Title: Exact Classical Simulation of the Quantum-Mechanical GHZ Distribution

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Abstract: John Bell has shown that the correlations entailed by quantum mechanics cannot be reproduced by a classical process involving non-communicating parties. But can they be simulated with the help of bounded communication? This problem has been studied for more than twenty years and it is now well understood in the case of bipartite entanglement. However, the issue was still widely open for multipartite entanglement, even for the simplest case, which is the tripartite Greenberger-Horne-Zeilinger (GHZ) state. We give an exact simulation of arbitrary independent von Neumann measurements on general n-partite GHZ states. Our protocol requires O(n^2) bits of expected communication between the parties, and O(n log n) expected time is sufficient to carry it out in parallel. Furthermore, we need only an expectation of O(n) independent unbiased random bits, with no need for the generation of continuous real random variables nor prior shared random variables. In the case of equatorial measurements, we improve earlier results with a protocol that needs only O(n log n) bits of communication and O(log^2 n) parallel time. At the cost of a slight increase in the number of bits communicated, these tasks can be accomplished with a constant expected number of rounds.

Exact simulation of the GHZ distribution with finite expected communication

Gilles Brassard
Université de Montréal

Luc Devroye
McGill University

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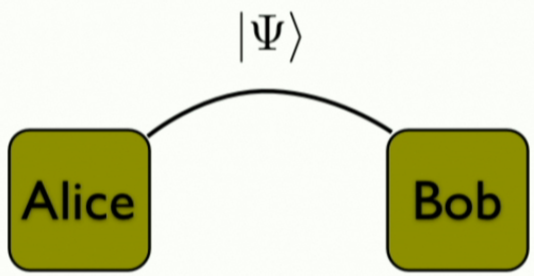
Université de Montréal

Quantum nonlocality



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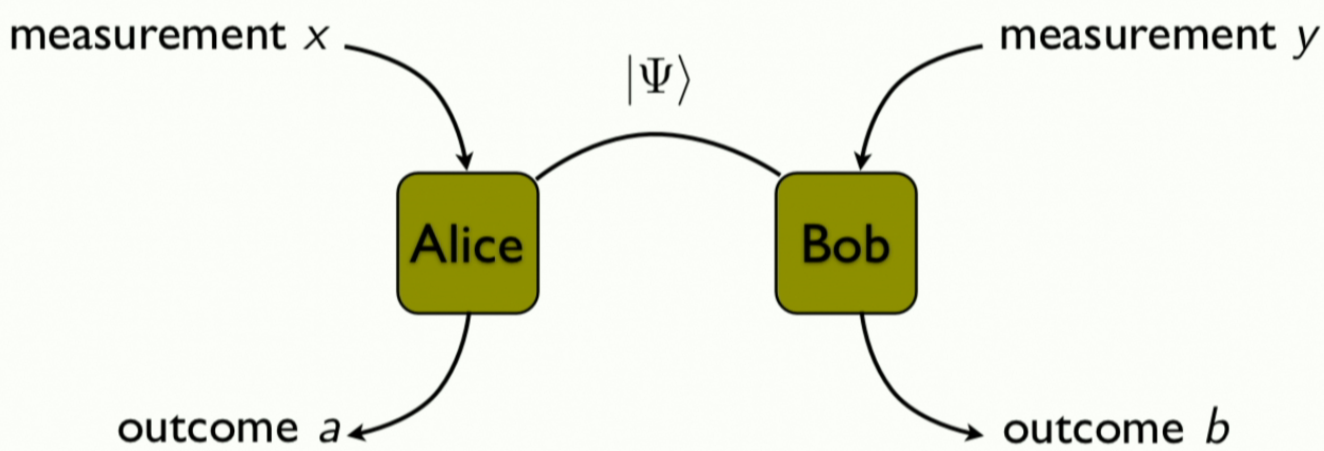

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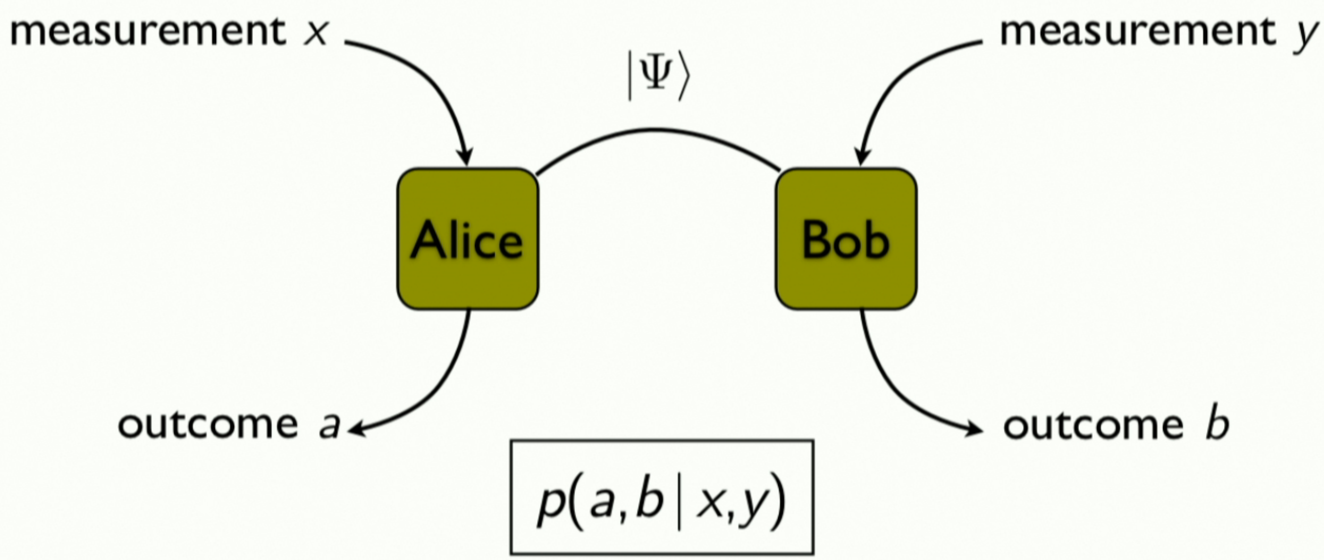

