Title: Renormalization group approach to 3d group field theory

Date: Apr 25, 2014 02:30 PM

URL: http://pirsa.org/14040112

Abstract: I will start with a brief overview of tensorial group field theories with gauge invariant condition and their relation to spin foam models. The rest of the talk will be focused on the SU(2) theory in dimension 3, which is related to Euclidean 3d quantum gravity and has been proven renormalizable up to order 6 interactions. General renormalization group flow equations will be introduced, allowing in particular to understand the behavior of the relevant couplings in the neighborhood of the Gaussian fixed point. I will close with preliminary investigations about the existence of a non-trivial fixed point.

Renormalization group approach to 3d group field theory

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TGFTs and renormalization

Reminder: **TGFT = Tensorial Group Field Theory**

- Razvan's talk:
  - tensorial interactions and combinatorial tools from Tensor Track (**TT**);
  - scales through gentle violation of tensor invariance in the kinetic term (**F**);
  - group structure and constraints implementing LQG/SF geometric data (**G**).

- Several renormalizable models on the market:
  - 4d combinatorial model on $\mathrm{U}(1)$, with Laplace operators in the covariance [Ben Geloun, Rivasseau '11] (TFT);
  - 3d combinatorial model on $\mathrm{U}(1)$, with first-order derivative operators in the covariance [Ben Geloun, Rivasseau '11] (TFT);
  - $\mathrm{U}(1)$ models with gauge invariance condition, and Laplace operators in the covariance [Oriti, Rivasseau, SC '12] [Ousmane Samary, Vignes-Tourneret '12] [Ousmane Samary '13] (**TGFT**);
  - 3d $\mathrm{SU}(2)$ model with gauge invariance condition, and Laplace operators in the covariance [Oriti, Rivasseau, SC '13] (**TGFT**);
  - and more, with various kinds of derivative operators in the propagators... [Ben Geloun, Livine '12] [Ben Geloun '13] (TFT)

Motivations

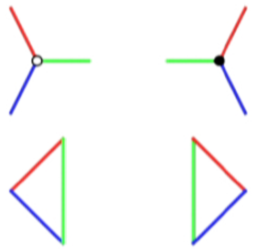
- Make GFTs well-defined as (perturbative) QFTs.
- Develop tools to deal with the dynamics of small chunks of space in an effective way.
- Later on: deal with very refined boundary states and find a continuous phase (Bianca's talk).
- Wilsonian RG flow perspective and look for new fixed points: condensed phase (Daniele's talk), resummation of infinite families of graphs (Razvan's talk).

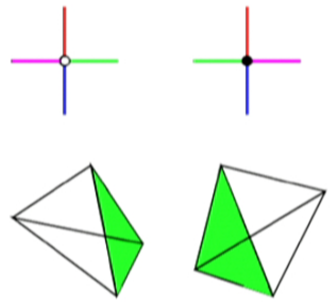
Outline

1. 3d euclidean quantum gravity TGFT model
2. Renormalization group flow equations
3. Properties of the Gaussian fixed point
4. TGFT in $D = 4 - \varepsilon$: towards a Wilson-Fisher fixed point?

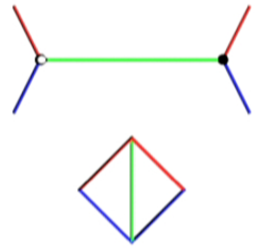
Colored graphs and triangulations

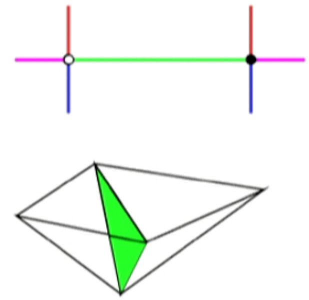
- Each node in a $(d+1)$-colored graph is dual to a d-**simplex**





- Each line represents the **gluing** of two d-simplices along their boundary $(d-1)$-**simplices**





$\Rightarrow$ **A $(d+1)$-colored graph represents a triangulation in dimension d.**

