Title: Tensor models in the large N limit

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Abstract: Tensor models generalize matrix models and provide a framework for the study of random geometries in arbitrary dimensions. Like matrix models they support a 1/N expansion, where N is the size of the tensor, with an analytically controlled large N limit. In this talk I will present some recent results in this field and I will discuss their implications for quantum gravity.

Tensor Models in the large N limit

Răzvan Gurău

RG for QG, PI 2014

The fundamental question

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How to quantize some gravity + matter action in D dimensions:

$$Z \sim \sum_{\text{topologies}} \int \mathcal{D}g_{(\text{metrics})}\, \mathcal{D}X_{\text{matter}}\, e^{-S}$$

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For instance, in $D = 2$ how do we quantize the Polyakov string action?

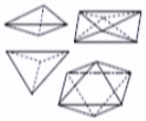
$$S \sim \kappa_R \int \sqrt{g}R - \kappa_V \int \sqrt{g} + \kappa_m \int d^2\xi \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$$

$$Z \sim \sum_{\text{topologies}} \int \mathcal{D}g_{(\text{worldsheet metrics})}\, \mathcal{D}X_{(\text{target space coordinates})}\; e^{-S}$$

Random Discrete Geometries

Quantum Gravity = summing random geometries.

Proposal: build the geometry by gluing discrete blocks, "space time quanta".



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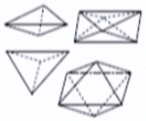

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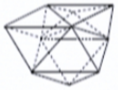
Fundamental interactions of few "quanta" lead to effective behaviors of an ensemble of "quanta".



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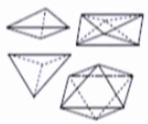

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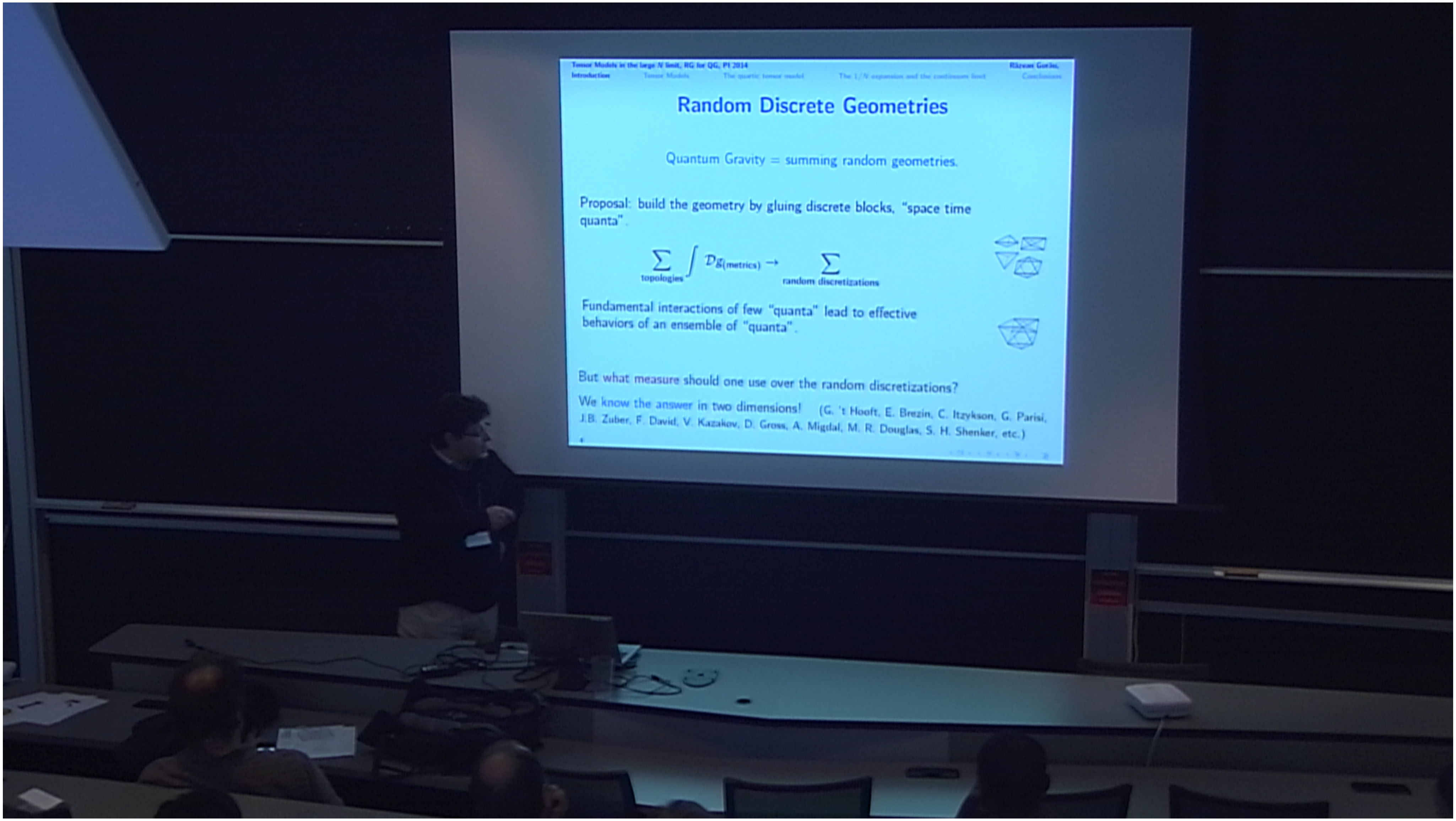
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We know the answer in two dimensions! (G. 't Hooft, E. Brezin, C. Itzykson, G. Parisi, J.B. Zuber, F. David, V. Kazakov, D. Gross, A. Migdal, M. R. Douglas, S. H. Shenker, etc.)

