Title: Grassmann tensor network renormalization and fermionic topological quantum field theory: a new route towards quantum gravity

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Abstract: Recently, the development of tensor network renormalization approach has provided us a powerful tool to construct new classes of topological quantum field theories(TQFTs) in discrete space-time. For example, the Turaev Viroâ€™s states sum constructions are fixed point tensor networks representing a special class of 2+1D TQFTs. Interestingly, the Grassmann variable generalization of tensor network renormalization approach leads to new classes of TQFTs for interacting fermion systems, namely, the fermionic TQFTs. In this talk, I will start with a fermionic topological nonlinear sigma model and discuss its corresponding new mathematics - group super-cohomology theory. Then I will explain the fermionic generalization of Dijkgraaf -Witten gauge theory by using group supercohomology theory. Finally I will show examples beyond fermionic Dijkgraaf -Witten gauge theory and discuss possible new routes towards quantum gravity.

Grassmann tensor network renormalization and fermionic topological quantum field theory:

A new route towards quantum gravity

Zheng-Cheng Gu (PI)

PI. April. 2014

Outline

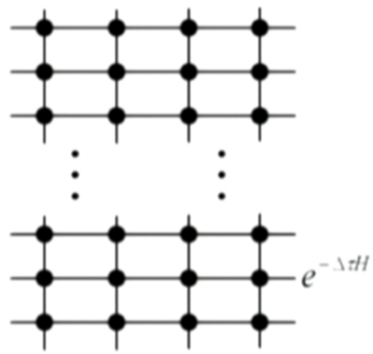
- Tensor network renormalization as quantum entanglement renormalization.
- A review of topological quantum phases described by bosonic tensor network.
- Fermionic topological nonlinear sigma model and a (special) group super cohomology theory.
- Fermionic Dijkgraaf-Witten and beyond: a new route towards quantum gravity.
- Summaries and outlook.

Tensor network renormalization: A first glance

Why tensor network?

- A natural local representation of partition function in discrete space time

$$Z = \sum_n \langle n | e^{-\beta H} | n \rangle =$$



$$\mathcal{L}^{(0)} \to \mathcal{L}^{(1)} \to \mathcal{L}^{(2)} \to \cdots \to \mathcal{L}^{(*)}$$

$$T^{(0)} \to T^{(1)} \to T^{(2)} \to \cdots \to T^{(*)}$$

What is tensor network renormalization?

- A numerical trick to compute phase diagram for condensed matter systems

What is the advantage of tensor network renormalization?

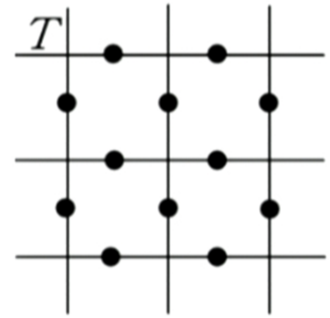
- It not only describes gapless fixed points that can be understood by quantum field theory, but also describes nontrivial fixed points beyond quantum field theory, e.g., topological phases, emergent fields.

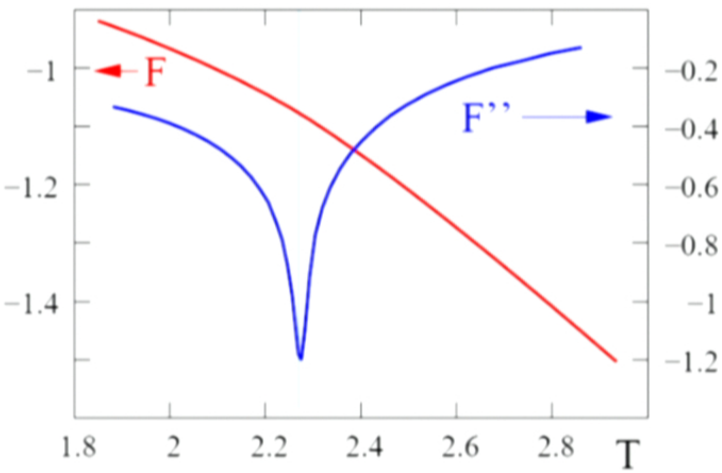
A simple example

2D statistical Ising model

$$H = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

$$\begin{aligned} T_{1122} &= T_{2211} = e^{-4\beta} \\ T_{1111} &= T_{2222} = e^{4\beta} \\ others &= 1 \end{aligned}$$





Apply tensor filtering RG up to 1024 sites

(Zheng-Cheng Gu, Xiao-Gang Wen Phys.Rev.B80:155131,(2009))

T>2.38 $T^{(0)}_{1111} = 1, \quad others = 0$

T<2.17 $T^{(Z_2)}_{1111} = 1, \quad T^{(Z_2)}_{2222} = 1, \quad others = 0$

$T^{(Z_2)} = T^{(0)} \oplus T^{(0)}$

Critical point:

c	h_1	h_2	h_3	h_4
0.49942	0.12504	0.99996	1.12256	1.12403
0.5	0.125	1	1.125	1.125

Tensor network renormalization: A deep thinking

Tensor network renormalization is a renormalization scheme with no energy scale and length scale!

Then what has been renormalized?

- The actual quantity that has been renormalized is quantum entanglement, or density matrix, and has nothing to do with energy or length scale.

What is encoded in fixed point tensor?