3d discrete gravity model

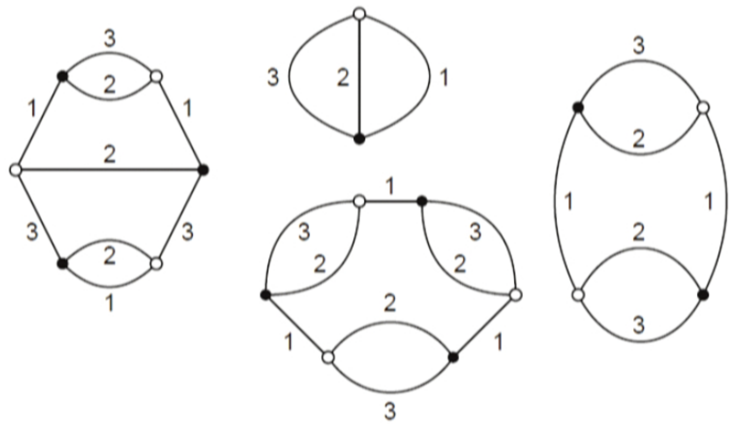
- Euclidean gravity in 3d $=$ BF theory
- To make sense of the formal quantum gravity path integral, one discretizes it on a cellular complex:

$$\mathcal{Z}_{\mathcal{M}} = \int [\mathcal{D}\omega]\delta(F(\omega)) \to \mathcal{Z}_{\triangle} = \int [\mathrm{d}h_l] \prod_{f \in \triangle^*} \delta\left(\overrightarrow{\prod_{l \in f}} h_l\right)$$

- Traditional construction: $\triangle$ = simplicial complex $\Rightarrow$ **Ponzano-Regge spin foam model**.
- Instead, one can use a **colored cellular complex**.
- Main advantages:
  - full **homology**;
  - good control over the **topology**;
  - tools from $1/N$ **expansion**.

3d discrete gravity model

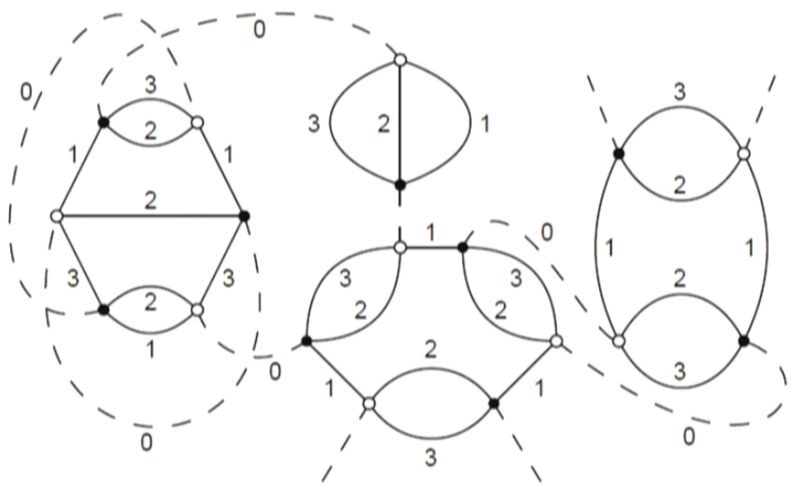
- Elementary building blocks = 3-**bubbles** = **3d cells with colored triangulated boundaries**...



- ...glued together along their boundary triangles.

3d discrete gravity model

- Elementary building blocks = 3-bubbles = 3d cells with colored triangulated boundaries...



- ...glued together along their boundary triangles.
- **Holonomy variables** associated to the dashed, color-0 lines.
- **Face of color** ℓ = connected set of (alternating) color-0 and color-ℓ lines.

Summing over graphs

- Two possible strategies to define the continuum limit: **refining** or **summing**.
- GFT strategy: summing using the QFT formalism.
- **Dynamical variable**: rank-3 complex field

$$\varphi : \mathrm{SU}(2)^3 \ni (g_1, g_2, g_3) \mapsto \mathbb{C}\,.$$

- **Partition function**:

$$\mathcal{Z} = \int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \mathrm{e}^{-S(\varphi,\overline{\varphi})}\,.$$

  - $S(\varphi, \overline{\varphi})$ is the interaction part of the action, and should be a sum of **connected tensor invariants**

$$S(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \overline{\varphi})$$

    which play the role of **local** terms.

