Title: Asymptotic safety in a pure matrix model

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Abstract: An attempt is made to define "lines of contant physics" in CDT and relate the corresponding picture to non-trivial UV fixed points as they appear in the asymptotic safety scenario.

RG and quantum geometry

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Work done in collaboration with L.Glaser, A. Gör lich, J. Jurkiewicz, A. Kreienbuhl and R. Loll.

RG and quantum geometry

Our modern understanding of QFT is Wilsonian

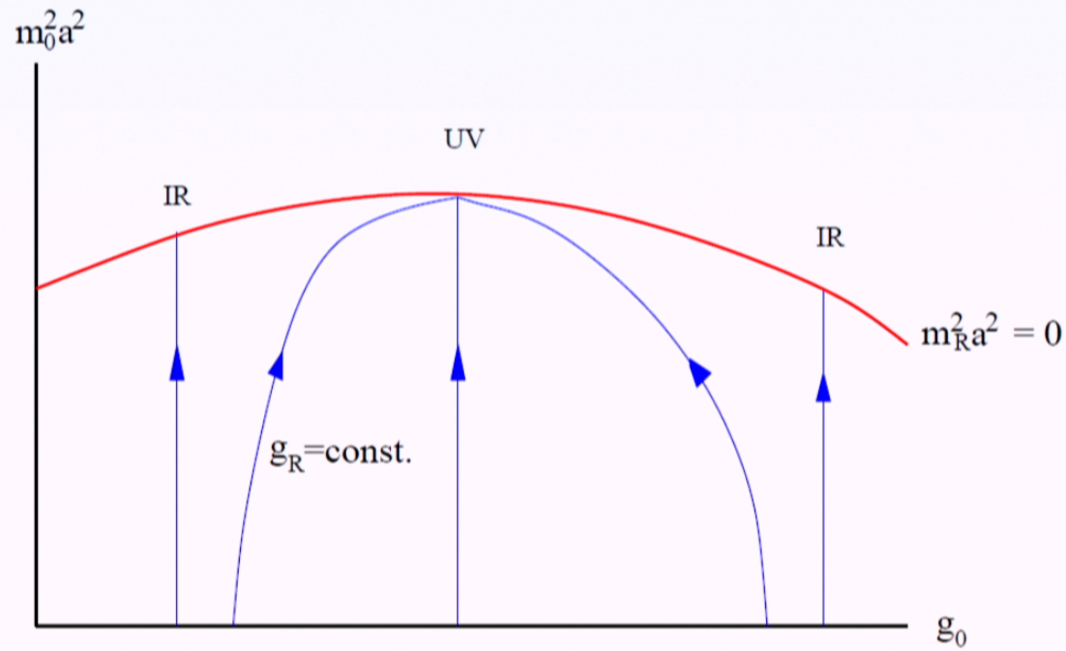
In this world GR only fits in uncomfortably (like the sigma model for pions and the 4-fermi interaction for β-decay).

Asymptotic safety is an attempt to extend the Wilsonian framework to theories with non-trivial UV fixed points and the framework of Dynamical Triangulations (DT) and Causal Dynamical Triangulations (CDT) are lattice theories designed to study quantum gravity or more generally "quantum theories of geometries" and explore if such fixed points exist.

Lattice field theories can be useful in deciding whether or not a quantum field theory exists and is non-trivial.