Can the theory be supplemented with local hidden variables (LHV)?

Can quantum-mechanical description of physical reality be considered complete?
Einstein, Podolsky and Rosen, 1935

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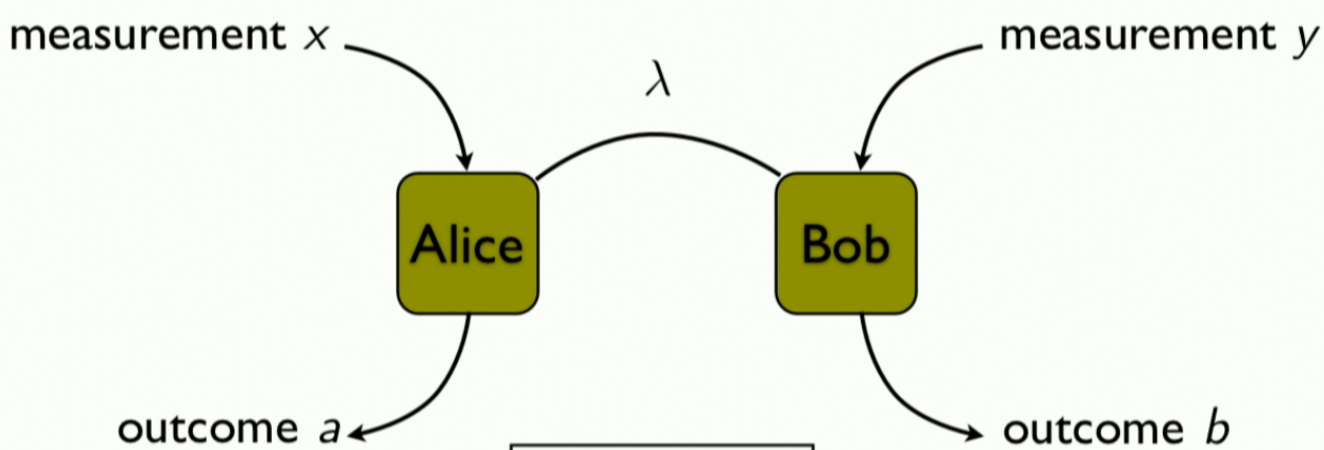

$$p(a,b\,|\,x,y)$$

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Einstein, Podolsky and Rosen, 1935

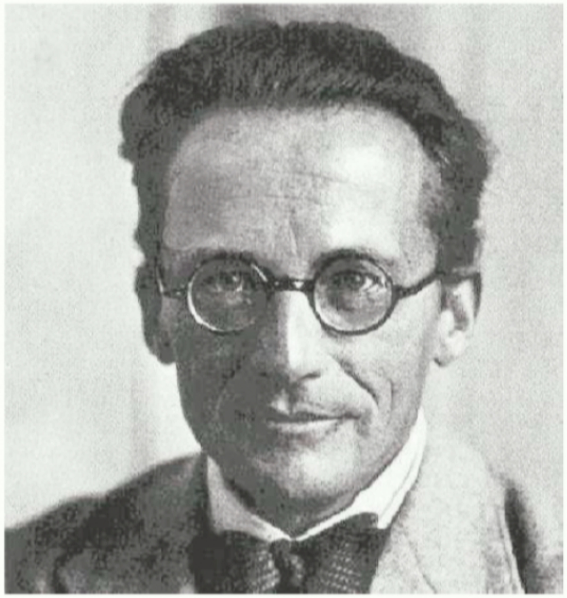
Einstein-Podolsky-Rosen

Albert Einstein

Boris Podolsky

Nathan Rosen

Entanglement



I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.

