Title: Quantum Gravity, Trees, and Polynomials

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Abstract: The perturbative series of colored group field theory are governed by a combinatorial 1/N-expansion. Controlling its coefficients is essential in order to understand the continuum limit. I will show how such a program is naturally related to higher-dimensional generalizations of trees in a colored Boulatov-Ooguri model, and present some partial results on the enumeration of such strucures in melonic graphs.

Quantum Gravity, Trees, and Polynomials

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Institut Camille Jordan (Lyon, France)

Perimeter Institute. April 3, 2014

Outline

1. Coloured GFT
2. Cellular Trees
3. An Enumeration Strategy

Group Field Theory

$\phi : G^{\times D} \to \mathbb{C}$, G (compact) Lie group

$$S_D[\phi] = \frac{1}{2}\int_{G^D} (\prod_{i=1}^{D} dg_i d\tilde{g}_i)\phi(g_1,\dots,g_D)C^{-1}(g_i\tilde{g}_i^{-1})\phi(\tilde{g}_1,\dots,\tilde{g}_D)$$

$$+\frac{\lambda}{(D+1)!}\int_{G^{D(D+1)}} (\prod_{i\neq j=1}^{D+1} g_{ij})\phi(g_{1j})\cdots\phi(g_{(D+1)j})K(g_{ij}g_{ji}^{-1}).$$

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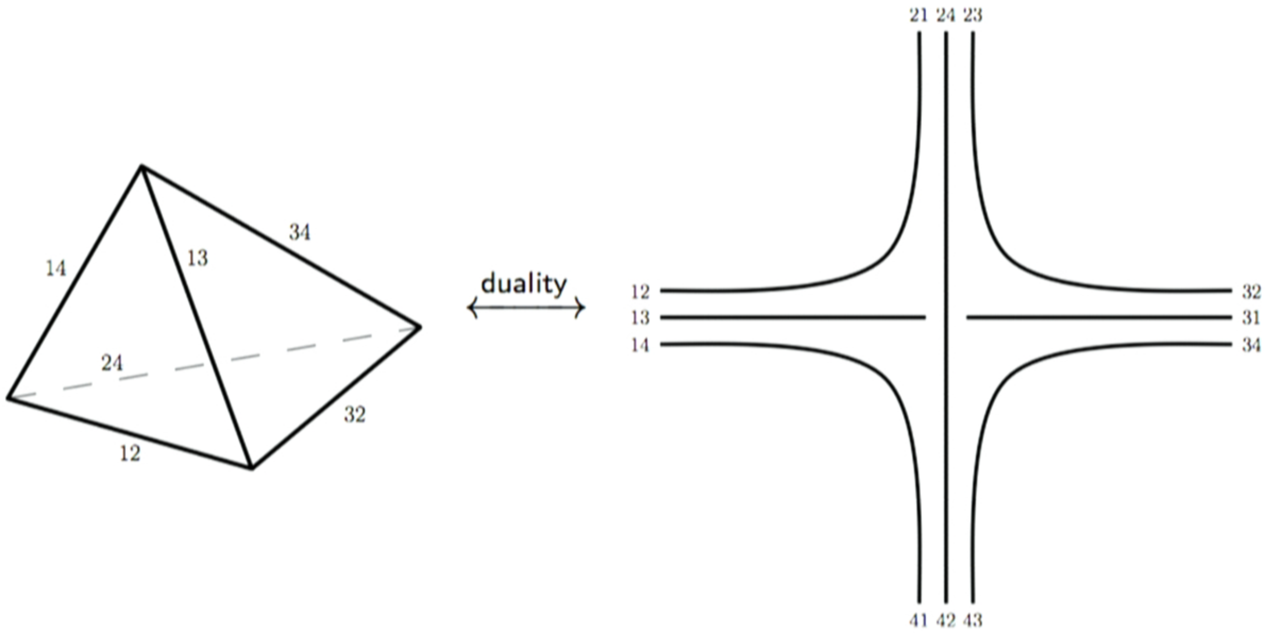
- $\phi \longrightarrow (D-1)$-simplex ($D=3$: a solid triangle)
- $g_i \longrightarrow (D-2)$-simplex ($D=3$: an edge)
- C "glues" two $(D-1)$-simplices ($D=3$: 2 triangles)
- $K \longrightarrow$ gluing of $D+1$ $(D-1)$-simplices to form a D-simplex ($D=3$: 4 triangles bound a tetrahedron)

Stranded graphs = gluing of simplices

$D = 3$



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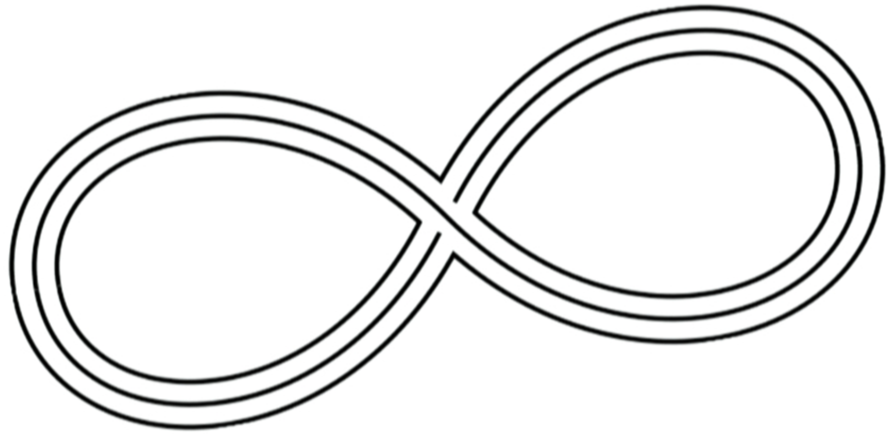

