Title: A non-commuting stabilizer formalism

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Abstract: We propose a non-commutative extension of the Pauli stabilizer formalism. The aim is to describe a class of many-body quantum states which is richer than the standard Pauli stabilizer states. In our framework, stabilizer operators are tensor products of single-qubit operators drawn from the group {\alpha I, X,S}, where \alpha=e^{i\pi/4} and S=diag(1,i). We provide techniques to efficiently compute various properties, related to e.g. bipartite entanglement, expectation values of local observables, preparation by means of quantum circuits, parent Hamiltonians etc. We also highlight significant differences compared to the Pauli stabilizer formalism. In particular we give examples of states in our formalism which cannot arise in the Pauli stabilizer formalism, such as topological models that supports non-abelian anyons. This is a joint work with O. Buerschaper and M. van den Nest.

A Non-commuting Stabilizer Formalism

Xiaotong Ni (MPQ)

Joint work with O. Buerschaper, M. van den Nest

Perimeter Institute, 2014/4/9

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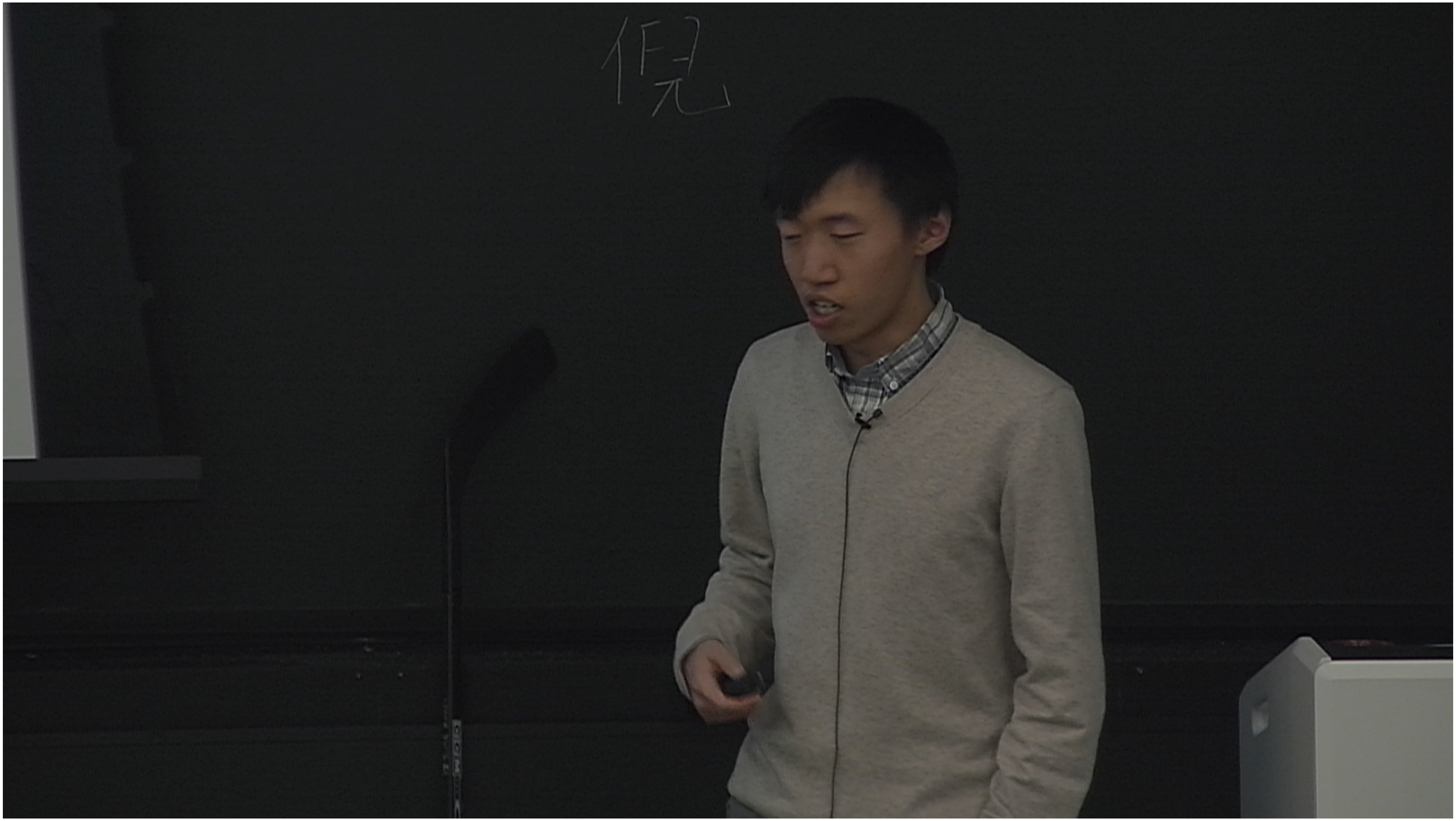
Pauli stabilizer formalism

