

Title: 13/14 PSI - Explorations in Condensed Matter - Lecture 6

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URL: <http://pirsa.org/14040029>

Abstract:

L6 Basics of entanglement

6.1 Introduction & preliminaries

A) n -level
quantum system

$$\mathcal{V} \cong \mathbb{C}^n$$

orthonormal
basis

$$\{|1\rangle, |2\rangle, \dots, |n\rangle\}$$

$$|\psi\rangle = \sum_{i=1}^n a_i |i\rangle$$

examples

$n=2$

qubit $|0\rangle, |1\rangle$

spin $|1/2\rangle, |1/2\rangle$

B) 2 such systems
 $A \otimes B$

$$\mathcal{V}^{AB} \cong \mathcal{V}^A \otimes \mathcal{V}^B$$

basis $|i,j\rangle \equiv |i\rangle^A \otimes |j\rangle^B$

$$|\psi\rangle = \sum_{i,j} a_{ij} |i,j\rangle$$

c) REDUCED DENSITY MATRIX

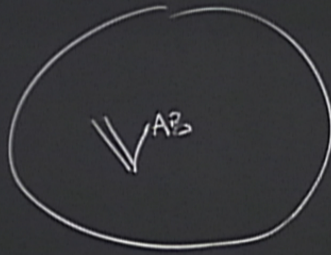
$$\hat{S}^A \equiv \text{tr}_B |\Psi\rangle\langle\Psi| = \sum_m \langle m^B | |\Psi\rangle\langle\Psi| |m^B\rangle = \sum (\hat{S}^A)_{ik} |i^A\rangle\langle k^A|$$

$$(\hat{S}^A)_{ik} = \sum_m a_{im} a_{km}^* = \uparrow (A A^\dagger)_{ik}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & & \\ \vdots & & \end{pmatrix}$$

6.2 ENTANGLEMENT IN BIPARTITE SYSTEMS

example



$|\psi\rangle$ is product state iff

$$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$$

entangled state iff

$$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$$

examples: $|1^A\rangle \otimes |1^B\rangle = |11\rangle$

$$|11\rangle + |22\rangle$$

$$|11\rangle + |12\rangle = |1\rangle \otimes (|1\rangle + |2\rangle)$$

$$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle)$$

$$|11\rangle + |12\rangle + |21\rangle$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |kk\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

maximally entangled state

B) Schmidt decomposition

Singular value decomposition (SVD)

$$|\psi\rangle \neq |\phi^A\rangle \otimes |\phi^B\rangle$$

$$|\phi_i^+\rangle \equiv$$

$$M = U S V^+$$

$m \times n$ $m \times m$ $m \times n$ $n \times n$

$$= \begin{bmatrix} \\ \\ \\ \end{bmatrix}_m \begin{bmatrix} s_1 & s_2 & 0 & \vdots & 0 \\ 0 & \ddots & s_n & & \\ & & & & \end{bmatrix}_m \begin{bmatrix} \\ \\ \\ \end{bmatrix}_n$$

Example

$$m \leq n$$

$$U U^+ = U^+ U = \mathbb{1}_m$$

$$V V^+ = V^+ V = \mathbb{1}_n$$

$$S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}$$

$$s_i \geq 0$$

Singular values
of M

$$|\psi\rangle = \sum_{i,j=1}^n a_{ij} |i^A j^B\rangle = \sum_{k=1}^n s_k \left(\sum_{i=1}^n u_{ik} |i^A\rangle \right) \otimes \left(\sum_{j=1}^n (V^+)_{kj} |j^B\rangle \right) = \sum_k s_k \left(|\phi_k^A\rangle \otimes |\varphi_k^B\rangle \right)$$