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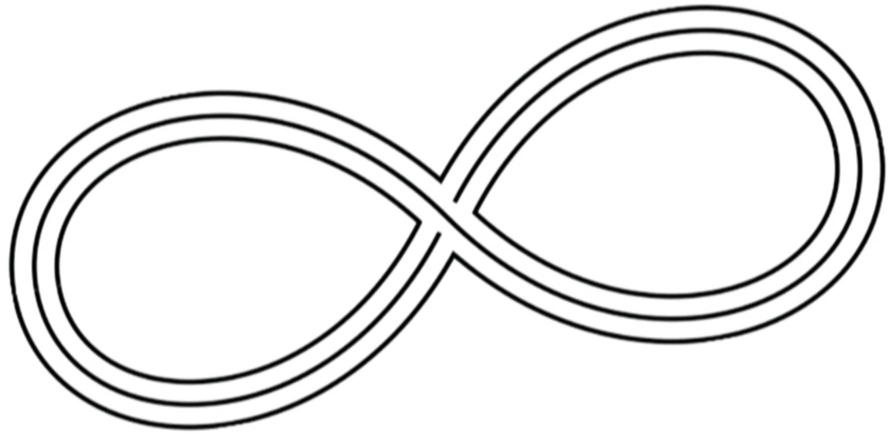

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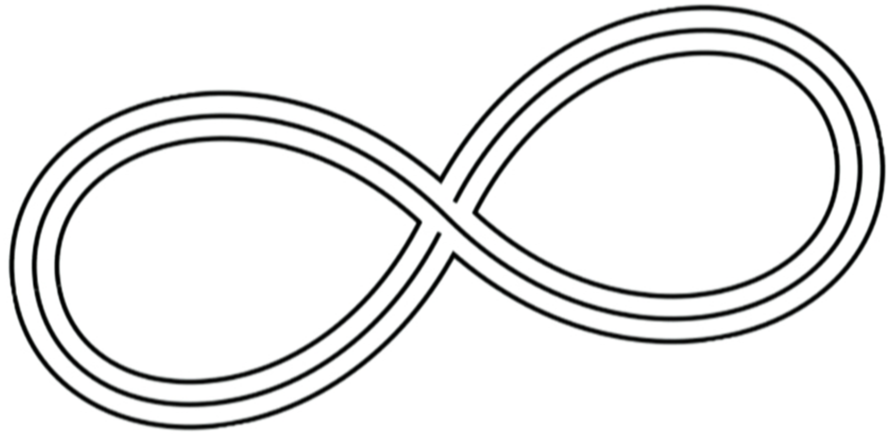
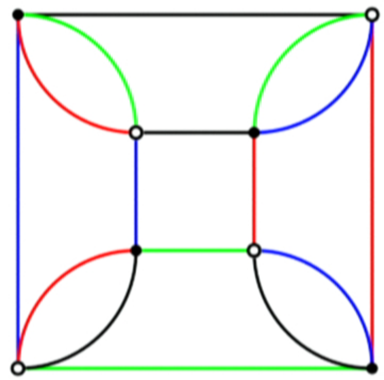


But not a *D*-complex!

Coloured Graphs

Definition

A k-coloured graph is a bipartite graph endowed with a proper edge-colouring with k colours.



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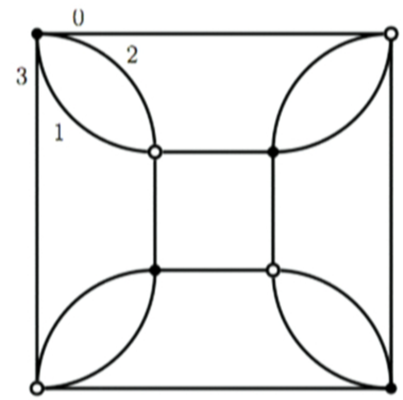
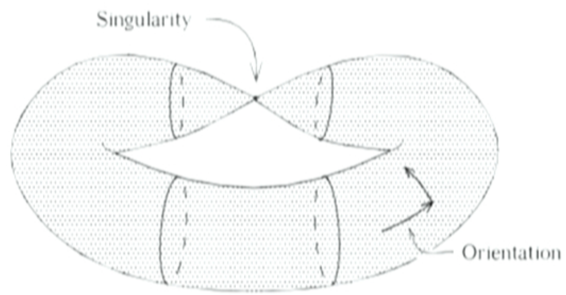


- Every bipartite regular graph is colourable.

Pseudo-manifolds

These are manifolds with singularities.



A pseudomanifold (the pinched torus)

A normal pseudo-manifold is such that the boundary of the neighbourhood of each of its points is a pseudo-manifold.

