Title: Renormalization of entanglement entropy and the gravitational effective action

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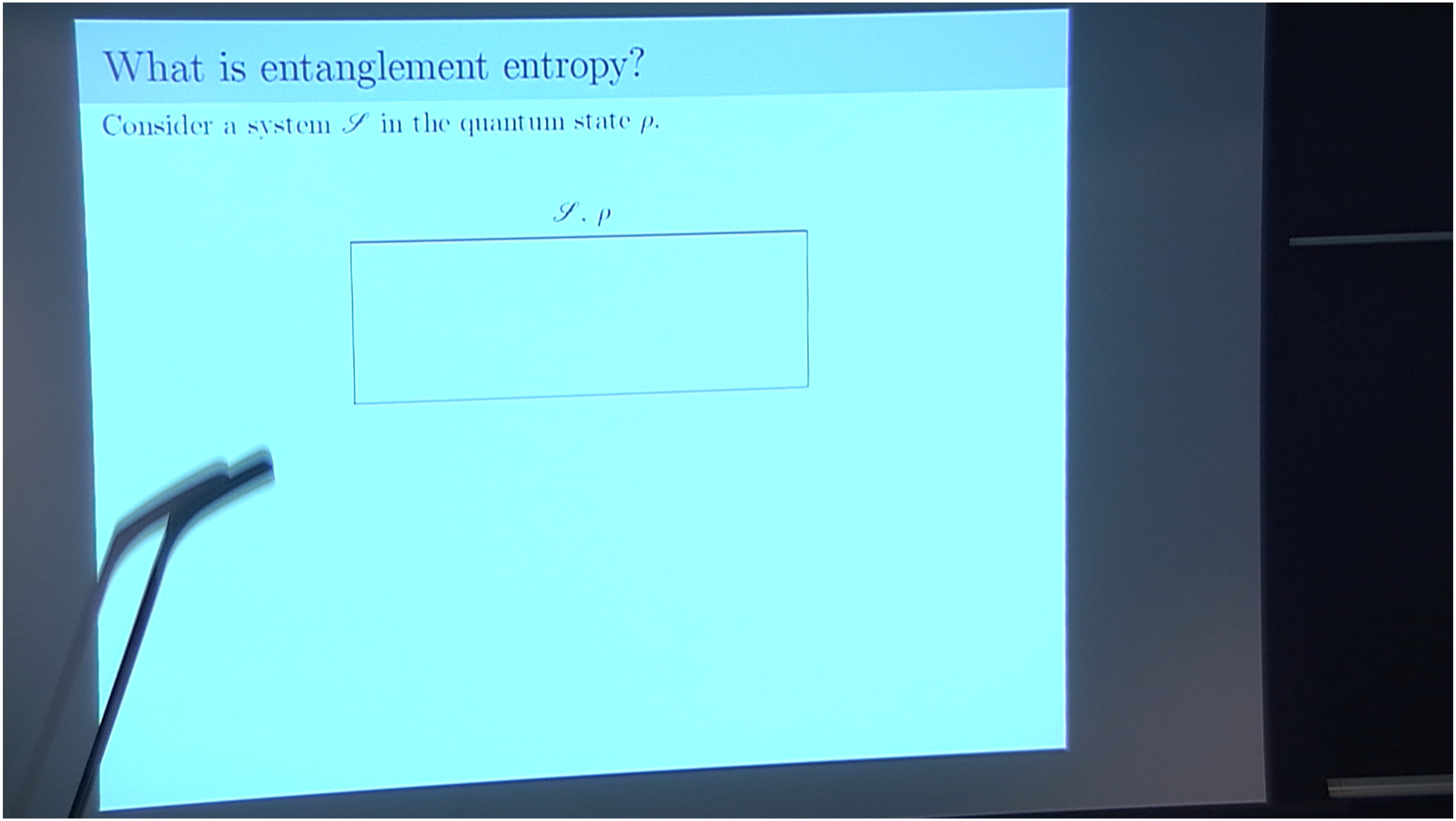
Abstract: The entanglement entropy associated with a spatial boundary in quantum field theory is ultraviolet divergent, its leading term being proportional to the area of the boundary. Callan and Wilczek proposed a geometrical prescription for computing this entanglement entropy as the response of the gravitational effective action to a conically singular metric perturbation. I argue that the Callan-Wilczek prescription is rigorously justified at least for a particular class of quantum states each expressible as a Euclidean path integral. I then show that the entanglement entropy is rendered ultraviolet finite by precisely the counterterms required to cancel the ultraviolet divergences in the gravitational effective action. In particular, the leading contribution to the entanglement entropy is given by the renormalized Bekenstein-Hawking formula. These results apply to a general quantum field theory coupled to a fixed background metric, holding for arbitrary entangling surfaces with vanishing extrinsic curvature in any dimension, to all orders in perturbation theory in the quantum fields, and for all ultraviolet divergent terms in the entanglement entropy. I also reconcile these results on the entanglement entropy with the existing literature, compare them to the Wald entropy, and speculate on their interpretation and implication.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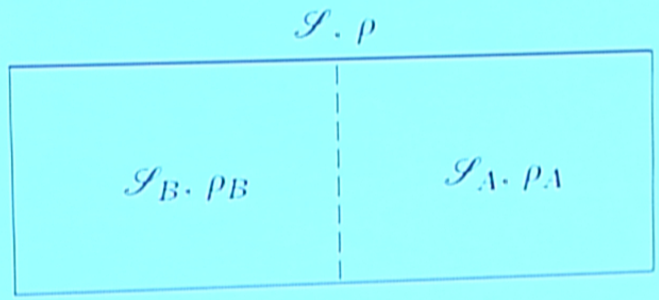