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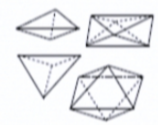

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A success story: Matrix Models provide a measure for random two dimensional surfaces. The theory of strong interactions, string theory, quantum gravity in $D = 2$, conformal field theory, invariants of algebraic curves, free probability theory, knot theory, the Riemann hypothesis, etc.

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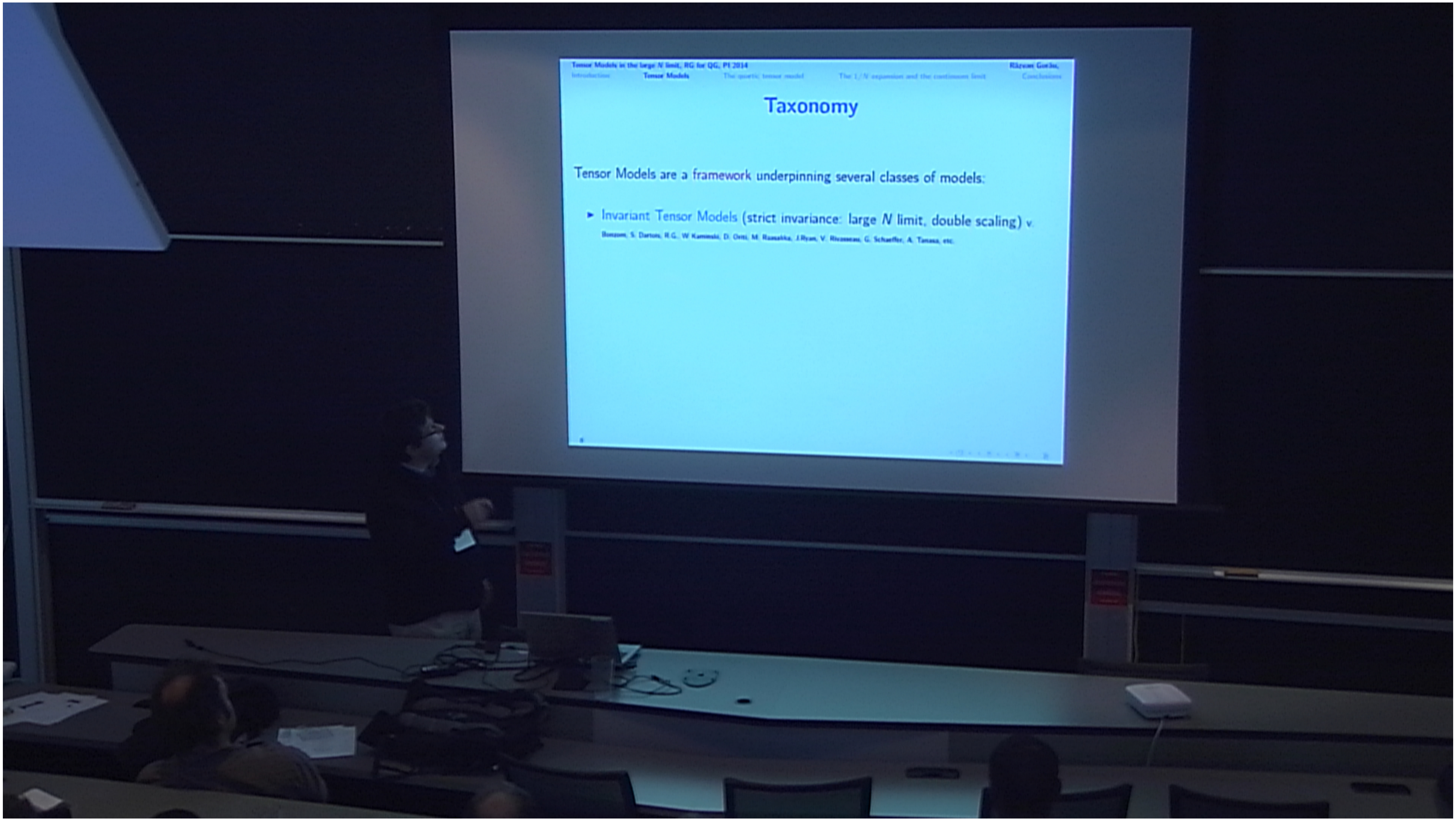
- Tensor invariance $\Rightarrow$ random discretizations.