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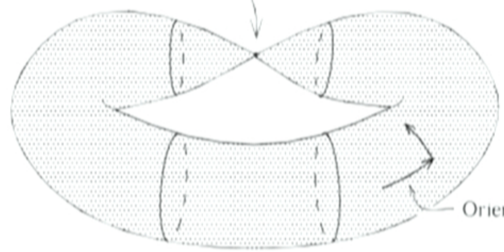

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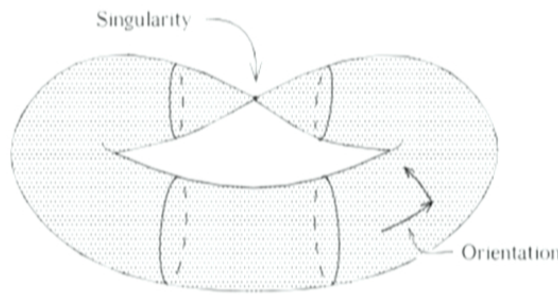

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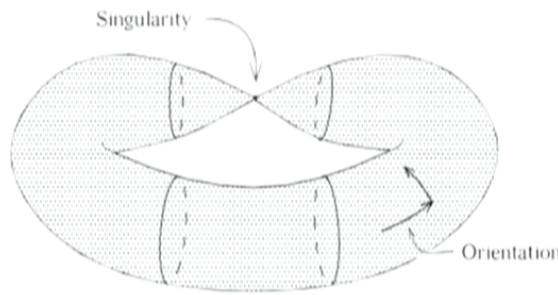

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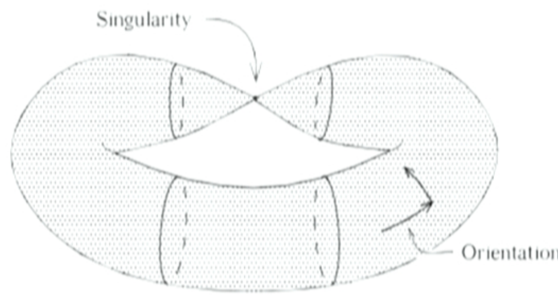

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- GEMs: a combinatorial and algorithmic approach to the classification of 3-manifolds [Ferri, Gagliardi, Lins etc '80].

Coloured Cellular Complex

Definition (Bubbles)

Let Γ *be a* $(D+1)$*-coloured graph and* $0 \leqslant k \leqslant D$*. A* k*-bubble of colours* $\{i_1, \dots, i_k\}$ *is a connected component of the subgraph of* Γ *induced by the edges of colours* $\{i_1, \dots, i_k\}$*. 0-bubbles are vertices.*

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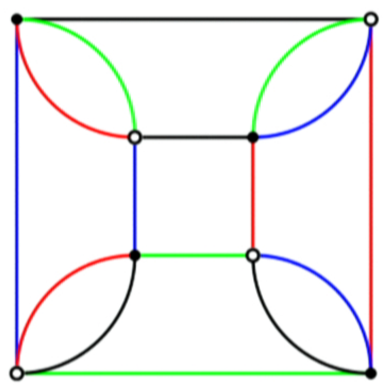

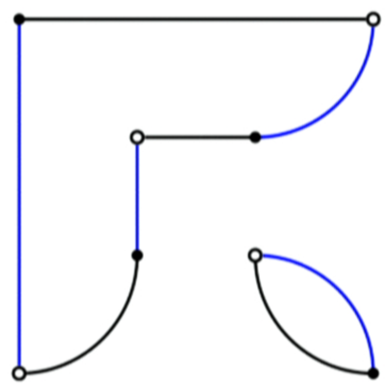

The 2-bubbles (or *faces*) of colors {blue, black}

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Definition (Cellular Complex)

For all $0 \leqslant k \leqslant D$, $C_k(\Gamma) :=$ free Abelian group generated by the k-bubbles.
$C_{D+1}(\Gamma) := \{0\}$ *and* $C_{-1}(\Gamma) := \mathbb{Z}$.
$\partial_k : C_k \to C_{k-1}$, $\partial_k b_{i_0,\dots,i_{k-1}} := \sum_{j=0}^{k-1} (-1)^j b'_{i_0,\dots,\hat{i}_j,\dots,i_{k-1}}$.
$\partial_{k-1}\partial_k = 0$.

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The colored Boulatov-Ooguri GFT

Mimicking 3-dimensional gravity

$\varphi_c : G^{\times D} \to \mathbb{C}$, $\quad c \in \{0, 1, \ldots, D\}$, G a compact Lie group
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Feynman graphs: edges bear D strands, bipartite, $(D+1)$-regular, proper edge-colouring.

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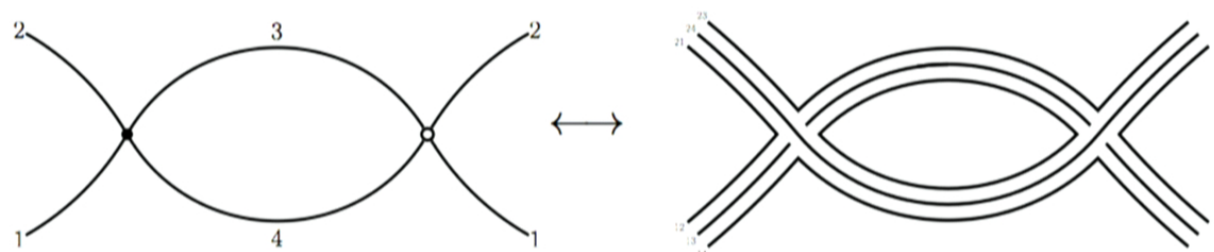
In bijection with $(D+1)$-coloured graphs.