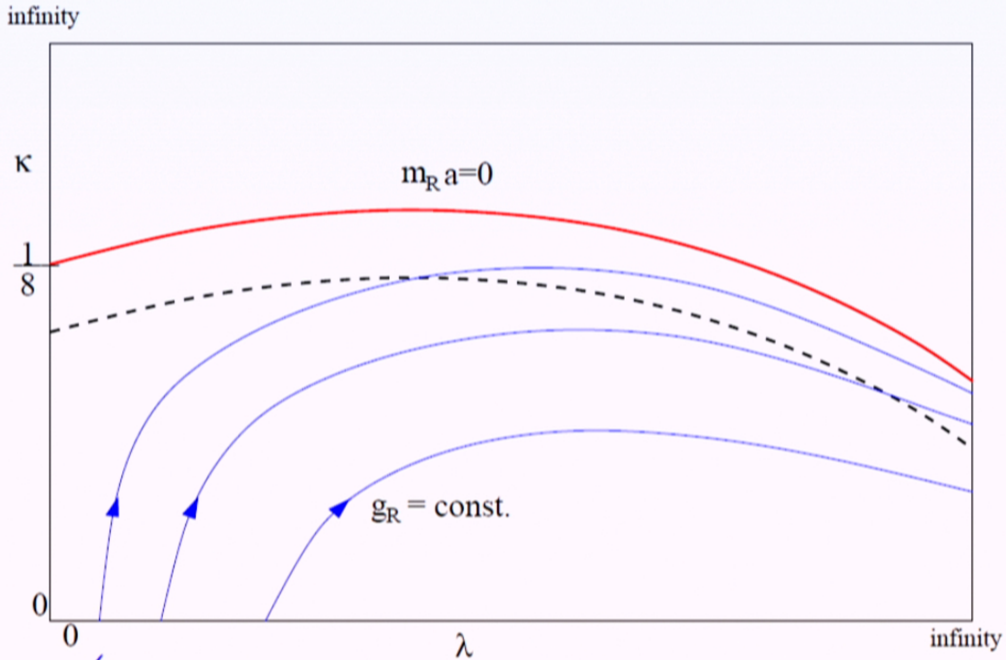
Here a “non-trivial QFT” means a quantum field theory which does not become a free field theory when the lattice cut off is removed.

Ex: ϕ^4 in 3d (Gaussian UV fixed point and non-trivial IR fixed point (Fisher-Wilson fixed point). Exists as a non-trivial QFT. QCD in 4d (Gaussian UV fixed point and non-trivial IR limit) Exists as a non-trivial QFT, and finally ϕ^4 in 4d (IR Gaussian fixed point, and no non-trivial UV fixed point). Does not exist as a non-trivial QFT when UV cut off is removed.



$$S = \sum_x a^4 \left(\frac{1}{2} \sum_\mu (\Delta_\mu \phi_0(x))^2 + \frac{1}{2} m_0^2 \phi_0^2(x) + \frac{g_0}{4!} \phi_0^4(x) \right).$$

$$a \frac{\partial g_0}{\partial a}\Big|_{g_R} = -\beta(g_0).$$



$$S=\sum_x\left(-2\kappa\sum_\mu\phi(x)\phi(x+\mu)+\phi^2(x)+\lambda(\phi^2(x)-1)^2-\lambda\right).$$

$$a\phi_0(x)=\sqrt{2\kappa}\phi(x),\quad a^2m_0^2=\frac{1-2\lambda}{\kappa}-8,\quad g_0=\frac{6\lambda}{\kappa^2}$$

J. Ambjørn Lattice Gravity and RG

Implementing this in a finite size scaling setting:

$$\langle \phi(x_n)\phi(x_m)\rangle \sim\sim e^{-m_R|x_n - x_m|} = e^{-m_R a|n-m|} = e^{-|n-m|/\xi}.$$

$m_R a \to 0$ means $\xi \to \infty$ and we want to approach this situation while keeping g_R fixed. We can study finite volume physics by insisting that $V = N_4 a^4$ stays constant. We ensure the physical interpretation of V constant, while changing the bare coupling constant (κ, λ) such that g_R is constant, by demanding in addition that for the linear size $L := N^{1/4}$

$$\frac{\xi(\kappa,\lambda)}{L} = const., \quad \text{i.e.} \quad V^{1/4} m_R = const.$$

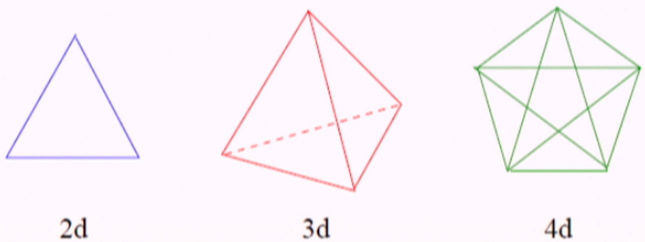
Thus effectively $N_4 \to \infty$ when we move along a $g_R(\kappa, \lambda) = const.$ trajectory towards the putative UV fixed point. If this UV point does not exist we will never reach $N_4 = \infty$ moving along the trajectory.

