Title: What happens to the SchrÃ¶dinger solution in quantum corrected gravity?

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Abstract:

What Happens to the Schwarzschild Solution in Quantum Corrected Gravity?

K.S. Stelle

Imperial College London

"Renormalization group approaches to quantum gravity"

Perimeter Institute, Waterloo
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K.S.S., Gen.Rel.Grav. 9 (1978) 353
W. Nelson, Phys.Rev. D82 (2010) 104026, arXiv:1010.3986
H. Lü, A. Perkins, C.N. Pope & K.S.S., in preparation

Quantum Context

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of $\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \rho R^2)$ terms ab initio in the gravitational action leads to a renormalizable $D = 4$ theory, but at the price of a loss of *unitarity* owing to the modes arising from the $\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

K.S.S., *Phys. Rev.* **D16**, 953 (1977).

[In $D = 4$ spacetime dimensions, this $(\mathrm{Weyl})^2$ term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination $\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$].

Classical gravity with higher derivatives

We shall not try here to settle philosophical debates about various attitudes that can be taken towards the implementation of quantum corrections (Wilsonian, or other), but shall simply adopt a point of view taking the higher-derivative terms and their consequences for gravitational field-theory solutions seriously.

Accordingly, we shall consider the gravitational action $I = -\int d^4x\sqrt{-g}(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \gamma\kappa^{-2}R)$, which can also be rewritten $I = -\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + (\frac{\alpha}{3} - \beta)R^2 + \gamma\kappa^{-2}R)$, so in terms of the earlier parametrization one has $\rho = \frac{\alpha}{3} - \beta$.

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} \;=\;& (\alpha - 2\beta)\nabla_\nu\nabla_\mu R - \alpha\nabla^\eta\nabla_\eta R_{\mu\nu} - (\frac{\alpha}{2} - 2\beta)g_{\mu\nu}\nabla^\eta\nabla_\eta R \\ & +2\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} - 2\beta RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\alpha R^{\eta\lambda}R_{\eta\lambda} - \beta R^2) \\ & +\gamma\kappa^{-2}R_{\mu\nu} - \frac{1}{2}\gamma\kappa^{-2}g_{\mu\nu}R = T_{\mu\nu} \end{aligned}$$

Separation of modes in the linearized theory

Solving the full nonlinear field equations is clearly a challenge. One can make initial progress by restricting the metric to infinitesimal fluctuations about flat space, defining $h_{\mu\nu} = \kappa^{-1}(g_{\mu\nu} - \eta_{\mu\nu})$ and restricting attention to field equations linearized in $h_{\mu\nu}$, or equivalently by restricting attention to quadratic terms in $h_{\mu\nu}$ in the action.

The action then becomes

$$
\begin{aligned}
I_{\text{Lin}} &= \int d^4x\{-\frac{1}{4}h^{\mu\nu}(\alpha\kappa^2\Box - \gamma)\Box P^{(2)}_{\mu\nu\rho\sigma}h^{\rho\sigma} \\
&\quad + \frac{1}{2}h^{\mu\nu}[2(3\beta - \alpha)\kappa^2\Box - \gamma]\Box P^{(0;s)}_{\mu\nu\rho\sigma}h^{\rho\sigma}\}
\end{aligned}
$$

$$
\begin{aligned}
P^{(2)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - P^{(0;s)}_{\mu\nu\rho\sigma} \\
P^{(0;s)}_{\mu\nu\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \qquad \omega_{\mu\nu} = \partial_\mu\partial_\nu/\Box\ ,
\end{aligned}
$$

where the indices are lowered and raised with the background metric $\eta_{\mu\nu}$.

Static and spherically symmetric solutions

Now we come to the question of what happens to spherically symmetric gravitational solutions in the higher-curvature theory. One may choose to work in traditional Schwarzschild coordinates, for which the metric is given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In the linearized theory, one then finds the general solution to the source-free field equations $H^L_{\mu\nu} = 0$, where $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are integration constants:

$$A(r) =$$

$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$

$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made *via* standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$A(r) = 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{48\pi\gamma}\frac{e^{-m_0 r}}{r}$$

$$B(r) = 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r}$$

with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

What about the Schwarzschild solution?

Returning to the full nonlinear field equations in the source-free case $H_{\mu\nu} = 0$, one notes directly that any solution to the source-free Einstein equation $R_{\mu\nu} = 0$ will also be a solution to the higher-curvature theory's *source-free* equations. But do we really want such solutions now?

In the above toy scalar higher-derivative model, the source-free field equation is $(\Box - m^2)\Box\phi = 0$. While it is true that any genuine solution to $\Box\phi = 0$ satisfies the source-free higher-derivative equations, things go wrong when one considers the standard q/r solution to the *sourced* static problem $\nabla^2\phi = q\delta^3(\vec{x})$.

In order for this to be a solution to the higher-derivative theory, the source on the right-hand side of the field equation would need to be of the form $q(\nabla^2 - m^2)\delta^3(\vec{x})$. This is a highly singular distribution, and is *not even positive* when integrated with a generic profile function. In other words, the attempt to claim solutions to the second-order $\Box\phi = 0$ theory as solutions for the higher-derivative theory implies couplings to other "matter" fields without energy positivity.