Tensor Models in the large N limit; RG for QG; PI 2014
Tensor Models
Taxonomy
Tensor Models are a framework underpinning several classes of models:
Invariant Tensor Models (strict invariance: large N limit, double scaling) v.
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Tensor invariants as Colored Graphs

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Rank D complex tensor, no symmetry, transforming under the external tensor product of D fundamental representations of $U(N)$

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Invariants ("traces") $\sum_{a^1,q^1} \delta_{a^1q^1} \dots T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \dots$

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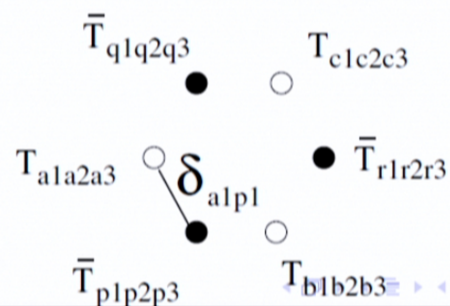
White (black) vertices for T ($\bar T$).

Edges for $\delta_{a^cq^c}$



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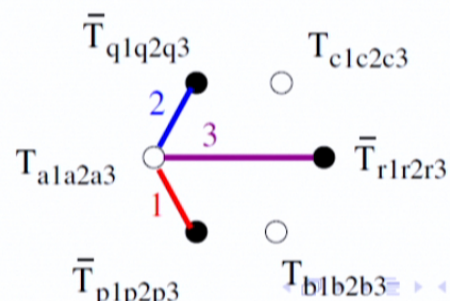

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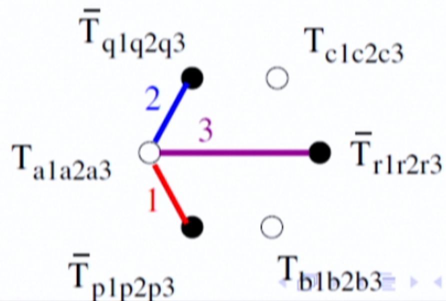