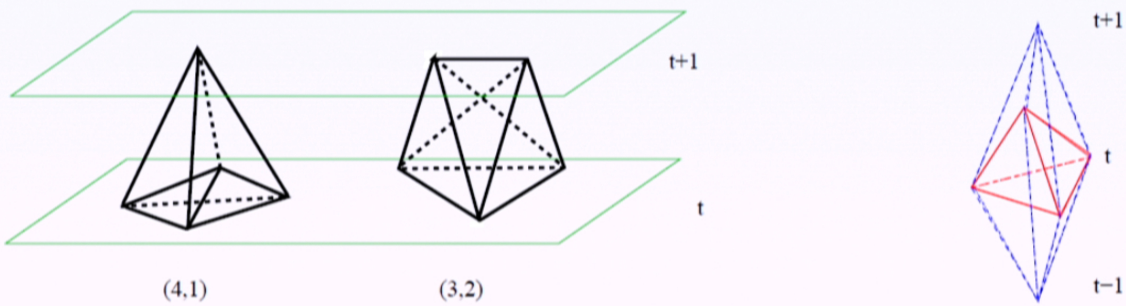
We now attempt to define QG using a lattice theory. DT and CDT are two specific ways to do this by making the lattice itself reflect the space-time geometry. The idea is that a triangulation uniquely specifies a piecewise linear geometry if we are given the lengths of all edges. For these geometries there is a geometric prescription of the Einstein-Hilbert action introduced by Regge. Thus summing or integrating over a suitable class of piecewise linear geometries using the Regge action might provide us with a regularized path integral, the UV cut off being the minimal edge length allowed.

DT and CDT implement this in the simplest possible way, using building blocks.

showcasing piecewise linear geometries via building blocks:

⟶





CDT slicing in proper time. Topology of space preserved.

$$a_t^2 = -\alpha a_s^2, \qquad iS_L[\alpha] = -S_E[-\alpha]$$

$$S_E[-\alpha] = -(\kappa_0 + 6\Delta)N_0 + \kappa_4\left(N_4^{(2,3)} + N_4^{(1,4)}\right) + \Delta\left(N_4^{(2,3)} + 2N_4^{(1,4)}\right)$$

$$S_E^{DT} = -\kappa_0 N_0 + \kappa_4 N_4$$

$$\int \mathcal{D}[g_{\mu\nu}]\, e^{-S[g_{\mu\nu}]} \to \sum_{T_a} e^{-S[T_a]}$$

where a is the link length which serves as a UV cut off.

The Regge prescription works in any dimension and using identical building blocks it becomes really simple:

$$\begin{aligned} S[g] &= -\frac{1}{16\pi G}\int d^Dx \sqrt{g(x)}\, R(x) + \frac{2\Lambda}{16\pi G}\int d^Dx \sqrt{g} \\ S[T] &= -\kappa_{D-2} N_{D-2}(T) + \kappa_D N_D(T) \end{aligned}$$

Quantum gravity becomes a pure counting problem

$$\boxed{Z(x,y) = \sum_T e^{-S[T]} = \sum_{N_{D-2},N_D} \mathcal{N}(N_D, N_{D-2})\, x^{N_D} y^{N_{D-2}}} \qquad \begin{aligned} x &= e^{-\kappa_D} \\ y &= e^{\kappa_{D-2}} \end{aligned}$$

Does it work?

Testing gound: $D = 2$. No graviton, but still reparametrization invariant, and $D = 2$ is a theory of maximally fluctuating geometries:

$$S = \Lambda \int d^2x \sqrt{g(x)} = \Lambda V(g), \quad Z(\Lambda) = \int \mathcal{D}[g]\, e^{-\Lambda V(g)}.$$

$$Z(\Lambda) = \int_0^\infty dV\, e^{-\Lambda V} Z(V), \qquad \boxed{Z(V) = \int_{V(g)=V} \mathcal{D}[g]\; 1}.$$

In $D = 2$ "gravity" is a renormalizable theory and one can calculate $Z(V)$ using methods of continuum QFT. Also one can count the number of triangulations and let the lattice spacing $a \to 0$. Agreement.

Thus the scaling limit of the lattice theory reproduces the result of a fully reparametrization invariant continuum theory.

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Universality in the Wilsonian framework of lattice QFT is governed by a divergent correlation length. But how to think at all about a correlation length in a theory which involves QG and where we are integrating over the geometries which define the correlation length.