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Feynman Amplitudes

- Closure constraint: $P(g_1, \ldots, g_D; g'_1, \ldots, g'_D) = \int_G dh \prod_i \delta(hg_i {g'}_i^{-1})$.
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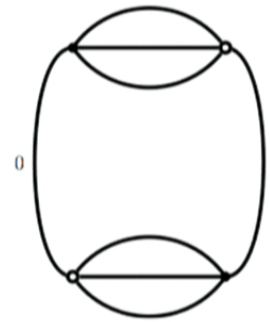
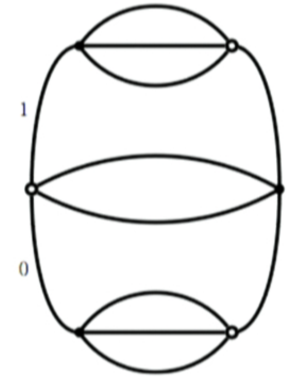
Melons

- They are coloured graphs dual to very specific triangulations of the sphere.
- They enjoy a recursive structure: "melons within melons".

A basic 0-melon



A melonic graph in M_2



Another one in M_3

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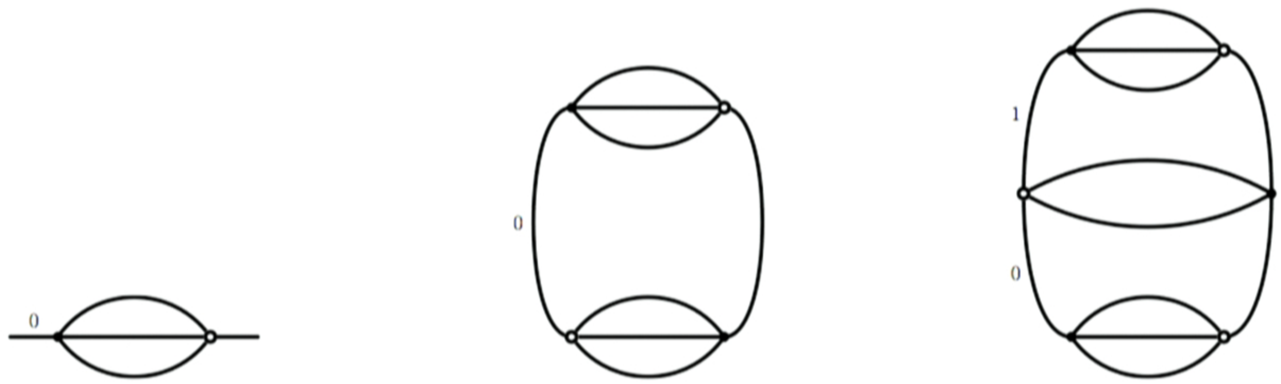
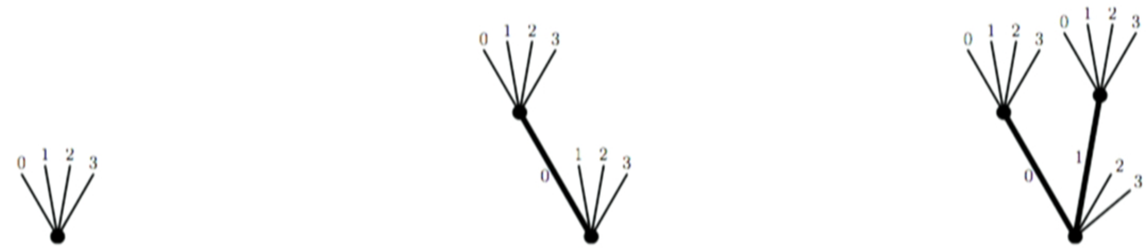


A basic 0-melon — A melonic graph in M_2 — Another one in M_3

- In bijection with colored $(D+1)$-ary trees.



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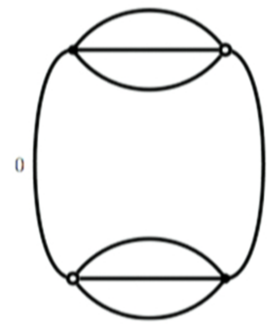

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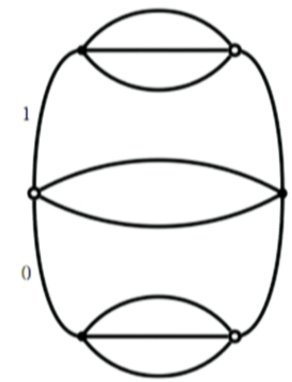

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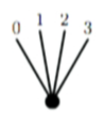
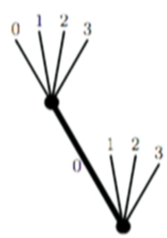
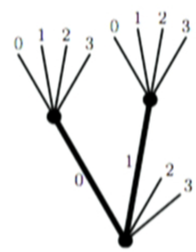






The colored Boulatov-Ooguri GFT

Feynman Amplitudes

- Closure constraint: $P(g_1, \ldots, g_D; g'_1, \ldots, g'_D) = \int_G dh \prod_i \delta(hg_i {g'}_i^{-1})$.
- Regularization: $\delta(g) = \lim_{\tau \to 0} K_\tau(g)$ (heat kernel).
- Convolution: $\int dg\, K_\tau(hg) K_\tau(g^{-1}h') = K_{2\tau}(hh')$.
- $N_\tau := \tau^{-1/2}$.

$$A_\tau(\mathcal{G}) = N_\tau^k \int \prod_{e \in \mathcal{E}} dh_e \prod_{f \in \mathcal{F}} K_{m_f \tau}\Big(\overrightarrow{\prod_{e \in \partial f}} h_e^{\epsilon_{fe}} \Big)$$

- $G^{|\mathcal{V}|}$-symmetry: $h_e \to k_{s(e)} h_e k_{t(e)}^{-1}$. Fix it ($h_{e \in \mathcal{T}} = \mathbb{1}$).
- Small τ expansion of K_τ.