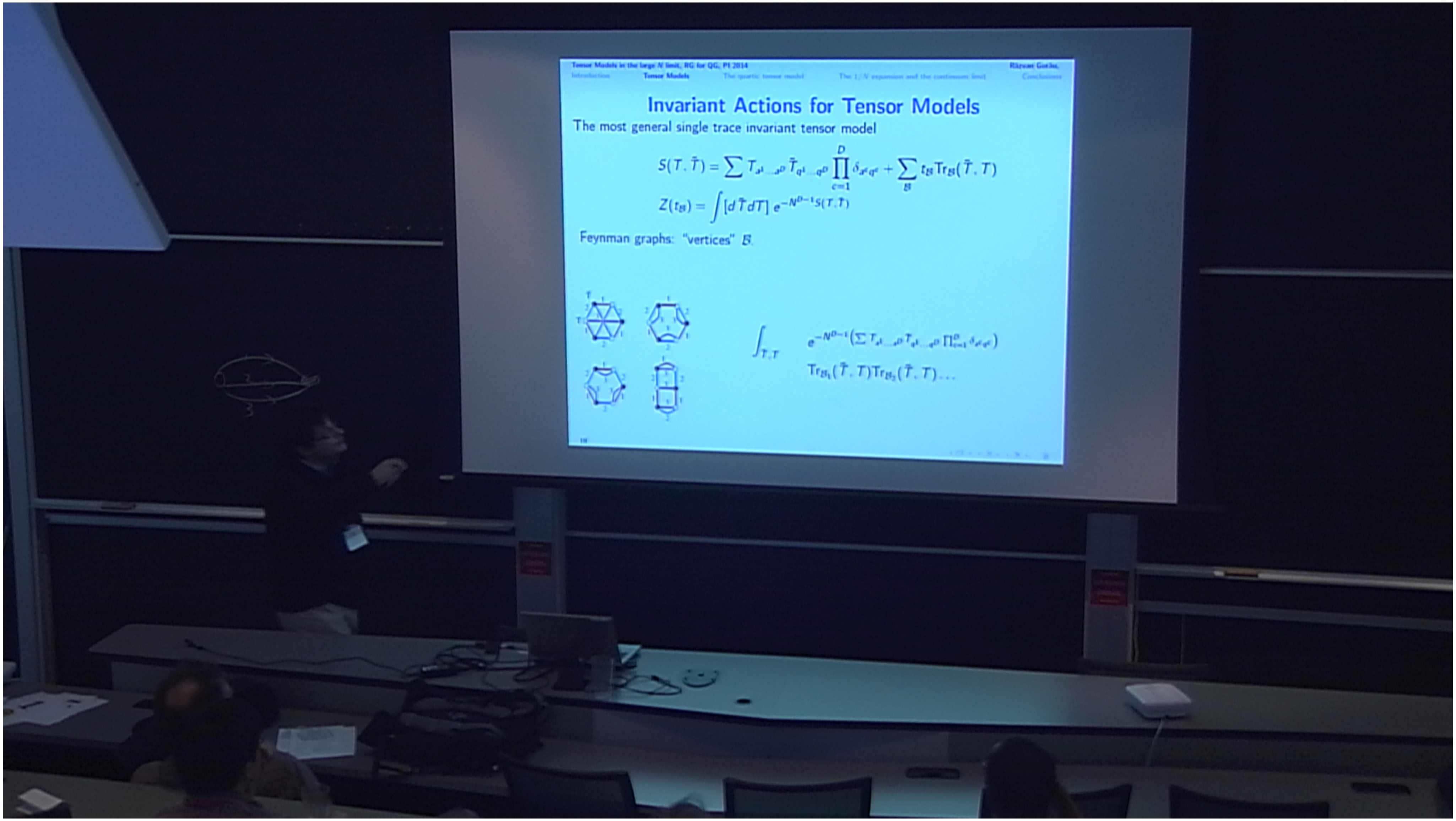
Invariant Actions for Tensor Models
The most general single trace invariant tensor model
$S(T,\bar{T}) = \sum T_{a^1 \dots a^D} \bar{T}_{q^1 \dots q^D} \prod_{c=1}^{D} \delta_{a^c q^c} + \sum_{\mathcal{B}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(\bar{T},T)$
$Z(t_{\mathcal{B}}) = \int [d\bar{T} dT] \, e^{-N^{D-1} S(T,\bar{T})}$
Feynman graphs: "vertices" $\mathcal{B}$.
$\int_{\bar{T},T}$
$\mathrm{Tr}_{\mathcal{B}_1}(\bar{T},T)\mathrm{Tr}_{\mathcal{B}_2}(\bar{T},T)\dots$

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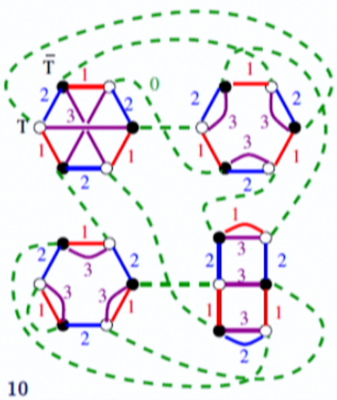
Feynman graphs: "vertices" $\mathcal{B}$. Gaussian integral: Wick contractions of T and $\bar{T}$ ("propagators") $\to$ dashed edges to which we assign the fictitious color 0.

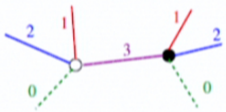


Graphs $\mathcal{G}$ with $D+1$ colors.

Represent triangulated D dimensional spaces.

Colored Graphs as gluings of colored simplices

White and black $D+1$ valent vertices connected by edges with colors $0, 1 \ldots D$.



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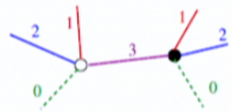
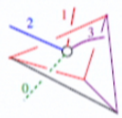
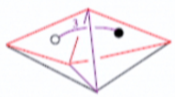


Vertex $\leftrightarrow$ colored D simplex .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



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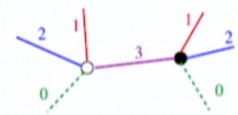

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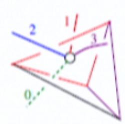

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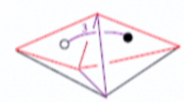

