Title: The computational power of quantum walk

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Abstract: Quantum computers have the potential to solve certain problems dramatically faster than classical computers. One of the main quantum algorithmic tools is the notion of quantum walk, a quantum mechanical analog of random walk. I will describe quantum algorithms based on this idea, including an optimal algorithm for evaluating Boolean formulas and one of the best known algorithms for simulating quantum dynamics. I will also show how quantum walk can be viewed as a universal model of quantum computation.

The computational power of quantum walk

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Why quantum computing?

Fast algorithms for classically hard problems

3107418240490043721350750035888567
9300373460228427275457201619488232
0644051808150455634682967172328678
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397547189678996851549366663853908
80271038021044989571912614655711

- Computing discrete logarithms
- Decomposing Abelian groups
- Computations in number fields
- Approximating Gauss sums
- Shifted Legendre symbol
- Counting points on algebraic curves
- Approximating the Jones polynomial (and other topological invariants)
- Simulating quantum systems
- Linear systems
- Computing effective resistance
- …

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- Formula evaluation
- Collision finding (k-distinctness, k-sum, etc.)
- Minimum spanning tree, connectivity, shortest paths, bipartiteness of graphs
- Network flows, maximal matchings
- Finding subgraphs
- Minor-closed graph properties
- Property testing (distance between distributions, bipartiteness/expansion of graphs, etc.)
- Checking matrix multiplication
- Group commutativity
- Subset sum
- …

What can be computed efficiently?

Nature is described by quantum mechanics, so to fully understand what can be computed in the real world, we have to understand the implications of quantum mechanics for computation.

Apparently nature can efficiently solve problems that a classical computer cannot.

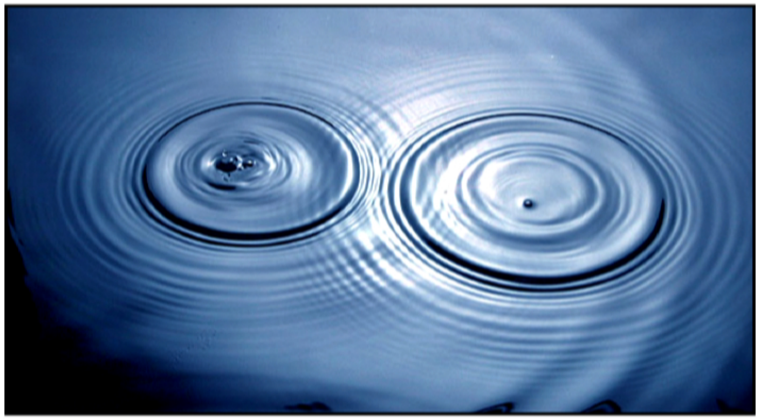
Only two alternatives:

- Classical computers can efficiently simulate quantum ones, or
- Quantum mechanics is not a good description of nature

Main goal of my research: Understand the advantages of quantum over classical computation

The origin of quantum speedup

Interference between computational paths



Arrange so that
- paths to the solution interfere constructively
- paths to non-solutions interfere destructively

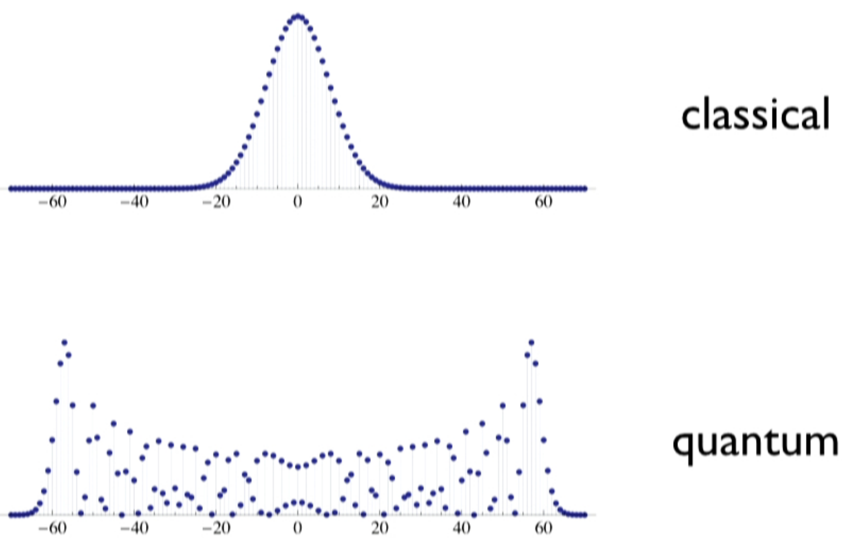
Quantum mechanics gives an efficient representation of complex interference phenomena