- Different fixed point tensors describe different kinds of quantum entanglement patterns, which characterize different types of quantum phases in the thermodynamic limit.

Background independent RG flow is possible, RG fixed point condition becomes topological invariant condition.

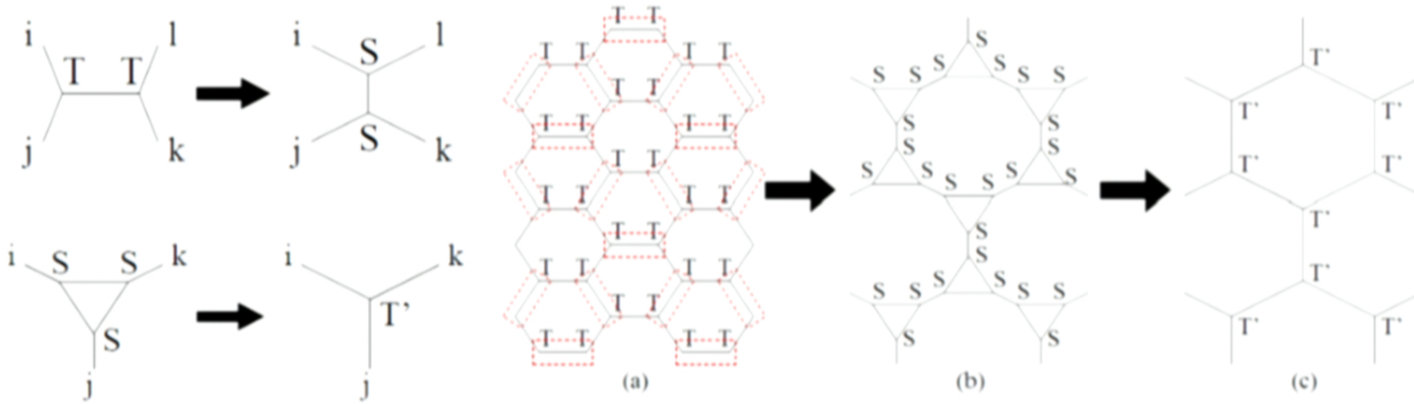
The most natural RG scheme for quantum gravity!

- For some cases, unitary condition also naturally emerges from RG fixed point/topological invariant condition, e.g., non-chiral topological phases described by unitary modular tensor category(UMTC) theory in 2+1D.

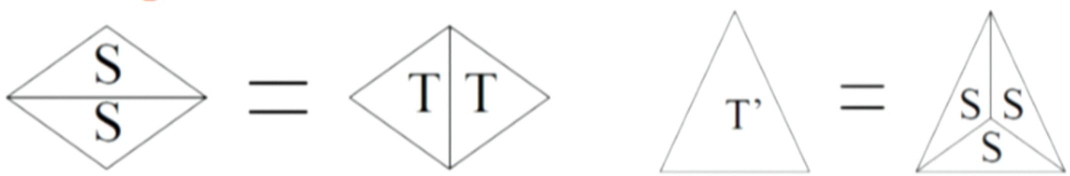
An example of background independent tensor network renormalization in 2D

A RG scheme for trivalent graph in 2D

(M. Levin and Cody P. Nave, Phys. Rev. Lett. 99, 120601 (2007))



The tensor network RG equations are indeed retriangulation conditions!



Fixed point condition: T'=T

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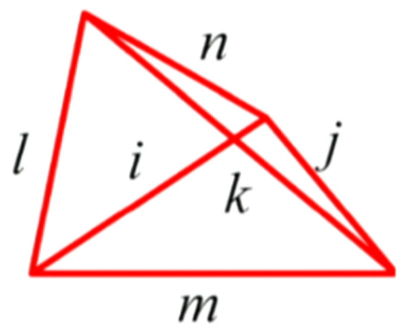
Tensor network/complex representation for topological phases in 2+1D

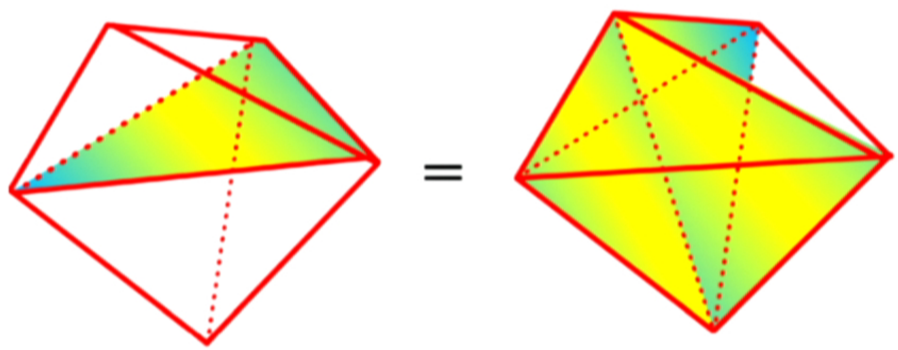
Turaev-Viro states sum invariants for 3D manifolds

$$Z = \frac{1}{D^{N_v}} \sum_{ijklmn...} \prod_{\text{link}} d_i \prod_{\text{tetrahedron}} G^{ijm}_{kln}$$

$G^{ijm}_{kln} =$ 

- d is the quantum dimension, D is the total quantum dimension and G is the (symmetric) 6j symbol