Erwin Schrödinger, 1935

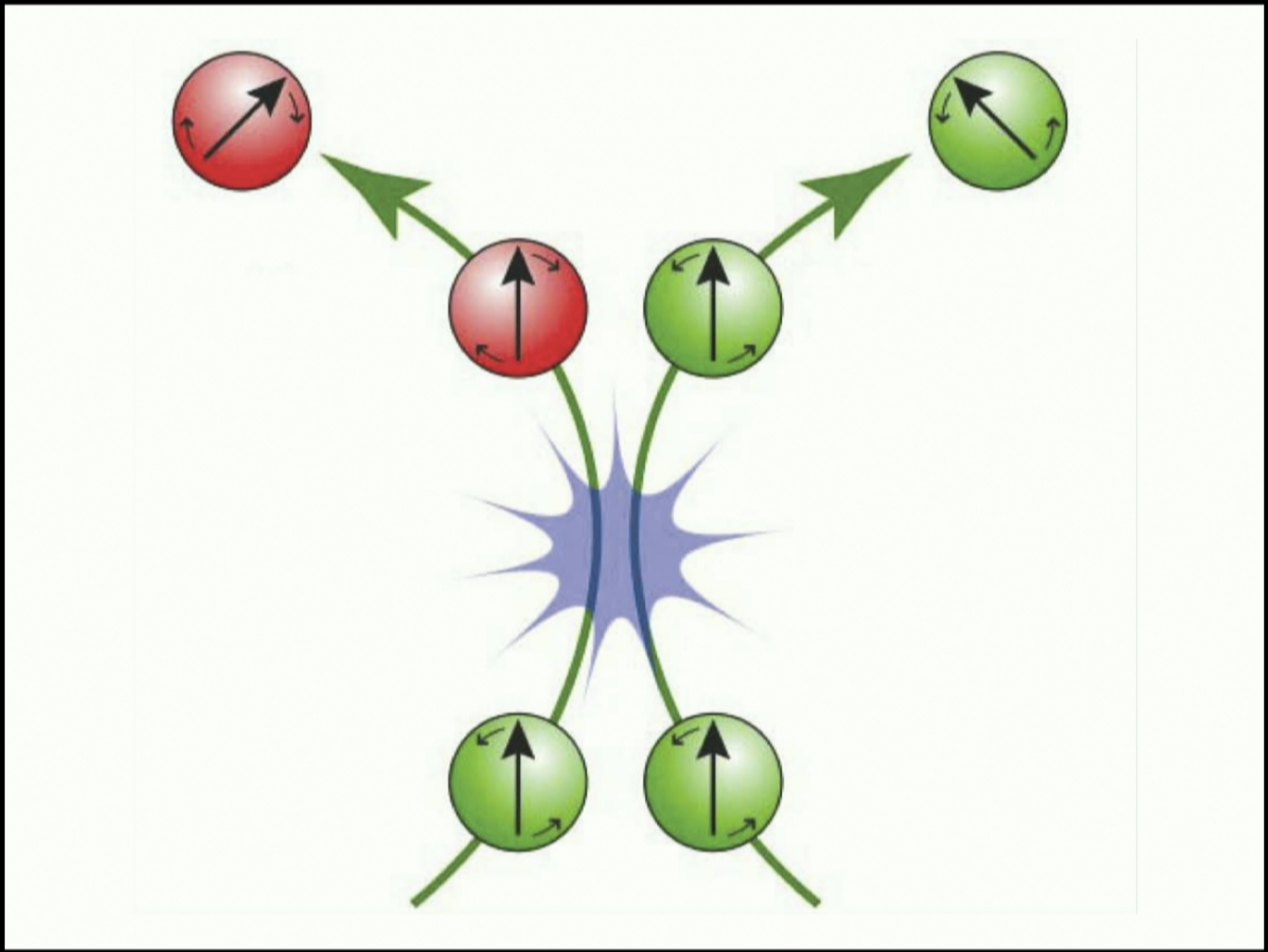
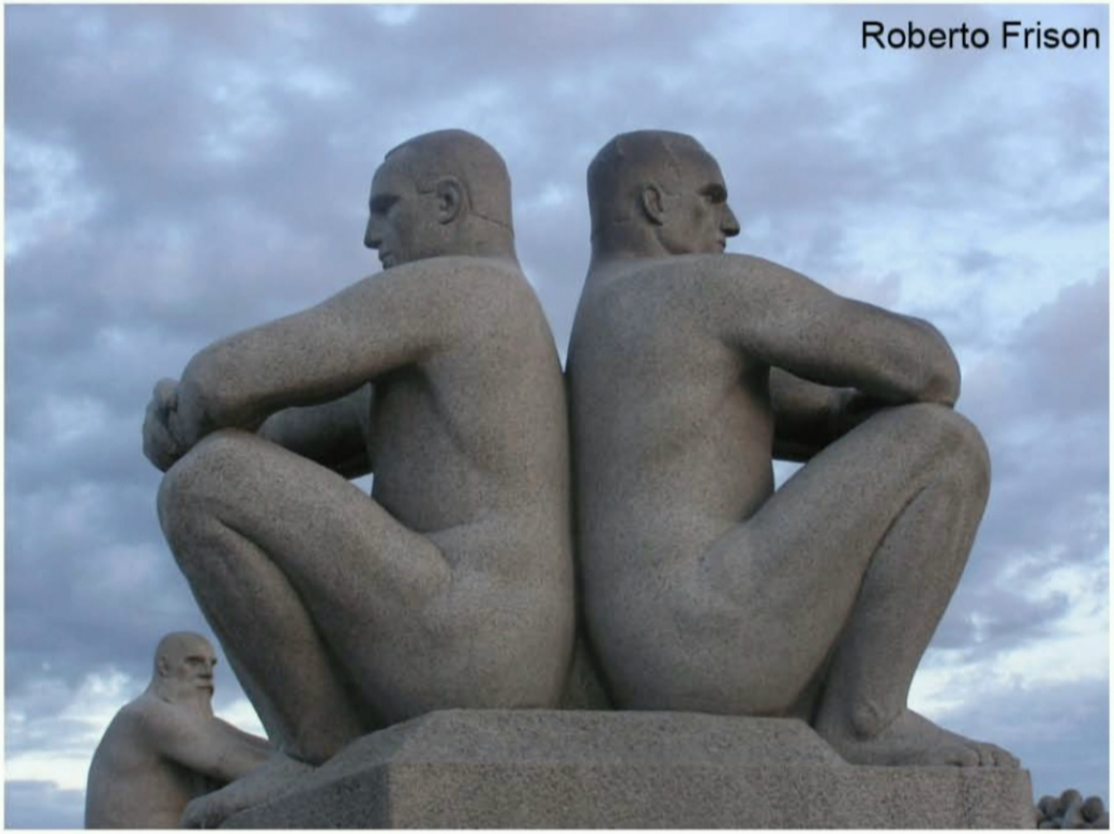


