Title: Between Matrices and Tensors

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Abstract: Quartic tensor models can be rewritten in terms of intermediate matrix fields. The corresponding expansion is not only simpler, it suggests also new bridges between matrices, strings and tensors.

Between Matrices and Tensors

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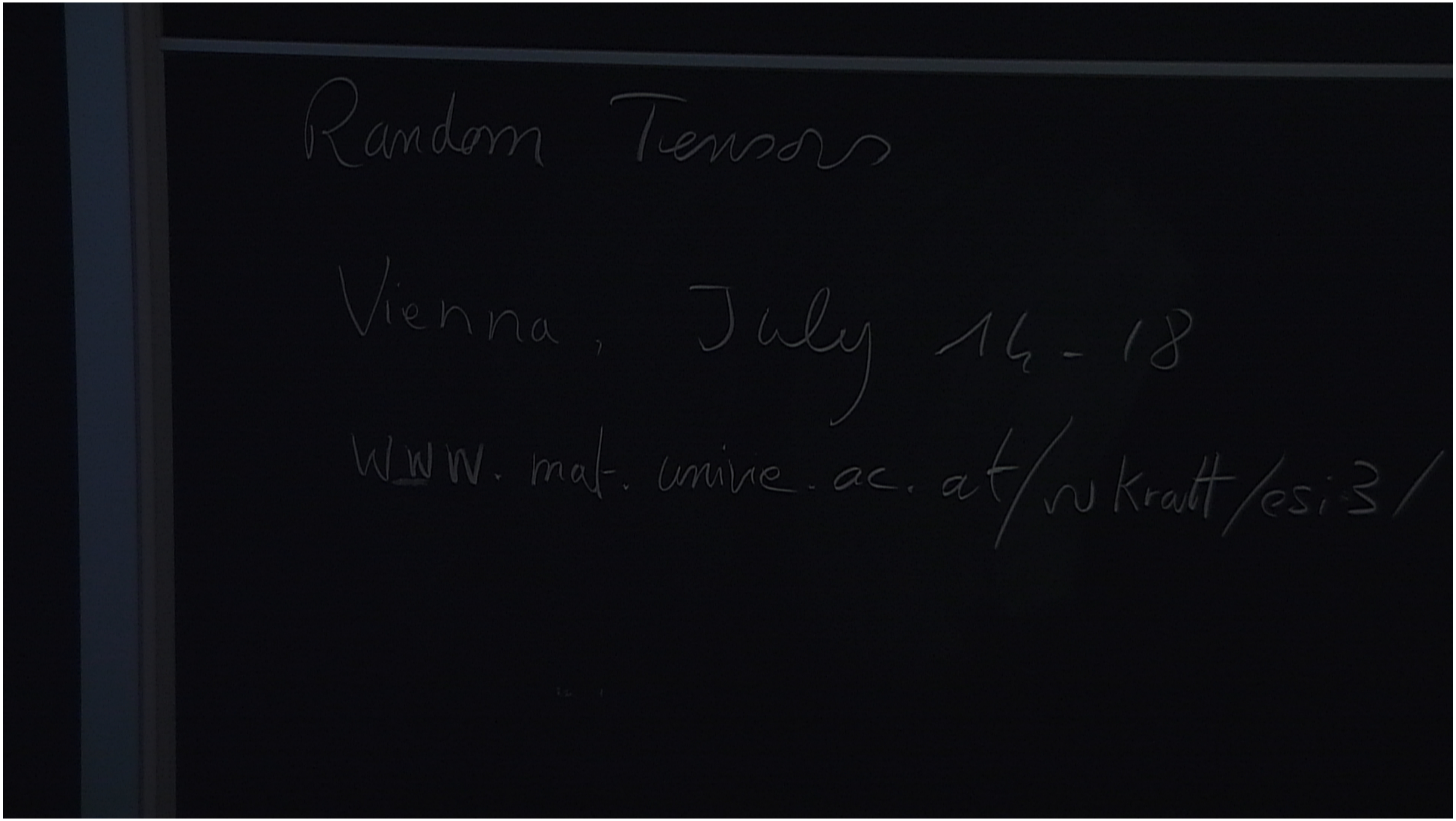
Renormalization Group Approaches To Quantum Gravity
Perimeter Institute, April 25 2014


Random Tensors
Vienna, July 14-18
www.mat.univie.ac.at/~kratt/esi3/

Tensor Track

Treasure to search: fixed points with a finite number of marginal and relevant directions. But in which theory space?

Tensor Track = candidate for a combinatoric, discretized "theory space" S.