Pentagon Equation = Tetrahedron move



$$GG = \sum_i d_i GGG$$

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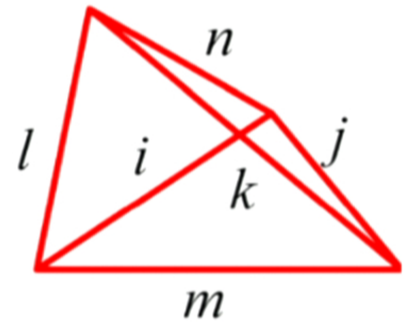

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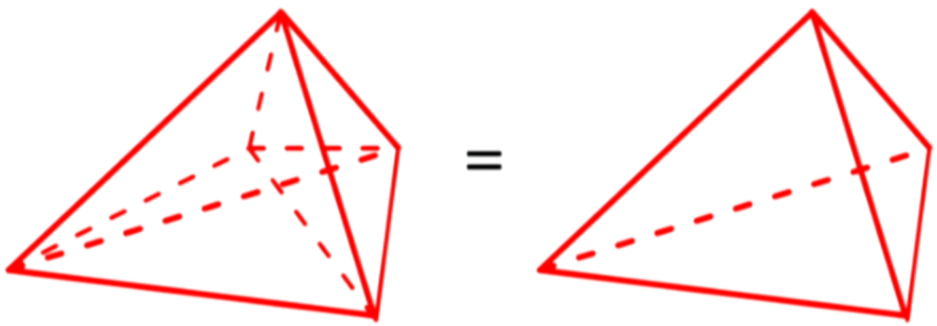
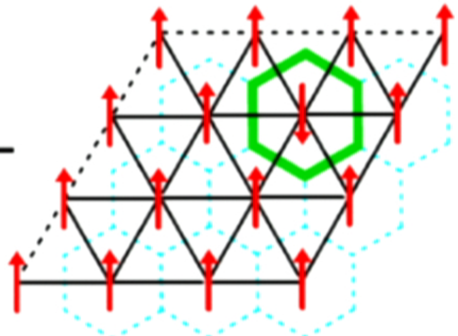
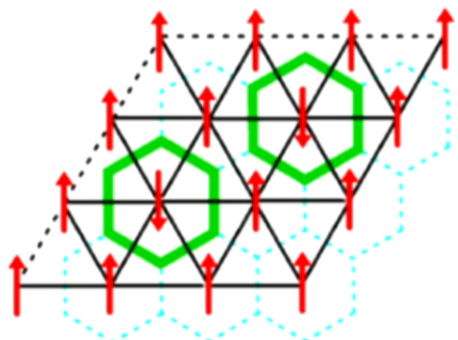


$$GG = \sum_i d_i GGG$$

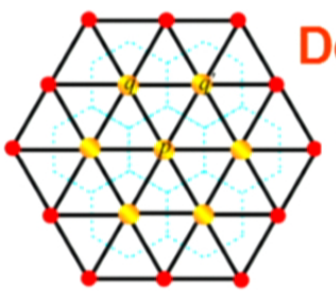
Symmetry protected topological(SPT) phases in arbitrary dimensions

How many different Ising paramagnetic phases? **Two!**

(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

$|\Psi_1\rangle = -$  $+$  $+ \dots$

$$\widetilde{\sigma}_p^x = \sigma_p^x \prod_{\langle qq'p \rangle} i^{\frac{1+\sigma_q^z \sigma_{q'}^z}{2}}; \qquad [\widetilde{\sigma}_p^x, \widetilde{\sigma}_{p'}^x] = 0$$