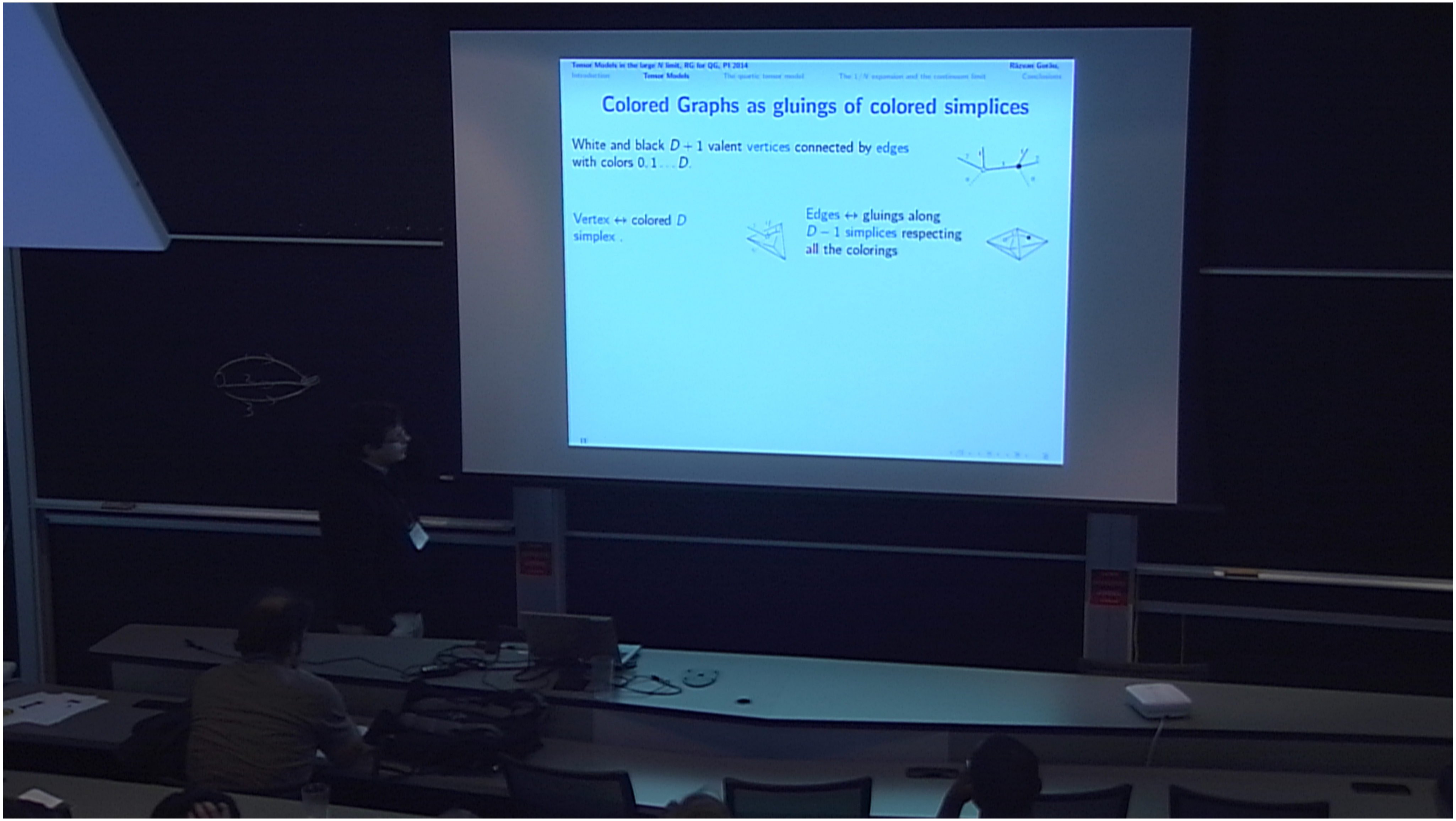
Tensor Models in the large N limit; RG for QG; PI 2014
Răzvan Gurău
Introduction
Tensor Models
The quartic tensor model
The 1/N expansion and the continuum limit
Conclusions
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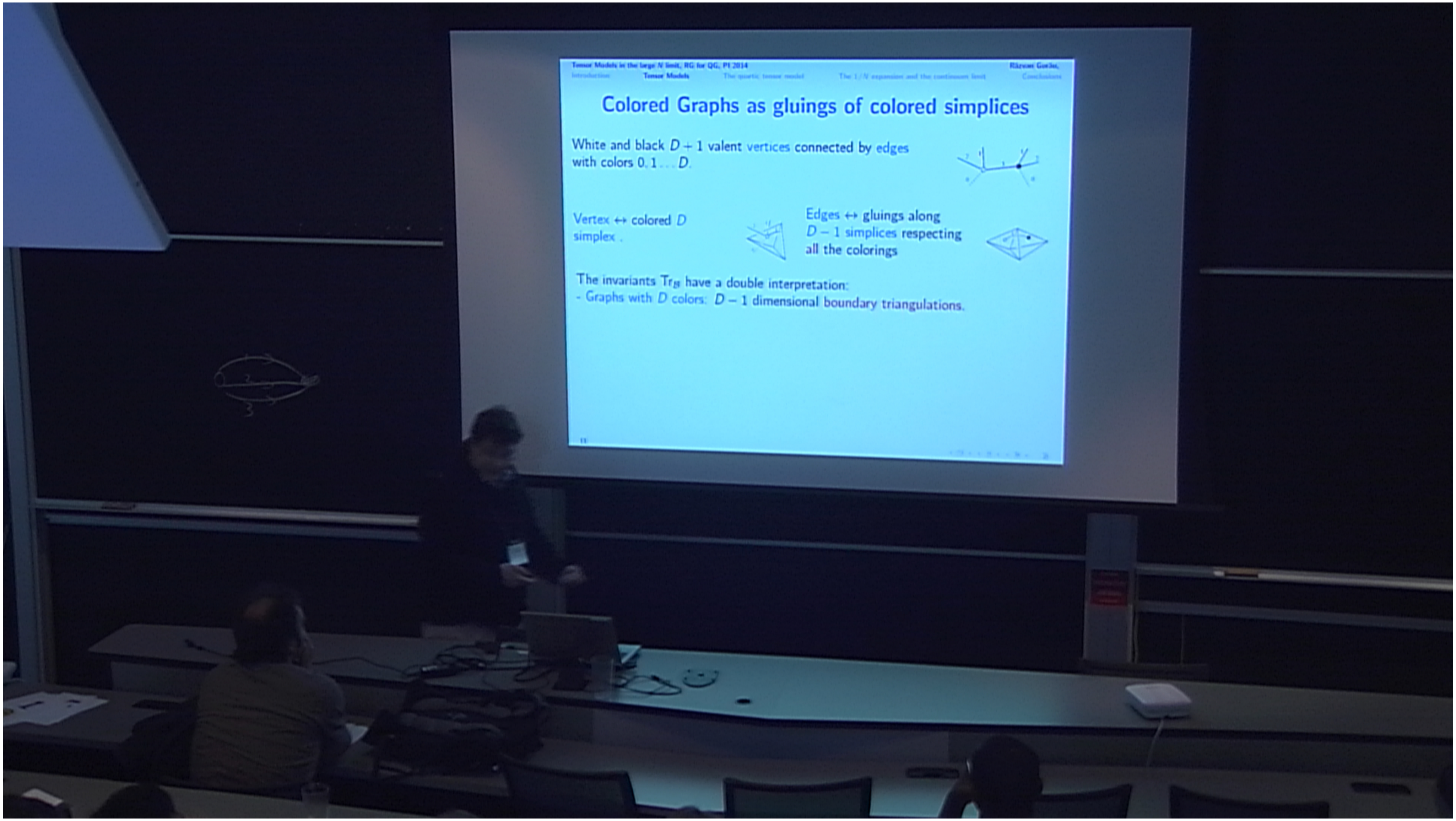

