

Title: 13/14 PSI - Explorations in Condensed Matter - Lecture 8

Date: Mar 26, 2014 10:15 AM

URL: <http://pirsa.org/14040031>

Abstract:

Lecture 8 ENTANGLEMENT IN SYSTEMS OF FREE FERMIONS

8.1 Systems that preserve fermion number

A) Review

N fermion modes $\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N$

$$\hat{N} \equiv \sum_i \hat{n}_i = \sum_i \hat{a}_i^\dagger \hat{a}_i$$

Hamiltonian $\hat{H}$
 $\left\{ \begin{array}{l} \text{quadratic} \\ [\hat{H}, \hat{N}] = 0 \end{array} \right.$

$$\hat{H} = \sum_{i,j} A_{ij} \hat{a}_i^\dagger \hat{a}_j$$

$$A = A^\dagger$$

$N \times N$ matrix

state $|\psi\rangle$
 $\left\{ \begin{array}{l} \text{- gaussian (Wick's theorem)} \\ [\hat{\psi}, \hat{N}] = 0 \end{array} \right.$

$$M_{ij} \equiv \delta_{ij} - 2 \langle \hat{a}_i^\dagger \hat{a}_j \rangle$$

correlation matrix

$$M = M^\dagger$$

$N \times N$ matrix

Ground state $|4\rangle$ of $\hat{H}$?

$$\hat{H} \longleftrightarrow A = U \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \ddots \\ & & & \varepsilon_N \end{bmatrix} U^\dagger$$

$$U \in U(N)$$

$\uparrow$
canonical transformation

$$|4\rangle \longleftrightarrow M = U \begin{bmatrix} \text{sign}(\varepsilon_1) & & \\ & \text{sign}(\varepsilon_2) & \\ & & \ddots \\ & & & \text{sign}(\varepsilon_N) \end{bmatrix} U^\dagger$$

Ground state $|4\rangle$ of $\hat{H}$?

$$\hat{H} \longleftrightarrow A = U \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \ddots \\ & & & \varepsilon_N \end{bmatrix} U^\dagger \quad \begin{array}{l} U \in U(N) \\ \uparrow \\ \text{canonical transformation} \end{array}$$

$$|4\rangle \longleftrightarrow M = U \begin{bmatrix} \text{sign}(\varepsilon_1) & & \\ & \text{sign}(\varepsilon_2) & \\ & & \ddots \\ & & & \text{sign}(\varepsilon_N) \end{bmatrix} U^\dagger$$

L modes

Part 1

Reduced density matrix

$$\hat{S}^A = \text{tr}_B |4 \times 4\rangle \quad 2^L \times 2^L$$

↓
M

$$A = \{2, 4\}$$

$$M = \begin{bmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \\ \text{---} & \text{---} \end{bmatrix}$$

$$MA = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$M = \begin{bmatrix} M^A & \\ & M^B \end{bmatrix}$$

$$\sum_{j=1}^L V_{kj} \hat{a}_j \quad k=1, \dots, L$$

$$\Rightarrow \hat{S}^A = \bigotimes_{k=1}^L S^{(k)}$$

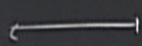
canonical transformation

$$\langle \hat{c}_k^\dagger \hat{c}_l \rangle = \delta_{k,l} \quad \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \delta_{k,l} \frac{1-M_k}{2}$$

$$\langle \hat{c}_k^\dagger \hat{c}_k \rangle = \frac{1-M_k}{2} \iff S^{(k)} = \begin{bmatrix} \frac{1+M_k}{2} & \\ & \frac{1-M_k}{2} \end{bmatrix}$$

Diagonalize

$$\hat{S}^A$$



$$M^A = V \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_L \end{bmatrix} V^\dagger$$

$$A \in \{1, 2, \dots, L\}$$

$$M^A = (M^A)^\dagger$$

$L \times L$

$$\hat{C}_k = \sum_i^L \hat{a}_i$$

$k=1, \dots, L$

transformation

$$\delta_{kl} \langle \hat{C}_k^\dagger \hat{C}_l \rangle = \delta$$

$$\hat{n}_k =$$

$$\frac{\mu_k}{2}$$

$$\longleftrightarrow \mathcal{F}(k)$$

$$\left[\begin{array}{c} \\ \vdots \\ \mu_k \end{array} \right] V^+$$

$$\hat{c}_k = \sum_{j=1}^L V_{kj} \hat{a}_j \quad k=1, \dots, L$$

$$\Rightarrow \hat{S}^A = \bigotimes_{k=1}^L S^{(k)}$$

canonical transformation

$$\langle \hat{c}_k^\dagger \hat{c}_l \rangle = \delta_{kl} \quad \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \delta_{k,l} \frac{1-\mu_k}{2}$$