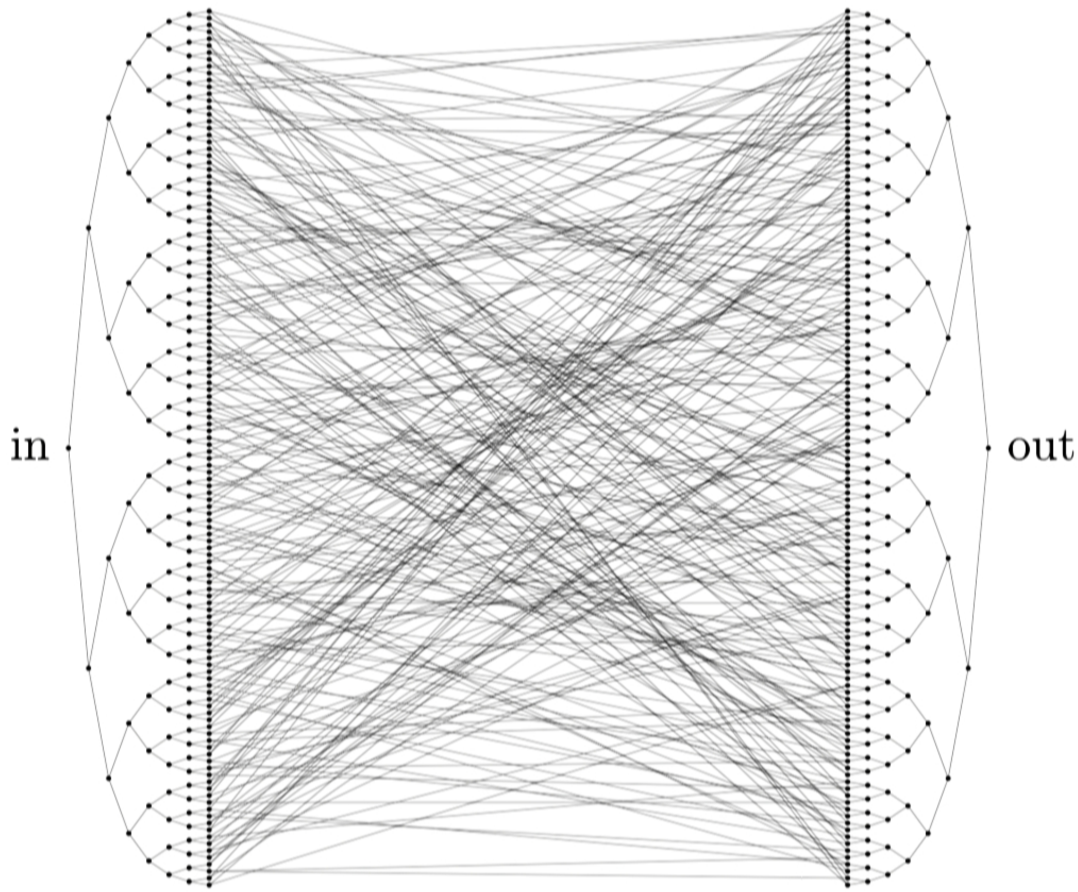
Quantum walk

Quantum walk

Quantum analog of a random walk on a graph.

Idea: Replace probabilities by quantum amplitudes.
Interference can produce radically different behavior!

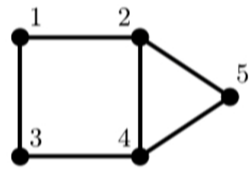




[Childs, Cleve, Deotto, Farhi, Gutmann, Spielman, STOC 2003]

From random walk to quantum walk

Graph G:



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

adjacency matrix *Laplacian*

Random walk on G

State: Probability $p_v(t)$ of being at vertex v at time t

Dynamics: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = L\vec{p}$

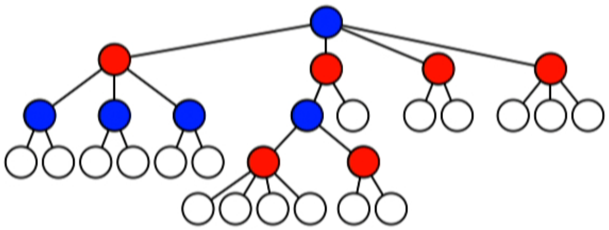
Quantum walk on G

State: Amplitude $a_v(t)$ to be at vertex v at time t

Dynamics: $i\frac{\mathrm{d}}{\mathrm{d}t}\vec{a} = L\vec{a}$

Outline

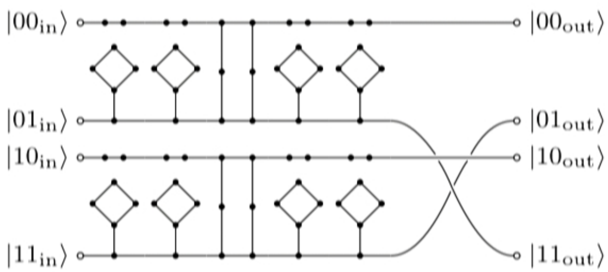
Formula evaluation



Quantum simulation

$$i \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = H \, \psi(t)$$

Universal computation

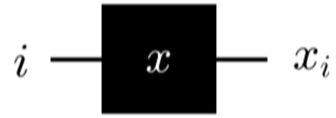


Formula evaluation

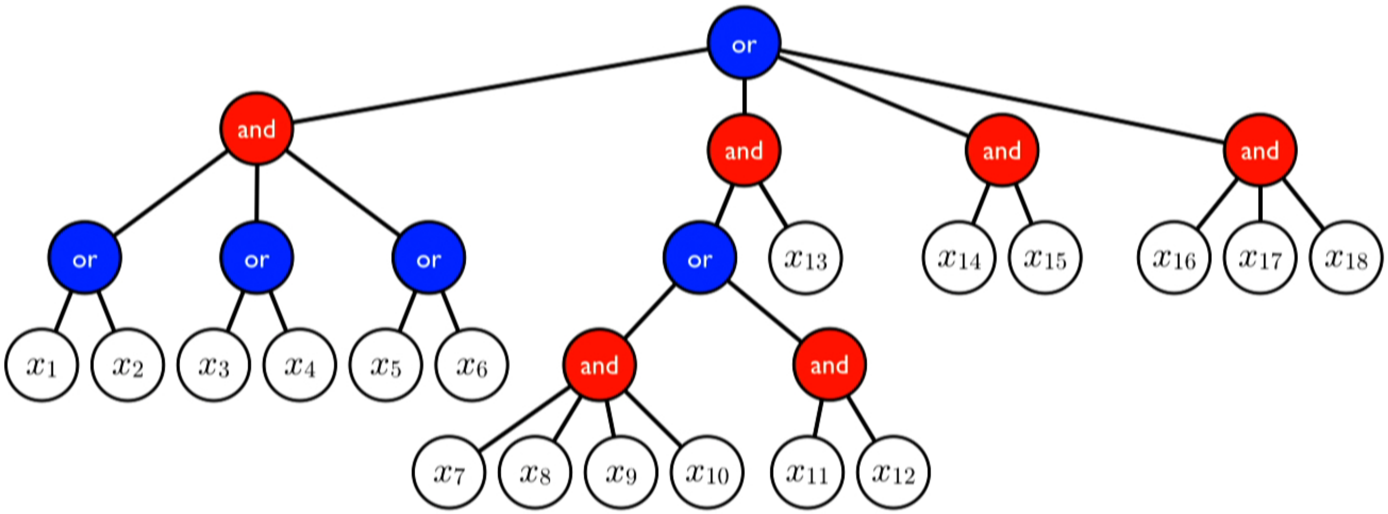
- Ambainis, Childs, Reichardt, Špalek, and Zhang, FOCS 2007, pp. 363–372; SIAM Journal on Computing **39**, 2513–2530 (2010)

Query complexity of formula evaluation

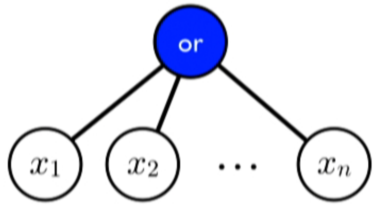
Query model: given a black box for a string $x \in \{0,1\}^n$