Colored Graphs as gluings of colored simplices
White and black D + 1 valent vertices connected by edges
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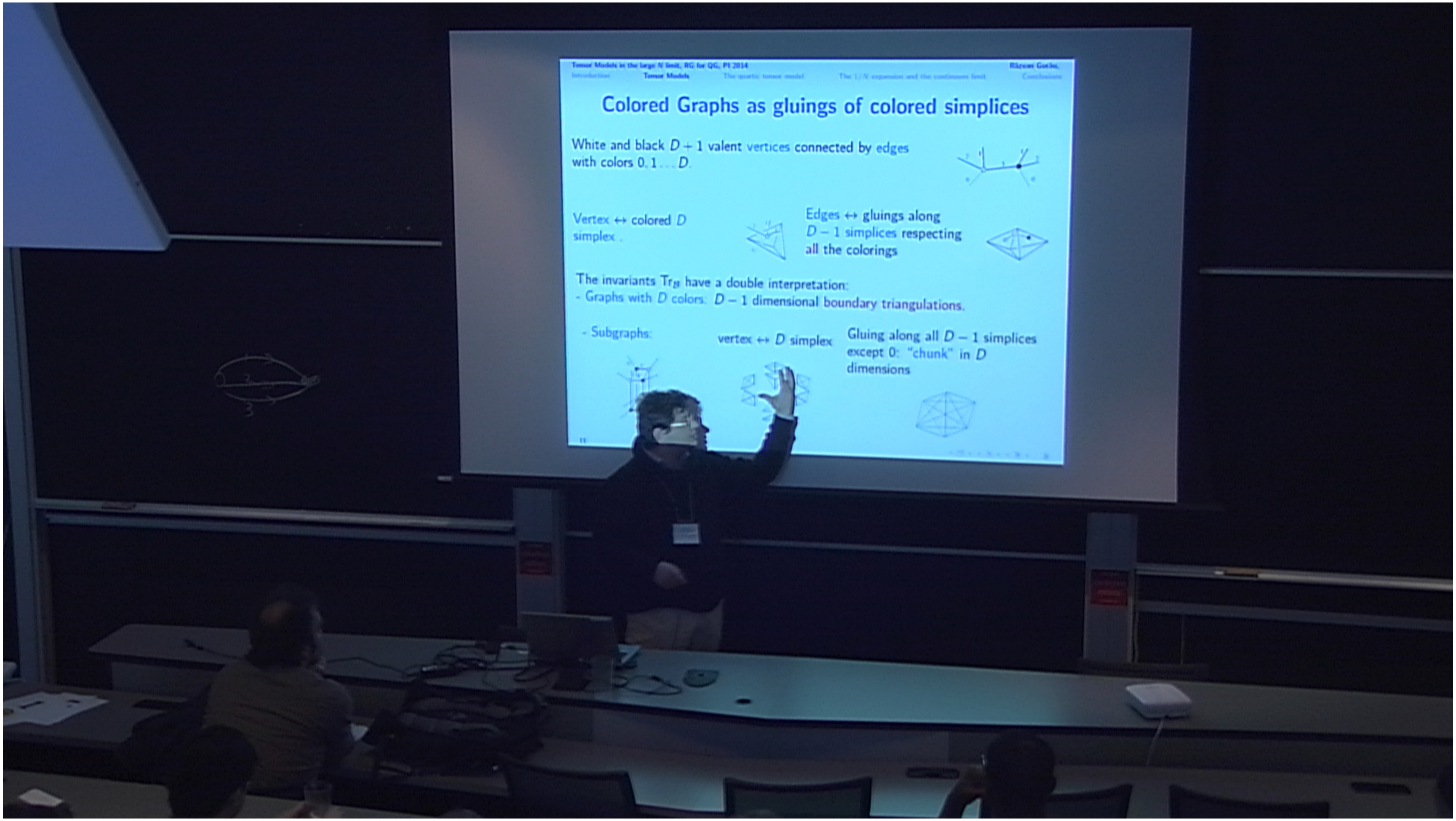
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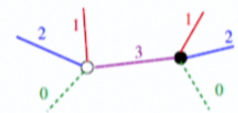
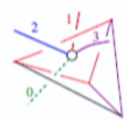
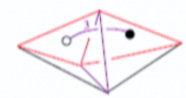