$$\hat{n}_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \langle \hat{n}_k \rangle = \frac{1-\mu_k}{2} \quad \Leftrightarrow S^{(k)} = \begin{bmatrix} \frac{1+\mu_k}{2} & 0 \\ 0 & \frac{1-\mu_k}{2} \end{bmatrix}$$

$$\sum_{j=1}^L V_{kj} \hat{a}_j \quad k=1, \dots, L$$

canonical transformation

$$\langle \hat{c}_k \hat{c}_l \rangle = \delta_{kl} \quad \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \delta_{k,l} \frac{1-M_k}{2}$$

$$\langle \hat{c}_k^\dagger \hat{c}_k \rangle = \frac{1-M_k}{2} \iff S^{(k)} = \begin{bmatrix} \frac{1+M_k}{2} & 0 \\ 0 & \frac{1-M_k}{2} \end{bmatrix}$$

$$\langle \hat{n}_k \rangle = \frac{1-M_k}{2}$$

$$\Rightarrow \hat{S}^A = \bigotimes_{k=1}^L S^{(k)}$$

c) Entanglement entropy of $\hat{S}^A$

$$S(\hat{S}^1 \otimes \hat{S}^2) = S(\hat{S}^1) + S(\hat{S}^2)$$

$$\sum_{j=1}^L V_{kj} \hat{a}_j \quad k=1, \dots, L$$

canonical transformation

$$\langle \hat{c}_k \hat{c}_l \rangle = \delta_{kl} \quad \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \delta_{k,l} \frac{1-M_k}{2}$$

$$\langle \hat{c}_k^\dagger \hat{c}_k \rangle = \frac{1-M_k}{2} \iff \rho^{(k)} = \begin{bmatrix} \frac{1+M_k}{2} & 0 \\ 0 & \frac{1-M_k}{2} \end{bmatrix}$$

$$\langle \hat{n}_k \rangle = \frac{1-M_k}{2}$$

$$\Rightarrow \hat{\rho}^A = \bigotimes_{k=1}^L \hat{\rho}^{(k)}$$

c) Entanglement entropy of $\hat{\rho}^A$

$$S(\hat{\rho}^1 \otimes \hat{\rho}^2) = S(\hat{\rho}^1) + S(\hat{\rho}^2)$$

$$S(\hat{\rho}^A) = \sum_{k=1}^L S(\hat{\rho}^k)$$

formation

$$\delta_{kl} \langle \hat{C}_k^\dagger \hat{C}_l \rangle = \delta_{k,l} \frac{1-M_k}{2}$$

$$\frac{1-M_k}{2} \longleftrightarrow \rho^{(k)} = \begin{bmatrix} \frac{1+M_k}{2} & 0 \\ 0 & \frac{1-M_k}{2} \end{bmatrix}$$

$$S(\hat{\rho}^A) = - \sum_{k=1}^L$$

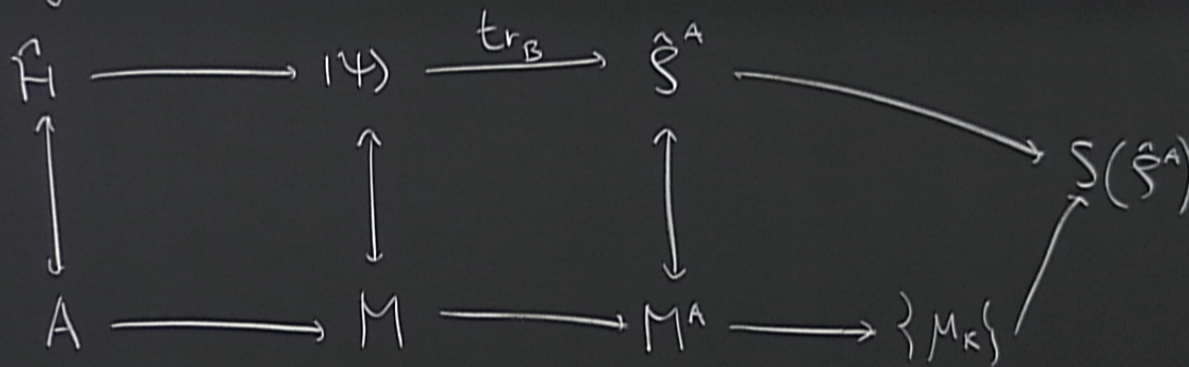
c) Entanglement entropy of $\hat{\rho}^A$