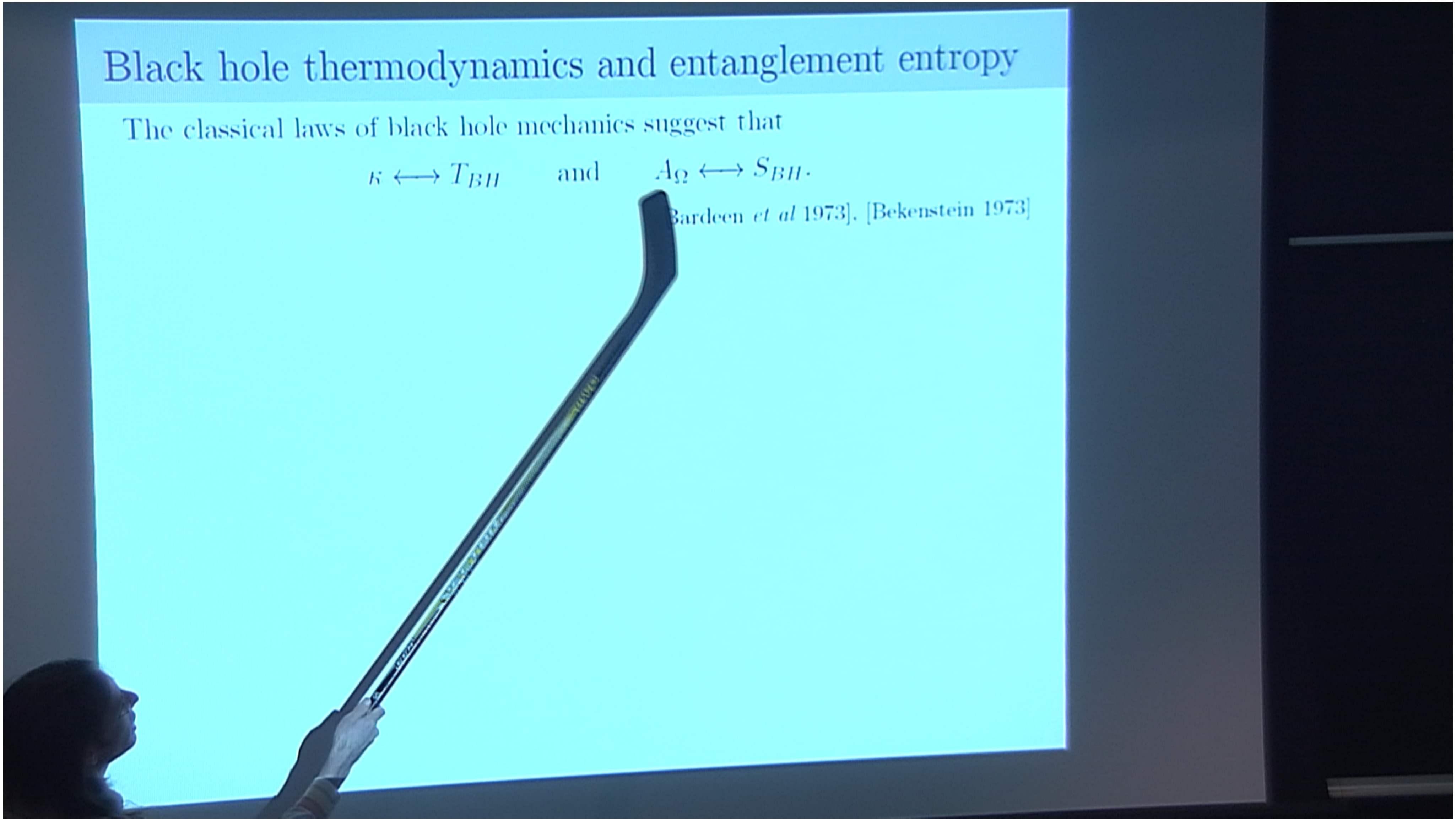
Reduced quantum states

$$\rho_B = \mathrm{Tr}_A \rho \qquad \rho_A = \mathrm{Tr}_B \rho$$

Entanglement entropies

$$S_{\mathrm{ent}}^{(B)} = -\mathrm{Tr}(\rho_B \ln \rho_B) \qquad S_{\mathrm{ent}}^{(A)} = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

- $S_{\mathrm{ent}}^{(A)}$ precisely quantifies the information that one disregards by ignoring subsystem $\mathscr{S}_B$ (and *vice versa*).
- If ρ is a pure quantum state, then $S_{\mathrm{ent}}^{(A)} = S_{\mathrm{ent}}^{(B)}$, so both $S_{\mathrm{ent}}^{(A)}$ and $S_{\mathrm{ent}}^{(B)}$ measure the amount of correlation between subsystems $\mathscr{S}_A$ and $\mathscr{S}_B$.


Black hole thermodynamics and entanglement entropy
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The entanglement entropy associated with quantum fields outside of the horizon is

$$S_{\text{ent}}[\Lambda] = -\text{Tr}\left(\rho_{\text{out}} \ln \rho_{\text{out}}\right) = c\Lambda^2 A_\Omega + \cdots .$$

[Sorkin 1983], [Bombelli et *al* 1986], [Srednicki 1993], [Frolov and Novikov 1993]

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The Susskind-Uglum conjecture

Susskind and Uglum conjectured that renormalization of the gravitational effective action also results in renormalization of the entanglement entropy.