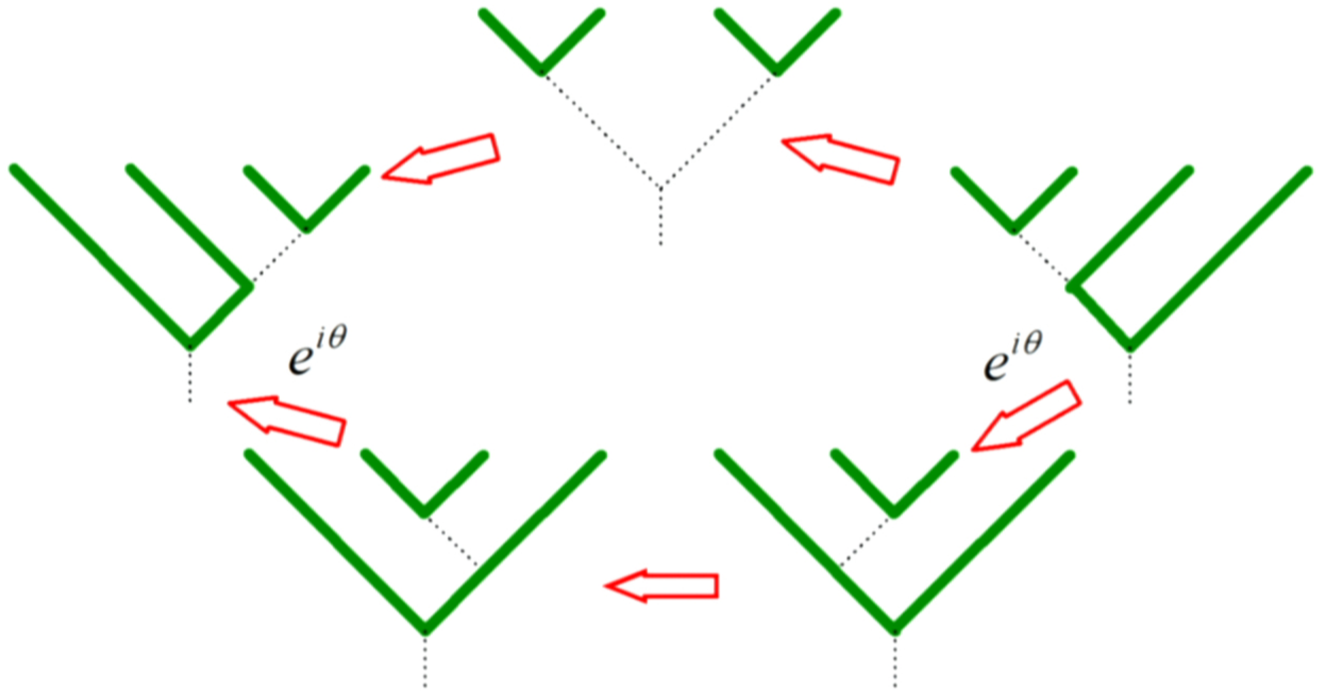
Domain deformation rule

 $= -$ 

But why not?

 $= e^{i\theta}$ 

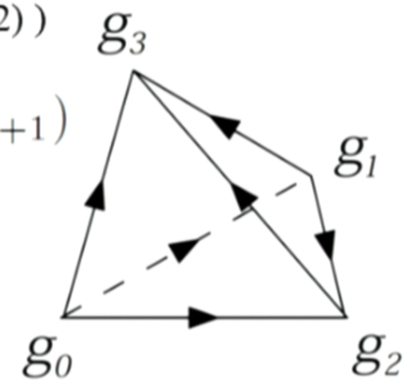
Topologically consistent condition for fixed point wavefunction



$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

Discrete space-time topological nonlinear sigma model

(X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen (Science 338, 1604 (2012))



$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_i\}} \prod_{d+1-\text{simplex}} \nu_{d+1}^{s_{01\cdots d}}(g_0, g_1, \cdots, g_{d+1})$$

- Branched(vertex ordered) d+1-simplex

$$\nu_{d+1} : G \times G \times \cdots \times G \mapsto U(1)$$

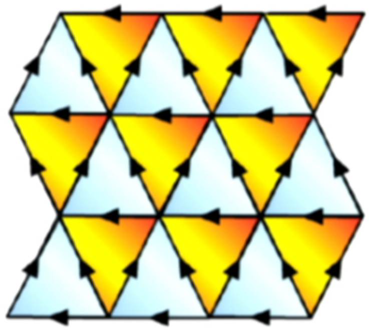
$$\nu_{d+1}(gg_0, gg_1, \cdots, gg_{d+1}) = \nu_{d+1}(g_0, g_1, \cdots, g_{d+1})$$

co-cycle condition: $\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{d+2}) = 1$

- **SPT phases in bosonic systems are classified by d+1 group cohomology $\mathcal{H}^{1+d}[G, U(1)]$ in d spacial dimension.**

- Each element gives rise to an exactly solvable hermitian Hamiltonian with a unique ground state on closed manifold.

An example of 1+1D case

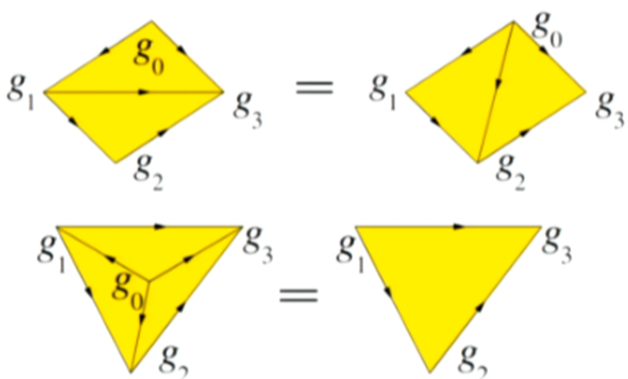


$$Z = |G|^{-N_v} \sum_{\{g_i\}} \mathrm{e}^{-S(\{g_i\})}$$

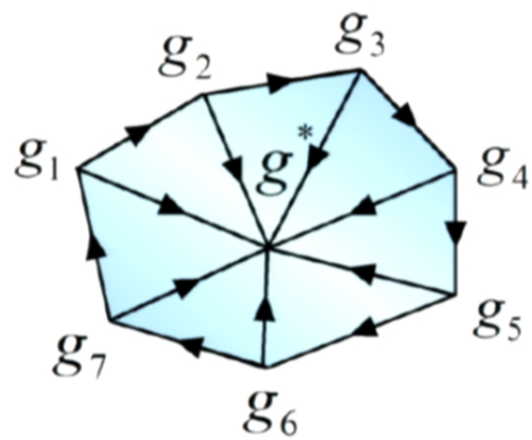
$$\mathrm{e}^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$