Vertex $\leftrightarrow$ colored D simplex .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



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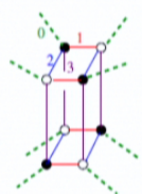

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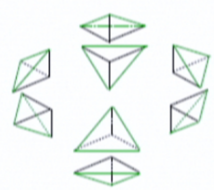

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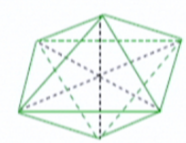

The general framework

Observables = invariants $\mathrm{Tr}_{\mathcal{B}}$ encoding boundary triangulations.
Expectations =

$$\Big\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \dots \mathrm{Tr}_{\mathcal{B}_q} \Big\rangle = \frac{1}{Z(t_{\mathcal{B}})} \int [d\bar{T}\, dT]\; \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \dots \mathrm{Tr}_{\mathcal{B}_q}\; e^{-N^{D-1} S(T,\bar{T})}$$

correlations between boundary states given by sums over all bulk triangulations compatible with the boundary states

- $\Big\langle \mathrm{Tr}_{\mathcal{B}} \Big\rangle$: $\mathcal{B}$ to vacuum amplitude
- $\Big\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \Big\rangle_c = \Big\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \Big\rangle - \Big\langle \mathrm{Tr}_{\mathcal{B}_1} \Big\rangle \Big\langle \mathrm{Tr}_{\mathcal{B}_2} \Big\rangle$: transition amplitude between the boundary states $\mathcal{B}_1$ and $\mathcal{B}_2$

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Tensor Models compute correlations

$$S(T,\bar{T}) = \sum T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \prod_{c=1}^{D} \delta_{a^c q^c} + \sum_{\mathcal{B}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(\bar{T},T)$$

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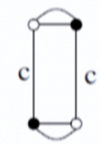

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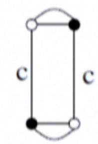

The simplest interacting theory: coupling constants $t_{\mathcal{B}} = \begin{cases}\frac{\lambda}{2}\,, & \mathcal{B}=\mathcal{B}^{(4),c}\\ 0\,, & \text{otherwise}\end{cases}$

Amplitudes and Dynamical Triangulations

Expand in λ (Feynman graphs):

$$\left\langle \frac{1}{N} \mathrm{Tr}_{\mathcal{B}(2)} \right\rangle = \sum_{D+1 \text{ colored graphs } \mathcal{G}} A^{\mathcal{G}}(N)$$

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Where is the RG flow?