From the above discussion, we conclude that, although the Schwarzschild solution is an apparent solution to the source-free higher-derivative equations $H_{\mu\nu} = 0$, it will not be a good solution arising from normal minimal coupling of gravity to matter fields. The sought-for solution should, in the weak-field linearized limit, display Yukawa corrections to the Newtonian $1/r$ potential at spatial infinity.

Now consider the full nonlinear field equations for the spherically symmetric case, once again source-free. They are somewhat frightful. Initially, one gets one third-order equation and one fourth-order equation. However, the system can then be rearranged into a system with two third-order equations for the two metric variables $A(r)$ and $B(r)$.

An Israel Theorem

Hopes for an analytic solution to the static spherically symmetric equations are clearly rather slim. In the end, it will be necessary to explore such solutions by numerical means. However, some definite conclusions can be reached by analytical methods. A key tool in this analysis is an extension of Werner Israel's "no-hair" theorem.

W. Nelson, Phys.Rev. D82 (2010) 104026; arXiv:1010.3986; H. Lü, A. Perkins, C.N. Pope & K.S.S. to appear.

This theorem extends the classic Israel-Lichnerowicz theorem of GR to the Einstein-plus-quadratic-curvature gravity theories for static and spherically symmetric solutions. The approach is a standard one for "no-hair" theorems: find an appropriate tensorial factor to contract with the $H_{\mu\nu}$ field equations and then integrate out from a presumed horizon null-surface to asymptotically flat infinity. Provided that contraction with the right tensorial factor has been made, integration by parts then yields an integrand formed from a sum of squares all with the same sign, plus boundary terms and one more type of term that will have the same sign as the sum of squares provided two inequalities are respected.

In the general non-tachyonic case for $\alpha > 0$ and $3\beta - \alpha > 0$, one needs to complete the discussion using the non-trace part of the field equation. The derivation then goes similarly, with a surface term that vanishes on a null-surface and at asymptotically flat spatial infinity. One again obtains a requirement for the vanishing of an integral over the spatial slice of a sum of squares with the same (negative) sign, plus two final terms that are also of the same sign provided certain inequalities are obeyed.

From the required vanishing of this integral, one finds that, provided the following inequalities are satisfied

$$\left.\begin{aligned} m_2^2 - {}^{(3)}R \geq 0 \\ m_2^2 \bar{R}^a{}_b \bar{R}^b{}_a + 2\bar{R}^a{}_b \bar{R}^b{}_c \bar{R}^c{}_a \geq 0 \end{aligned}\right\} \begin{array}{l} \text{Become trivial for } m_2 \text{ large.} \\ \text{How large is relevant? See later.} \end{array}$$

one must have $\bar{R}_{ab} = 0$ and ${}^{(3)}R = 0$, where $\bar{R}_{ab}$ is the pull-back of the $D = 4$ Ricci tensor to the $D = 3$ spatial slice. Together, these imply $R_{\mu\nu} = 0$, requiring the solution to be Schwarzschild.

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Indicial Analysis

A type of asymptotic analysis of the field equations complementary to the linearized analysis at $r \to \infty$ spatial infinity is study of the *indicial equations* for behavior as $r \to 0$. K.S.S. 1978 Let

$$A(r) = a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \cdots$$
$$B(r) = b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots$$

and analyze the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general α, β theory:

$(s,t) = (1,-1)$ with 4 free parameters
$(s,t) = (0,0)$ with 3 free parameters
$(s,t) = (2,2)$ with 6 free parameters

However, for the $(1,-1)$ and $(0,0)$ cases, the Israel theorem can once again be used to rule out these cases as candidates for solutions that match to the Yukawa-corrected asymptotically-flat solutions at infinity. This leaves the $(2,2)$ behavior at the origin as the unique remaining candidate for such solutions.

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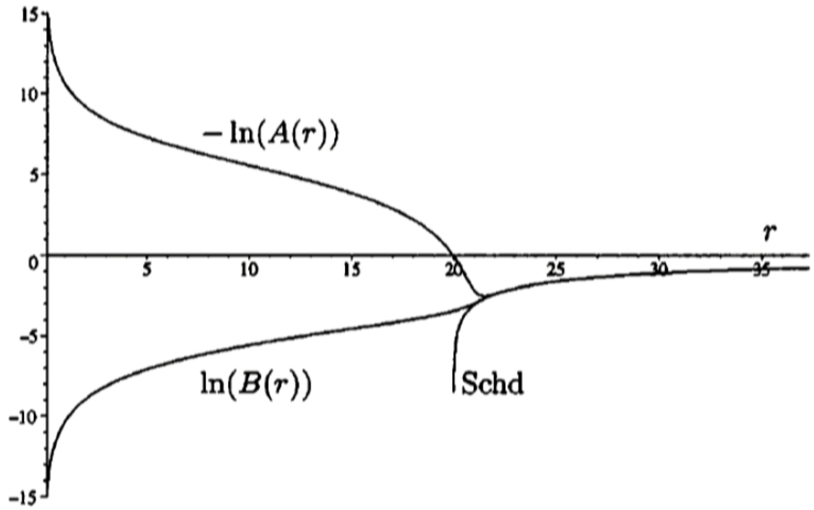
However, for the $(1,-1)$ and $(0,0)$ cases, the Israel theorem can once again be used to rule out these cases as candidates for solutions that match to the Yukawa-corrected asymptotically-flat solutions at infinity. This leaves the $(2,2)$ behavior at the origin as the unique remaining candidate for such solutions.