Fixed point condition

$$\frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1$$



Fixed point wavefunction



$$\Psi(\{g_i\}_M) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$$

Generalization of Dijkgraaf-Witten gauge theory into arbitrary dimensions

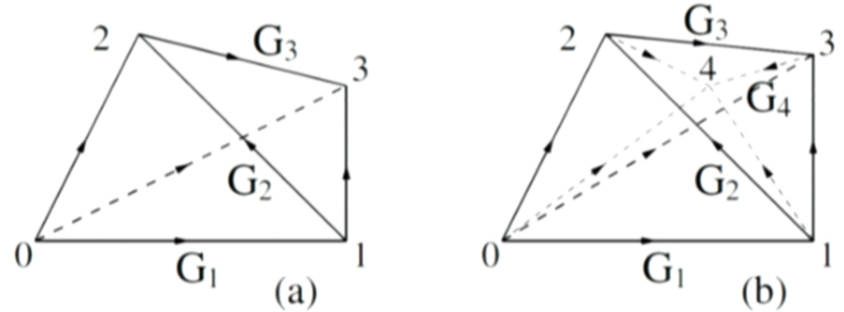
By gauging the global symmetry in SPT phases, we can generalize Dijkgraaf-Witten gauge theory into arbitrary dimensions

$$Z = |G|^{-N_v} \sum_{\{G_{ij}\}} \prod_{[ij...k]} \alpha^{s(i,j,...,k)}(\{G_{ij}\})$$

Example in 2+1D

M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012)

L-Y Huang, X.G. Wen, arXiv:1211.2767(2012)



$$\frac{\alpha(G_1,G_2,G_3)\alpha(G_1,G_2G_3,G_4)\alpha(G_2,G_3,G_4)}{\alpha(G_1G_2,G_3,G_4)\alpha(G_1,G_2,G_3G_4)} = 1$$

$$\alpha(G_1,G_2,G_3) = \nu_3(1,G_1,G_1G_2,G_1G_2G_3)$$

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Grassmann tensor network renormalization: a new world of fermionic topological phases

Why Grassmann number?

- Grassmann path integrals are natural formalisms for fermionic quantum systems.

Why fermions?

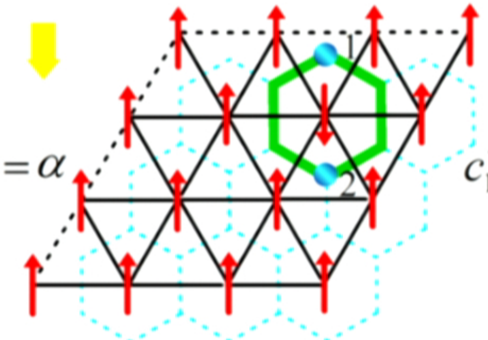
- Fundamental particles, e.g., quarks and leptons, are all fermions.
- Bosons can be regarded as pair of fermions, therefore topological phases in interacting fermion systems are strictly richer than topological phases in boson systems.
- There are intrinsic fermionic topological phases that can not be realized in any bosonic system, e.g. 1/3 fractional quantum hall state.

A new route towards quantum gravity!

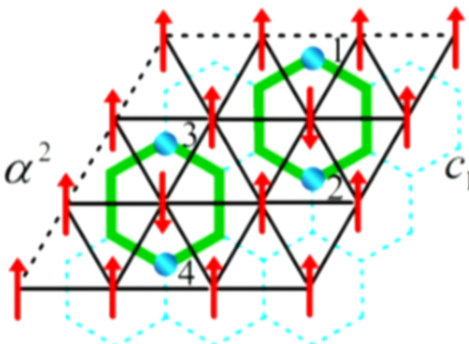
- Although classical space-time is described by bosonic variables, there is no justification whether quantum space-time is described by bosonic variables or not at cut-off scale.
- Vacuum energy cancellation requires fermionic degree of freedoms for quantum gravity.

An example of intrinsic fermionic Ising SPT phase in 2D

$|\Psi_f\rangle = \alpha$



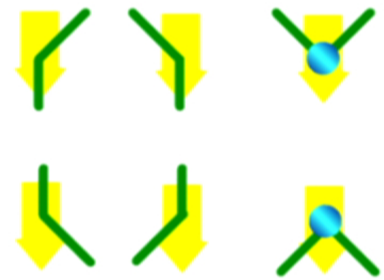
$c_1^+ c_2^+ + \alpha^2$



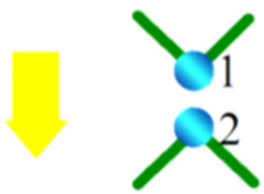
$c_1^+ c_2^+ c_3^+ c_4^+ + \ldots$

$\alpha = \pm i$

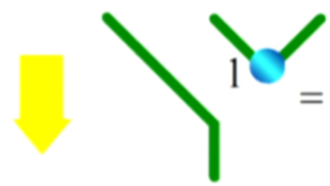
Domain decoration rule:



Domain deformation rule:

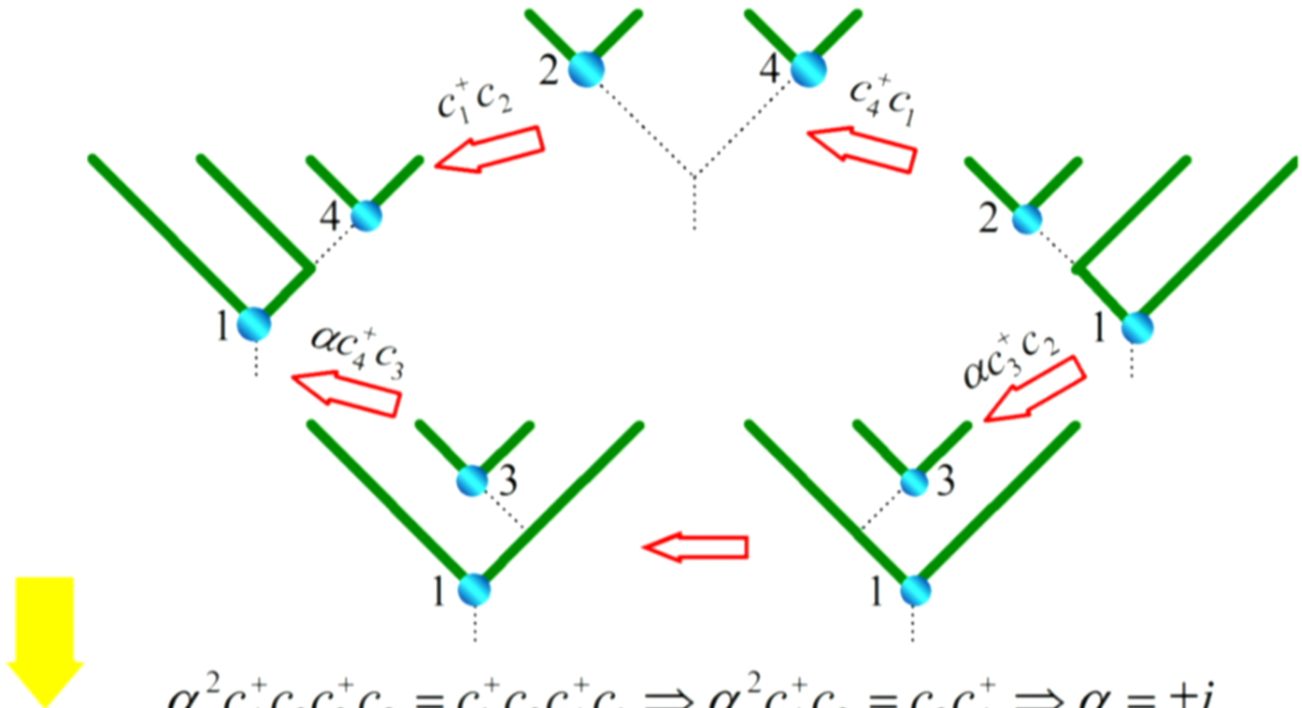


$= \alpha c_2^+ c_1^+$



$= \alpha c_1^+ c_2$

Topological consistent condition for fixed point wavefunction

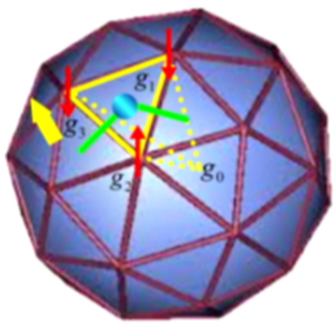


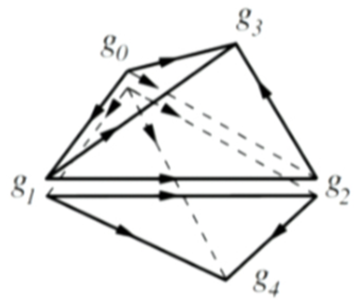
$$\alpha^2 c_4^+ c_3 c_3^+ c_2 = c_1^+ c_2 c_4^+ c_1 \Rightarrow \alpha^2 c_4^+ c_2 = c_2 c_4^+ \Rightarrow \alpha = \pm i$$

The two intrinsic fermionic Ising SPT phases can be viewed as the square root of bosonic Ising SPT phase.

The concept of Grassmann valued topological Berry phase

The domain decoration picture for wavefunction implies Grassmann graded amplitude for partition function





$$\mathcal{V}_3^+(g_0,g_1,g_2,g_3)=\nu_3^+(g_0,g_1,g_2,g_3)\times \theta_{(1,2,3)}^{n_2(g_1,g_2,g_3)}\theta_{(0,1,3)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(0,2,3)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(0,1,2)}^{n_2(g_0,g_1,g_2)}$$

$$\mathcal{V}_3^-(g_0,g_1,g_2,g_4)=\nu_3^-(g_0,g_1,g_2,g_4)\times \theta_{(0,1,2)}^{n_2(g_0,g_1,g_2)}\theta_{(0,2,4)}^{n_2(g_0,g_2,g_4)}\bar\theta_{(0,1,4)}^{n_2(g_0,g_1,g_4)}\bar\theta_{(1,2,4)}^{n_2(g_1,g_2,g_4)}$$

Arbitrary dimension

$$\mathcal{V}_d^+(g_0,g_1,...,g_d)\in M_f,\quad g_i\in G_b\quad G_b^{1+d}\to M_f$$

$$\mathcal{V}_d^+(g_0,g_1,...,g_d)=\nu_d^+(g_0,g_1,g_2,g_3,g_4,...,g_d)\times \theta_{(1234...d)}^{n_{d-1}(g_1,g_2,g_3,g_4,...,g_d)}\theta_{(0134...d)}^{n_{d-1}(g_0,g_1,g_3,g_4,...,g_d)}...\times \bar\theta_{(0234...d)}^{n_{d-1}(g_0,g_2,g_3,g_4,...,g_d)}\bar\theta_{(0124...d)}^{n_{d-1}(g_0,g_1,g_2,g_4,...,g_d)}...$$