- Pauli group on *n* qubits: $\mathcal{P}_n = \langle i, X, Z \rangle^{\otimes n}$
- Pauli stabilizer state (code): consider $\langle g_1, \ldots, g_m \rangle \subset \mathcal{P}_n$, We call the state (subspace) satisfies $g_j|\psi\rangle = |\psi\rangle$ Pauli stabilizer state (code)

Gottesman 1997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- An example: $g_1 = ZZI, g_2 = IZZ, g_3 = XXX$
- The state being $|000\rangle + |111\rangle$

Gottesman 1997

Usage

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- Etc.

- Benefits from the (computational) efficiency
- Can we generalize the formalism while keep the efficiency?

Overview

- A (strictly) larger class compared to PSF?
- Yes. More topological phases and LU-inequivalence

- Much larger?
- Entanglement properties are very similar

- Computational efficiency?
- In general NP-Complete. Exist a restricted class where (almost) everything is efficient

An example

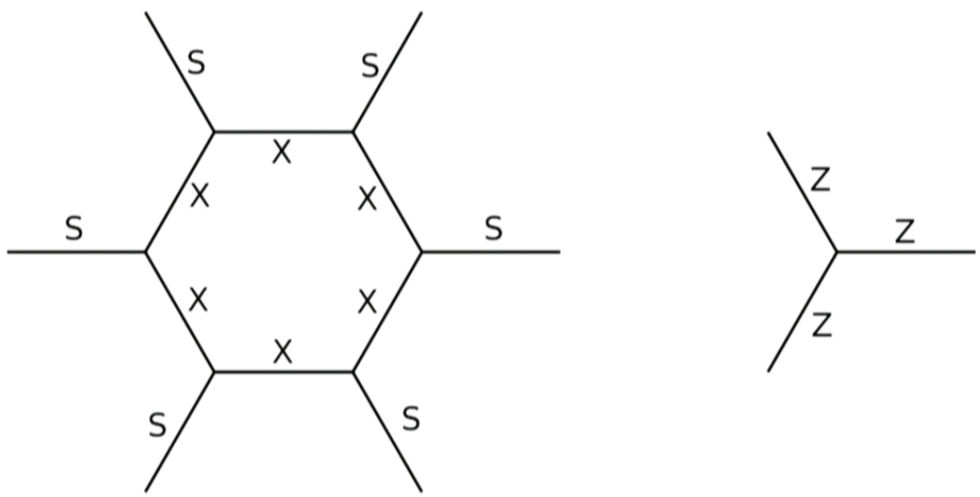
$$g_1 = X \otimes S^3 \otimes S^3 \otimes S \otimes X \otimes X,$$
$$g_2 = S^3 \otimes X \otimes S^3 \otimes X \otimes S \otimes X,$$
$$g_3 = S^3 \otimes S^3 \otimes X \otimes X \otimes X \otimes S.$$

$$|\psi\rangle = \sum_{x_j=0}^{1} (-1)^{x_1 x_2 x_3} |x_1, x_2, x_3, x_1 \oplus x_2, x_2 \oplus x_3, x_3 \oplus x_1\rangle.$$

- The phase $(-1)^{x_1 x_2 x_3}$ means it cannot be stabilized by Pauli stabilizers.
- Not equivalent to Pauli stabilizer states even with single qubit basis changes.

The double semion model

- Defined on a honeycomb lattice.
- Support different anyons compared to toric code.



Levin, Wen 2004

Computational complexity

- Given

$$G = \langle g_1, \ldots, g_m \rangle \subset \mathcal{P}_n^s$$

- Asking whether there is a state stabilized by *G* is NP-complete.

The outline of NP-Hard proof

XS-stabilizer problem

Assume all g_j has the form $i^3 S_k S_l S_p$.

Then there is a state stabilized by all g_j

⇕

there is a computational basis $|x\rangle$ stabilized by all g_j.