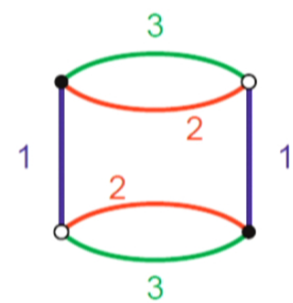
  - **Closure constraint** or **gauge invariance condition** imposed by the Gaussian measure $\mathrm{d}\mu_C$:

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int \mathrm{d}h \prod_{\ell=1}^{3} \delta\left(g_\ell h g_\ell'^{-1}\right)\,.$$

Bubbles and tensor invariants

Correspondence between colored graphs b and tensor invariants $I_b(\varphi, \overline{\varphi})$:

- white (resp. black) **node** $\leftrightarrow$ **field** (resp. complex conjugate field);
- **edge** of color ℓ $\leftrightarrow$ **convolution** of ℓ-th indices of φ and $\overline{\varphi}$.



$$\int [\mathrm{d}g_i]^6 \, \varphi(g_1, g_2, g_3)\overline{\varphi}(g_1, g_4, g_5)\varphi(g_6, g_4, g_5)\overline{\varphi}(g_6, g_2, g_3)$$

Divergences and scales

- The amplitudes are **generically divergent**.
- Two strategies to cure the divergences: $1/N$ **expansions** or **renormalization**.
- **Problem**: no notion of scale, the covariance is a projector (analogue of an ultralocal QFT).
- **Solution**: compose the original projector with a **non-trivial differential operator**. For instance

$$\left(m^2 - \sum_{\ell=1}^{d} \Delta_\ell\right)^{-1},$$

which is a conservative choice also suggested by study of radiative corrections [Ben Geloun, Bonzom '11].

This defines the new Gaussian measure $\mathrm{d}\mu_C$:

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int_0^{+\infty} \mathrm{d}\alpha\, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{3} K_\alpha(g_\ell h g_\ell'^{-1}),$$

where K_α is the **heat kernel** on $\mathrm{SU}(2)$ at time α.

Floating cut-off formalism

Slicing of scales: fix $M > 1$ and write

$$C = \int_0^{+\infty} \mathrm{d}\alpha \, \ldots = \sum_{i=0}^{+\infty} C_i \,, \qquad C_i = \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \ldots$$

$\Rightarrow$ **momentum slices** labeled by the integer $i \in \mathbb{N}$.

- Effective action:

$$S_i(\varphi, \overline{\varphi}) = \sum_b t_{b,i} \frac{I_b(\varphi, \overline{\varphi})}{k(b)}$$

- Discrete flow upon integration of the shell $\alpha \in \left[M^{-2i}, M^{-2(i-1)} \right]$:

$$\log \left(\int \mathrm{d}\mu_{C_i}(\varphi, \overline{\varphi}) \, \mathrm{e}^{-S_i(\Phi+\varphi, \overline{\Phi}+\overline{\varphi})} \right) \underset{+\,wave-function\,ren.}{\longrightarrow} S_{i-1}(\Phi, \overline{\Phi})$$

$$t_{b,i} \longrightarrow t_{b,i-1}$$

Power-counting and dimensionless couplings

- **Power-counting** in a slice:

$$|\mathcal{A}_{\mathcal{G}}| \leq K M^{\omega(\mathcal{G}) i},$$

where the degree of divergence ω is

$$\omega = 3 - \frac{N}{2} + \sum_{k \in \mathbb{N}} (3-k) n_{2k} - 3\rho \,.$$

N = number of external legs.
n_{2k} = number of vertices with valency $2k$.
$\rho = 0$ for a melonic graph, and $\rho \leq -1$ otherwise.

- In order to kill the n_{2k} dependence, we need to introduce **dimensionless coupling constants**

$$u_{b,i} \equiv t_{b,i} M^{-d_b i}, \qquad d_b = [t_b] \equiv 3 - \frac{N_b}{2},$$

where N_b is the valency of the bubble b.

Melonic flow equations

- Intermediate coupling constants defined by a sum over melonic graphs:

$$\tilde{u}_{b,i-1} M^{-d_b} = u_{b,i} - \sum_{\mathcal{G} \in \mathcal{M}(b)} \frac{k(b)}{k(\mathcal{G})} \left(\prod_{b'} (-u_{b',i})^{n_{b'}(\mathcal{G})} \right) a(u_{2,i}, \mathcal{G}) \,.$$

- Rescaling due to **wave-function renormalization**:

$$u_{b,i-1} \equiv \frac{\tilde{u}_{b,i}}{(1 + CT_{\varphi,i-1})^{N_b}} \,.$$

In this framework, **all melonic graphs** contribute to the flow equations.