Numerical Analysis

In the absence of a suitably general analytic solution to the higher-derivative equations $H_{\mu\nu} = 0$, one must have recourse to numerical studies. This has been investigated by Bob Holdom.

B. Holdom, Phys.Rev. D66 (2002) 084010; hep-th/0206219

Here is a graph of his results, showing, indeed, r^2 behavior for both $A(r)$ and $B(r)$ as $r \to 0$, but connecting on to a Yukawa-corrected approximation to the Schwarzschild solution as $r \to \infty$:



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Taking this numerical study together with the implications of the Israel theorem, a coherent picture emerges:

- The link between behavior near the origin, $r \to 0$, to asymptotically-flat Yukawa-corrected solutions at infinity happens in the $(s, t) = (2, 2)$ class of solutions to the higher-derivative theory. Note that the number of free parameters at the origin for this class matches precisely the number of parameters in the linearized solution. (Of course, rising Yukawa terms need to be excluded from the asymptotically flat solution set, but they are still solutions to the linearized theory.)
- *There is no horizon* in this set of minimally-coupled, Yukawa-corrected solutions. Solutions asymptotically approach the Schwarzschild solution for large r, but differ strikingly in what would have been the inner-horizon region.

- Although there is a curvature singularity at the origin in the (2, 2) class of solutions (*e.g.* for this class, one has $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 20a_2^{-2}r^{-8} + \cdots$), this is a *timelike* singularity, unlike the *spacelike* singularity of the Schwarzschild solution.

Although one might complain that this non-Schwarzschild behavior occurs in a theory with a massive spin-two ghost, the limit as $\alpha \to 0$ removes this ghost as well as the complications of the m_2-dependent inequalities. The $R + R^2$ theory at $\alpha = 0$ is ghost-free, and yet has the same horizonless structure for its spherically symmetric static solutions as in the general α, β case, when its spherically-symmetric solution is derived from minimal coupling to non-gravitational matter.

Stability Issues

So, what does one make of all this for real black holes? The above spherically-symmetric static analysis does not consider the issue of stability, *i.e.* what happens to dynamical solutions evolving from small perturbations away from the static solutions. Since no closed-form version of the exact (2,2) solutions is available, this is not an easy question to address. However, one can get some information by considering the stability of the classic Schwarzschild solution within the higher-derivative theory.

- In the $R + R^2$ theory, study of the normal modes about the Schwarzschild solution shows it to be stable. This is perhaps not surprizing, since that theory is classically equivalent to ordinary Einstein gravity plus a scalar field with a peculiar potential, for which the ordinary GR stability considerations and no-hair theorem should apply. Whitt, Starobinsky

- When the $(\text{Weyl})^2$ term is present in the action, however, the stability situation is different: there may be a phase structure, depending on the value of $\mu = \frac{m_2 M}{M_{\rm Pl}^2}$, where m_2 is the spin-two particle mass, M is the mass of the black hole and $M_{\rm Pl}$ is the Planck mass. For $\mu \gg 1$, *i.e.* "largeish" black holes, one obtains stability for the Schwarzschild solution. For $\mu \leq 1$, on the other hand, stability is not guaranteed.

Starobinsky, private communication

- This was also studied by Brian Whitt Phys. Rev. D32 (1985) 379 , who showed that the $R + (\text{Weyl})^2$ theory should be stable for $\mu \geq 0.44$ and raised the question of whether an instability could set in for $\mu < 0.44$. Indeed, he suggested that there could be a bifurcation of the spherically symmetric solution set into two branches at this value.

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- Whitt's detailed calculation seemed to show, nonetheless, that for $\mu < 0.44$ there was still no instability, at least in a static perturbation analysis (*i.e.* for $k = 0$ momentum modes).
- This analysis has, however, been challenged in a more recent paper by Y.S. Myung Phys.Rev. D88 (2013) 2, 024039; arxiv:1306.3725 who claims that Whitt did not do the Schwarzschild stability analysis properly and instead does find, from a nonstatic $k \neq 0$ analysis, an instability of the Schwarzschild solution for $\mu < \mathcal{O}(1)$, which he compares to the Gregory-Laflamme instability of the $D = 5$ black string solution.

Accordingly, for microscopic spherically symmetric solutions in Einstein-plus-quadratic-curvature gravity, the most stable solution for *small* erstwhile black holes may very well be the kind of solution found in the above classical analysis, with a (classically) naked singularity and with no horizon.