Illustration of Entanglement



Vigeland Park, Olso, Norway


Do you believe in God?
Yes!
No!

Applications of quantum nonlocality

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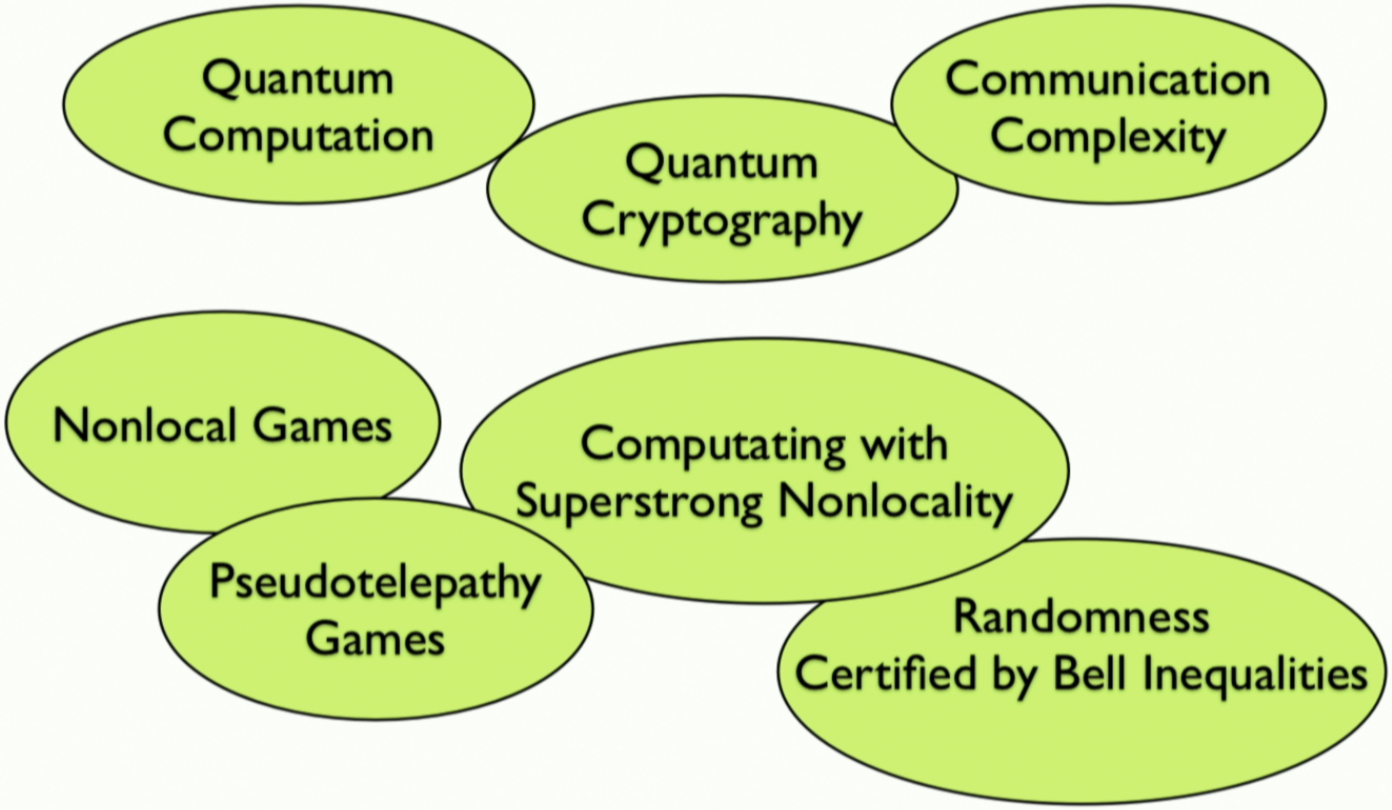
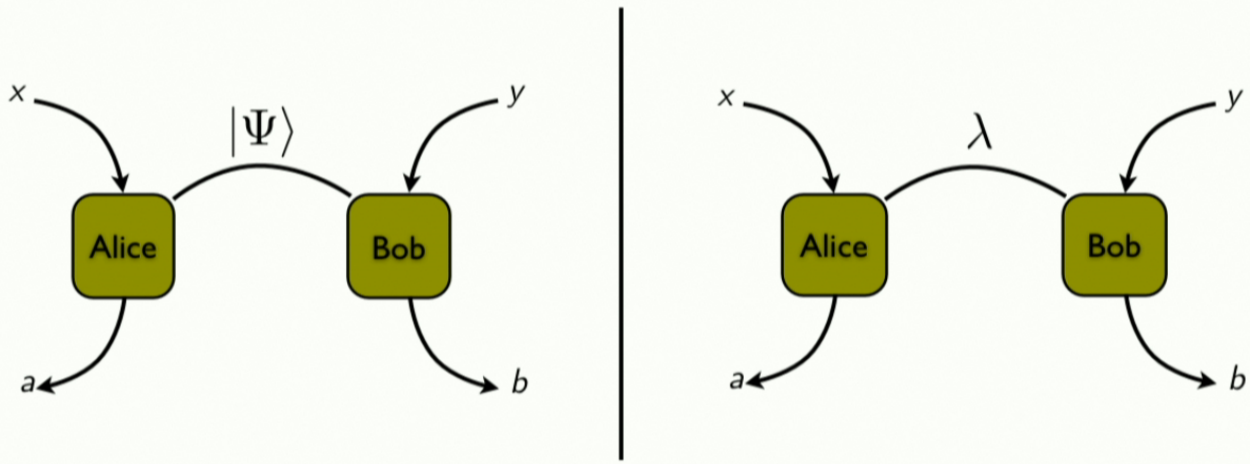
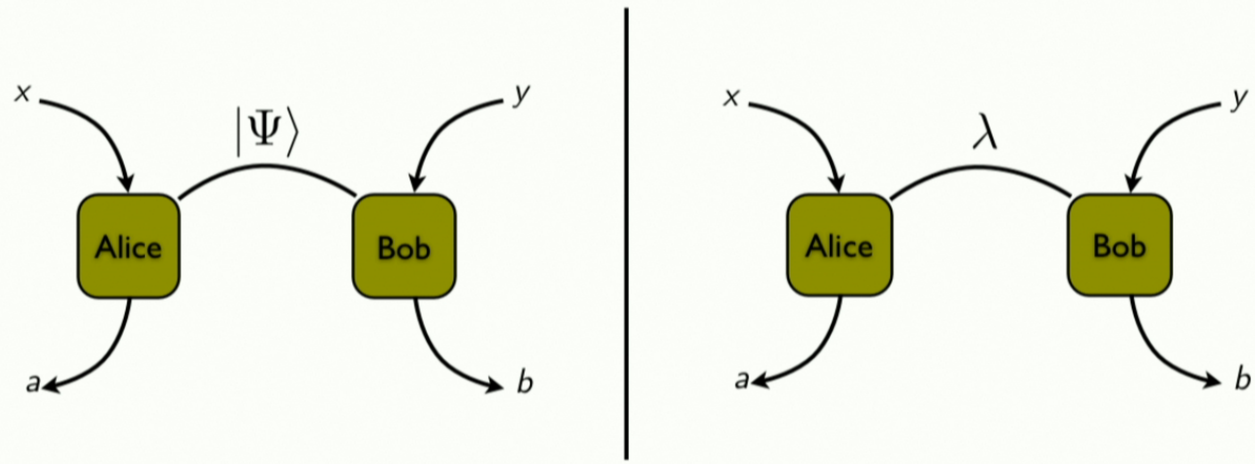