$$a_{ij} = \sum_k u_{ik} s_k (V^+)_{kj}$$

$$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{kk'}$$

$$\langle \varphi_k^B | \varphi_{k'}^B \rangle = \delta_{kk'}$$

$$|\psi\rangle = \sum_{i,j=1}^n a_{ij} |i^A j^B\rangle = \sum_{k=1}^n s_k \left(\sum_{i=1}^n u_{ik} |i^A\rangle \right) \otimes \left(\sum_{j=1}^n (V^\dagger)_{kj} |j^B\rangle \right) = \sum s_k$$

$$a_{ij} = \sum_k u_{ik} s_k (V^\dagger)_{kj}$$

$$\eta = \langle \psi | \psi \rangle = \sum_{k=1}^n s_k^2$$

$$s_k \geq 0$$

$$\frac{1}{\sqrt{p_k}}$$

$$p_k = s_k^2$$

$$\boxed{p_k \geq 0}$$

$$\boxed{\sum p_k = 1}$$

$$|\psi\rangle = \sum s_k \left(|\phi_k^A\rangle \otimes |\varphi_k^B\rangle \right) = \sum_{k=1}^n \sqrt{p_k} |\phi_k^A\rangle \otimes |\varphi_k^B\rangle \quad \text{Schmidt decomposition}$$

$$s_k \geq 0$$

$$\frac{1}{\sqrt{p_k}}$$

$$p_k = s_k^2$$

$$\boxed{\begin{array}{l} p_k \geq 0 \\ \sum p_k = 1 \end{array}}$$

$$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{k,k'}$$

$$\langle \varphi_k^B | \varphi_{k'}^B \rangle = \delta_{k,k'}$$

$$\left(\sum_{j=1}^n (V^+)_{kj} |j^B\rangle \right) = \sum_{k=1}^{\infty} s_k \left(|\phi_k^A\rangle \otimes |\psi_k^B\rangle \right) = \sum_{k=1}^{\infty} \sqrt{p_k} \left(|\phi_k^A\rangle \otimes |\psi_k^B\rangle \right) \quad \text{Schmidt decomposition}$$

$$s_1, s_2, s_3, \dots, s_n, 0 \quad s_k \geq 0$$

$$s_1, s_2, s_3, \dots, s_n, \underbrace{0, 0, \dots}_{\infty} \quad \sqrt{p_k}$$

$$p_k = s_k^2$$

$$\boxed{p_k \geq 0, \sum p_k = 1}$$

$$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{k,k'}$$

$$\langle \psi_k^B | \psi_{k'}^B \rangle = \delta_{k,k'}$$

$\chi \equiv \neq$ non-zero singular values

$$|\psi\rangle = \sum_{k=1}^{\infty} s_k |\phi_k^A\rangle \otimes |\psi_k^B\rangle = \sum_{k=1}^{\infty} \sqrt{p_k} |\phi_k^A\rangle \otimes |\psi_k^B\rangle \quad \text{Schmidt decomposition}$$

$$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{k,k'}$$

$$\langle \psi_k^B | \psi_{k'}^B \rangle = \delta_{k,k'}$$

$\chi \equiv \neq$ non-zero singular values

Schmidt rank

Schmidt coefficients

Schmidt bases

$$s_k \geq 0$$

$$\sum_{k=1}^{\infty} \sqrt{p_k} = 1$$

$$p_k = s_k^2$$

$$\boxed{\begin{array}{l} p_k \geq 0 \\ \sum p_k = 1 \end{array}}$$

$$|\psi\rangle = \sum_{ij} a_{ij} |i\rangle |j\rangle$$

$$|11\rangle + |22\rangle$$

$$|11\rangle + |12\rangle = |1\rangle \otimes (|1\rangle + |2\rangle)$$

$$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle)$$

$$|11\rangle + |12\rangle + |21\rangle \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|\phi_n^+\rangle \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^n |k\rangle |k\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

maximally entangled state

$$0 \quad \left[\begin{array}{c} m \\ n \end{array} \right]$$

Examples

$$|\psi\rangle = |\phi\rangle \otimes |\varphi\rangle$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