$$S(\hat{\rho}^1 \otimes \hat{\rho}^2) = S(\hat{\rho}^1) + S(\hat{\rho}^2)$$

$$S(\hat{\rho}^A) = \sum_{k=1}^L S(\hat{\rho}^k)$$

$$\left\{ \frac{1+M_k}{2} \log_2 \left(\frac{1+M_k}{2} \right) + \frac{1-M_k}{2} \log_2 \left(\frac{1-M_k}{2} \right) \right\}$$

D) Diagram



8.2 Systems that preserve

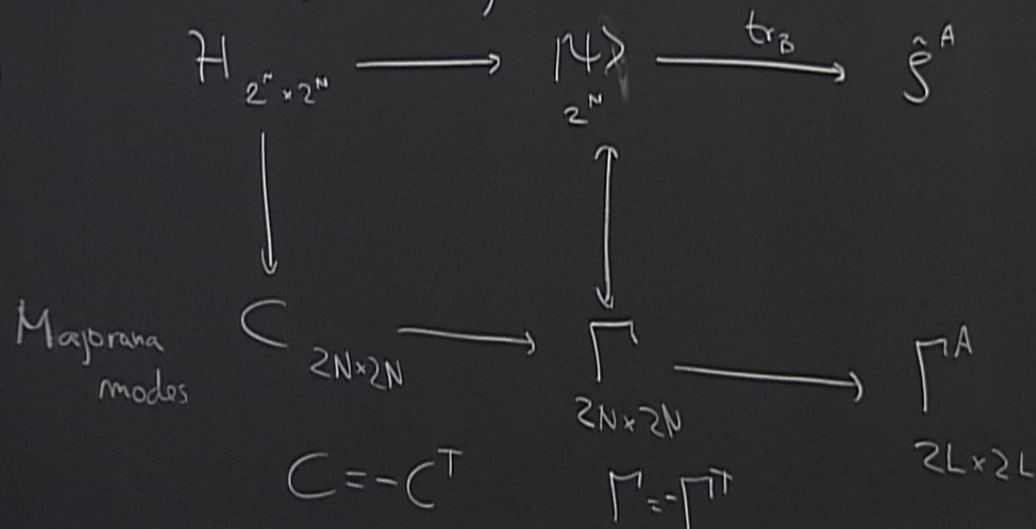
8.2 Systems that preserve fermion parity

$$\frac{1 \pm M_1}{2} \leftarrow \frac{1 \pm M_2}{2} \leftarrow \dots \leftarrow \frac{1 \pm M_L}{2}$$