Compute some function of x using as few queries as possible



A single OR gate

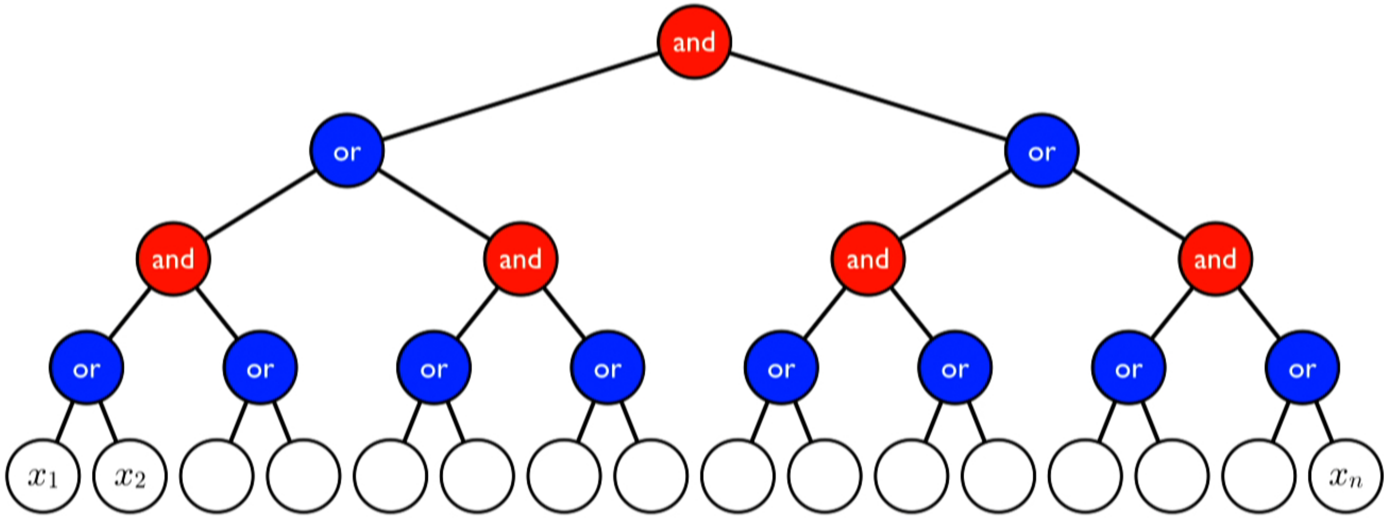


Classical complexity: $\Theta(n)$

Quantum algorithm [Grover 1996]: $O(\sqrt{n})$

Quantum lower bound [BBBV 1996]: $\Omega(\sqrt{n})$

Balanced binary AND-OR trees

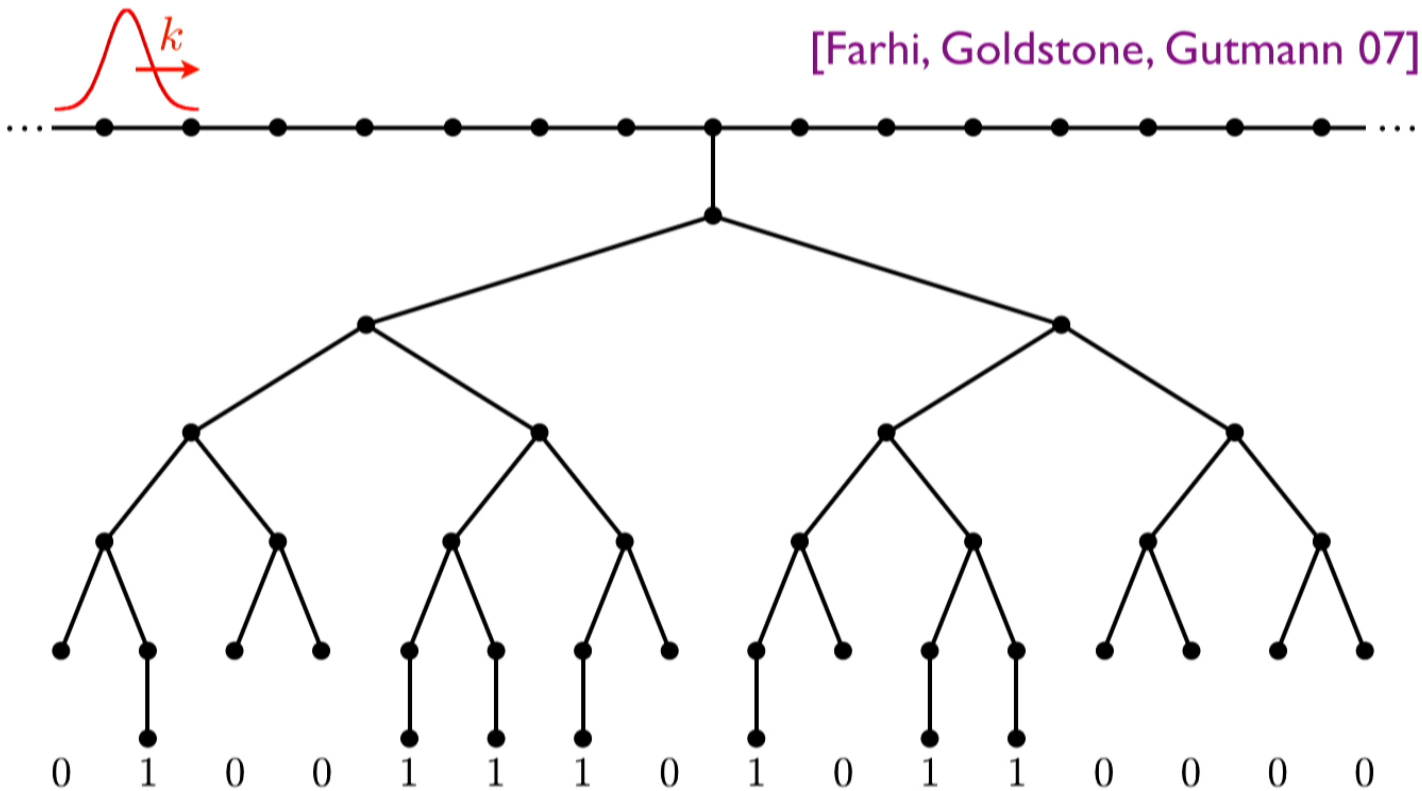


Classical complexity [Snir 85; Saks, Wigderson 86; Santha 95]: $\Theta(n^{0.753})$