$\chi = 1 \iff$ product state
Schmidt
rank

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle)$$

$\chi = 2 \Rightarrow$ entangled

$$\vec{p} = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right)$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |kk\rangle$$

$\chi = n$

$$\vec{p} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$$

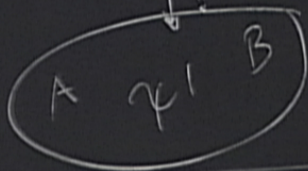
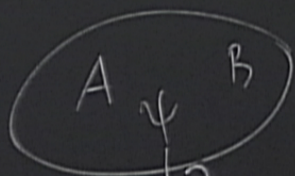
6.3 MEASURES OF ENTANGLEMENT

$|\psi\rangle$
 ↗ product state
 ↘ entangled state

$$|\psi\rangle \neq |\phi^A\rangle \otimes |\phi^B\rangle$$

(A)

Alice



convertibility under LOCC (local operation & classical communication)

Bob

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\psi'\rangle$$

?
 if YES then

$|\psi\rangle$ is more entangled than $|\psi'\rangle$

$$E(\psi) \geq E(\psi')$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

Schmidt rank

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle)$$

$\chi=2 \rightarrow$ entangled

$$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |kk\rangle$$

$\chi=n$

$$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$$

$|\psi\rangle$ product state
entangled state

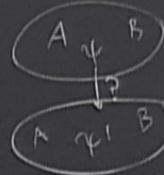
$$|\psi\rangle \neq |\phi^A\rangle \otimes |\phi^B\rangle$$

(A)

Alice

convertibility under LOCC

Bob



LOCC (local operation & classical communication)

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\psi'\rangle$$

? if YES then

$|\psi\rangle$ is more entangled than $|\psi'\rangle$

$$\mathcal{E}(\psi) \geq \mathcal{E}(\psi')$$

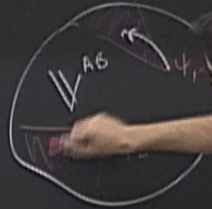
$$\psi_1 \xrightarrow{\text{LOCC}} \psi_2$$

$$\mathcal{E}(\psi_1) \geq \mathcal{E}(\psi_2)$$

$$\psi_1 \xleftarrow{\text{LOCC}} \psi_2$$

$$\mathcal{E}(\psi_1) \leq \mathcal{E}(\psi_2)$$

$$\psi_1 \xleftrightarrow{\text{LOCC}} \psi_2$$



$$\vec{P} = (1, 0, 0, \dots, 0)$$

Schmidt rank

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle) \quad \chi=2 \rightarrow \text{entangled}$$

$$\vec{P} = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |kk\rangle \quad \chi=n$$

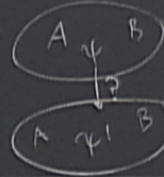
$$\vec{P} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$$

$|\psi\rangle$
 $\nearrow$ product state
 $\searrow$ entangled state

$$|\psi\rangle \neq |\phi^A\rangle \otimes |\phi^B\rangle$$

(A)

Convertibility under LOCC (local operation & classical communication)



LOCC (local operation & classical communication)

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\psi'\rangle$$

? if YES then

$$\mathcal{E}(\psi) \geq \mathcal{E}(\psi')$$

$|\psi\rangle$ is more entangled than $|\psi'\rangle$

$$\psi_1 \xrightarrow{\text{LOCC}} \psi_2$$

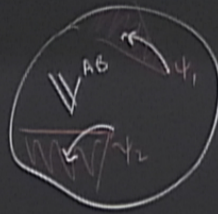
$$\mathcal{E}(\psi_1) \geq \mathcal{E}(\psi_2)$$