Quantifying nonlocality



How far are these two models?

Quantifying nonlocality



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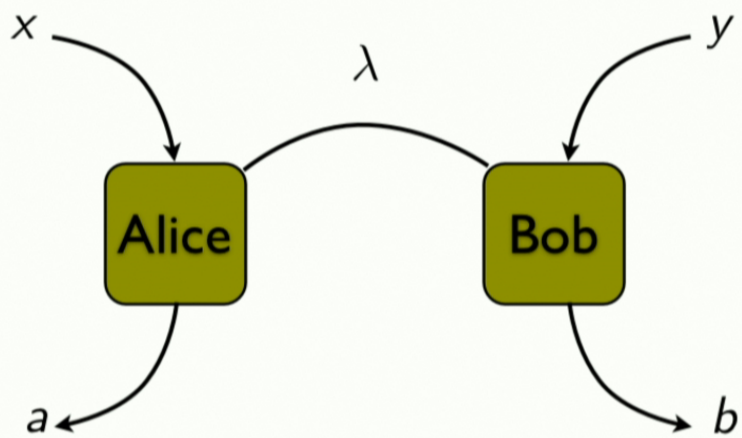

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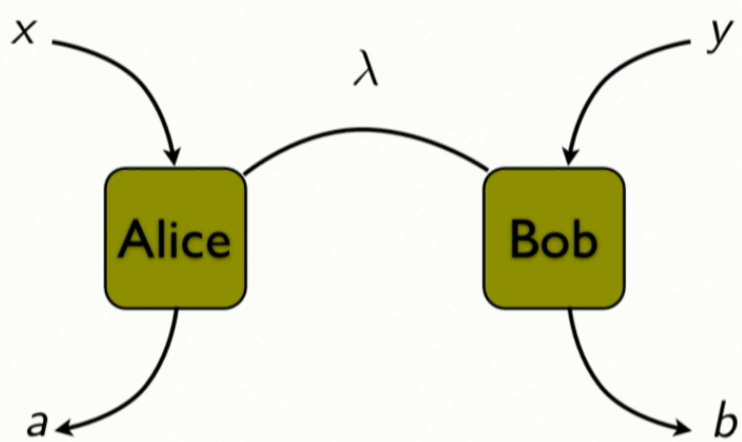

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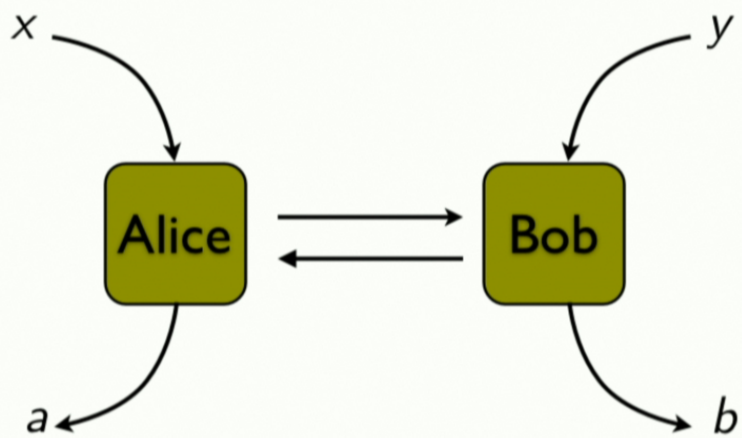

How much communication is required to reproduce the predictions of quantum mechanics?

What do we want to simulate?

We can't hope to simulate classically everything made possible by entanglement.

Quantum Teleportation

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We can't simulate quantum teleportation!

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But can we simulate the effect of measurements?

Quantifying nonlocality

Simulating binary observables on bipartite states

[Maudlin'92] 1.17 expected bits

[Brassard-Cleve-Tapp'99] 8 bits <u>worst case</u>

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Quantifying nonlocality

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Quantifying nonlocality

Simulating binary observables on bipartite states

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[Toner-Bacon'03] 1 bit worst case (↥two maximally entangled qubits↥)

[Regev-Toner'07] 2 bits worst case (arbitrary bipartite state)

How about multipartite states?

What do we want to simulate?

We can't hope to simulate classically everything made possible by entanglement.

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But can we simulate the effect of measurements?

Von Neumann measurement

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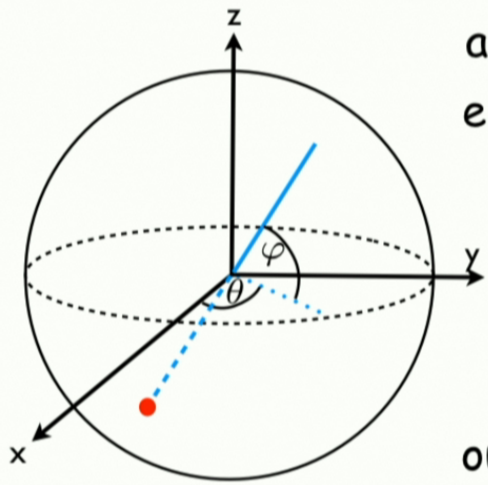


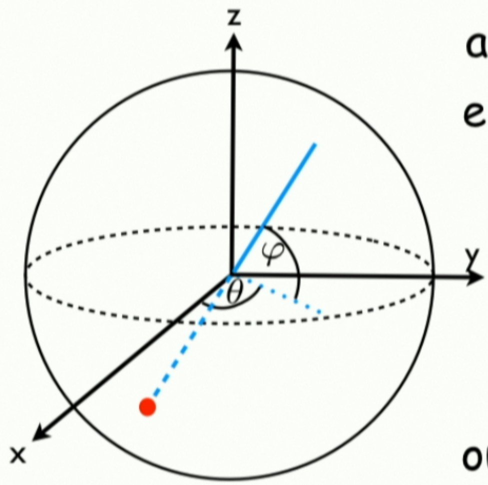
azimuthal angle $0 \leq \theta < 2\pi$

elevation angle $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$$x = \cos\theta\cos\varphi$$
$$y = \sin\theta\cos\varphi$$
$$z = \sin\varphi$$

outcome of measurement is +1 or -1

Von Neumann measurement



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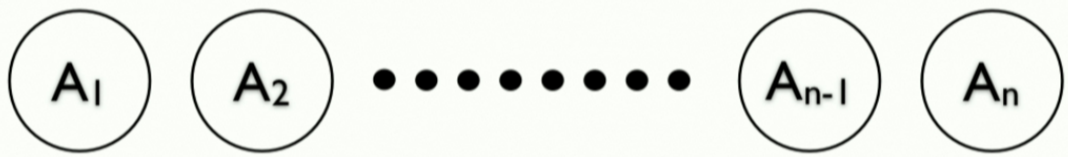
GHZ States