Quantum lower bound [Barnum, Saks 02]: $\Omega(\sqrt{n})$
(holds for arbitrary AND-OR formulas)

Formula evaluation by scattering

[Farhi, Goldstone, Gutmann 07]



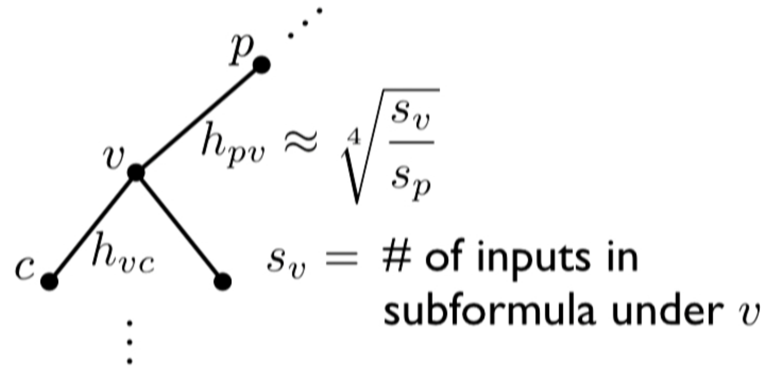
Claim: For small k, the wave is transmitted if the formula (translated into NAND gates) evaluates to 0, and reflected if it evaluates to 1.

General formulas

This simple strategy does not work for general formulas.

To get a general algorithm:

- Rewrite the formula to be "approximately balanced"
- Assign weights to the edges of the tree
- Show that eigenvectors are related to the function value

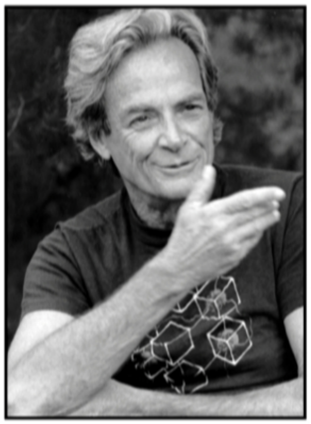


Lemma: If the formula evaluates to 0, then the tree has an eigenstate with eigenvalue 0 that has constant overlap on the root. If the formula evaluates to 1, then all eigenstates with eigenvalue $O(1/\sqrt{n})$ have no overlap on the root.

The quantum query complexity of evaluating any AND-OR formula is $O(n^{\frac{1}{2}+\epsilon})$ (subsequently improved to $O(\sqrt{n})$ [Reichardt 10])

Quantum simulation

- Childs, Communications in Mathematical Physics **294**, 581–603 (2010)
- Berry and Childs, Quantum Information and Computation **12**, 29–62 (2012)



"… nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."

Richard Feynman
Simulating physics with computers (1981)

Quantum dynamics

The dynamics of a quantum system are determined by its *Hamiltonian*.

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\,\psi(t)$$

Quantum simulation problem: Given a description of the Hamiltonian H, an evolution time t, and an initial state $\psi(0)$, produce the final state $\psi(t)$ (to within some error tolerance)

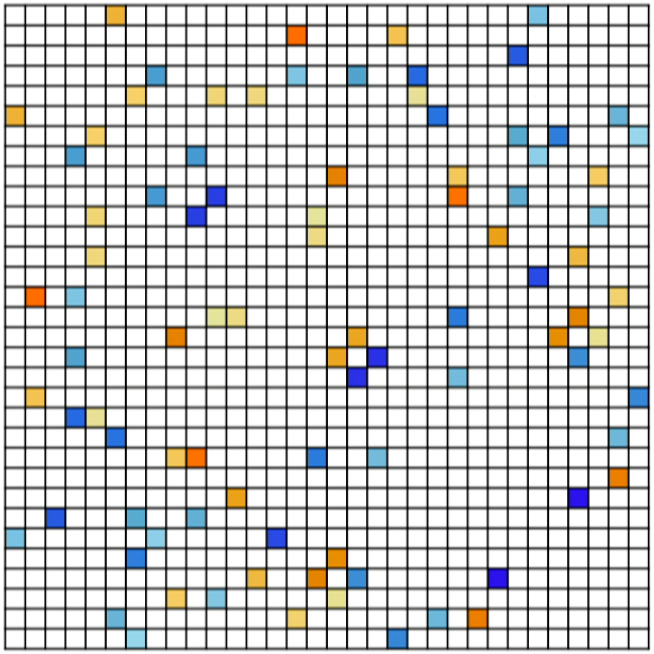
A classical computer cannot even represent the state efficiently