Ordinary QFT: Assume the volume V is sufficiently large and rotation and translational invariance except for boundary effects. ($S(R)$ "area" of spherical shell)

$$\langle \phi\phi(R)\rangle_V = \int \mathcal{D}\phi \, e^{-S[\phi]} \iint \frac{dx\, dy}{S(R)\, V} \, \phi(x)\phi(y)\, \delta(R - |x-y|).$$

Generalization to a diffeomorphism invariant, metric theory

$$\boxed{\langle \phi\phi(R)\rangle_V = \int_{V(g)=V} \mathcal{D}[g]\, \mathcal{D}\phi \, e^{-S[g,\phi]} \iint \frac{dx dy \sqrt{g(x)g(y)}}{S_g(y,R) V} \phi(x)\phi(y)\delta(R - D_g(x,y))}$$

$D_g(x,y)$ is the geodesic distance between x and y.

Main question: does it make sense to think about a divergent "diffeomorphism invariant" correlation length in a lattice theory? Does the above definition provide us with the wanted correlation length?

If not, it is difficult to believe that one can apply any Wilsonian way of thinking...

Again this can be tested in 2d quantum gravity coupled to matter. Example: the Ising model put on the random lattices of DT. It has a critical point (even if we average over random triangulations), a higher order phase transition, and one has a divergent spin-spin correlation length, measured in terms of the "diffeomorphism invariant geodesic distance" R defined above. Again agreement between lattice and continuum calculations.

We expect the following behavior for a conformal theory coupled 2d Euclidean QG:

$$\langle \phi\phi(R) \rangle_V = R^{-d_h \Delta} \, F\left(\frac{R}{V^{1/d_h}}\right),$$

$$\langle \phi\phi(R) \rangle_V = V^{-\Delta} \; \frac{F(x)}{x^{d_h \Delta}}, \qquad x = \frac{R}{V^{1/d_h}}$$

Here $F(0) = const. > 0$, and $F(x)$ falls off for $x > 1$.

Since the geodesic distance R is a complicated, composite quantum field operator, it can scale anomalously. And it does! Thus d_h (the so-called Hausdorff dimension) is not necessarily equal to 2 (as in flat space). Also the scaling dimension Δ is not equal to the scaling dimension of the conformal theory in flat spacetime.
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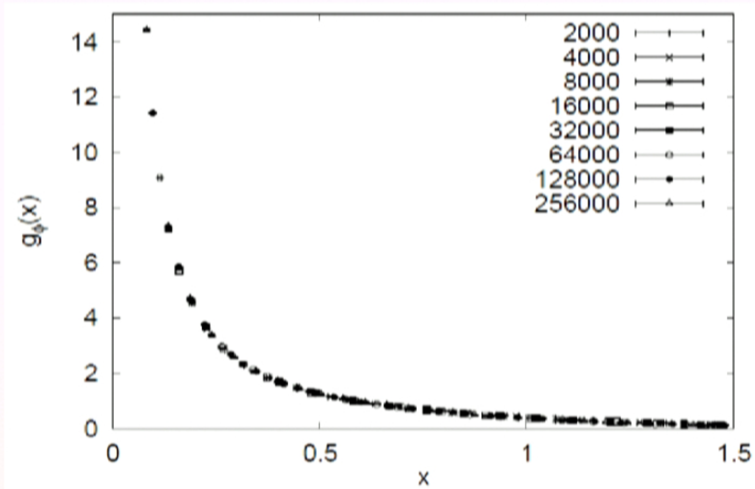
One can test the prediction for d_h and Δ using DT.

UV cut off: links of length a. Spacetimevolume $V \sim Na^2$.
Geodesic distance $\ell \approx$ link distance or dual link distance (moving between centers of triangles).

$$\langle \phi\phi(\ell) \rangle_N = N^{-\Delta} \, \frac{F(x)}{x^{d_h \Delta}} \qquad x = \frac{\ell}{N^{1/d_h}} \qquad \boxed{\text{FSS!}}$$



Conclusion: If quantum gravity in four dimensions exists as a "stand alone" QFT there seems to be no conceptual problems regularizing it using a lattice in the way described (DT, CDT).

Last main question:
If one only uses the Einstein-Hilbert action and rotate to Euclidean signature, the Euclidean E-H action is unbounded from below. Any reasonable regularization of the action will suffer from the same problem up to cut off effects. In DT and CDT the finite edge lenghts provide the cut off. In the path integral it is thus impossible to take a semiclassical limit. It will always be dominated by the unboundedness of the conformal mode, which in the DT model manifests itself as so-called branched polymers, and in CDT as disconnected universes of "Planck scale".