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1-in-3 SAT problem

$$g_j|x_k x_l x_p\rangle = |x_k x_l x_p\rangle$$

is equivalent to require only one of x_k, x_l, x_p equal to 1.

This is the 1-in-3 SAT problem. It is NP-complete.

Towards a restricted class

- Note that the hardness come from ~~diagonal operators that contain S.~~

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- Define: Regular XS-stabilizer group G if the diagonal subgroup of G can be generated by Z-type operators.
- Theorem: For a regular G, we can find the stabilizer state(s) efficiently. (In fact, almost all properties of the state can be computed efficiently)

Example (of regular stabilizer)

$$g_1 = X \otimes S^3 \otimes S^3 \otimes S \otimes X \otimes X,$$
$$g_2 = S^3 \otimes X \otimes S^3 \otimes X \otimes S \otimes X,$$
$$g_3 = S^3 \otimes S^3 \otimes X \otimes X \otimes X \otimes S.$$

- The diagonal subgroup of $G = \langle g_1, g_2, g_3 \rangle$ is generated by

$$I \otimes Z \otimes Z \otimes Z \otimes I \otimes I$$
$$Z \otimes I \otimes Z \otimes I \otimes Z \otimes I$$
$$Z \otimes Z \otimes I \otimes I \otimes I \otimes Z$$

Justification of the restriction

- Many examples are regular XS-stabilizer, (e.g. the six qubit example and double semion)
- For non-regular G, we show that the code (ground) space have a basis

$\mathcal{H}_G \rightarrow$

$|\psi_1\rangle$ with regular group G_1

$\vdots$

$|\psi_k\rangle$ with regular group G_k

So the properties of regular XS-stabilizer state also applies to the basis.

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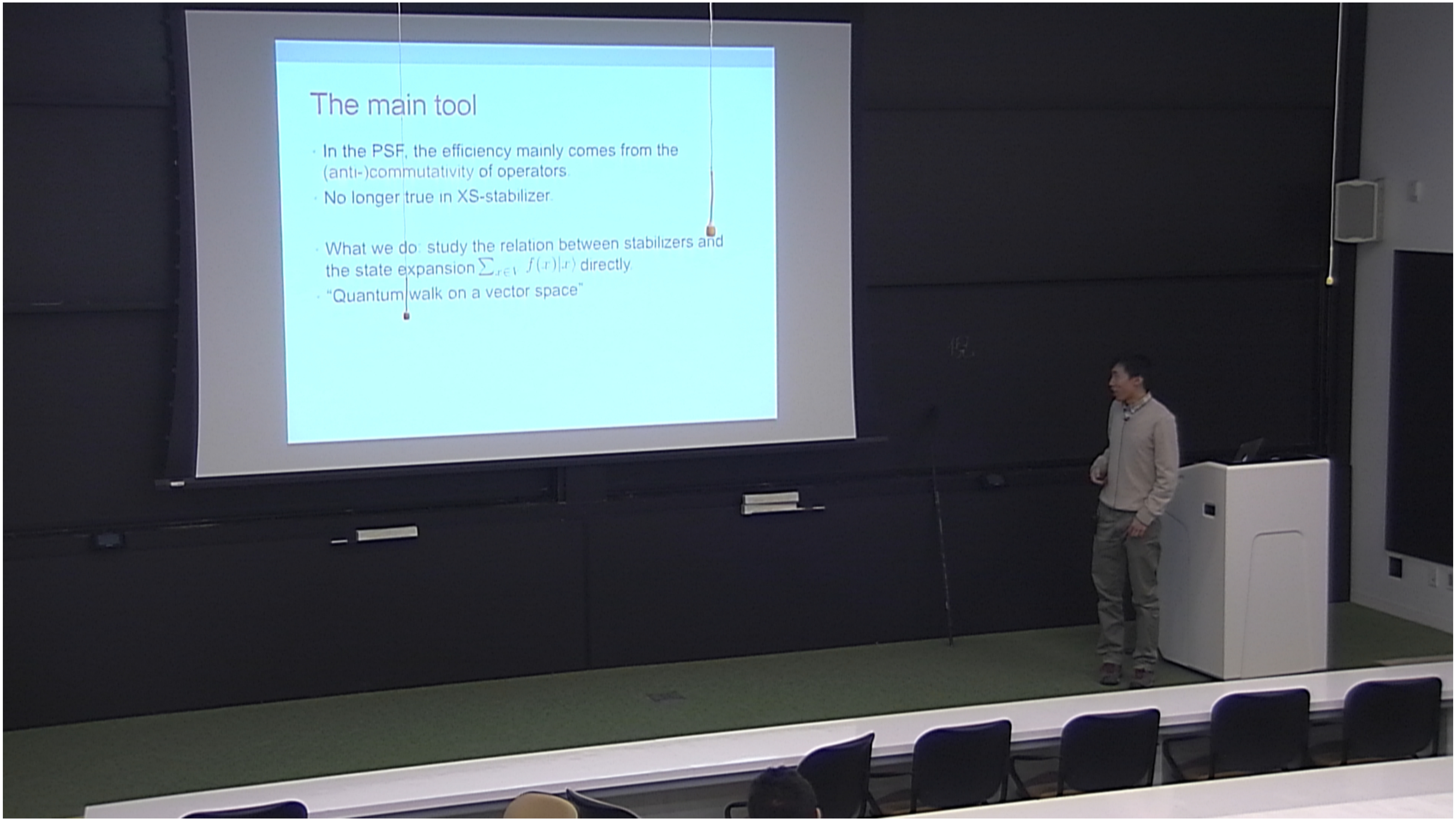
BASIC MECHANISM