Being field theories, RG techniques are natural in Tensor Models.

A genuine RG flow is obtained only for models with a soft breaking of tensor invariance of the quadratic part (i.e. ITM and GFT do not flow, only TFT and TGFT do).

The $1/N$ expansion

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Two parameters: λ and N.

1) Feynman expansion: $K_2 = 1 - D\lambda - \frac{1}{N^{D-2}} D\lambda + \sum_{\mathcal{G}} A^{\mathcal{G}}(N) \quad A^{\mathcal{G}}(N) \sim \lambda^2$

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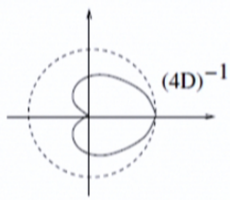
$\mathcal{R}_N^{(p)}(\lambda)$ analytic in $\lambda = |\lambda| e^{\imath\varphi}$ in the domain



$$|\mathcal{R}_N^{(p)}(\lambda)| \leq \frac{1}{N^{p(D-2)}} \frac{|\lambda|^p}{\left(\cos \frac{\varphi}{2}\right)^{2p+2}} \, p! \, A \, B^p$$

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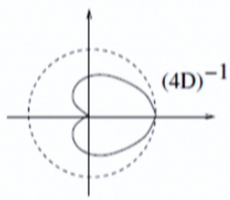

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$\lim_{N\to\infty} |\mathcal{R}_N^{(1)}(\lambda)| = 0$, hence

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- Give up the field theory framework: CDT, spin foams, etc.
- Add holonomies, change the propagator (GFT, TFT, TGFT)

Beyond branched polymers

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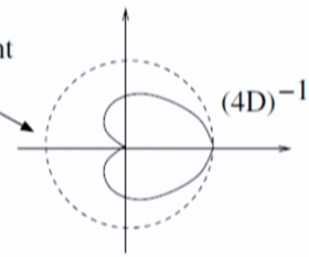
Loop edges decorate this tree $\rightarrow$ emergent extended space.
Loop effects: fine tunning the approach to criticality (double scaling, triple scaling, etc.)

But the critical point is on the wrong side!



Major (nonperturbative) challenge: extend the analyticity domain of $\mathcal{R}_N^{(p)}(\lambda)$ to the disk of radius $(4D)^{-1}$ minus the negative real axis!

The Double Scaling Limit

In perturbative sense the graphs can be reorganized as

$$K_2 = \sqrt{(4D)^{-1} + \lambda} \sum_{p \geq 0} \frac{c_p}{\left(N^{D-2} \left[(4D)^{-1} + \lambda \right] \right)^p} + Rest$$

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Subleading terms in $1/N$ are more singular (hence enhanced) when tunning to criticality! Uniform when we keep $x = N^{D-2}\left[(4D)^{-1} + \lambda\right]$ fixed.

Double scaling $N \to \infty$, $\lambda \to -\frac{1}{4D}$ like $\lambda = -\frac{1}{4D} + \frac{x}{N^{D-2}}$,

$$K_2 = N^{1-\frac{D}{2}} \sum_{p \geq 0} \frac{c_p}{x^{p-\frac{1}{2}}} + Rest \qquad Rest < N^{1/2-D/2}$$


Tensor Models in the large N limit, RG for QG, PI 2014
The 1/N expansion and the continuum limit
Flowing to continuum geometries
In TFT and TGFT similar results are obtained by a genuine RG flow.
TFTs and TGFTs are generically asymptotically free.
Like in QCD, the coupling constant grows in the IR, and eventually one develops bound states (hadronic physics).
The bound states of TFT and TGFT are the resummation of the melonic sector. Such theories naturally flow into a phase of extended geometry.
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Advantages vs. Questions

We have an analytic framework to study random discrete geometries!

- canonical path integral formulation.
- built in scales (tensors of size N^D).
- sums over discretized geometries.
- with weights the discretized (Einstein Hilbert, $B \wedge F$, etc.) action.
- non perturbative predictions

Question: Is space truly discrete? what we know for sure is that the universe has a large number of degrees of freedom $\Rightarrow$ the universe must be composed of a large number of quanta.

Conclusions

The tensor track is largely open and begs to be explored!

A personal list of open questions:

- non perturbative results
  - extend the non perturbative treatment to other models.
  - extend the analyticity domain of the rest and study the discontinuity of the rest on the negative real axis (non perturbative cut effects are crucial for unitarity and the role of time)
- study the geometry of the space emerging under multiple scalings.
  - algebra of constraints, Hausdorff and spectral dimensions, geodesics.
- Effective field theory description of the confined phase.
- Phenomenological implications.
- ....