$$F_{\tau, \lambda\bar{\lambda}} \underset{\tau \to 0}{\sim} N_\tau^{(\dim G)(D-1)} F^{(0)}_{\lambda\bar{\lambda}}, \qquad F^{(0)}_{\lambda\bar{\lambda}} = \sum_{p \in \mathbb{N}} \frac{(\lambda\bar{\lambda})^p}{p} \sum_{\mathcal{G} \in \boldsymbol{M}_p} a(\mathcal{G}),$$

$$a(\mathcal{G}) = \Big(\mathbf{det}(\tilde{\boldsymbol{L}}) \prod_f m_f\Big)^{-\frac{\dim G}{2}}, \qquad L_{ee'} = \sum_f \frac{1}{m_f} \epsilon_{ef} \epsilon^T_{fe'}.$$

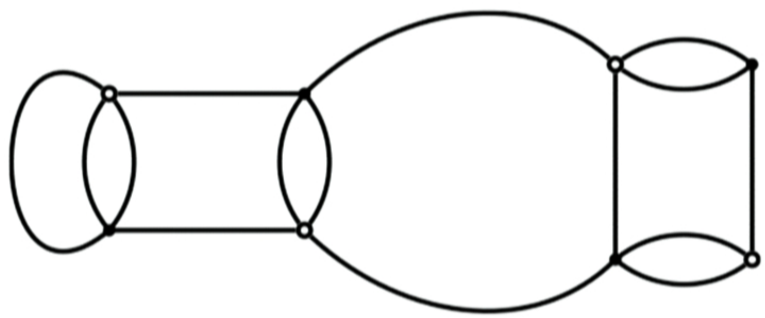
From Graph Theory to Algebraic Topology

Homological Definition of (1-)trees

Definition

Let G be a graph. A spanning (1-)tree of G is a subgraph g of G such that:

1. $V(g) = V(G)$ *(spanning),*
2. g *is acyclic,*
3. g *is connected,*
4. $|E(g)| = |V(g)| - 1$.



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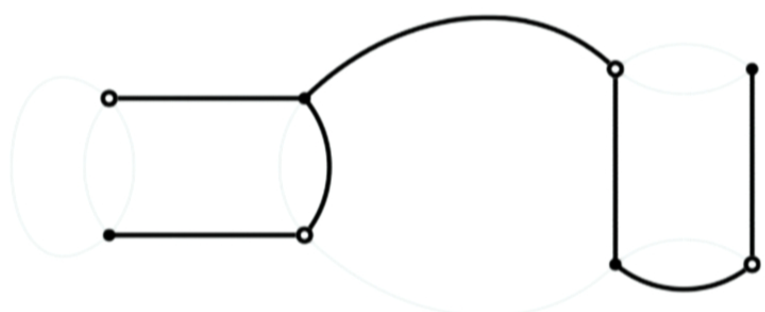

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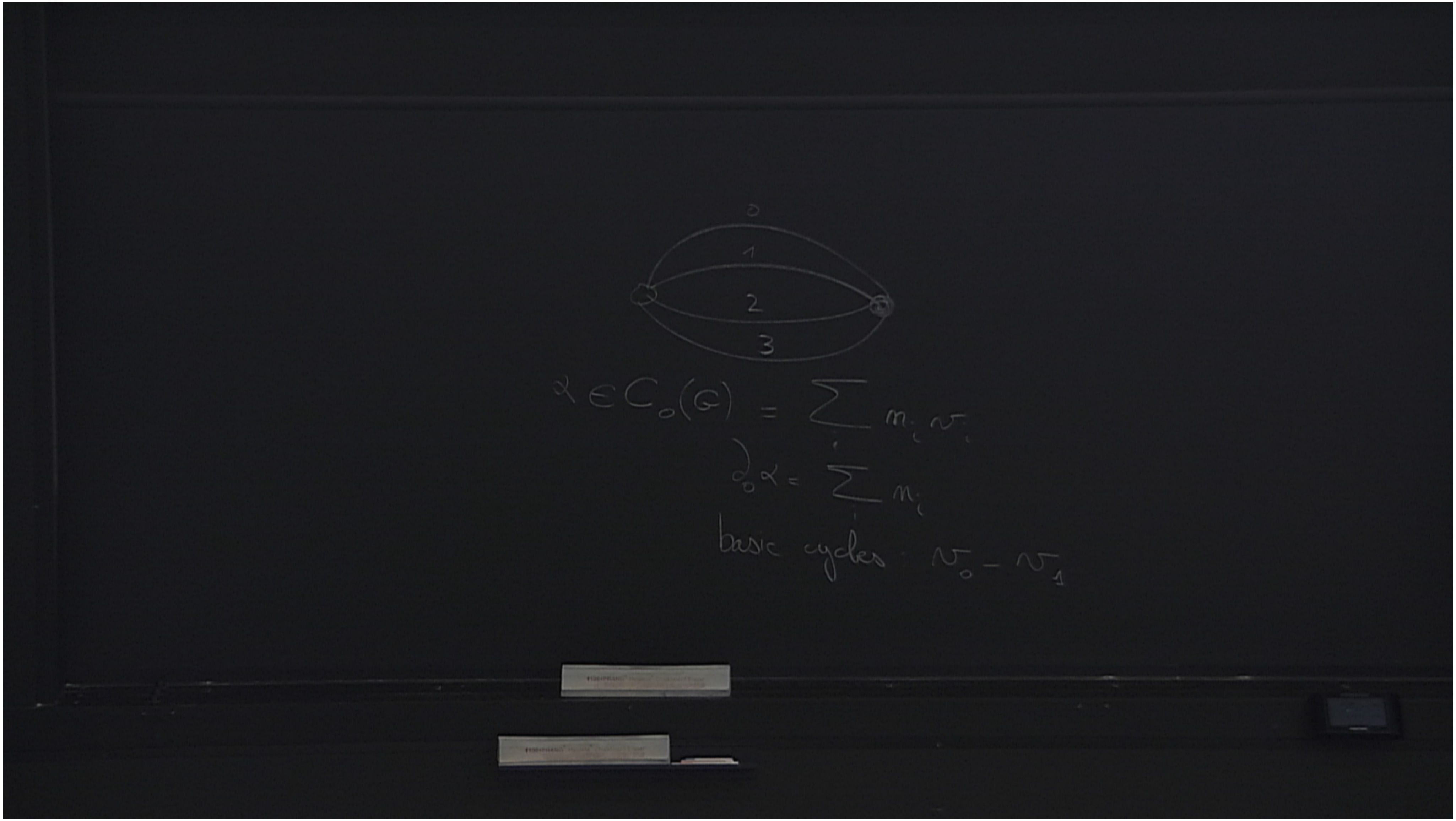
Any two of the conditions $2, 3$ and 4 imply the third one.