By performing measurements on the final state, a quantum computer can efficiently answer questions that (apparently) a classical computer cannot

Sparse Hamiltonians

At most d nonzero entries per row (here $d = 4$)

$$H =$$



Sparse Hamiltonians

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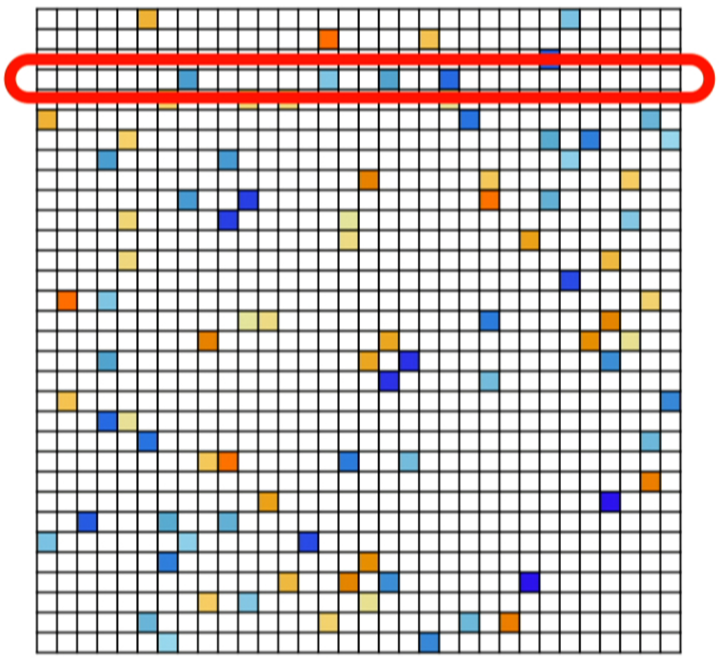
$$H =$$ 

Assumption: we can efficiently compute locations and values of nonzero entries in any given row

Simulation via product formulas

Original approach to sparse Hamiltonian simulation:

- Decompose $H = \sum_j H_j$ where each H_j is 1-sparse (distributed edge coloring)
- Recombine terms (product formulas, e.g., $e^{-i(A+B)t} \approx (e^{-iAt/r}e^{-iBt/r})^r$)

Running time of the best approach of this kind:

- Superlinear in evolution time t
- Cubic in sparsity d

[AT 03, CCDFGS 03, BACS 07, CK 10]

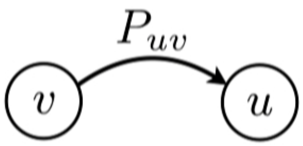
Discrete-time quantum walk

Can we define a quantum walk that takes discrete steps?

In general, locality and unitarity are incompatible

Workaround: define a walk on the *directed edges* (a "coined walk")

Szegedy 05: For a stochastic transition matrix P,



- Reflect about $\mathrm{span}\{\psi_v : v \in V\}$

  where $(\psi_v)_{(w,u)} = \begin{cases} \sqrt{P_{uv}} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$

- Swap the edge direction

Hamiltonian simulation by quantum walk

1. Define an analog of Szegedy's walk for any Hamiltonian H (in place of the stochastic matrix P)
2. Show how to perform steps of this walk using queries to the sparse Hamiltonian
3. Relate the spectrum of the walk to the spectrum of H
4. Infer information about the spectrum of the walk (and hence of H) using quantum phase estimation
5. Introduce the appropriate phase $e^{-i\phi t}$ for each eigenstate of H with eigenvalue ϕ

Theorem: This running time of this approach is $O(dt)$.

This algorithm is optimal with respect to either d or t alone

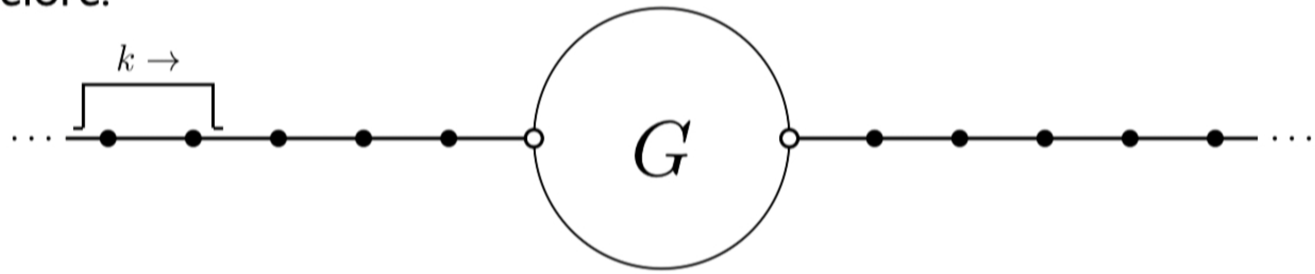
Universal computation

- Childs, Physical Review Letters **102**, 180501 (2009)
- Childs, Gosset, and Webb, Science **339**, 791–794 (2013)