Total symmetry

$$G=G_b\otimes Z_2^f$$

Z_2 graded structure

$$n_{d-1}(g_i,g_j,...,g_k)=0,1$$

$$\sum_{i=0}^{d}n_{d-1}(g_0,...,\hat g_i,...,g_d)=\text{ even}$$

Zheng-Cheng Gu and Xiao-Gang Wen arXiv:1201.2648(2012)

Fermionic topological nonlinear sigma model

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}$$

$$\equiv \int \prod_{(ij...k)} d\theta_{(ij...k)}^{n_{d-1}(g_i,g_j,...,g_k)} d\bar{\theta}_{(ij...k)}^{n_{d-1}(g_i,g_j,...,g_k)} \times$$

$$\prod_{\{xy...z\}} (-)^{m_{d-2}(g_x,g_y,...,g_z)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}(g_a,g_b,...,g_c)$$

$$n_{d-1}(g_1,g_2,...,g_d) = \sum_{i=1}^{d} m_{d-2}(g_1,...,\hat{g}_i,...,g_d) \bmod 2$$

Super co-cycle condition (consistent domain deformation rules)

Topological invariant conditions enforce $\nu_{d+1}^{\pm}$ can be expressed by m_{d-1} and ν_{d+1} that satisfies:

$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0,\cdots,g_{i-1},g_{i+1},\cdots,g_{d+2}) = (-)^{f_{d+2}}$$

Example in 2+1D:

$$f_1(g_0,g_1) = 0;$$
$$f_2(g_0,g_1,g_2) = 0;$$
$$f_3(g_0,g_1,...,g_3) = 0;$$

$$\nu_3^{+}(g_0,g_1,g_2,g_3) = (-)^{m_1(g_0,g_2)}\nu_3(g_0,g_1,g_2,g_3),$$
$$\nu_3^{-}(g_0,g_1,g_2,g_3) = (-)^{m_1(g_1,g_3)}/\nu_3(g_0,g_1,g_2,g_3)$$

$$f_4(g_0,g_1,...,g_4) = n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)$$

A (special) group super-cohomology theory

A (special) group super-cohomology theory

f_{d+2} is the Steenrod square Sq^2 of n_d, which maps:
$$n_d \in \mathcal{H}^d(G_b, \mathbb{Z}_2) \to f_{d+2} \in \mathcal{H}^{d+2}(G_b, \mathbb{Z}_2)$$

- The Steenrod square, one of the most novel structures in algebraic topology, finally came into physics since its discovery by Steenrod 50 years ago!

Compute group super-cohomology class by using short exact sequence

d_{sp}	short exact sequence
0	$0 \to \mathcal{H}^1[G_b, U_T(1)] \to \mathscr{H}^1[G_f, U_T(1)] \to \mathbb{Z}_2 \to 0$
1	$0 \to \mathcal{H}^2[G_b, U_T(1)] \to \mathscr{H}^2[G_f, U_T(1)] \to \mathcal{H}^1(G_b, \mathbb{Z}_2) \to 0$
2	$0 \to \mathcal{H}^3[G_b, U_T(1)] \to \mathscr{H}^3[G_f, U_T(1)] \to B\mathcal{H}^2(G_b, \mathbb{Z}_2) \to 0$
3	$0 \to \mathcal{H}^4_{\text{rigid}}[G_b, U_T(1)] \to \mathscr{H}^4[G_f, U_T(1)] \to B\mathcal{H}^3(G_b, \mathbb{Z}_2) \to 0$

A valid graded structure must be obstruction free:

$$B\mathcal{H}^d[G_b, \mathbb{Z}_2] \equiv \{n_d | n_d \in \mathcal{H}^d[G_b, \mathbb{Z}_2] \text{ and } (-)^{f_{d+2}} \in \mathcal{B}^{d+2}[G_b, U(1)]\}$$

$$\mathcal{H}^d_{\text{rigid}}[G_b, U_T(1)] = \mathcal{H}^d[G_b, U_T(1)]/\Gamma$$

Γ is a subgroup of $\mathcal{H}^d[G_b, U_T(1)]$ generated by $(-)^{f_d}$

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Fermionic Dijkgraaf-Witten gauge theory and beyond

By gauging the global symmetry in fermionic SPT phases, we can obtain fermionic Dijkgraaf-Witten gauge theory in arbitrary dimensions.

New classes of fermionic topological phases with topological Majorana modes on open manifold.

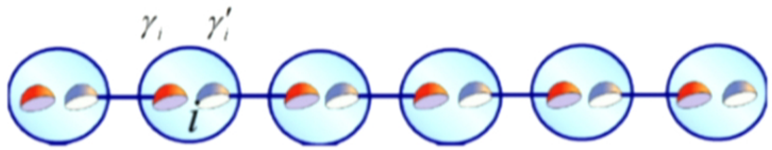
$$Z_f = \frac{1}{|G|^{N_v} D^{N_v}} \sum_{\{g_i\},\{m_{ij\cdots}\}} \int \prod_{d-\text{simplex}} \mathrm{d}\theta^{+}_{(i,j,\cdots,\bar{l},\cdots,k)} \mathrm{d}\theta^{-}_{(i,j,\cdots,\bar{l},\cdots,k)} \prod_{d-\text{simplex}} \left[1 - \theta^{+}_{(i,j,\cdots,\bar{l},\cdots,k)} \theta^{-}_{(i,j,\cdots,\bar{l},\cdots,k)}\right] (-)^{m_{ij\cdots}} \times \mathcal{V}^{s_{ij\ldots k}}_{d+1} \left(g_i, g_j, \cdots, g_k; \{m_{ij\ldots}\}\right), \tag{82}$$

