Title: Categories of Convex Sets and C*-Algebras

Date: Apr 15, 2014 03:30 PM

URL: http://pirsa.org/14040140

Abstract: The start of the talk will be an outline how the ordinary notions of quantum theory translate into the category of C*-algebras, where there are several possible choices of morphisms. The second half will relate this to a category of convex sets used as state spaces. Alfsen and Shultz have characterized the convex sets arising from state spaces C*-algebras and this result can be applied to get a categorical equivalence between C*-algebras and state spaces of C*-algebras which is a generalization of the equivalence between the Schroedinger and Heisenberg pictures.

Categories of Convex Sets and C^* Algebras

Robert Furber

Radboud University Nijmegen

15th April, 2014

Introduction

- Relating C^*-algebra quantum mechanics to convex sets.

C*-algebras

- In this context "algebra" means a vector space A in which the vectors can be multiplied.
- Additionally the structure of a map $\text{-}^* : A \to \overline{A}$ that reverses the order of multiplication.
- Norm that works like the operator norm.
- An axiomatization of rings of operators on a Hilbert space. Enough axioms to make things work.
- The usual terminology for operators can be translated, *e.g.* unitary, positive, self-adjoint, projection etc.
- Commutative refers to this multiplication.
- Examples: $B(\mathcal{H})$, $\mathbb{C}^n$, M_n, $C(X)$.

Possible choices of morphisms

- *-homomorphisms.
- Completely positive maps preserving unit.
- Positive maps preserving unit.

Ordinary quantum stuff in terms of (maps of) C*-algebras

- Each unitary u defines a map $B(\mathcal{H}) \to B(\mathcal{H})$ by

$$a \mapsto u^* a u$$

and this map is a *-homomorphism. This is the usual time evolution in the Heisenberg picture.

Ordinary quantum stuff in terms of (maps of) C*-algebras

- Each unitary u defines a map $B(\mathcal{H}) \to B(\mathcal{H})$ by

$$a \mapsto u^* a u$$

  and this map is a *-homomorphism. This is the usual time evolution in the Heisenberg picture.
- POVMs on a finite set correspond to positive unital maps $\mathbb{C}^n \to B(\mathcal{H})$
- POVMs on a space X correspond to positive unital maps $C(X) \to B(\mathcal{H})$.
- PVM on a finite set correspond to *-homomorphisms $\mathbb{C}^n \to B(\mathcal{H})$.

Ordinary quantum stuff in terms of (maps of) C*-algebras 2

- If we have a basis, we also have a positive unital map $B(\mathcal{H}) \to \mathbb{C}^n$ that is a one-sided inverse to the other map.

Relationship to convex sets

- On each C^*-algebra A, we can take its state space, the positive unital maps $A \to \mathbb{C}$.

Definition of convex set

- Usually defined as a subset of a vector space.

Definition of convex set

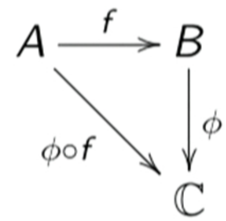
- Usually defined as a subset of a vector space.
- Can also be defined using an operation $[0,1] \times X \times X \to X$.

Convexity monad

- An advantage of this is that we can define algebraic structures in other categories than **Set**.
- To each (compact Hausdorff) space X, we define $\mathcal{R}(X)$ to be the space of measures on X. This is a monad.
- If on a convex subset X of a vector space we can define the centre of mass of a measure, we can define a continuous map $\alpha : \mathcal{R}(X) \to X$ that maps a measure to its centre of mass.
- In general we take "convex set" to refer to an algebra of $\mathcal{R}$, i.e. a map $\alpha : \mathcal{R}(X) \to X$.
- Every algebra can be concretely represented on a topological vector space such that α becomes centre of mass. [Ś74].

State spaces

- If we take the state space $S(A)$ of a C^*-algebra A, we get a convex set of this type.
- Given a positive unital map $f : A \to B$, we get a $\mathcal{R}$-algebra map $S(f) : S(B) \to S(A)$ as follows:



- We have a functor.
- For the time evolution in the Heisenberg picture $f : B(\mathcal{H}) \to B(\mathcal{H})$, $S(f)$ gives the corresponding time evolution on states, in the Schrödinger picture (on density matrices rather than unit vectors).
- For a P(O)VM $f : C(X) \to B(\mathcal{H})$, $S(f)$ turns states into measures on X. This is the Born rule.
- For the projection maps $p : B(\mathcal{H}) \to C(X)$, $S(p)$ turns

Freeness

- For commutative C^*-algebras, we can use the notion of free algebras that is intrinsic to the monad $\mathcal{R}$.

Freeness

- For commutative C^*-algebras, we can use the notion of free algebras that is intrinsic to the monad $\mathcal{R}$.
- A free algebra is one of the form $T(X)$, where the map $\mu_X : T^2(X) \to T(X)$ is part of the structure of the monad.

General C*-algebras

- Alfsen, Shultz and Hanche-Olsen [AHOS80] characterized the convex sets arising from C*-algebras.