vee)$.
  - $\boldsymbol{\rho}$ can be classified, up to conjugation, by the nilpotent orbits of $\mathrm{Lie}(G^\vee)$.
- For a classical group G, $T_\rho(G)$ is a quiver theory.
- The quiver for $T_\rho(SU(N))$ is

$$[U(N)] - (U(N_1)) - (U(N_2)) - \cdots (U(N_d))$$

with $\boldsymbol{\rho} = (N - N_1,\ N_1 - N_2,\ \ldots,\ N_{d-1} - N_d,\ N_d)$ and $\boldsymbol{\rho}$ is non-increasing: $N - N_1 \geq N_1 - N_2 \geq \ldots \geq N_{d-1} - N_d \geq N_d > 0.$

- A brane configuration [Hanany-Witten '97]:



$\boldsymbol{\rho}$ is the set of linking numbers of each NS5-brane.

The Coulomb branch of $T_\rho(G)$

- The monopole formula for the Coulomb branch Hilbert series works well in most cases for $T_\rho(G)$, except that
  1. For an exceptional group G, the Lagrangian description (and quiver) is not available! No starting point for the monopole formula!
  2. For a theory such as
     $T_{(1,1,1,1)}(SO(5)) : \; [SO(5)] - (USp(4)) - (O(3)) - (USp(2)) - (O(1))$
     the monopole formula blows up to infinity, because the dimension $\Delta(\boldsymbol{m})$ for the monopole operator in $USp(2)$ vanishes when $\boldsymbol{m} \neq 0$:
     'Bad theory' in the sense of [Gaiotto-Witten '08].
- The $SO(4)$ UV R-symmetry is **not** a subgroup of the R-symmetry of the SCFT in the IR. [Gaiotto-Witten '08; Yaakov'13; Bashkirov '13]
- In general, a theory $T_\rho(G)$ is 'bad' occurs if it contains
  - $U(N_c)$ gauge group with $N_f < 2N_c - 1$;
  - $SO(N_c)$ gauge group with $N_f < N_c - 1$;
  - $USp(2N_c)$ gauge group with $N_f < 2N_c + 1$.

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     'Bad theory' in the sense of [Gaiotto-Witten '08].
- The $SO(4)$ UV R-symmetry is **not** a subgroup of the R-symmetry of the SCFT in the IR. [Gaiotto-Witten '08; Yaakov'13; Bashkirov '13]
- In general, a theory $T_\rho(G)$ is 'bad' occurs if it contains
  - $U(N_c)$ gauge group with $N_f < 2N_c - 1$;
  - $SO(N_c)$ gauge group with $N_f < N_c - 1$;
  - $USp(2N_c)$ gauge group with $N_f < 2N_c + 1$.

The Hall-Littlewood formula for Coulomb branch of $T_{\rho}(G)$

[Cremonesi, Hanany, NM, Zaffaroni '14]

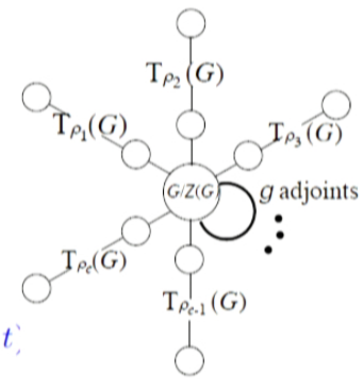
The HL formula for the Coulomb branch HS of $T_{\rho}(G)$:

$$H[T_{\boldsymbol{\rho}}(G^\vee)](t;\boldsymbol{z};\boldsymbol{n}) = t^{\delta_{G^\vee}(\boldsymbol{n})}(1-t^2)^{r_G} K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t)\Psi^{\boldsymbol{n}}_G(\boldsymbol{a}(t,\boldsymbol{x});t)$$