The E-H DT action has two coupling constants:

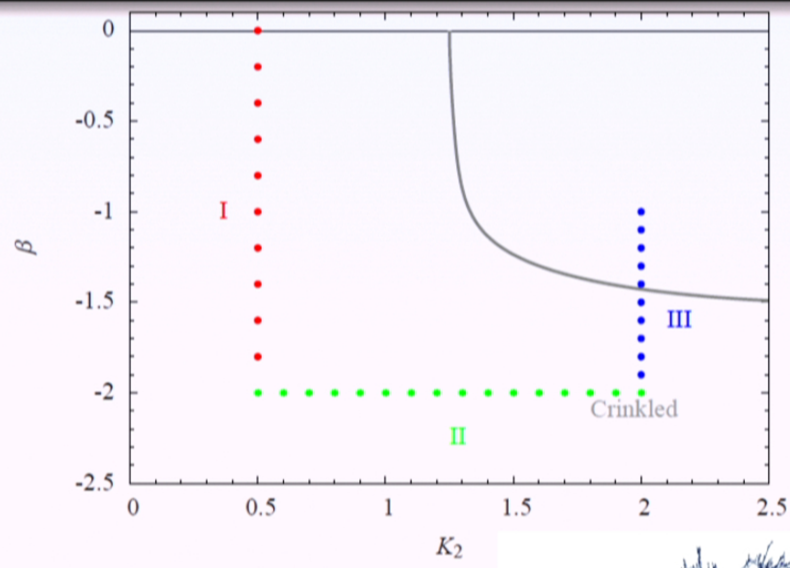
$$S_E^{DT} = -\kappa_0 N_0 + \kappa_4 N_4$$

In the computer simulations we keep N_4 fixed (but simulate with different N_4's). Thus effectively we have one coupling constant κ_0, which is the inverse of the gravitational coupling constant. We find a phase transition at a critical value κ_0^c. Unfortunately the transition is a first order transition and it seems difficult to associate any interesting continuum physics with the model. For $\kappa_0 > \kappa_0^c$ one obtains branched polymers and for $\kappa_0 < k_0^c$ one obtains a crumpled universe of no extention ($d_h = \infty$). Adding higher curvature terms to the E-H action using the Regge formalism was unsuccessful (the DT formalism not convenient for this). However, recently Jack Laiho added indirectly higher curvature terms to the theory and saw some evidence that a new phase occurred which showed some similarity with one of the CDT phases.

Why was this exciting? Because the CDT theory is formally a unitary theory (it has a reflection positive transfer matrix) and thus it might be that this particular realization of the higher curvature terms actually lead to a theory which is both renormalizable and unitary! Unfortunately it is not true.

$$\sum_T \frac{1}{C(T)} \to \sum_T \frac{1}{C(T)} \prod_{t=1}^{N_2} o_t^{\beta},$$

Thus negative β leads to suppression of high curvature terms.