- Consider G as a 1-dim. cell complex Δ: 0-cells = vertices, 1-cells = edges.
- A *spanning* subgraph is a subcomplex δ such that $\delta_{(0)} = \Delta_{(0)}$.
- Choose an orientation of δ to get a chain complex:

$$0 \xrightarrow{0} C_1(\delta) \xrightarrow{\partial_1} C_0(\delta) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{0} \{0\}.$$


0
1
2
3
$\alpha \in C_0(G) = \sum_i n_i v_i$
$\partial_0 \alpha = \sum_i n_i$
basic cycles: $v_0 - v_1$

Cellular Trees

Definition

Definition (Duval, Klivans, Martin '09)

Let Δ be an n-dimensional cell complex. A spanning k-tree of Δ ($k \leqslant n$) is a subcomplex δ of $\Delta_{(k)}$ such that:

1. $\delta_{(k-1)} = \Delta_{(k-1)}$, *spanning*
2. $\widetilde{H}_k(\delta) = \{0\}$, *acyclic*
3. $\widetilde{\beta}_{k-1}(\delta) = 0$, *connected*
4. $f_k(\delta) = f_k(\Delta) - \widetilde{\beta}_k(\Delta_{(k)}) + \widetilde{\beta}_{k-1}(\Delta_{(k)})$.

- Any two of the conditions 2, 3 and 4 imply the third one.
- A cell complex contains a k-tree iff $\widetilde{\beta}_{k-1}(\Delta) = 0$.

- Examples: a triangulation of $\mathbb{S}^2$ - one 2-cell is a 2-tree. $\mathbb{R}P^2$ is a 2-tree.

Matrix-Tree Theorem

Theorem ($D = 1$)

Let G be a graph and $\partial_1 : C_1(G) \to C_0(G)$ be its boundary operator. Then, for all $v \in V(G)$,

$$\det(\partial_1 \partial_1^T)_v = \#\{\text{spanning trees in } G\}\,.$$

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Theorem (Duval, Klivans, Martin '09)

Let Δ be a cell complex of dimension n and $k \leqslant n$. If $\widetilde{\beta}_{k-1}(\Delta) = \widetilde{\beta}_{k-2}(\Delta) = 0$, then

$$\det(\partial_k \partial_k^T)_U = \frac{|\widetilde{H}_{k-2}(U)|^2}{|\widetilde{H}_{k-2}(\Delta)|^2} \sum_{\delta \in \mathcal{T}_k(\Delta)} |\widetilde{H}_{k-1}(\delta)|^2$$

where U is any spanning $(k-1)$-tree of Δ.

An Enumeration Strategy

1. Coloured GFT
2. Cellular Trees
3. An Enumeration Strategy

The Tutte Polynomial

- Remark: $D = 3$, 2-trees in Δ in bijection with 1-trees in Δ^*.
- **Aim**: Counting spanning 1-trees in the dual complex of 3 dim. melons.

Definition ((Multivariate) Tutte Polynomial)

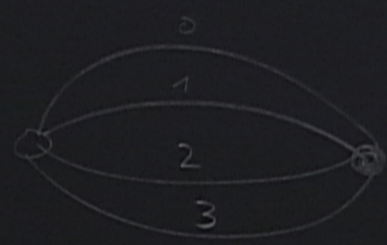
Let G be a graph, $c(G)$ its number of connected components.

$$Z_G(q;\mathbf{v}) \coloneqq \sum_{\text{spann. subgr. } g} q^{c(g)} \prod_{e\in E(g)} v_e.$$

$$X_G(\mathbf{v}) \coloneqq \lim_{\lambda\to 0} \lambda^{c(G)-v(G)} \lim_{q\to 0} q^{-c(G)} Z_G(q;\lambda\mathbf{v})$$

Proposition

If G is connected, $X_G(\mathbf{1}) = \#\{\text{spann. trees of } G\}$.



$$X_G(\underline{v}) = \sum_{\substack{\text{spann.}\\ \text{trees } g \subseteq G}} \prod_{e \in g} v_e$$

$$\alpha \in C_0(G) = \sum_i n_i v_i$$

$$\partial_0 \alpha = \sum_i n_i$$

basic cycles: $v_0 - v_1$

How to Count Trees in Melons?