- inspired by the success of matrix models
- background independent
- sums over all topologies
- represents easily branched polymers and Brownian spheres (2d gravity)
- include (in the field theory version) both AS and AF fixed points

Locality as a Symmetry

- no background space $\rightarrow$ simple algebraic setting.
- Many-component field $(\bar{\phi}, \phi) \in C^N$,
- Replaces locality by invariance under change of basis $(GL(N)$ or $U(N))$.
- S_0 = Local terms = space of connected invariants which are polynomial in the components of $(\bar{\phi}, \phi)$.
- Breaking this locality at propagator level defines tensor (group) field theories, with quasi local observables S_i.

From Vectors to Tensors

What distinguishes vector models from matrix models and tensor models?

An N component vector field ϕ has a single connected polynomial invariant, namely the scalar product $s =< \bar{\phi}.\phi >$. $S_0(vectors)$ is then restricted to polynomials in the single variable s.

Suppose $N = N_1 N_2$. Matrix models means symmetry breaking from $U(N_1 N_2)$ to $U(N_1) \otimes U(N_2)$, hence they have more invariants.

Tensor models simply means further breaking of the symmetry:
$N = N_1 N_2 N_3 \cdots N_D$, symmetry broken to $U(N_1) \otimes U(N_2) \otimes \cdots \otimes U(N_D)$, hence still more invariants.

Counting Tensor Invariants

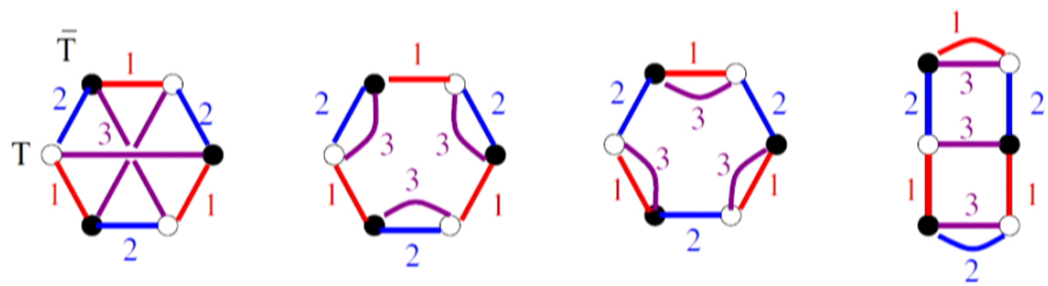
Rank D invariants = regular D-edge-colored connected bipartite graphs can be counted (J. Ben Geloun and S. Ramgoolam)

$$Z_1^c(n) = 1, 0, 0, 0, 0, \dots \qquad \bar{\Phi} \cdot \Phi$$

$$Z_2^c(n) = 1, 1, 1, 1, 1, 1, 1 \dots \qquad \mathrm{Tr}(MM^\dagger)^p$$

$$Z_3^c(n) = 1, 3, 7, 26, 97, 624, 4163 \dots$$

$$Z_4^c(n) = 1, 7, 41, 604, 13753 \dots$$



Solving Vector Models

Simplest interacting complex ϕ^4 vector model: conjugate vector fields $(\phi = \{\phi_i\}, \bar{\phi} = \{\bar{\phi}_i\}, i = 1, \cdots, N$, $(\lambda/2N)(\bar{\phi}\cdot\phi)^2$ interaction.

$$Z(\lambda, N) = \frac{1}{(2i\pi)^N}\int e^{-(\bar{\phi}\cdot\phi)-(\lambda/2N)(\bar{\phi}\cdot\phi)^2} d\bar{\phi} d\phi$$

$N \to \infty$ regime? Use intermediate field σ:

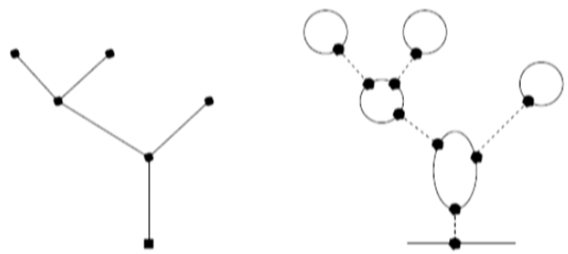
$$\begin{aligned} Z(\lambda, N) &= \frac{1}{(2i\pi)^N}\int d\sigma \frac{e^{-\sigma^2/2}}{\sqrt{2\pi}} \int e^{-(\bar{\phi}\cdot\phi)+i\sqrt{\lambda/N}(\bar{\phi}\cdot\phi)\sigma} d\bar{\phi} d\phi \\ &= \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\sigma^2/2 - N\log(1-i\sqrt{\lambda/N}\sigma)} \end{aligned}$$

Saddle point at $\tau_c = \frac{1}{2\sqrt{z}}[1-\sqrt{1-4z}]$, where $z = \sqrt{\lambda}$.