0
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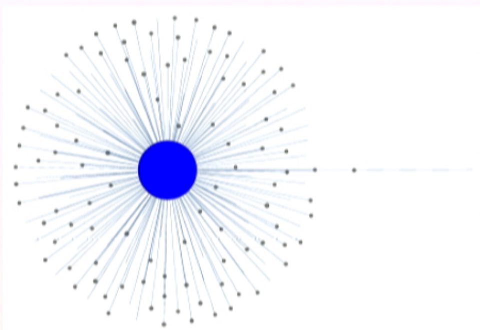
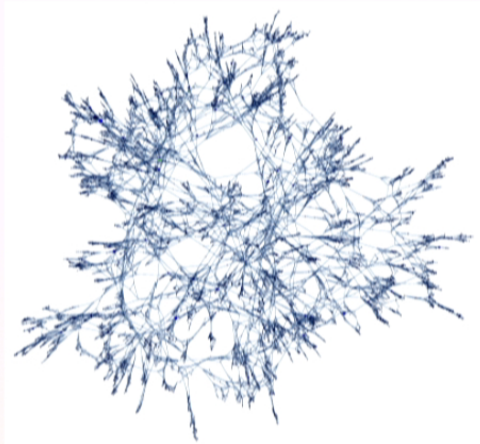
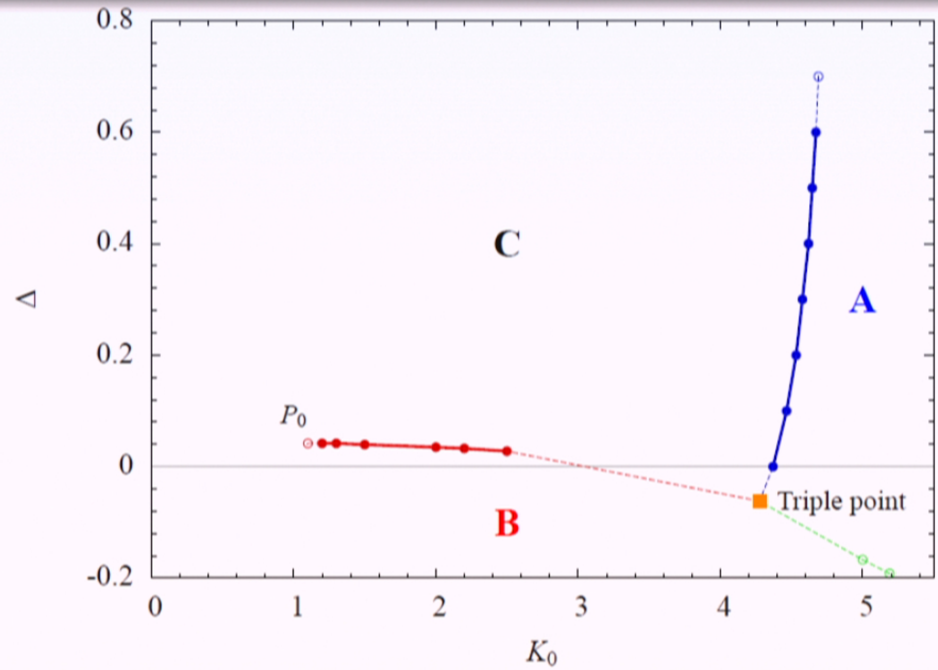


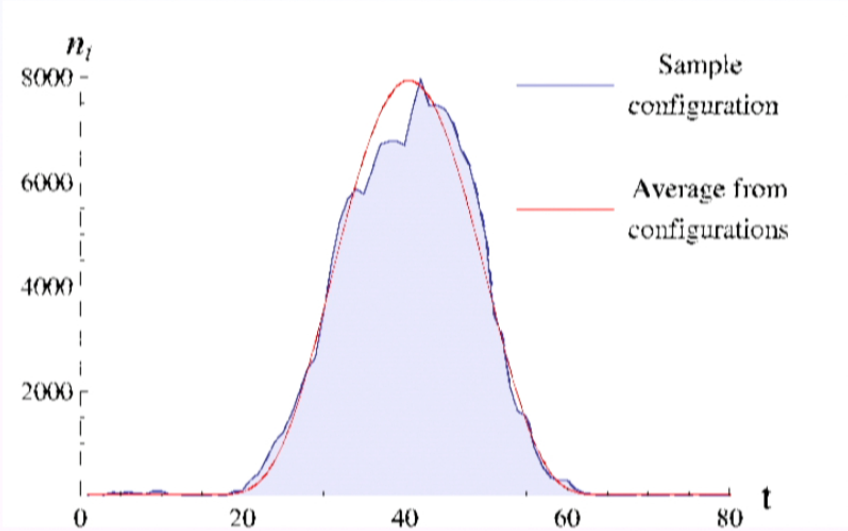


CDT phase diagram



Lifshitz-like diagram....