- Gravitational effective action

$$W = -i \ln Z \qquad \text{for} \qquad Z = \int \mathcal{D}\Phi \, e^{iS[\Phi,g]}$$

  - Regularized

$$W[\Lambda] = \int d^D x \sqrt{-g} \left[c_0 \Lambda^D + c_2 \Lambda^{D-2} R + c_{4,1} \Lambda^{D-4} R^2 + \cdots \right]$$

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$$W^{(CT)} = \int \mathrm{d}^D x \sqrt{-g} \left[c_0^{(CT)} + c_2^{(CT)} R + c_{4,1}^{(CT)} R^2 + \cdots \right]$$

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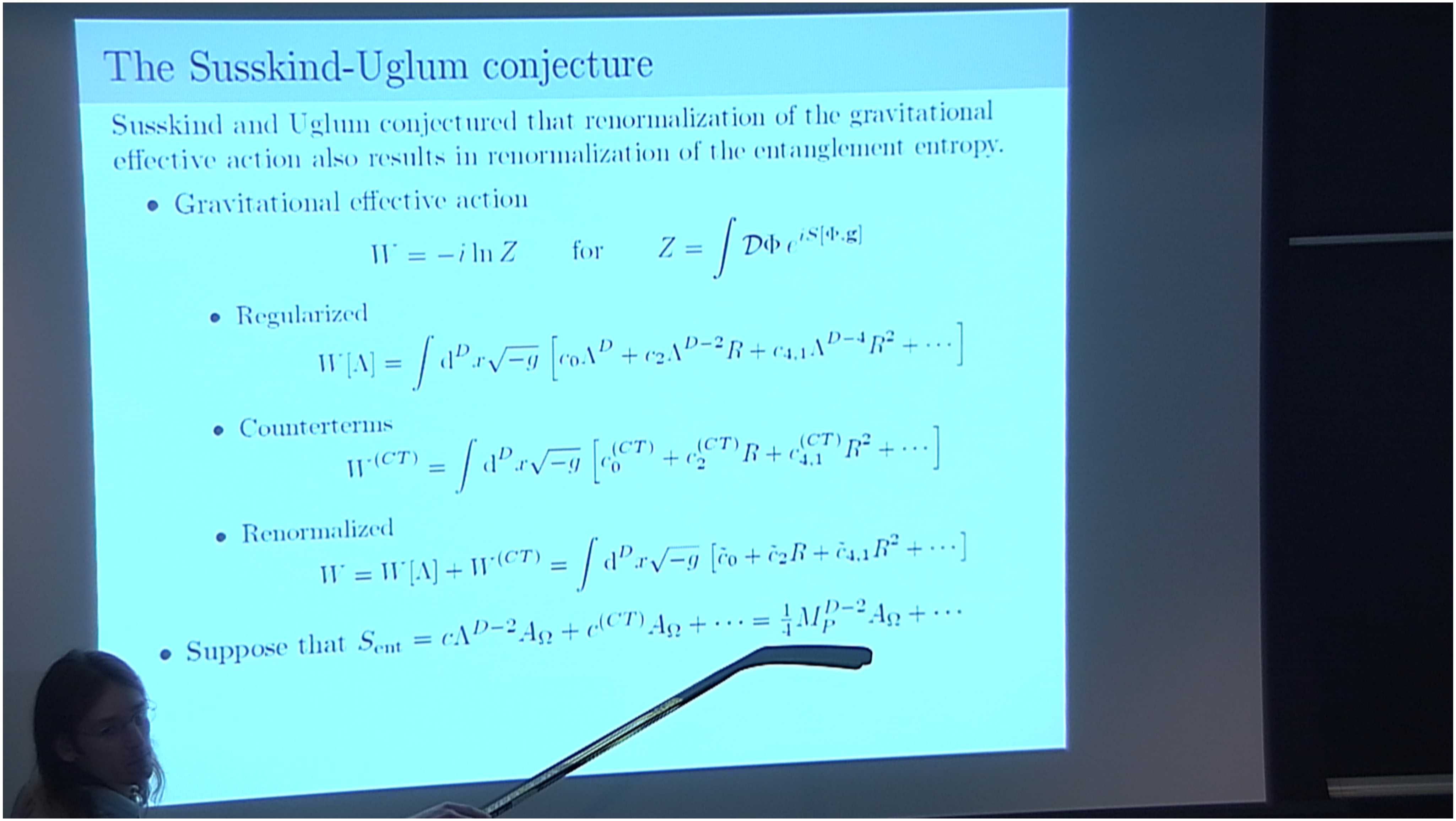
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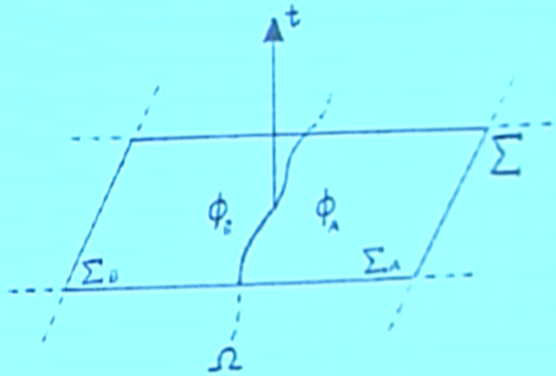
Entanglement entropy in quantum field theory