Linearization

- **Linearized flow equations**:

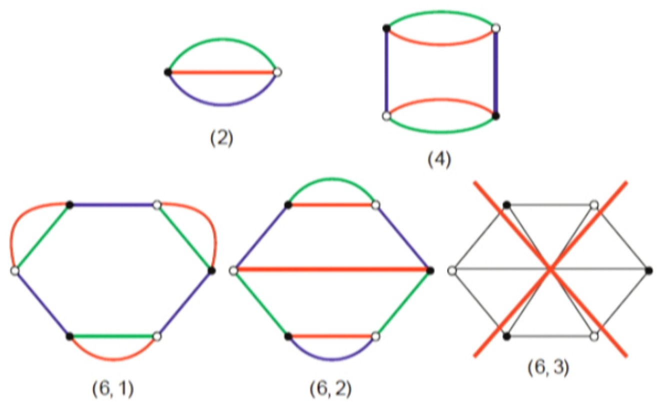
$$\forall b \in \mathcal{B}\,, \qquad u_{b,i-1} = M^{d_b} \left[u_{b,i} + \sum_{b' > b} \lambda(b, b') u_{b',i} \right] + \mathcal{O}(u^2)\,.$$

- Triangular system of equations, with diagonal $= (M^{d_b})_{b \in \mathcal{B}} \Rightarrow$ (formally) diagonalizable.

- Coupling constants with $N_b > 6$ only contribute to eigendirections with eigenavalues $M^{d_b} < 1$. They are therefore stable directions, or in other words irrelevant.

Linearization

- Bubbles with $N_b \leq 6$:

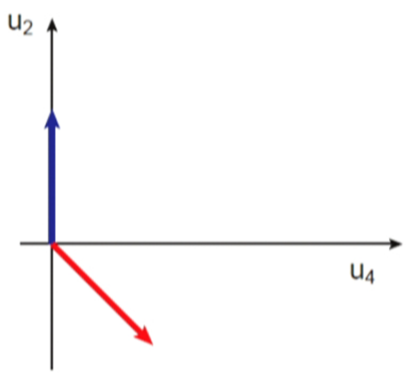


(6, 3) bubbles completely decouple and can also be ignored.
Imposing **color permutation invariance** of the action, we are left with 4 independent coupling constants: $u_{2,i}$, $u_{4,i}$, $u_{(6,1),i}$ and $u_{(6,2),i}$.

Eigendirections

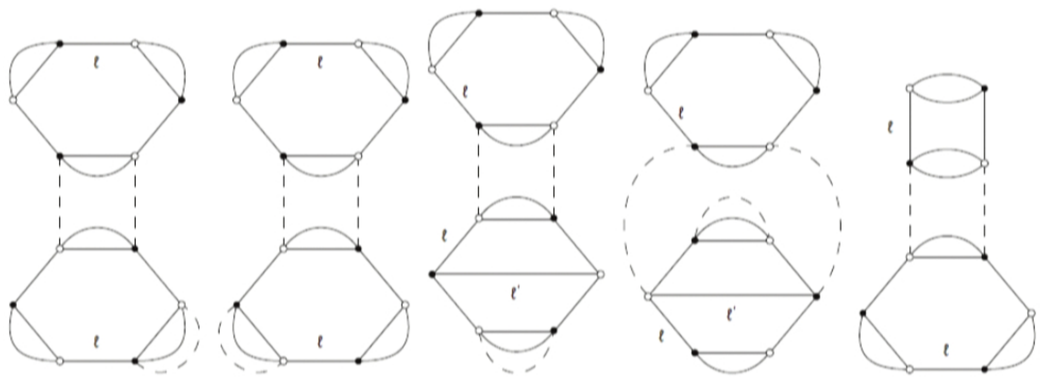
The **relevant eigendirections** in the (u_2, u_4)−plane can be computed:

$$\sigma_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad \sigma_4 = \begin{pmatrix} -6\sqrt{\pi} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

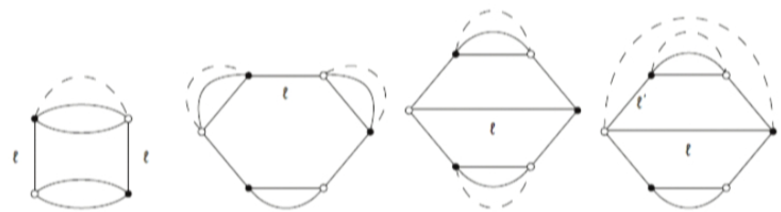


Marginal couplings

In order to determine the fate of φ^6 **couplings**, one needs to compute the second order.