2^L

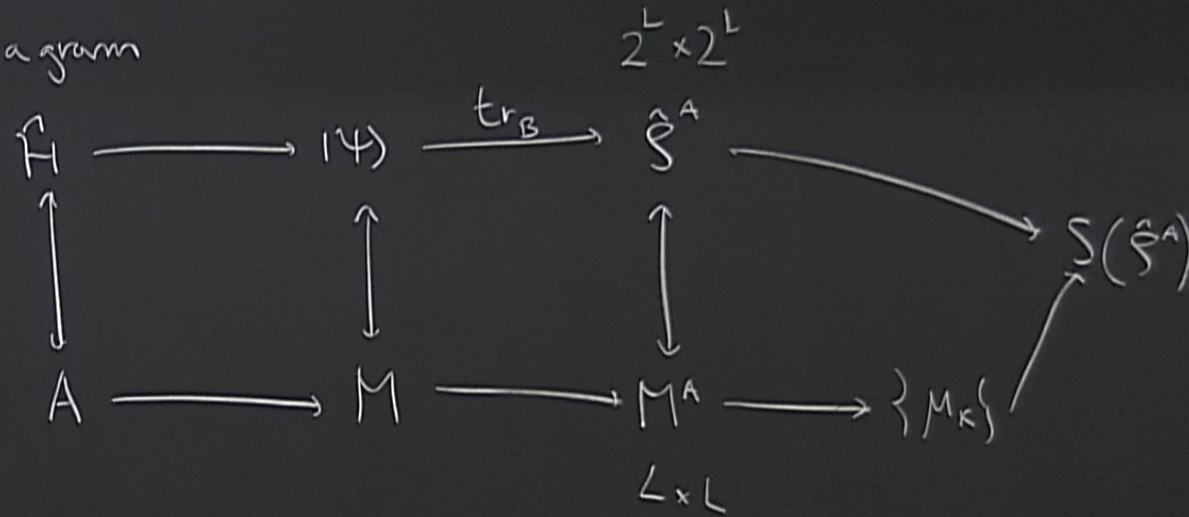
$$\hat{S}^{(1)} \otimes \hat{S}^{(2)} \\ \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} \otimes \begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix} = \begin{pmatrix} p_1 q_1 & & & \\ & p_1 q_2 & & \\ & & p_2 q_1 & \\ & & & p_2 q_2 \end{pmatrix}$$

8.2 Systems that preserve fermion parity (tutorial)



$$\begin{aligned}
 & \frac{1 \pm M_1}{2} \leftarrow \frac{1 \pm M_2}{2} \leftarrow \dots \leftarrow \frac{1 \pm M_L}{2} \\
 & 2^L \\
 & \hat{\mathcal{S}}^{(1)} \otimes \hat{\mathcal{S}}^{(2)} \\
 & \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} \otimes \begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix} = \begin{pmatrix} p_1 q_1 & & & \\ & p_1 q_2 & & \\ & & p_2 q_1 & \\ & & & p_2 q_2 \end{pmatrix}
 \end{aligned}$$