GHZ States

$$|\Psi_n\rangle = \frac{1}{\sqrt{2}}|00\ldots 0\rangle + \frac{1}{\sqrt{2}}|11\ldots 1\rangle$$



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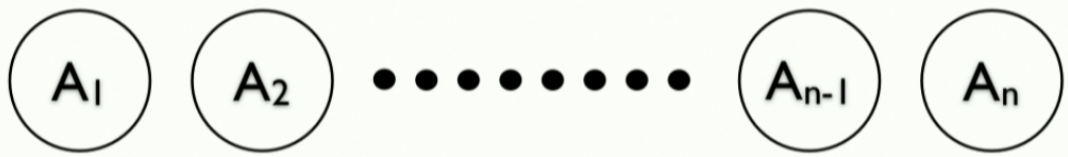
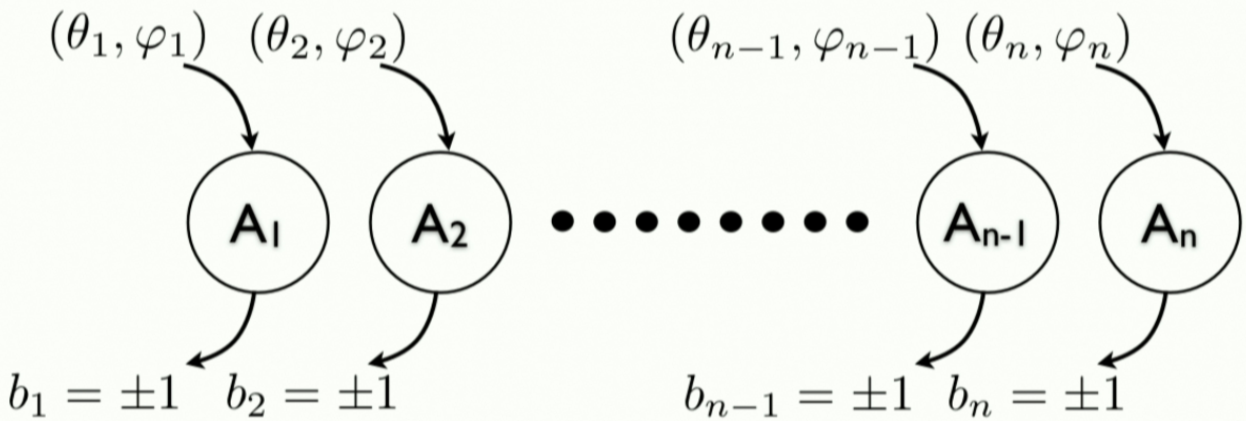


GHZ States

Classical communication



$$p\Big((b_1, b_2, \ldots, b_n) \;\Big|\; (\theta_1, \varphi_1), (\theta_2, \varphi_2), \ldots, (\theta_n, \varphi_n)\Big)$$

Simulation of GHZ distribution

Simulating equatorial measurements

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[Bancal-Branciard-Gisin'10]:
10 expected bits for n=3, 20 expected bits for n=4.

Simulation of GHZ distribution

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Simulation of GHZ distribution

Simulating <u>equatorial</u> measurements

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[Brassard-Kaplan'12]: $O(n^2)$ expected bits for arbitrary n.

[Broadbent-Chouha-Tapp'08]:
$\Omega(n \log n)$ lower bound in the worst case.

Simulating <u>arbitrary</u> von Neumann measurements?

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Simulating arbitrary von Neumann measurements?

$O(n^2)$ expected bits for arbitrary n

$O(n \log n)$ expected bits for equatorial measurements

Simulation of GHZ distribution

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Simulating arbitrary von Neumann measurements?

$O(n^2)$ expected bits for arbitrary n; $O(n \log n)$ parallel time.

$O(n \log n)$ expected bits for equatorial measurements;
$O(\log^2 n)$ parallel time.

GHZ probability distribution

$$p(b) = \cos^2\left(\frac{\theta}{2}\right) p_1(b) + \sin^2\left(\frac{\theta}{2}\right) p_2(b)$$

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$$p(b) = \boxed{\cos^2\left(\frac{\theta}{2}\right)} p_1(b) + \boxed{\sin^2\left(\frac{\theta}{2}\right)} p_2(b) \qquad \theta = \sum_{j=1}^{n} \theta_j$$

It is a convex decomposition

Key observation

It is possible to sample exactly
a probability distribution whose
parameters are given as
arbitrarily precise approximations

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This would not be possible
if we wanted to perform
actual measurements!

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Example: Bernoulli (heads or tails)
with probability p of heads=0

Bernoulli(p)

Choose $U \in_r [0, 1)$

Bernoulli(p)

Choose $U \in_r [0, 1)$

If $U < p$ then return 0 else return 1

$$
\begin{aligned}
p &= 0.01010101010101 = \tfrac{1}{3} \\
U &= 0.01101101010100
\end{aligned}
$$

Bernoulli(p)

Choose $U \in_r [0,1)$

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$$k + \lceil \lg n \rceil + 4 \text{ bits of each } \frac{\theta_j}{2} \implies k \text{ bits of } \cos^2 \frac{1}{2}\left(\sum \theta_j\right)$$