Contrary to ordinary scalar field theories, the wave-function renormalization plays a crucial role at this order, with contributions from:



Marginal couplings

- One finds:

$$
\begin{aligned}
u_{6,1,i-1} \quad = \quad & u_{6,1,i} + 3\left(\mathbb{e} - \mathbb{c}\right) u_{6,1,i} u_{4,i} + 3\left(\mathbb{f} - 2\mathbb{a}\mathbb{c}\right) {u_{6,1,i}}^2 \\
& + 6\left(\mathbb{a}\mathbb{e} + \mathbb{g} - \mathbb{a}\mathbb{c} - \mathbb{d}\right) u_{6,1,i} u_{6,2,i} + \mathcal{O}(u^3)\,,
\end{aligned}
$$

  and

$$
\begin{aligned}
u_{6,2,i-1} \quad = \quad & u_{6,2,i} + \left(3\mathbb{e} - 2\mathbb{c}\right) u_{6,2,i} u_{4,i} + 2\left(3\mathbb{a}\mathbb{e} + 3\mathbb{g} - 2\mathbb{d}\right) {u_{6,2,i}}^2 \\
& + \left(3\mathbb{f} - 4\mathbb{a}\mathbb{c}\right) u_{6,1,i} u_{6,2,i} + \mathcal{O}(u^3)\,.
\end{aligned}
$$

- All the positive contributions are due to wave-function counter-terms.
- It turns out that:

$$
\mathbb{e} > \mathbb{c}\,, \qquad \mathbb{f} > 2\mathbb{a}\mathbb{c}\,, \qquad \mathbb{g} > \mathbb{d}\,.
$$

Conclusion

The φ^6 couplings are marginally relevant $\Rightarrow$ Asymptotic Freedom.

TGFT in $D = 4 - \varepsilon$: towards a Wilson-Fisher fixed point?

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Changing the group or the rank

It is possible to consider similar gauge invariant models with different groups and tensor ranks. Necessary conditions for their renormalizability were derived, in terms of the rank d, the dimension of the group D, and the maximal valency of the relevant bubbles v_{max}:

Type	d	D	v_{max}	ω
A	3	3	6	$3 - N/2 - 2n_2 - n_4 + 3\rho$
B	3	4	4	$4 - N - 2n_2 + 4\rho$
C	4	2	4	$4 - N - 2n_2 + 2\rho$
D	5	1	6	$3 - N/2 - 2n_2 - n_4 + \rho$
E	6	1	4	$4 - N - 2n_2 + \rho$

[Oriti, Rivasseau, SC '13]

- $d = D = 3$ is the only case for which the combinatorial dimension can match the dimension of space-time inferred from the symmetry group G.
- Analogy with ordinary scalar field theory: at fixed $d = 3$
  - φ^6 model in $D = 3$;
  - φ^4 model in $D = 4$.

TGFT in $D = 4 - \varepsilon$

- One way of analytically continue the group dimension D

$$\mathrm{SU}(2) \mapsto \mathrm{SU}(2) \times \mathrm{U}(1)^{1-\varepsilon}$$

- By analogy with scalar field theory in dimension $4 - \varepsilon$, there should be a new non-trivial **Wilson-Fisher fixed point**:

$$u_2^* \sim a\,\varepsilon + \mathcal{O}(\varepsilon^2)\,, \qquad u_4^* \sim b\,\varepsilon + \mathcal{O}(\varepsilon^2)\,.$$

- **Question**: what are the properties of this putative fixed point?
  - eigendirections?
  - scaling dimensions?

A non-trivial fixed point in 3d quantum gravity?

- Does this Wilson-Fisher fixed point survive in the limit $\varepsilon \to 1$?
- How to study this question?
  - Resummation of the ε-expansion;
  - FRGE approach: Tim's talk [Eichhorn, Koslowski '13]
- Possible interesting consequences:
  - Flow towards massive theory in the infrared, where $\sum_\ell \Delta_\ell$ term is effectively suppressed: this would provide a dynamical realization of the initial cut-off model.
  - All the melonic coupling constants might be switch-on in a controlled way at the new fixed point. If yes, can we resum the melons (Razvan's talk)? Perturbations around this new vacuum?

Summary

There is a TGFT realization of the spin foam Ponzano-Regge model for **3d quantum gravity**.

- It is **perturbatively renormalizable** (at all orders in perturbation theory).
- It is **asymptotically free**.
- Preliminary calculations suggest the existence of a **non-trivial UV fixed point**.

More generally: Renormalization in GFT is imposing itself as a central tool, which hopefully will help addressing key difficulties, especially in four dimensions: **continuum limit**, **universality** etc.