D) Diagram



8.2 System
(tuto

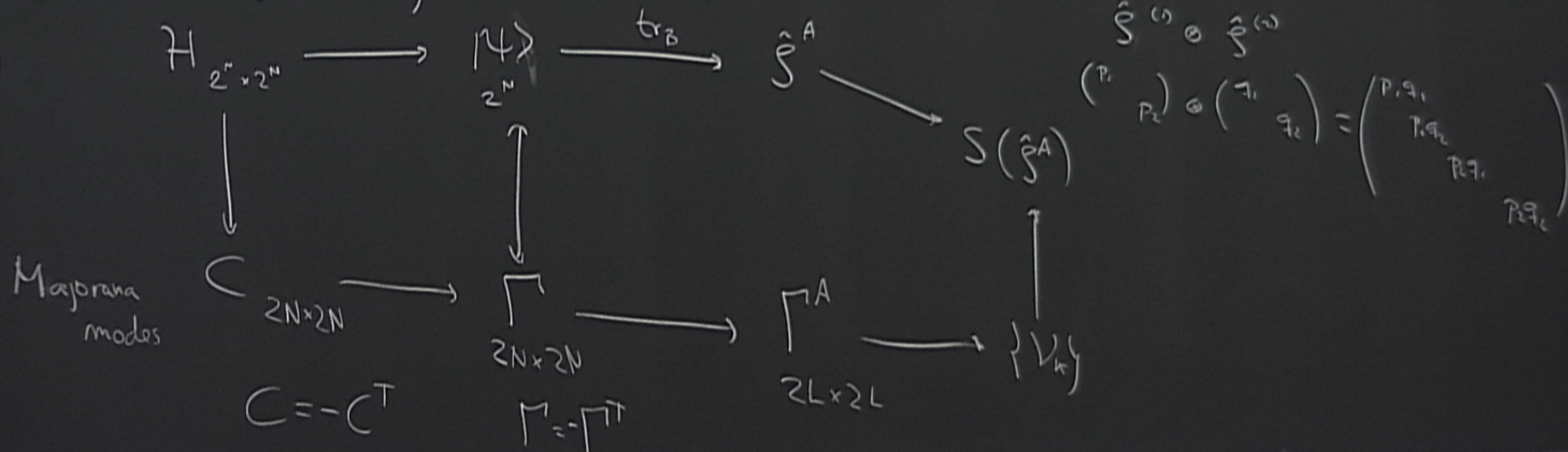
$$H_{2^n \times 2^n} \rightarrow$$

Majorana
modes

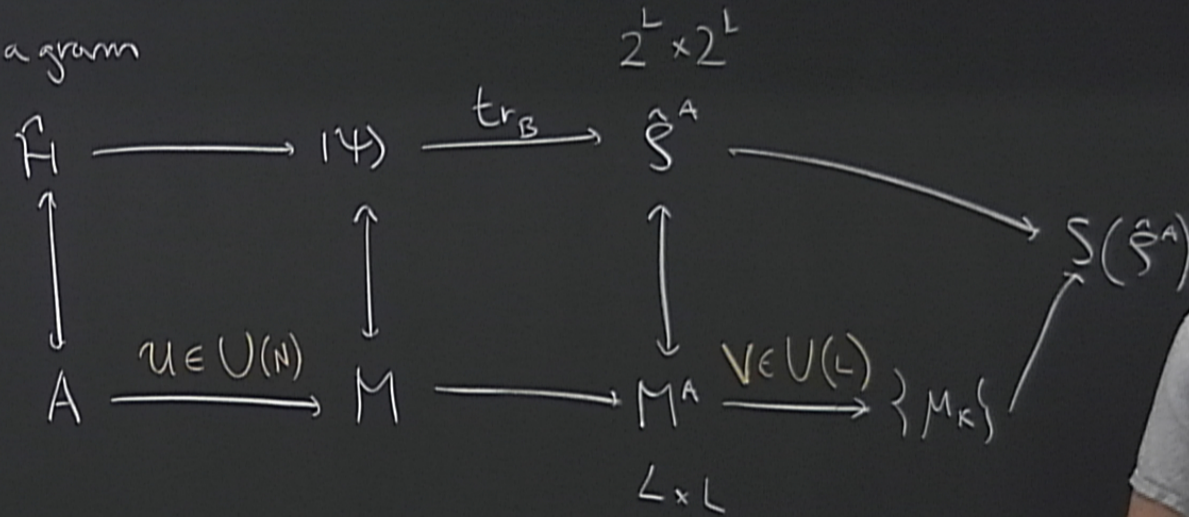
$$C_{2N \times 2N}$$

$$C = -C$$

8.2 Systems that preserve fermion parity (tutorial)



D) Diagram



8.2 System

Editor - C:\Users\Guifre Vida\Documents\Matlab\PSI 2014 Session V\FreeFermionsParticleNumberEntropy.m

EDITOR PUBLISH VIEW

New

Open

Save

Compare

Print

Find Files

Insert

Comment

Indent

Go To

Find

Breakpoints

Run

Run and Advance

Run Section

Advance

Run and Time

FILE EDIT NAVIGATE BREAKPOINTS RUN

FreeFermionsParticleNumberEntropy.m

```

1 % PSI 2014 Session V FreeFermionsParticleNumberEntropy.m
2 - clear all
3 - N=1000;
4 - Lmax = 100;
5 - lambda = 0.0; % lambda = 0, 0.3, 0.4
6
7 % Build the Hamiltonian A
8 - A = zeros(N);
9 - for n=1:N
10 -     A(n,n) = -2*lambda;
11 - end
12 - for n=1:N-1
13 -     A(n,n+1) = 1;
14 - end
15 - A(N,1) = 1; % periodic boundary conditions
16 - A = (A+A')/2;
17
18 % Diagonalize H
19 - [U,D] = eig(A); % eigenvalue decomposition A = U*D*U'
20
21 % ground state correlation matrix M
22 - M = U*sign(D)*U';
23
24 % compute entropies of reduced density matrix of up to Lmax sites

```

Ln 1 Col 1

Preferences

Set Path

Help

Community

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ENVIRONMENT RESOURCES

Workspace

Name	Value
Started.	