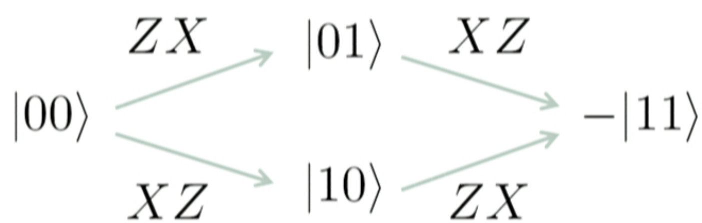

The main tool
In the PSF, the efficiency mainly comes from the (anti-)commutativity of operators.
No longer true in XS-stabilizer.
What we do: study the relation between stabilizers and the state expansion $\sum_{x\in V} f(x)|x\rangle$ directly.
"Quantum walk on a vector space"

Example (PSF)

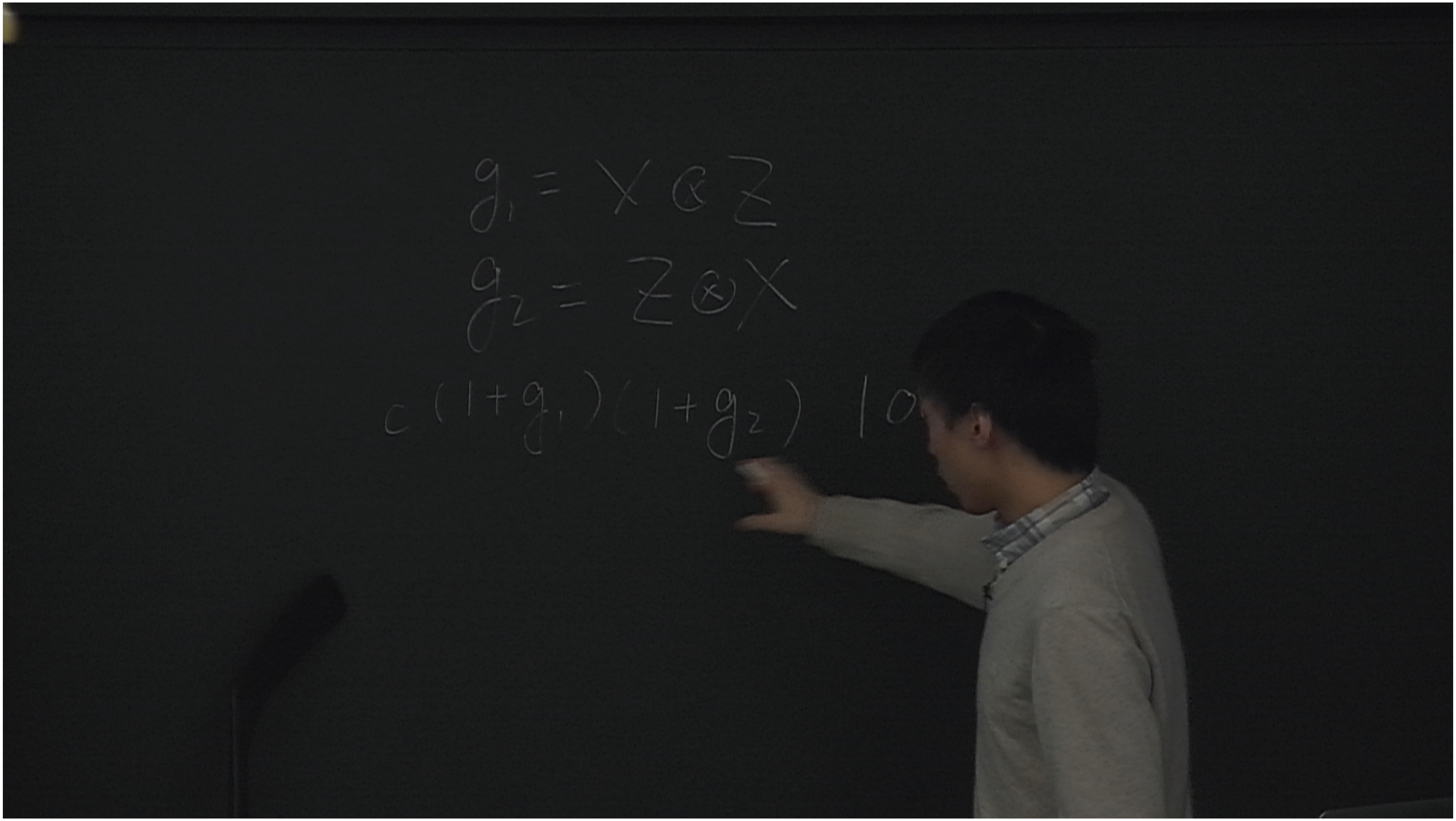
- Consider the 2-qubit example $\sum(-1)^{x_1 x_2}|x_1 x_2\rangle$