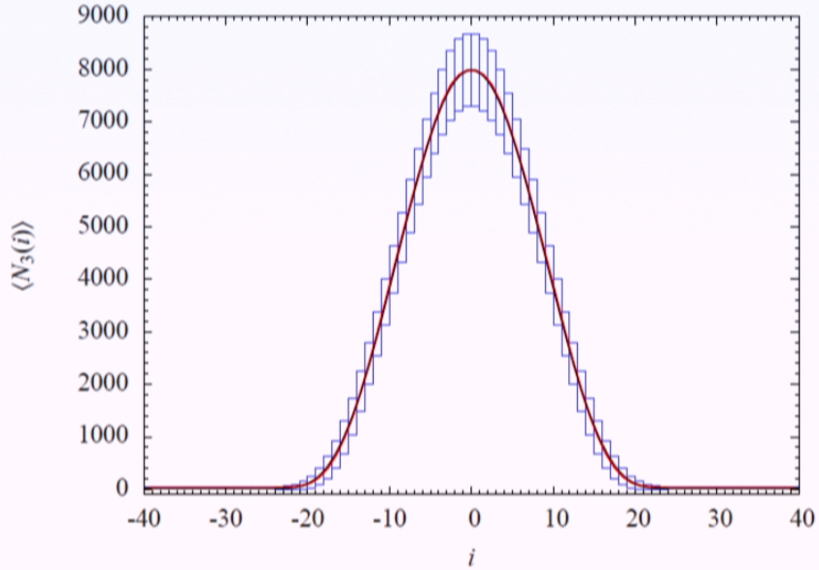
Phase C: Constant 4d geometry (constant magnetization)
Phase B: no 4d geometry (zero magnetization)
Phase A: conformal mode (oscillating magnetization)



A typical configuration. Distribution of a spatial volume $N_3(t)$ as a function of (imaginary) time t.

Quantum fluctuation around a semiclassical background?

Configuration consists of a "stalk" of the cut-off size and a "blob". Center of the blob can shift. We fix the "center of mass" to be at zero time.



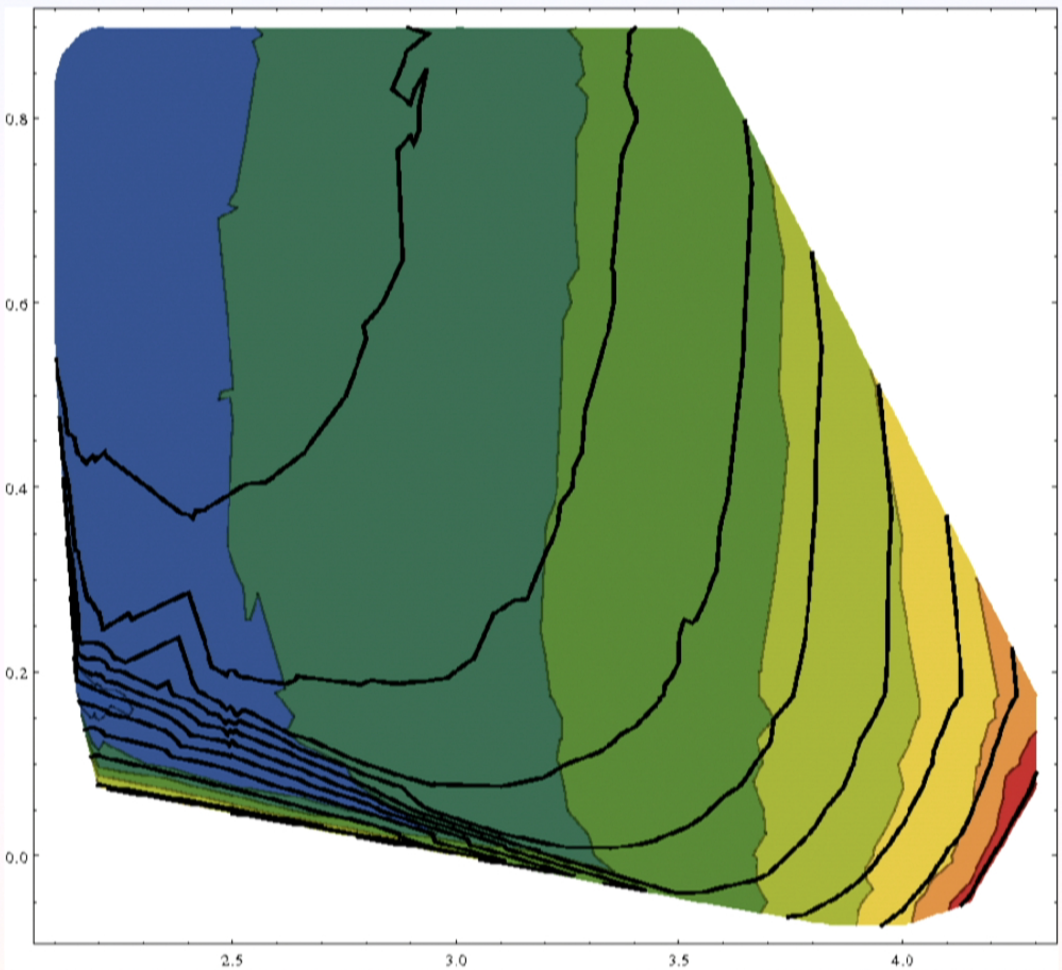
$$\langle N_3(i)\rangle_{N_4} = N_4\,\frac{3}{4}\,\frac{1}{\omega N_4^{1/4}}\,\cos^3\left(\frac{i}{\omega N_4^{1/4}}\right), \quad |i| < \frac{\pi}{2}\omega\, N_4^{1/4}$$