Let $\mathcal{G}$ be a 3-dim. melon, Δ its associated cell complex.
We need to compute $X_{\Delta^{\star}_{(1)}}(\mathbf{1})$.

Use the recursive structure of melons!

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X_G obeys the following reduction relations

$$X_G = \begin{cases} v_e X_{G/e} & \text{if } e \text{ is a bridge,} \\ X_{G-e} & \text{if } e \text{ is a loop,} \\ v_e X_{G/e} + X_{G-e} & \text{if } e \text{ is ordinary,} \\ X_{G'}(v_e + v_{e'}) & \text{if } e \parallel e'. \end{cases}$$

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Contracting Melons

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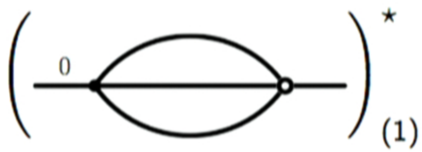
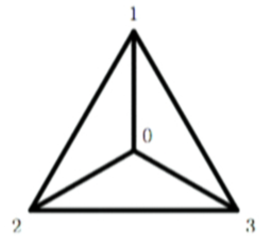
$=$ 3-ball made of the gluing of 2 tetrahedra along 3 facets.

- 

$=$ 

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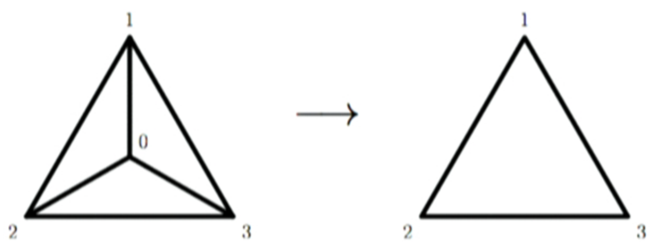

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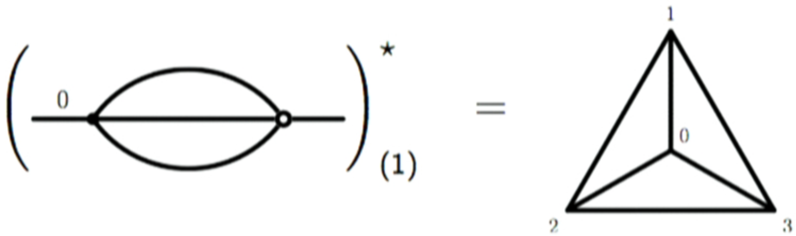
- 

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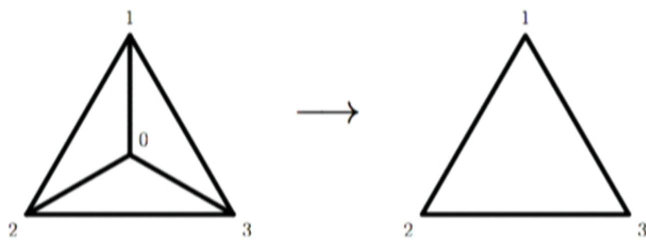

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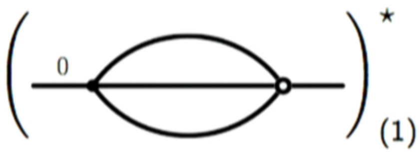
- 

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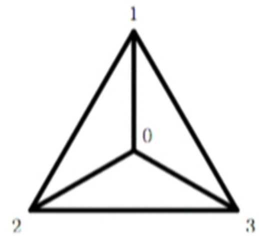

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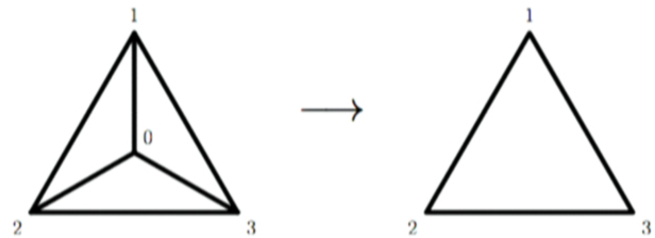

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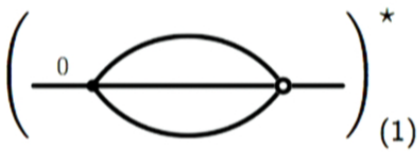
- 

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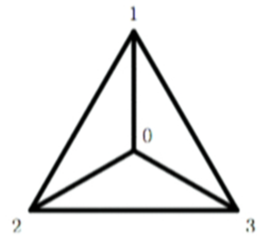

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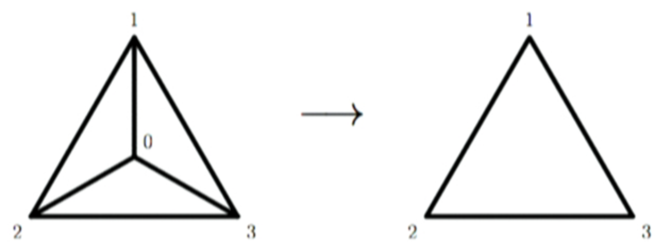

A Recursion

Proposition

$$X_{\Delta^\star_{(1)}}(v_{01}, v_{02}, v_{03}, v_{12}, v_{13}, v_{23}, \dots) = v_{01} X_{(\Delta/M_0)^\star_{(1)}}(v_{12}+v_{02}, v_{13}+v_{03}, v_{23}, \dots)$$
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This is *not* a recursion for the number of spanning trees!