Consider a quantum field Φ propagating on a fixed background spacetime $\mathbf{g}$.

- Spacetime geometry
  - Time slice Σ
  - $\phi = \Phi|_\Sigma$
  - Entangling surface Ω dividing Σ into two regions Σ_A and Σ_B
  - $\phi_A = \phi|_{\Sigma_A}$ and $\phi_B = \phi|_{\Sigma_B}$



- Quantum state
  - $\rho = |\Psi\rangle\langle\Psi|$ on Σ
  - In the field operator basis $\{|\phi_A \phi_B\rangle\}$

$$\rho(\phi_A, \phi_B, \phi'_A, \phi'_B) = \langle \phi'_A \phi'_B|\Psi\rangle\langle\Psi|\phi_A \phi_B\rangle$$

- Entanglement entropy
  - Reduced quantum state on Σ_A

$$\rho_A(\phi_A, \phi'_A) = \mathrm{Tr}_B(\rho) = \int \mathcal{D}\phi_B \, \langle \phi'_A \phi_B|\Psi\rangle\langle\Psi|\phi_A \phi_B\rangle.$$

  - Entanglement entropy

$$S^{(A)}_{\rm ent} = -\mathrm{Tr}\,(\rho_A \ln \rho_A) = -\int \mathcal{D}\phi_A \, \rho_A(\phi_A, \phi_A) \ln \rho_A(\phi_A, \phi_A).$$

The Callan-Wilczek formula for entanglement entropy

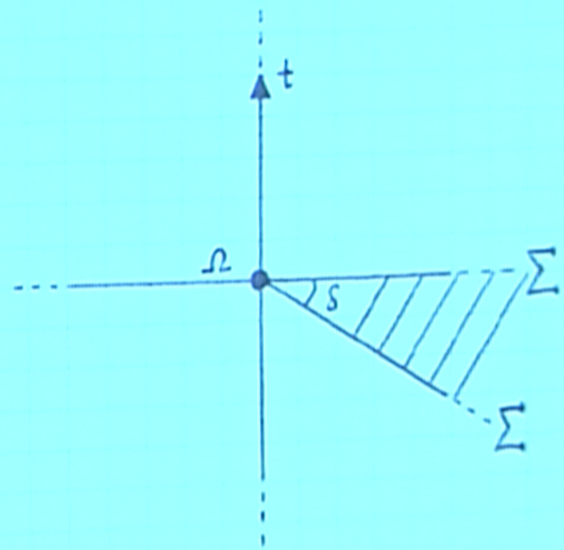
Callan and Wilczek proposed a formula for the entanglement entropy:

$$S_{\mathrm{ent}} = -\lim_{\delta\to 0}\left(2\pi\frac{\mathrm{d}}{\mathrm{d}\delta}+1\right)W_E^{(\delta)}.$$

$W_E^{(\delta)}$ is the Euclidean gravitational effective action for a spacetime $\mathbf{g}$ with conical singularity of deficit angle δ on the entangling surface Ω:

$$W_E^{(\delta)} = \int \mathcal{D}\Phi\, e^{-S_E^{(\delta)}[\Phi,\mathbf{g}]}.$$