$$\langle \delta N_3(i)\rangle_{N_4} = \gamma\, N_4^{1/2}\, F\left(\frac{i}{\omega N_4^{1/4}}\right).$$

We now want to define a path of constant continuum physics in the (κ_0, Δ) coupling constant space. $\omega(\kappa_0, \Delta)$ determines the "shape" of the universe. Firstly we want to keep $V_4 \propto N_4 a^4$ fixed. This is a way to take the lattice spacing $a \to 0$ by changing N_4 (which we control). However, the shape of the emergent background will change with ω. Thus we need to keep $\omega(\kappa_0, \Delta)$ fixed. This does still not ensure that the "emergent" continuum universe is unchanged when $N_4 \to \infty$ because we have that the for continuum three-volume $V_3(\tau_i) = a^3 N_3(i)$:

$$\frac{\delta V_3(\tau_i)}{V_3(\tau_i)} = \frac{\delta N_3(i)}{N_3(i)} \propto \frac{\gamma(\kappa_0, \Delta)\, \omega(\kappa_0, \Delta)}{N_4^{1/4}},$$

$$\gamma(\kappa_0(N_4), \Delta(N_4)) \propto N_4^{1/4}, \qquad \omega(\kappa_0(N_4), \Delta(N_4)) = const.$$


0.8
0.6
0.4
0.2
0.0
2.5
3.0
3.5
4.0

The following effective action seems to describe well the data from the computer simulations

$$S_{discr} = k_1 \sum_i \left(\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + \tilde{k} N_3^{1/3}(i) \right).$$

If we assume a minisuperspace description

$$ds^2 = N^2(t)dt^2 + a^2(t)d\Omega_3^2,$$

the following quadratic action

$$S_{cont} = \tilde{\kappa} \int dt d^3x N(t) \sqrt{g} \left((K_{ij}K^{ij} - \lambda K^2) + \tilde{\delta} R_3 \right),$$

which has the solution

$$V_3(\tau) = V_4 \frac{3}{4} \left(\frac{8\pi^2}{3\xi^3 V_4} \right)^{1/4} \cos^3 \left(\left(\frac{8\pi^2}{3\xi^3 V_4} \right)^{1/4} \tau \right), \quad N = const.$$

This corresponds to the metric

$$ds^2 = d\tau^2 + R^2 \cos^2\left(\frac{\tau}{\xi R}\right) d\Omega_3^2, \quad R = \left(\frac{V_4}{\xi}\right)^{1/4}$$

(a deformed sphere) with

$$\xi^2 = \frac{1 - 3\lambda}{2\tilde{\delta}}.$$

and leads to the discretized action above with the identification

$$\xi^{3/4} \propto \omega, \qquad \text{i.e.} \quad k_1 \propto (1 - 3\lambda)\, a^2 \tilde{\kappa}, \quad \tau_i \propto i \cdot a.$$

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A UV fixed point should have $a^2\tilde{\kappa} \to const$. Quadratic fluctuations lead to

$$\gamma^2 \propto \frac{1}{k_1} \propto \frac{1}{(1-3\lambda)\, a^2\tilde{\kappa}}$$

Thus $\gamma \propto N_4^{1/4}$ and $a^2\tilde{\kappa} \to const$ leads to $\lambda \to 1/3$: Conformal invariant point of HL lagrangian. This as a candidate of the UV fixed point is however not consistent with our definition of constant physics. Clearly one has to define constant physics by some anomalous scaling of time relative to space.

Work in progress: we have to study more seriously the region close to the B-C line. Our effective minisuperspace action is not reliable. Presently we are trying to use the transfer matrix of CDT.

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