Command History

```

-- 25/03/2014 10:11 AM --
RandomStateEntropyScali
GroundStateEntropyScali
figure(2)
GroundStateEntropyScali
figure(3)
GroundStateEntropyScali
figure(4)
GroundStateEntropyScali
-- 25/03/2014 6:23 PM --
-- 26/03/2014 9:28 AM --

```

VLC media player

100APPLE

FreeFermionsParticleNumberEntropy.m

Ready

10:52 AM

26/03/2014

$$A \in \{1, 2, \dots, L\} \quad M^A = (M^A)^+ \\ L \times L$$

$$\hat{H}(\lambda) = \frac{1}{2} \sum_{n=1}^N (\hat{a}_n^\dagger \hat{a}_{n+1} + \text{h.c.})$$

$$-2\lambda \sum \hat{a}_n^\dagger \hat{a}_n$$

$$A = \begin{bmatrix} -2\lambda & 1/2 & & \\ 1/2 & -2\lambda & 1/2 & \\ & 1/4 & -2\lambda & \ddots \\ & & & -2\lambda \end{bmatrix}$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{U \in U(N)} & M & \longrightarrow & M^A \xrightarrow{V \in U(L)} \{M_K\} \\ & & & & L \times L \end{array}$$

Majorana
mod

The screenshot displays the MATLAB IDE interface. The main editor window shows the script `EntropyScalingFFNumber.m` with the following code:

```

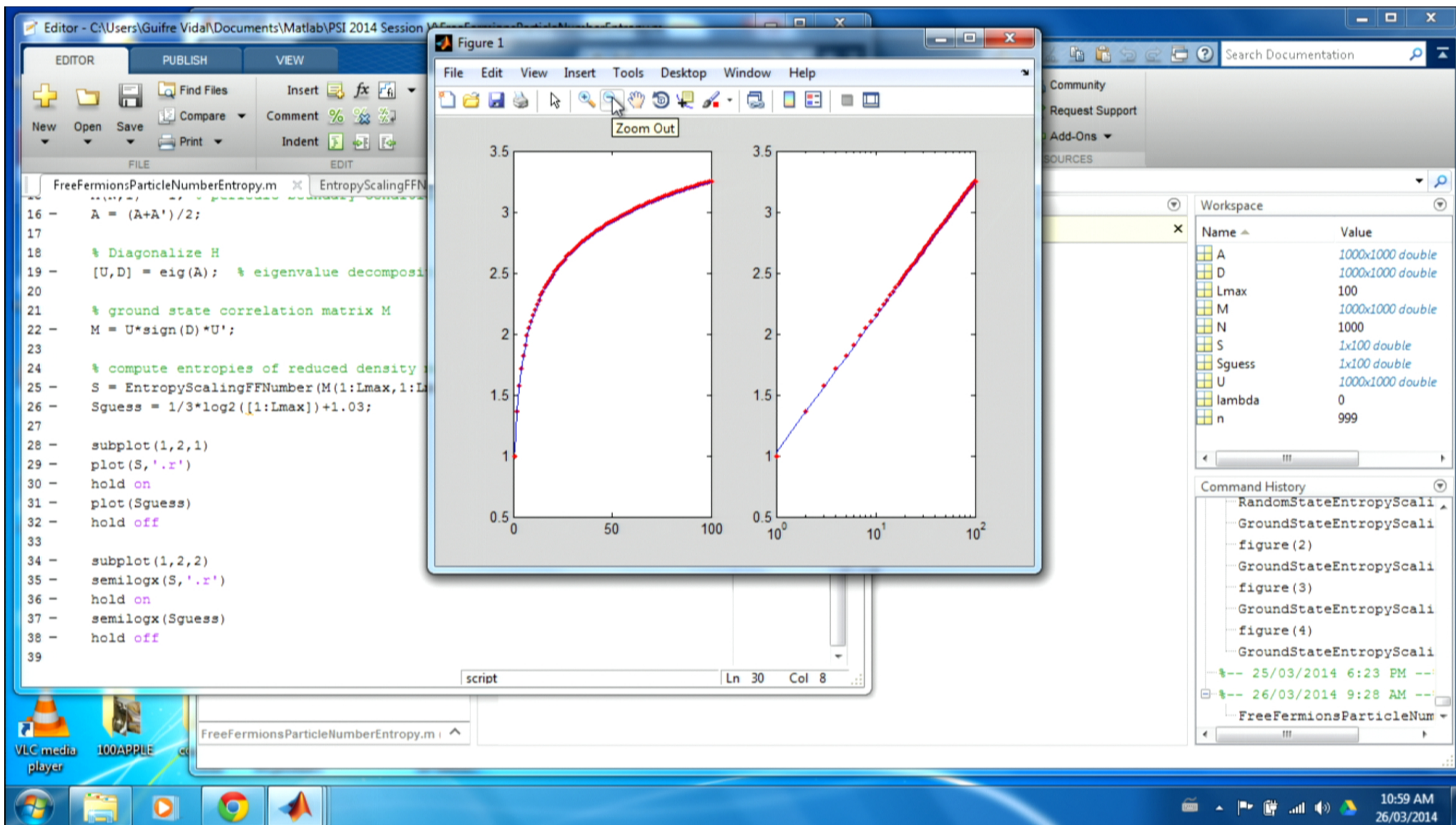
1 % PSI 2014 Session V EntropyScalingFFNumber.m
2 function S = EntropyScalingFFNumber(M, Lmax);
3 for L = 1:Lmax
4     S(L) = ComputeEntropyFFNumber(M(1:L, 1:L), L);
5 end
6 end
7
8 function S = ComputeEntropyFFNumber(MA, L)
9     Mu = eig(MA);
10    S = 0;
11    for k=1:L
12        if abs(Mu(k)) < 1 % avoid problems with zero and (finite precision) negative p or q
13            p = (1+Mu(k))/2;
14            q = (1-Mu(k))/2;
15            S = S - p*log2(p) - q*log2(q);
16        end
17    end
18    S = real(S);
19 end

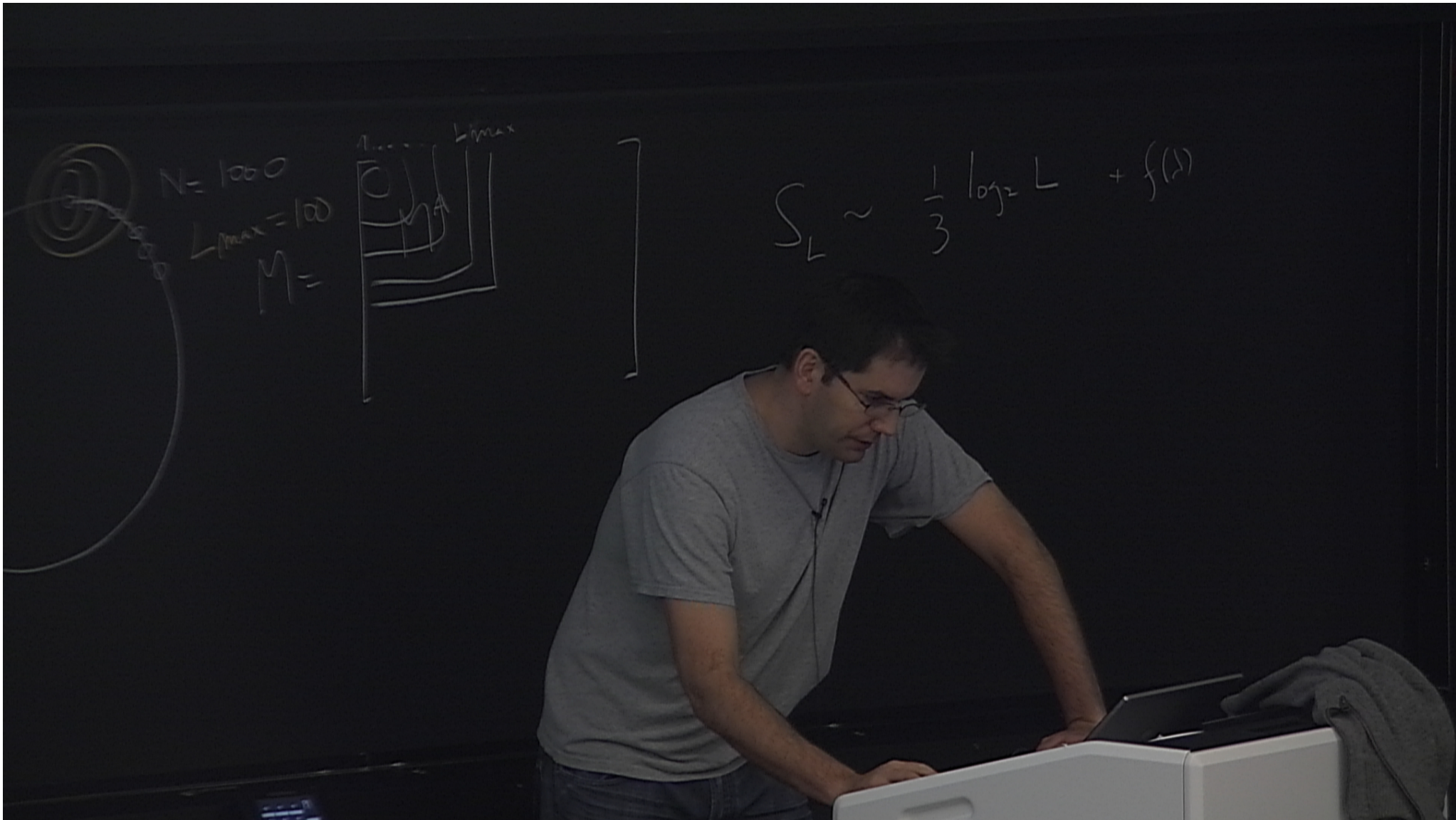
```