[Callan and Wilczek 1994]



Justification of the Callan-Wilczek formula

Calculation of the entanglement entropy [JHC and Luty 2013]

Regularize the conical singularity by introducing a scale $\ell \ll \frac{1}{\Lambda}$

$$ds^2 = dr^2 + \alpha^2(r,\sigma)\left[1 - \frac{\delta}{2\pi} f(r,\ell)\right]^2 d\theta^2 + \gamma_{ij}(r,\sigma) d\sigma^i d\sigma^j$$

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Compute the entanglement entropy from the Callan-Wilczek formula

$$\begin{aligned} S_{\mathrm{ent}}[\Lambda] &= -\lim_{\ell\to 0}\lim_{\delta\to 0}\left(2\pi\frac{\mathrm{d}}{\mathrm{d}\delta} + 1\right) W_E^{(\delta)}[\Lambda,\ell] \\ &= -2\pi \int_\Omega \mathrm{d}^{D-2}\sigma\sqrt{\gamma}\sum_{\mathcal{F}} c_{\mathcal{F}} \Lambda^{D-\dim\mathcal{F}} I_1[\mathcal{F}] \end{aligned}$$

- Algorithm for computing the geometric invariant $I_1[\mathcal{F}]$ from $O(\delta^0)$ piece of $\mathcal{F}$

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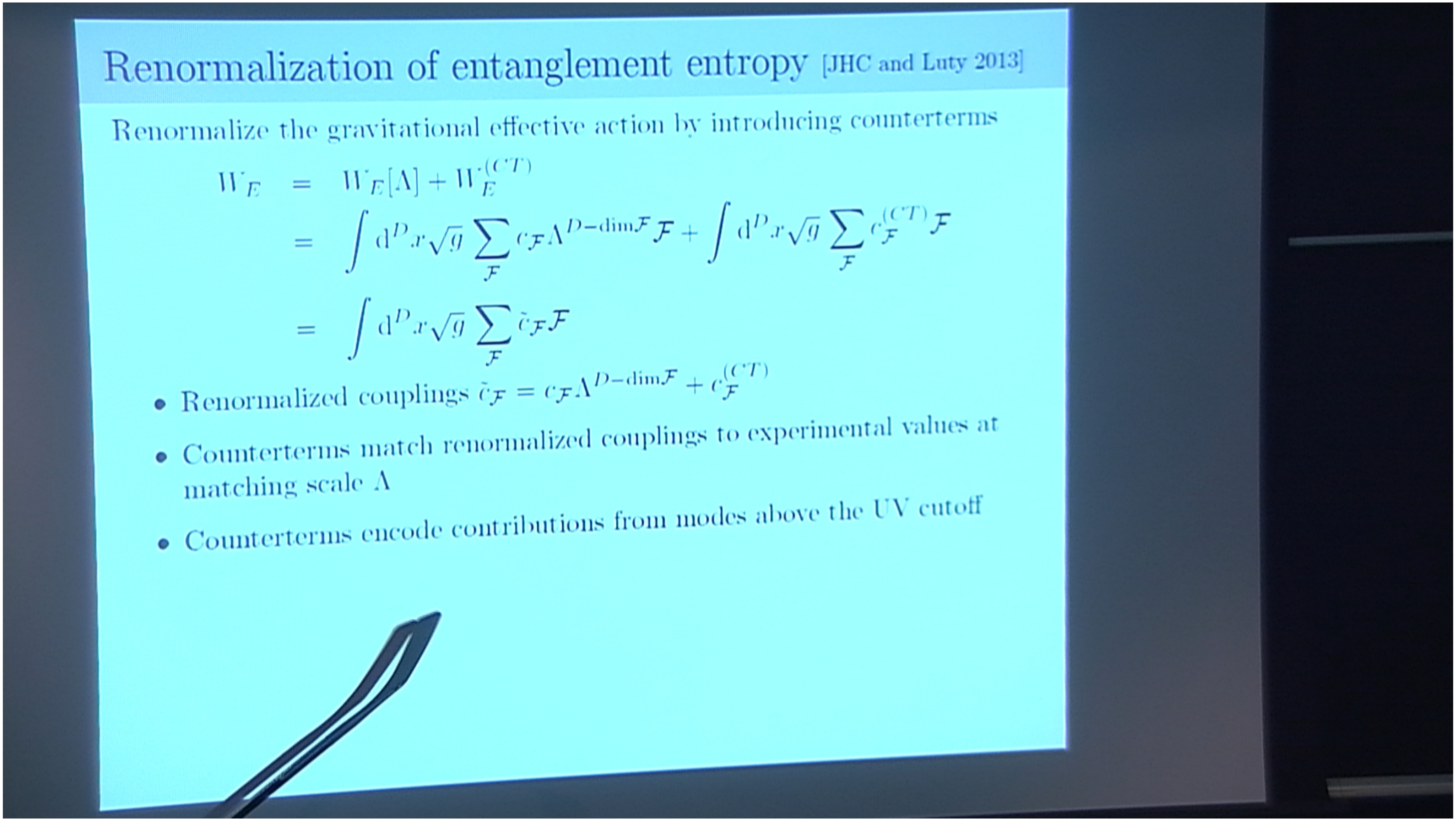
- Algorithm for computing the geometric invariant $I_1[\mathcal{F}]$ from $O(\delta^0)$ piece of $\mathcal{F}$
- Entanglement entropy depends on $O(\delta^1)$ terms in $W_E^{(\delta)}[\Lambda, \ell]$
- Divergences on the entangling surface Ω enter at $O(\delta^2)$ in $W_E^{(\delta)}[\Lambda, \ell]$