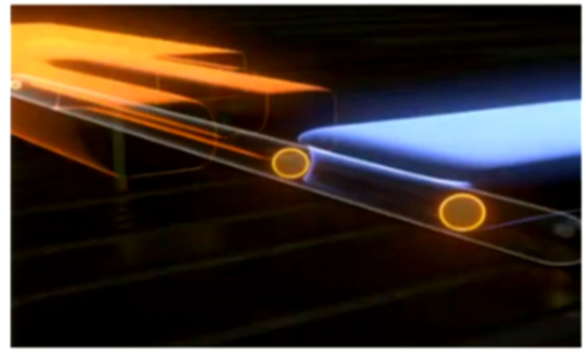
$$\mathcal{V}^{s_{ij\ldots k}}_{d+1}(g_i, g_j, \cdots, g_k) = \sum_{\{n_{(ij\cdots,\bar{l}\cdots k)}\}} \nu^{s_{ij\ldots k}}_{d+1} \left(g_i, g_j, \cdots, g_k; \{n_{(ij\cdots\bar{l}\cdots k)}\}\right) \prod_{l=ij\ldots k}^{s_{ij\ldots k}} \left[\theta^{s_{ij\cdots\bar{l}\cdots k}}_{(ij\cdots\bar{l}\cdots k)}\right]^{n_{(ij\cdots\bar{l}\cdots k)}}$$

$$\sum_{l=ij\cdots k} n_{(ij\cdots\bar{l}\cdots k)} \mod 2 = 0 \qquad n_{(ij\cdots\bar{l}\cdots k)} = \sum_{d-1-\text{simplex}} m_{ij\cdots}$$

Topological Majorana mode in 1+1D

Topological Majorana mode in the ends of Kitaev's Majorana Chain

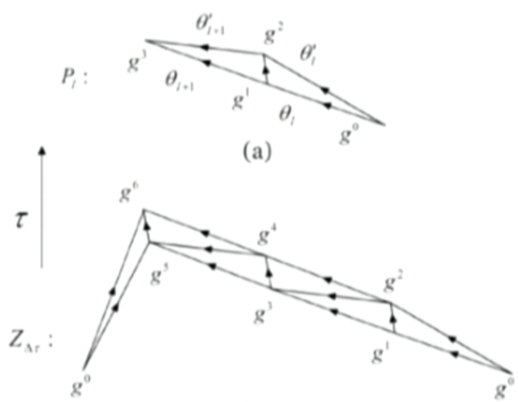




$$H = \sum_{i=1}^{N} i\gamma_i'\gamma_{i+1} = \sum_{i=1}^{N}\left(c_i - c_i^\dagger\right)\left(c_{i+1} + c_{i+1}^\dagger\right); \quad c_i = \frac{1}{2}(\gamma_i + i\gamma_i')$$

A. Kitaev (2001)
S. D.Sarma(2010)

Topologically invariant partition function



$$\mathcal{V}_2^+(g^i, g^j, g^k) = \frac{1}{2}\sum_{n_{ij}, n_{jk}} (\theta_{ij}^+)^{n_{ij}} \left(\theta_{jk}^+\right)^{n_{jk}} (\theta_{ik}^-)^{|n_{ij}-n_{jk}|}$$

$$\mathcal{V}_2^-(g^i, g^j, g^k) = \frac{1}{2}\sum_{n_{ij}, n_{jk}} (\theta_{ik}^+)^{|n_{ij}-n_{jk}|} \left(\theta_{jk}^-\right)^{n_{jk}} (\theta_{ij}^-)^{n_{ij}}$$

Topological Majorana mode in 3+1D

Fermionic topological phase with topological Majorana modes on open manifold also exists in 3D!

Conjecture: after summing over topology, the partition function describing topological Majorana modes is gapless and might describe Einstein gravity.

- Einstein gravity can be derived in a saddle approximation of a fermionic topological quantum field theory with Lorentz gauge symmetry. (Z. C. Gu, 2014, to appear)

Testable predictions

- By further assuming a Majorana neutrino is made up of four topological Majorana zero modes at cutoff scale, we naturally explained the origin of three generations of neutrinos and obtained the neutrino mass mixing matrix from a first principle.

- Mixing angles are intrinsically close to experimental data. Exact neutrino mass can be predicted according to current neutrino oscillation data, and CP violation angle is also predicted. (Z C Gu, arXiv:1308.2488, arXiv: 1403.1869)

Conclusions and future work

- Tensor network renormalization is a concept of entanglement renormalization, which is background independent and leads to topological invariance at fixed point.

- Grassmann tensor network renormalization gives rise to new classes of topological phases in fermion systems.

- Fermionic topological phase with topological Majorana modes naturally explains the origin of three generations of neutrinos and their mass mixing angles, therefore such a topological phase might also describe quantum gravity.

- Topological Majorana modes could be fundamental block of all matter fields, and U(1)*SU(2)*SU(3) gauge fields in Standard Model will arise naturally.(Z. C. Gu, 2014, in preparing)

- Since the boundary theory of fermionic topological phase with topological Majorana modes should be described by conformal field theory(CFT), and therefore it might provide us a first principle derivation of ADS-CFT duality!