The Command History window on the right shows the following commands:

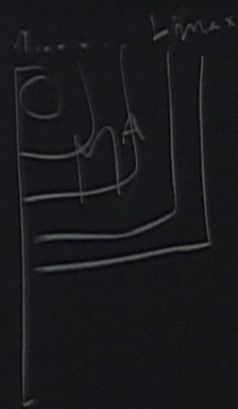
- Started.
- RandomStateEntropyScali
- GroundStateEntropyScali
- figure(2)
- GroundStateEntropyScali
- figure(3)
- GroundStateEntropyScali
- figure(4)
- GroundStateEntropyScali

The system clock at the bottom right indicates the time is 10:59 AM on 26/03/2014.





$N = 1000$
 $L_{max} = 100$
 $M =$

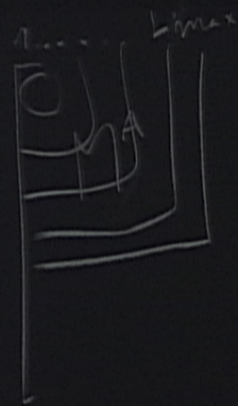


$$S_L \sim \frac{1}{3} \log_2 L + f(L)$$

$$S_L \sim \frac{c}{3} \log L + \text{const}$$

$c=1$ $\nearrow$ universal
 $\nwarrow$ non-universal

$N = 1000$
 $L_{\max} = 100$
 $M =$



$$S_L \sim \frac{1}{3} \log_2 L + f(L)$$

$$S_L \sim \frac{c}{3} \log L + \text{const}$$

$c=1$ non-universal
 universal

$$S_L \sim \frac{c}{3} \log L + f(\lambda)$$

c = central charge
of CFT

$$S_L \sim \frac{c}{3} \log L + f(\lambda)$$

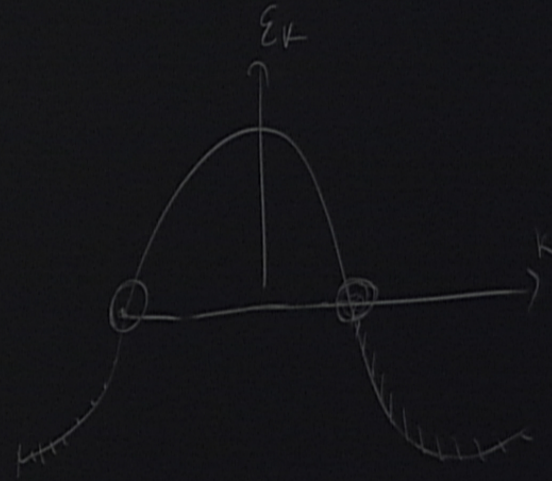
c = central charge
of CFT

$c=1$ free boson CFT

$$S_L \sim \frac{c}{3} \log_2 L + f(\lambda)$$

c = central charge
of CFT

$c=1$ free boson CFT



$S_L \sim L^{D-1}$
 $D=1$
 $S_L \sim L^{1-1} = L^0 = \text{constant}$
 $S_L \sim L^{D-1} \log L$

$|bulk| \sim L^2$
 $|boundary| \sim L$

$c=1$ $un, versal$