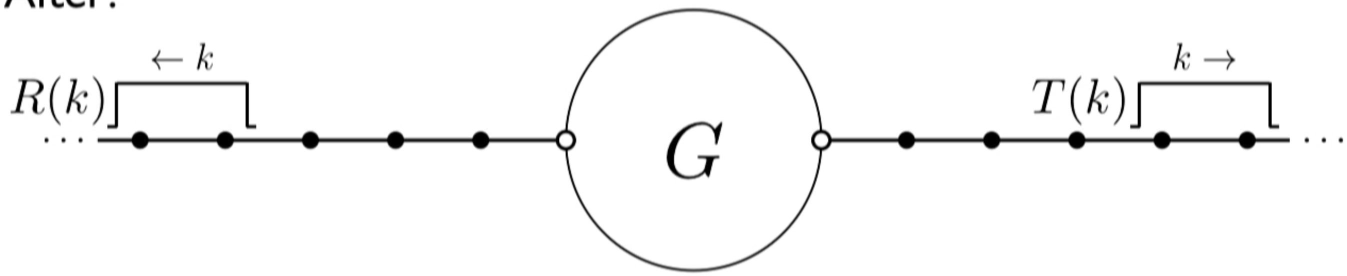
Scattering on graphs

Attach semi-infinite paths to two vertices of an arbitrary finite graph.

Before:

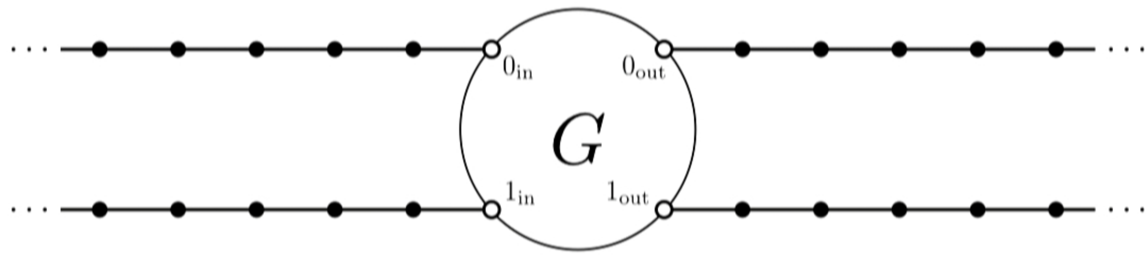


After:



More generally, attach any number of semi-infinite paths. The scattering behavior is described a unitary matrix called the *S-matrix*.

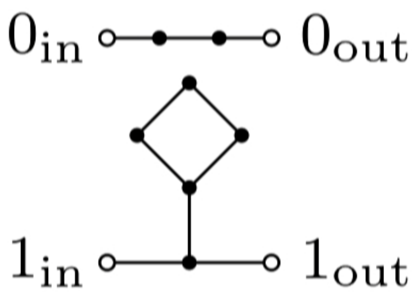
Implementing a gate

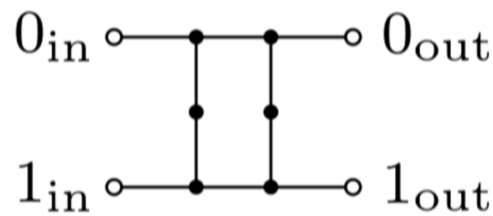


To perform a gate, design a graph whose S-matrix implements the desired transformation U at the momentum used for the encoding.

$$S(k) = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$$

Universal set of single-qubit gates





$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix} \qquad -\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

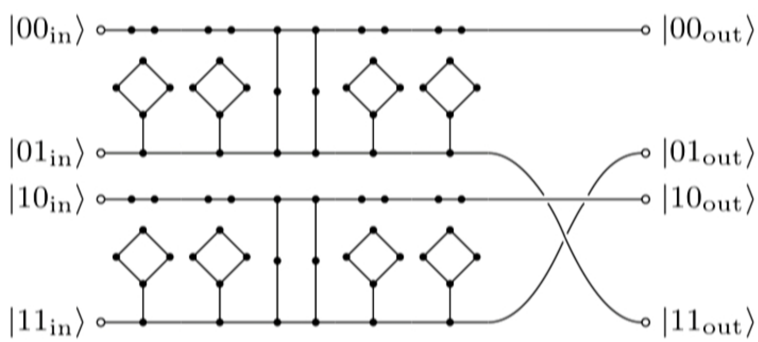
momentum for logical states: $k = \pi/4$

Universality construction

With an appropriate encoding of n-qubit states, two-qubit gates are trivial.