- $\delta_{G^\vee}(\boldsymbol{n}) := \sum_{\boldsymbol{\alpha}\in\Delta_+(G^\vee)} |\boldsymbol{\alpha}(\boldsymbol{n})|$.
- The Hall-Littlewood (HL) polynomial associated with a group G:
$$\Psi^{\boldsymbol{n}}_G(x_1,\ldots,x_{r_G};t) := \sum_{w\in W_G} \boldsymbol{x}^{w(\boldsymbol{n})} \prod_{\boldsymbol{\alpha}\in\Delta_+(G)} \frac{1-t\boldsymbol{x}^{-w(\boldsymbol{\alpha})}}{1-\boldsymbol{x}^{-w(\boldsymbol{\alpha})}}$$
- The argument $\boldsymbol{a}(t,\boldsymbol{x})$ of the HL poly comes from the decomposition
$$\chi^G_{\mathbf{fund}}(\boldsymbol{a}) = \sum_k \chi^{G_{\rho_k}}_{\mathbf{fund}}(\boldsymbol{x}_k)\chi^{SU(2)}_{\rho_k\text{-dim irrep}}(t)$$
where $G_{\rho_k} \subset G_{\boldsymbol{\rho}}$ associated with the number k in $\boldsymbol{\rho}$ that appears r_k times.
- $K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t)$ is fixed by the decomposition of adjoint rep of G into those of $SU(2)\times G_{\boldsymbol{\rho}}$:
$$\chi^G_{\mathbf{Adj}}(\boldsymbol{a}) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}} \chi^{SU(2)}_{[2j]}(t)\chi^{G_{\boldsymbol{\rho}}}_{\mathbf{R}_j}(\boldsymbol{x}_j)\,, \quad K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t) = \mathrm{PE}\left[\sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}} t^{2j+2}\chi^{G_{\boldsymbol{\rho}}}_{\mathbf{R}_j}(\boldsymbol{x}_j)\right].$$

Coulomb branch of a $3d$ Sicilian theory of the A type

Gluing: The Coulomb branch HS of a mirror of the Sicilian theory of the A_{N-1} type on a genus g surface with e punctures $\{\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_e\}$.

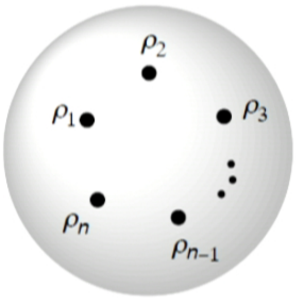


$$H[\text{mirror } g, \{\boldsymbol{\rho}_i\}_{i=1}^{e}](t; \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(e)})$$
$$= \sum_{n_1 \geq \cdots \geq n_{N-1} \geq 0} t^{(e+2g-2)\sum_{j=1}^{N-1}(N+1-2j)n_j} (1-t^2)^{eN+1} \times$$
$$P_{U(N)}(t; n_1, \ldots, n_{N-1}, 0) \prod_{j=1}^{e} K_{\boldsymbol{\rho}_j}(\boldsymbol{x}^{(j)}; t) \Psi_{U(N)}^{(n_1, \ldots, n_{N-1}, 0)}(\boldsymbol{x}^{(j)} t^{\boldsymbol{w}_{\boldsymbol{\rho}_j}}; t)$$

where

- $\Psi_{U(N)}^{\boldsymbol{n}}(\boldsymbol{x} t^{\boldsymbol{w}_{\boldsymbol{\rho}}}; t) := \Psi_{U(N)}^{(n_1, \ldots, n_N)}(x_1 t^{\boldsymbol{w}_{\rho_1}}, x_2 t^{\boldsymbol{w}_{\rho_2}}, \ldots, x_{d+1} t^{\boldsymbol{w}_{\rho_{d+1}}}; t)$.
- $t^{\boldsymbol{w}_r} = (t^{r-1}, t^{r-3}, \ldots, t^{-(r-3)}, t^{-(r-1)})$.
- $P_{U(N)}(t; \boldsymbol{n})$ is the generating function for the indep Casimirs in the residue group left unbroken by $\boldsymbol{n}$.

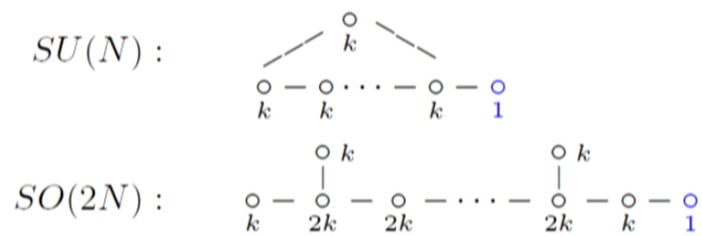
Agreement with the Hall-Littlewood limit of the superconformal index



- For genus $g = 0$, this formula agrees with the Hall-Littlewood index for the $4d$ Sicilian theory with punctures $\{\rho_i\}_{i=1}^{e}$. [Gadde, Rastelli, Razamat, Yan '11]
  - This is equal to the **Higgs branch HS** for the corresponding $3d$ Sicilian theory.
  - Agrees with mirror symmetry.
- This Coulomb branch HS can also be computed for mirrors of D-, twisted A- and twisted D-type Sicilian theories. Agree with the Higgs branch computations from [Lemos, Peelaers, Rastelli '12; Chacaltana, Distler, Tachikawa '12; Chacaltana, Distler, Trimm '13]