$$X \otimes Z$$

$$Z \otimes X$$



- (formally on the blackboard)


g1 = X ⊗ Z
g2 = Z ⊗ X
c (1+g1)(1+g2) |0

- We can compute an XS-stabilizer state in the same way.
- One difference:

$$Z|x_1 \oplus x_2\rangle = (-1)^{x_1+x_2}|x_1 \oplus x_2\rangle$$
$$S|x_1 \oplus x_2\rangle = i^{x_1+x_2}(-1)^{x_1 x_2}|x_1 \oplus x_2\rangle$$

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↓

Pauli: $\sum_{x\in V} i^{l(x)}(-1)^{q(x)}|x\rangle$

XS: $\sum_{x\in V} \sqrt{i}^{l(x)} i^{q(x)}(-1)^{c(x)}|x\rangle$

(e.g. $c(x) = x_1x_2x_3 + x_2x_4x_5 + \dots$)

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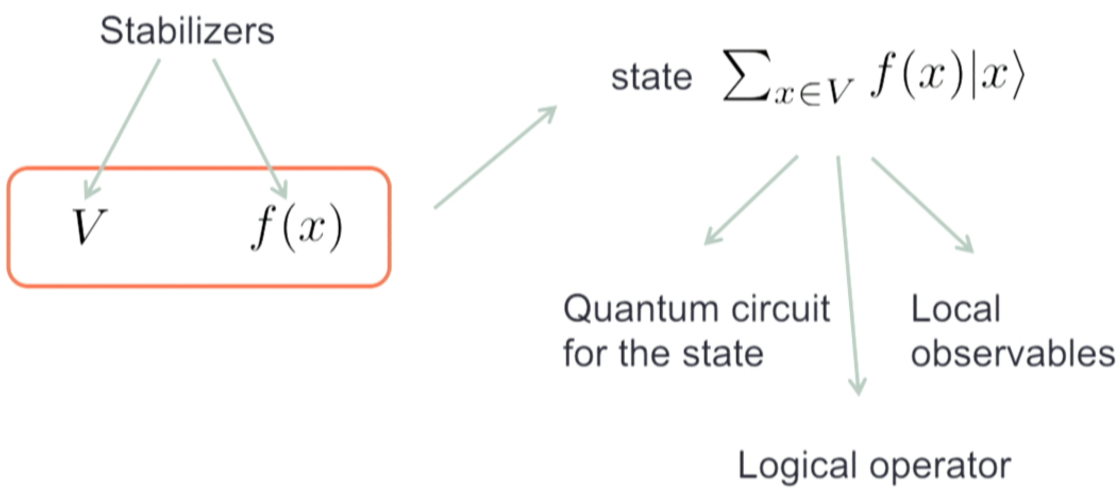