Implement sequences of gates by concatenation.

Result: Any n-qubit circuit can be simulated by some graph.



Quantum walks with many walkers

Consider a quantum walk with many walkers that interact locally

With m walkers on an n-vertex graph, there are n^m states

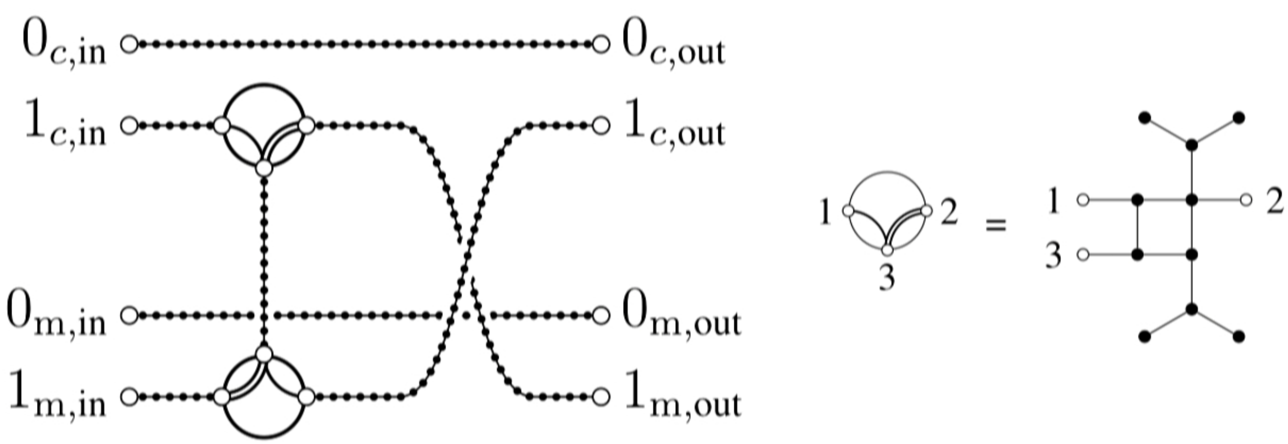
Theorem: Any n-qubit, g-gate quantum circuit can be simulated by a quantum walk with $n+1$ walkers interacting for time $\mathrm{poly}(n,g)$ on an unweighted planar graph with $\mathrm{poly}(n,g)$ vertices.

Consequences:

- Quantum walks with many interacting walkers (on small graphs) are also computationally powerful
- New architecture for a quantum computer (with no time-dependent control)
- Simulating the dynamics of interacting many-body systems is BQP-hard (e.g., the "Bose-Hubbard model" on a sparse, unweighted, planar graph)

Universal computation with many walkers

Main new idea: a gadget that implements a two-qubit interaction via momentum-dependent routing

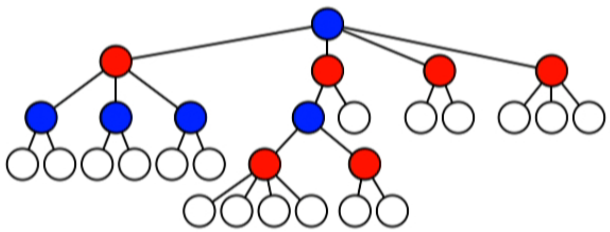


(+ extensive analysis to show the strategy works on a finite graph with small error)

Summary

Quantum walk is a powerful algorithmic tool.

Formula evaluation



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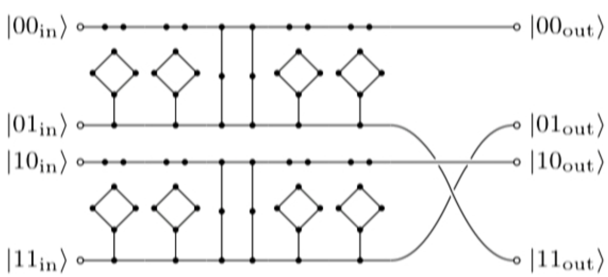
Universal computation



Outlook

When will we have large-scale quantum computers?

"Prediction is very difficult, especially about the future." –Niels Bohr

We can (and should!) address many crucial questions now:

- How can we design cryptosystems that resist quantum attacks?
- How efficiently can quantum computers simulate quantum systems?
- What other problems have fast quantum algorithms?
- What other tools are useful for building quantum algorithms?
- What problems are hard even for quantum computers?