Renormalization of entanglement entropy [JHC and Luty 2013]
Renormalize the gravitational effective action by introducing counterterms
$$W_E = W_E[\Lambda] + W_E^{(CT)}$$
$$= \int \mathrm{d}^D x \sqrt{g} \sum_{\mathcal{F}} c_{\mathcal{F}} \Lambda^{D-\dim\mathcal{F}} \mathcal{F} + \int \mathrm{d}^D x \sqrt{g} \sum_{\mathcal{F}} c_{\mathcal{F}}^{(CT)} \mathcal{F}$$
$$= \int \mathrm{d}^D x \sqrt{g} \sum_{\mathcal{F}} \tilde{c}_{\mathcal{F}} \mathcal{F}$$
• Renormalized couplings $\tilde{c}_{\mathcal{F}} = c_{\mathcal{F}} \Lambda^{D-\dim\mathcal{F}} + c_{\mathcal{F}}^{(CT)}$
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Leading UV-divergent term in entanglement entropy

Einstein-Hilbert term in the gravitational effective action

$$W_E = \int \mathrm{d}^D x \sqrt{g}\, c_2 \Lambda^{D-2} R$$

Entanglement entropy from the Einstein-Hilbert term

$$S_{\mathrm{ent}}[R,\Lambda] = -2\pi \int_\Omega \mathrm{d}^{D-2}\sigma \sqrt{\gamma}\, 2c_2\Lambda^{D-2} = -4\pi c_2 \Lambda^{D-2} A_\Omega$$

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Counterterm for the Einstein-Hilbert term

$$W_E^{(CT)} = -\int \mathrm{d}^D x \sqrt{g}\, \frac{M_{P0}^{D-2}}{16\pi} R$$

Renormalized entanglement entropy from the Einstein-Hilbert term

$$S_{\mathrm{ent}}[R] = -4\pi c_2 \Lambda^{D-2} A_\Omega + \frac{1}{4} M_{P0}^{D-2} A_\Omega = \frac{1}{4} M_P^{D-2} A_\Omega$$