Stabilizers
V
$f(x)$
state $\sum_{x\in V} f(x)|x\rangle$
Quantum circuit for the state
Local observables
Logical operator

Entanglement

- Given a regular XS-stabilizer state $|\psi\rangle$ and any bipartition (A,B), we can efficiently find $U_A \otimes U_B$, such that $U_A \otimes U_B|\psi\rangle$ is a Pauli stabilizer state .
- This has the following consequences:

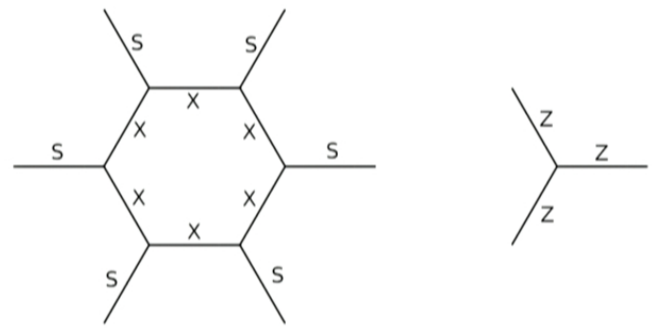
Summary so far

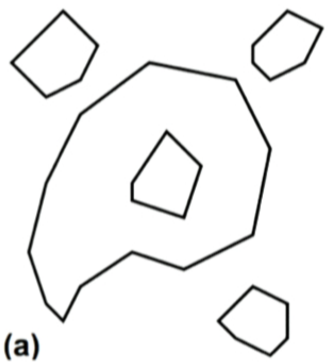
- Pauli-S group: $\mathcal{P}_n^s = \langle \alpha, X, S \rangle^{\otimes n}$
- In general finding the stabilized state(s) is NP-complete
- Regular: the diagonal subgroup of *G* don't contain S. This can be checked efficiently given $G = \langle g_j \rangle$.

TWISTED QUANTUM DOUBLE MODEL

The double semion

- Up to a local basis change, the double semion ground state is
  $$\sum_{x \in V} (-1)^{f(x)} |x\rangle$$
- $f(x)$ is the number of loops $|x\rangle$ has. And now we know it is a cubic polynomial $c(x)$.



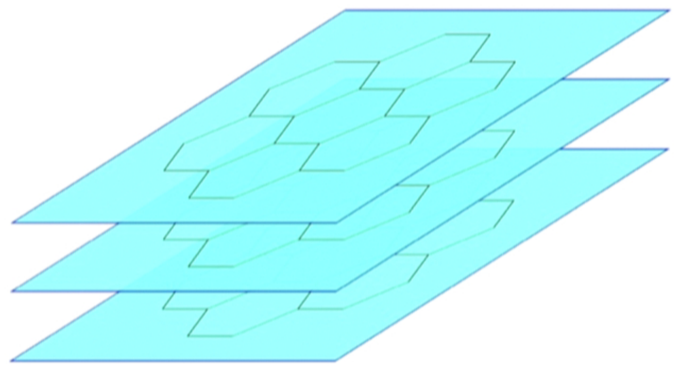

(a)

Twisted double of $Z_2 \times Z_2 \times Z_2$

- Consider *V* to be the subspace of closed loops on 3 layers.

  $$\sum_{x \in V} (-1)^{f(x)} |x\rangle$$

- There are "interesting" $f(x)$ that have local Hamiltonian.
- Tool: group cohomology



Y. Hu, etc. 2012

Interesting cases

- On the $Z_2 \times Z_2 \times Z_2$, there are twisted double models that have degeneracy 22 on a torus, while others have 64 (for example, 3 independent toric code).
- Those models support non-abelian anyons.

Example:

- The double semion model: projecting each stabilizer into the gauge invariant subspace
- 4 qubit example: with diagonal subgroup:

$$XXSS \qquad IIZZ$$
$$SSXX \qquad ZZII$$

- The subspace stabilized by diagonal subgroup is $|\tilde{0}\tilde{0}\rangle, |\tilde{0}\tilde{1}\rangle, |\tilde{1}\tilde{0}\rangle, |\tilde{1}\tilde{1}\rangle$, with $|\tilde{0}\rangle = |00\rangle$, etc.
- In the subspace, the stabilizers become (effectively)

$$XZ$$
$$ZX$$

- Hermitian and commuting!