Remarks

- The two point function is easiest to compute:

$$G_2^{\infty} = \frac{1}{2z}[1 - \sqrt{1-4z}] = T(z)$$



- Generating function for rooted plane trees, counted by Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$).
- The random geometry is that of branched polymers.

Planar Graphs and Matrix Models

Simplest interacting complex matrix model:

$$Z = \int dM d\bar{M} \, \exp(-\frac{1}{2}\mathrm{Tr} M^t \bar{M} + \frac{\lambda}{N}\mathrm{Tr} M^t \bar{M} M^t \bar{M}) = \sum_{n,g} a_{n,g} \lambda^n N^{2-2g}$$

$$2 - 2g = V - L + F = -V + F \quad , \quad \text{since} \quad L = 2V.$$

lead by planar graphs as $N \to \infty$ ('t Hooft, 1974).

Single and double scalings for matrix models

- Exact counting of planar graphs is possible (Tutte, 1962)

$$Q_n = 3^n \frac{2}{n+2} C_n \tag{1}$$

where Q_n is the number of rooted planar quadrangulations):

- Single Scaling $N \to \infty$-limit at fixed λ. Only spheres survive:

$$G_{2,planar}(\lambda) = \frac{-1 - 36\lambda + (1+24\lambda)^{3/2}}{216\lambda^2}.$$

- Random Geometry is the Brownian sphere (Le Gall, Miermont...)
- intermediate field method leads to N-uniform Borel summability
- interesting double scaling

$2d$ pure gravity

- $2d$ pure gravity (string theory on a point) can be probed through an invariant matrix model, which infinitely many invariant interactions $t_k \mathrm{Tr} M^k$
- or through a non-invariant model with simple interaction $\mathrm{Tr} M^3$ but non invariant propagators (Kontsevich model)

$$\log Z \simeq \log \int dM e^{-(1/2)\mathrm{Tr} M^2 \Lambda + (i/6)\mathrm{Tr} M^3} \qquad (2)$$

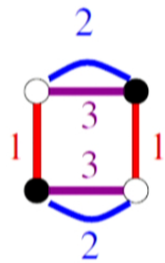
Grosse-Wulkenhaar

The Grosse-Wulkenhaar model is a Kontsevich-type model with $\Lambda_{ij} = 1 + i_1 + i_2 + j_1 + j_2$ and $\mathrm{Tr} M^4$ interaction.

- it is perturbatively renormalizable, asymptotically safe
- it has a $4d$ geometric interpretation
- the mechanism behind the ultraviolet fixed point is analytically well understood (SD + WI + LS)
- full Euclidean 4d invariance is restored in the $\theta \to \infty$ "Infra red limit".
- there is numerical evidence that it is unitary in a certain range of parameters.

The Quartic Melonic (Rank 3) Tensor Model

It is the simplest of all tensor models. There are three quartic invariants at rank 3



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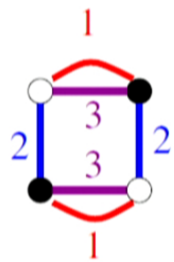

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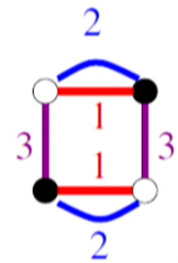

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The model has partition function

$$Z = \int dT d\bar{T} e^{-\frac{1}{2}\bar{T}.T-(\lambda/2N^2)\sum_{i=1}^{3}\mathcal{B}_i(\bar{T},T)}$$

where $\mathcal{B}_i = \mathrm{Tr} M_i^2$, the M's being the partial traces

$$[M_1]_{jk} := \sum_{l,m=1}^{N} T_{jlm}\bar{T}_{klm}, \quad [M_2]_{jk} := \sum_{l,m=1}^{N} T_{ljm}\bar{T}_{lkm}, \quad [M_3]_{jk} := \sum_{l,m=1}^{N} T_{lmj}\bar{T}_{lmk}$$