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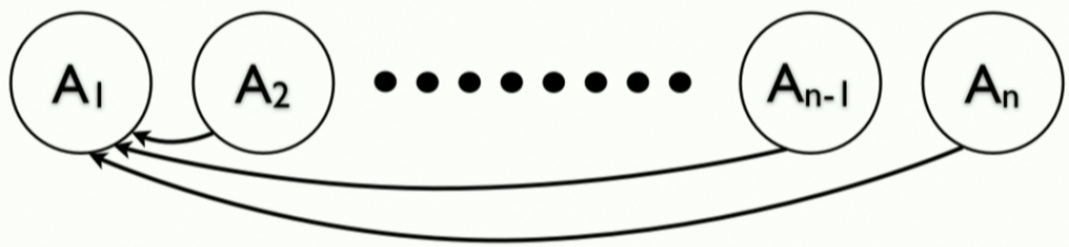

$O(n \log n)$ expected bits of communication suffice in order to decide whether to sample p_1 or p_2

GHZ probability distribution

$$p(b) = \cos^2\left(\tfrac{\theta}{2}\right) p_1(b) + \sin^2\left(\tfrac{\theta}{2}\right) p_2(b)$$

$$p_1(b) = \tfrac{1}{2}\big(\mathrm{a}_1(b) + \mathrm{a}_2(b)\big)^2 \qquad p_2(b) = \tfrac{1}{2}\big(\mathrm{a}_1(b) - \mathrm{a}_2(b)\big)^2$$

$$\mathrm{a}_1(b) = \prod_{j|b_j=+1} \alpha_j \prod_{j|b_j=-1} \beta_j \qquad \mathrm{a}_2(b) = \prod_{j|b_j=+1} \beta_j \prod_{j|b_j=-1} -\alpha_j$$

$$\alpha_j = \sin\left(\tfrac{1}{2}\left(\varphi_j + \tfrac{\pi}{2}\right)\right) \qquad \beta_j = \cos\left(\tfrac{1}{2}\left(\varphi_j + \tfrac{\pi}{2}\right)\right)$$

$$\alpha_j^2 + \beta_j^2 = 1$$

GHZ probability distribution

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$$p_1(b) \le p_1(b) + p_2(b) = \mathrm{a}_1^2(b) + \mathrm{a}_2^2(b) = 2\,\frac{\mathrm{a}_1^2(b)+\mathrm{a}_2^2(b)}{2}$$

Very easy to sample!

Von Neumann Rejection Algorithm

Say we want to sample distribution $f(x)$

We know how to sample $g(x)$

There exist c such that $(\forall x)\, f(x) \leq c\, g(x)$

Sample X according to distribution g

Sample U according to uniform distribution on $[0, 1)$

Von Neumann Rejection Algorithm

Say we want to sample distribution $f(b) = p_1(b)$

We know how to sample $g(b)$

There exist c such that $(\forall b)\, f(b) \leq c\, g(b)$

Sample B according to distribution g

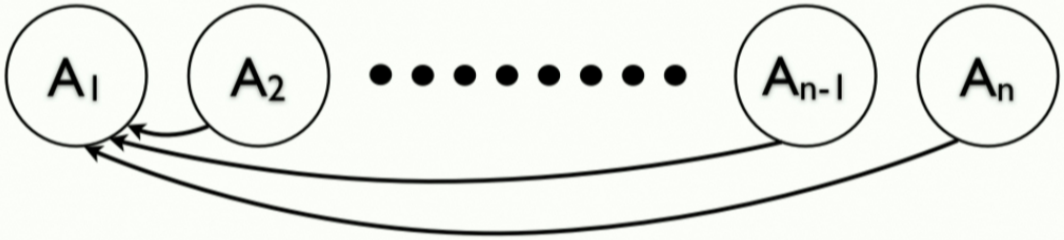
Sample U according to uniform distribution on $[0, 1)$

If $f(B) \leq c\ g(B)\, U$ go back to "Sample B"

B is now sampled according to distribution f !

The expected number of times round the loop is c

Parallel version



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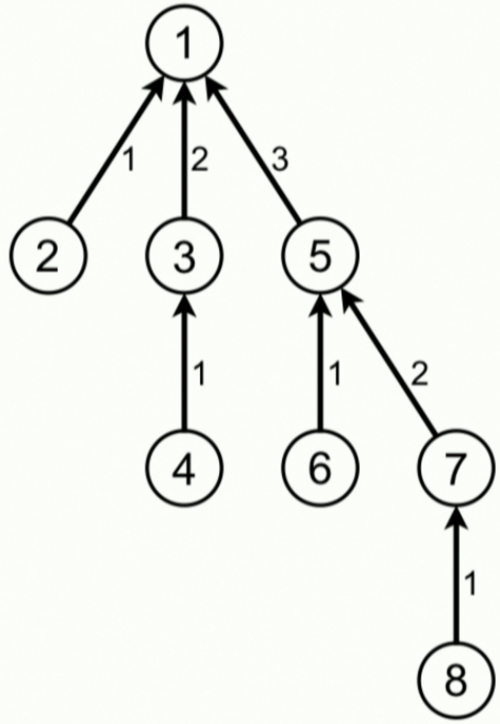

Open problems

Open problems

- $O(n \log n)$ expected communication?
- Worst case communication?
- More general multipartite states?