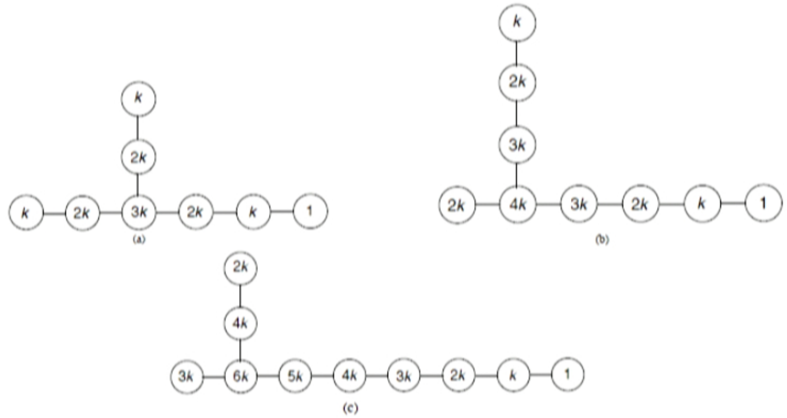
The moduli spaces of k G instantons on $\mathbb{C}^2$

- For a simple group G, such a moduli space can be realised from the Coulomb branch of the affine Dynkin diagram of G with a $U(1)$ node attached at the **extended node**.
- If G is simply-laced (ADE), such Dynkin diagrams correspond to quiver diagrams **with Lagrangian descriptions**. Each node ℓ denotes a $U(\ell)$ group and each line denotes a hypermultiplet. [Intriligator-Seiberg '97]



The moduli spaces of k $E_{6,7,8}$ instantons on $\mathbb{C}^2$

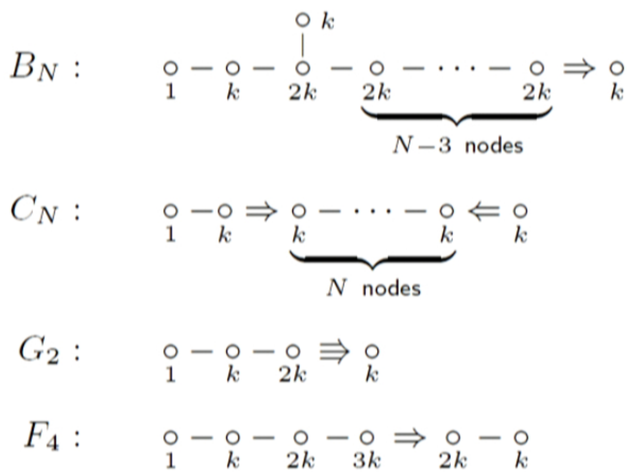
[Gaiotto-Razamat '12]



	$\boldsymbol{\rho}_1$	$\boldsymbol{\rho}_2$	$\boldsymbol{\rho}_3$
E_6	(k,k,k)	(k,k,k)	$(k,k,k-1,1)$
E_7	(k,k,k,k)	$(2k,2k)$	$(k,k,k,k-1,1)$
E_8	$(3k,3k)$	$(2k,2k,2k)$	$(k,k,k,k,k,k-1,1)$

The Hilbert series, i.e. $5d$ instanton partition functions, can be computed using the monopole formula or the HL formula + gluing technique.

Non-simply laced groups



- These diagrams do not correspond to theories with known Lagrangian descriptions.
- For G_2 and F_4, the theories themselves and their mirrors have no known Lagrangian. Mirror symmetry does not help in these cases!
- Nevertheless, these quivers have brane configurations in terms of D3, NS5 and ON-planes. Magnetically charged BPS objects can be identified. [Hanany, Troost '01]

Conclusions

- Study moduli spaces of $3d$ $\mathcal{N} = 4$ gauge theories using Hilbert series.
- Understand quantum corrections to the Coulomb branch chiral rings using the **monopole formula**.
- Compute Coulomb branch Hilbert series of $T_\rho(G)$ and $3d$ Sicilian theories using the **Hall-Littlewood formula**.
- Provide more tests for $3d$ mirror symmetry.
- Connections between Coulomb branches and instanton moduli spaces
- Connections between Hilbert series and $5d$ instanton partition functions.
- Hilbert series for k G-instantons on $\mathbb{C}^2$ can be computed for any simple group G.