Renormalized Planck mass

$$M_P^{D-2} = M_{P0}^{D-2} - 16\pi c_2 \Lambda^{D-2}$$

Previous research on the Susskind-Uglum conjecture

- Checks of the Susskind-Uglum conjecture for leading UV-divergent term at one loop
  - Spin 0 (minimally coupled) [Susskind and Uglum 1994], *etc.*
  - Spin 0 (nonminimally coupled) [Fursaev 1995], *etc.*
  - Spin $\frac{1}{2}$ [Kabat 1995], *etc.*
  - Spin 1 [Kabat 1995], *etc.*
  - Spin $\frac{3}{2}$ [Fursaev and Miele 1997]
  - Spin 2 [Iellici and Moretti 1996], [Fursaev and Miele 1997]
- Checks of the Susskind-Uglum conjecture for subleading UV-divergent terms at one loop
  - Spin 0 (minimally coupled) for 4-derivative terms [Fursaev and Solodukhin 1995, 1996]
  - Spin 1 for 4-derivative terms [de Nardo *et al* 1997]
- Novel results [JHC and Luty 2013]
  - Check of the Susskind-Uglum conjecture for all subleading UV-divergent terms in the entanglement entropy
  - Check of the Susskind-Uglum conjecture to all orders in perturbation theory
  - Resolution of the issue of nonminimally coupled fields

Comparison to the Wald entropy

Wald defined an entropy for a classical metric theory of gravity with Lagrangian density $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\tau R_{\mu\nu\rho\sigma}, \ldots)$. [Wald 1993]

$$S_W = 2\pi \int_\Omega \mathrm{d}^{D-2}\sigma \sqrt{\gamma} \left[\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\tau \frac{\partial \mathcal{L}}{\partial \nabla_\tau R_{\mu\nu\rho\sigma}} + \cdots \right] \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

- Applicable to asymptotically flat spacetimes possessing a bifurcate Killing horizon
- Classical thermodynamic entropy

Relation of Wald entropy to renormalized entanglement entropy computed *via* the Callan-Wilczek formula

- Equivalent for $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ [Jacobson *et al* 1995], [Fursaev and Solodukhin 1995]
- Check of equivalence for $\Delta\mathcal{L} = \nabla_\tau R_{\mu\nu\rho\sigma} \nabla^\tau R^{\mu\nu\rho\sigma}$ [JHC and Luty 2013]

Relation of Wald entropy to renormalized entanglement entropy computed directly

- Not equivalent for entangling surfaces that are not bifurcate Killing horizons [Hung, Myers, and Smolkin 2011]
- Check of equivalence for Jacobson-Myers generalization of Wald entropy [Jacobson and Myers 1993], [Hung, Myers, and Smolkin 2011]

Beyond the Callan-Wilczek formula

How do we give a completely general geometric definition of entanglement entropy in quantum field theory on a fixed background spacetime?

- How do we treat entangling surfaces that are not bifurcate Killing horizons?
  - Expectation from dimensional analysis

$$\Delta S_{\mathrm{ent}}[\Lambda] = \int_{\Omega} \mathrm{d}^{D-2}\sigma \left[a_4 \Lambda^{D-4} R_{D-2} + b_{4,1} \Lambda^{D-4} K^2 + b_{4,2} \Lambda^{D-4} K_{ij} K^{ij} + \cdots \right]$$

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  - Conjecture for conformal field theory on Minkowski spacetime with general entangling surface [Solodukhin 2008], [Fursaev, Patrushev, and Solodukhin 2013]
  - Computation for general field theory on Minkowski spacetime with spherical or cylindrical entangling surface [Casini, Huerta, and Myers 2011], *etc.*
  - Treatment of perturbations of the entangling surface [Rosenhaus and Smolkin 2014]
  - Holographic entanglement entropy [Ryu and Takayanagi 2006], [Hubeny, [illegible] and Takayanagi 2007], *etc.*