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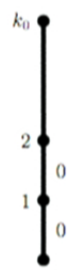
The number of 2-trees in  is: $3^{k_0}(k_0 + 4)^2$.

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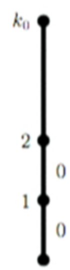
The number of 2-trees in 

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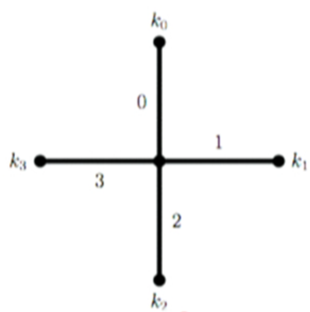
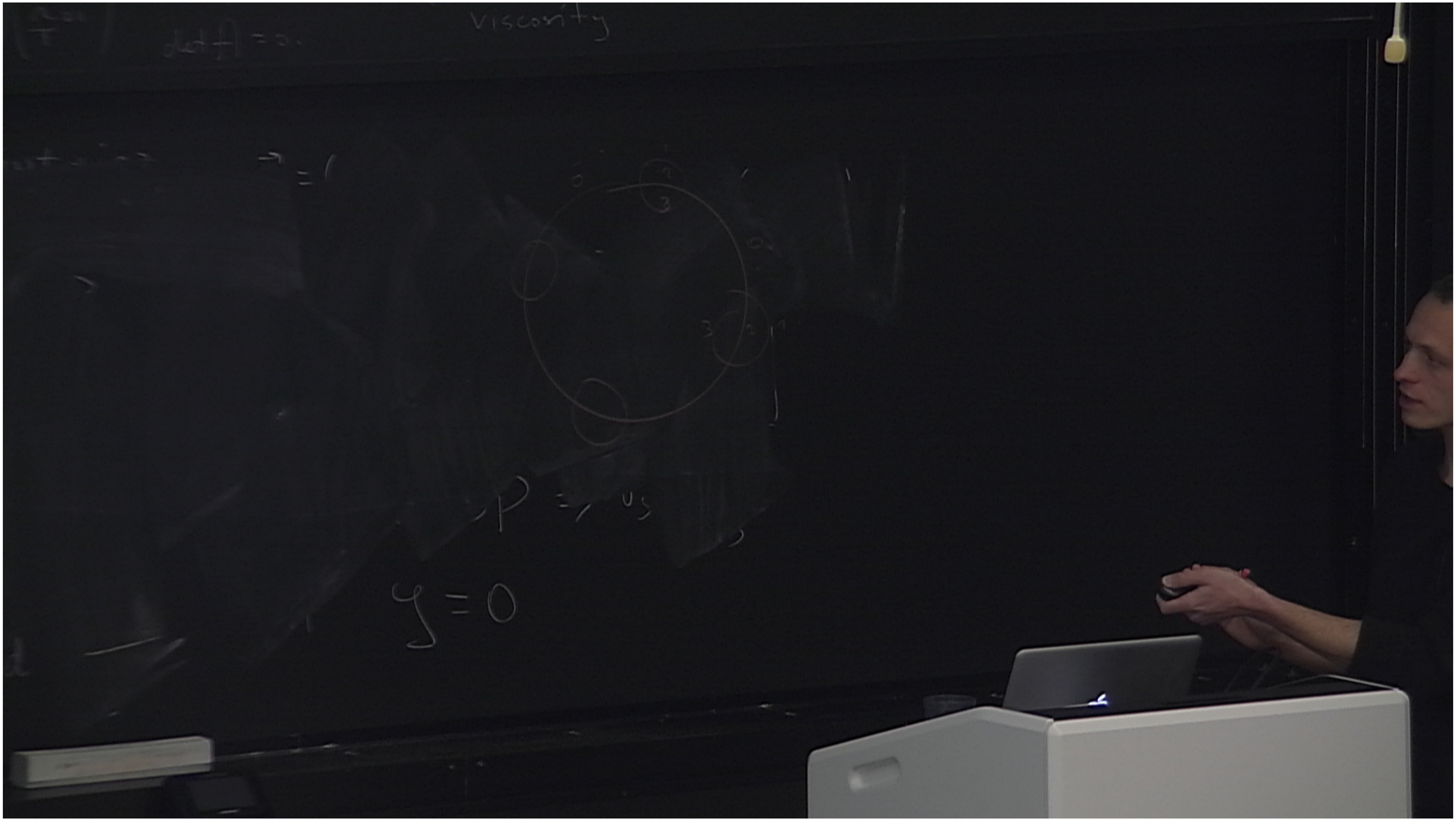
The number of 2-trees in  is:

$$3^{\sigma_1-3}(6\sigma_1\sigma_2 - 4\sigma_3 + 27\sigma_1^2 + 24\sigma_2 + 216\sigma_1 + 432),$$

with $\sigma_1 = \sum_{i=0}^{3} k_i$, $\sigma_2 = \sum_{i<j} k_i k_j$, $\sigma_3 = \sum_{i<j<l} k_i k_j k_l$.


viscosity
y = 0

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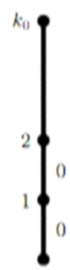
The number of 2-trees in 

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Summary and perspectives

- k-trees are k-dim. generalizations of trees.
- Such trees appear in the Feynman amplitudes of GFT.
- Their enumeration would help to characterize the critical UV behaviour of the model.

A lot remains to be done:

- Complete enumeration,
- Counting k-trees in dimension D (via the Tutte-Krushkal polynomial?),
- What about torsionful trees? Enumeration, characterization.
- After the melons, the cherry trees?