Intermediate Field Representation

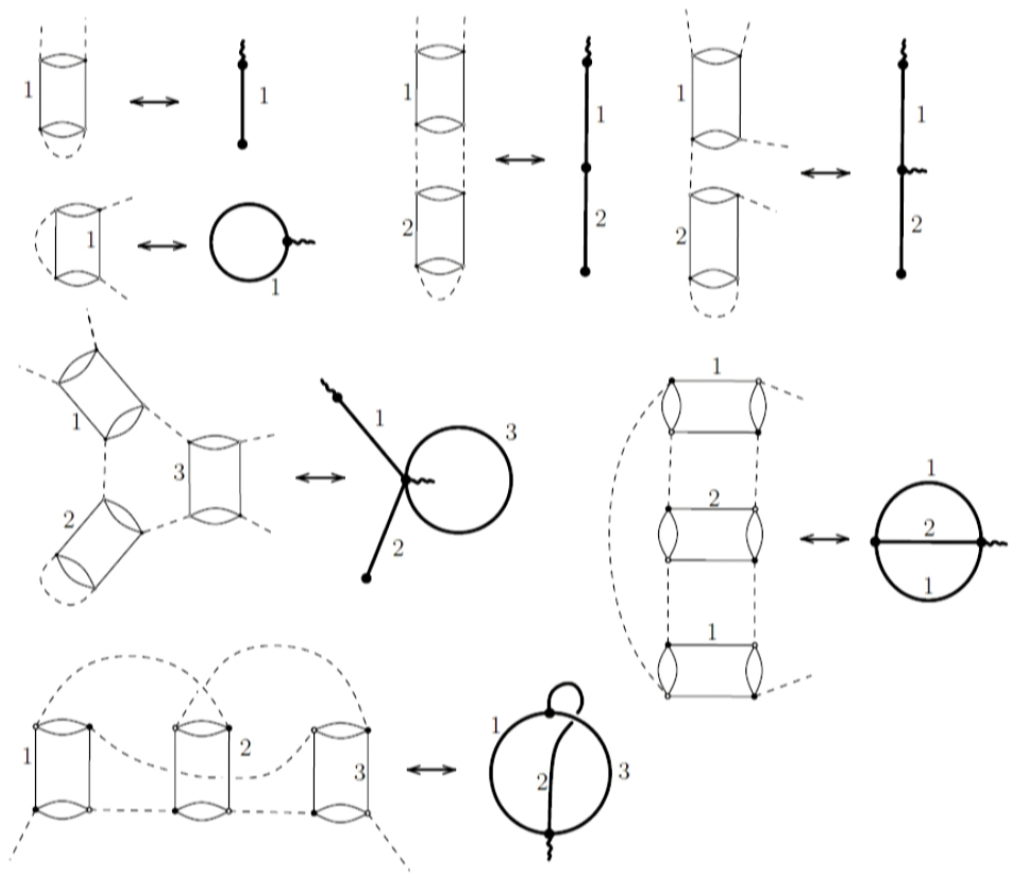
We can write easily an intermediate field representation with intermediate matrix fields σ_1, σ_2, σ_3, using

$$e^{-(\lambda/2N^2)\mathrm{Tr}M_j^2} = \int d\sigma_j e^{-\mathrm{Tr}\sigma_j^2 + i(\sqrt{\lambda}/N)\mathrm{Tr}\sigma_j}$$

$$Z = \int d\sigma_1 d\sigma_2 d\sigma_3 e^{-\sum_j \mathrm{Tr}\sigma_j^2 - \mathrm{Tr}^{\otimes} \log\left[1\otimes1\otimes1 + i(\sqrt{\lambda}/N)(\sigma_1\otimes1\otimes1 + 1\otimes\sigma_2\otimes1 + 1\otimes1\otimes\sigma_3)\right]}$$

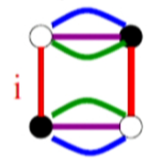
It simplifies the expansion, melonic graphs $\to$ trees. The melonic limit is a saddle point in the intermediate fields. This builds a bridge between tensor and matrix models, possibly extending towards string theory.

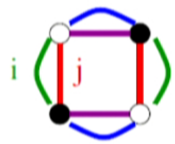
New (recolored..) Graphs



Quartic Tensor Model at rank 4

There are now 7 quartic invariants, 4 melonic $\mathcal{B}_i$ and 3 matrix-like $\mathcal{B}_{ij}$





- The method of intermediate fields still applies
- With standard scalings λN^{-3} in front of all $\mathcal{B}_i$ and $\mathcal{B}_{ij}$ we have the usual $1/N$ tensor expansion with melonic (branched polymer) $N \to \infty$ behavior.
- Scalings $\lambda_1 N^{-3} \sum \mathcal{B}_i + \lambda_2 N^{-2} \sum_{ij} \mathcal{B}_{ij}$ are more natural.
- With these scalings there is still $1/N$ expansion but with different $N \to \infty$ behavior; there is competition between branched polymers and Brownian sphere behavior, depending on the respective values of λ_1 and λ_2. At $\lambda_1 = K\lambda_2 = \lambda_c$ both singularities happen simultaneously.

Conclusion

- Tensor models include naturally background independence and topology change
- Anisotropic power counting seems easy to implement in the tensor formalism $\rightarrow$ HL, CDT
- they include analytically understood prototypes for AF and for AS fixed points with enhanced symmetry $\rightarrow$ Reuter et al.
- Joining efforts may lead to a more unified picture and hopefully to unexpected discoveries ...