$$\psi_1 \xleftarrow{\text{LOCC}} \psi_2$$

$$\mathcal{E}(\psi_1) \leq \mathcal{E}(\psi_2)$$

$$\psi_1 \xleftrightarrow{\text{LOCC}} \psi_2$$

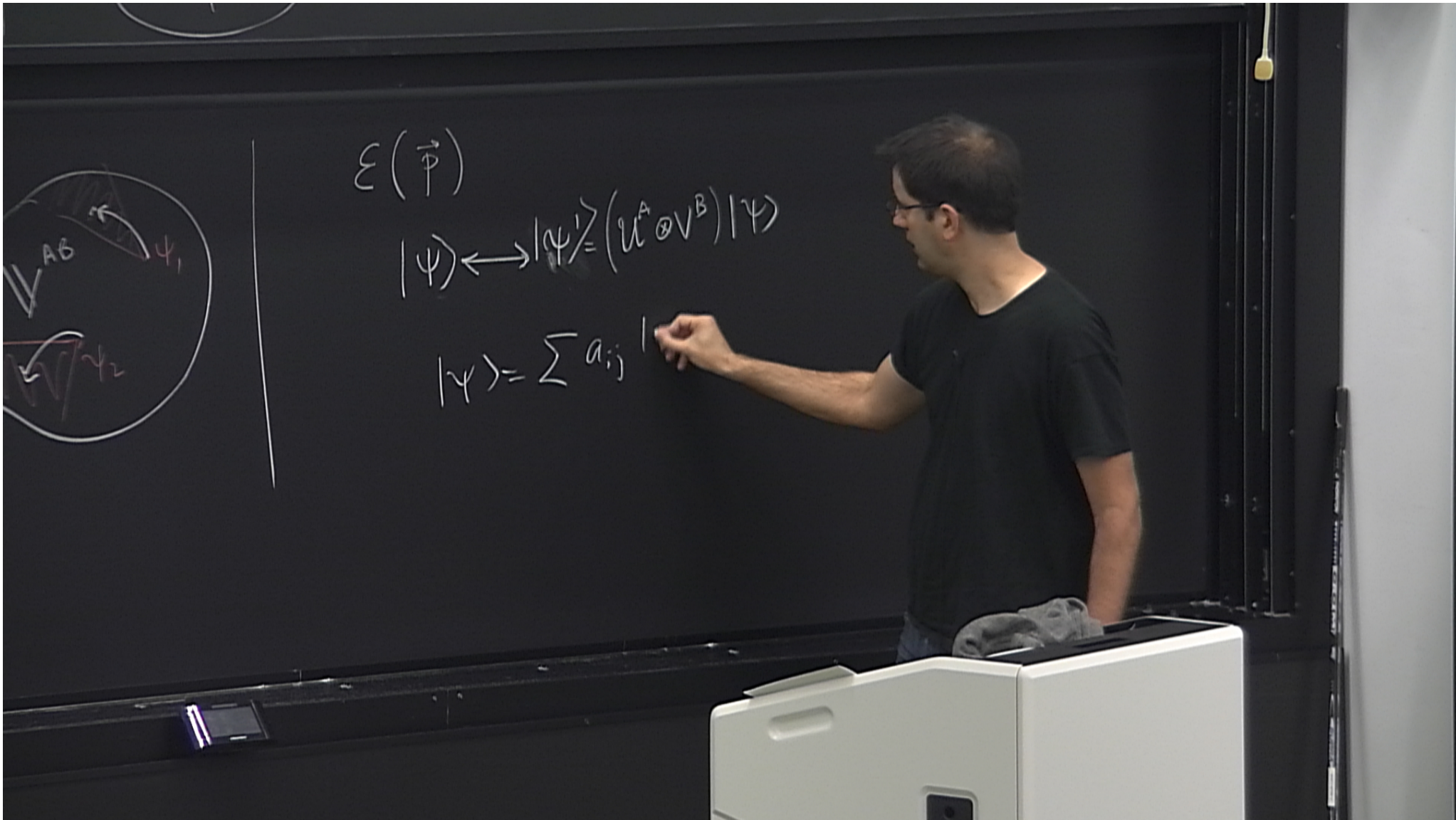
partial order

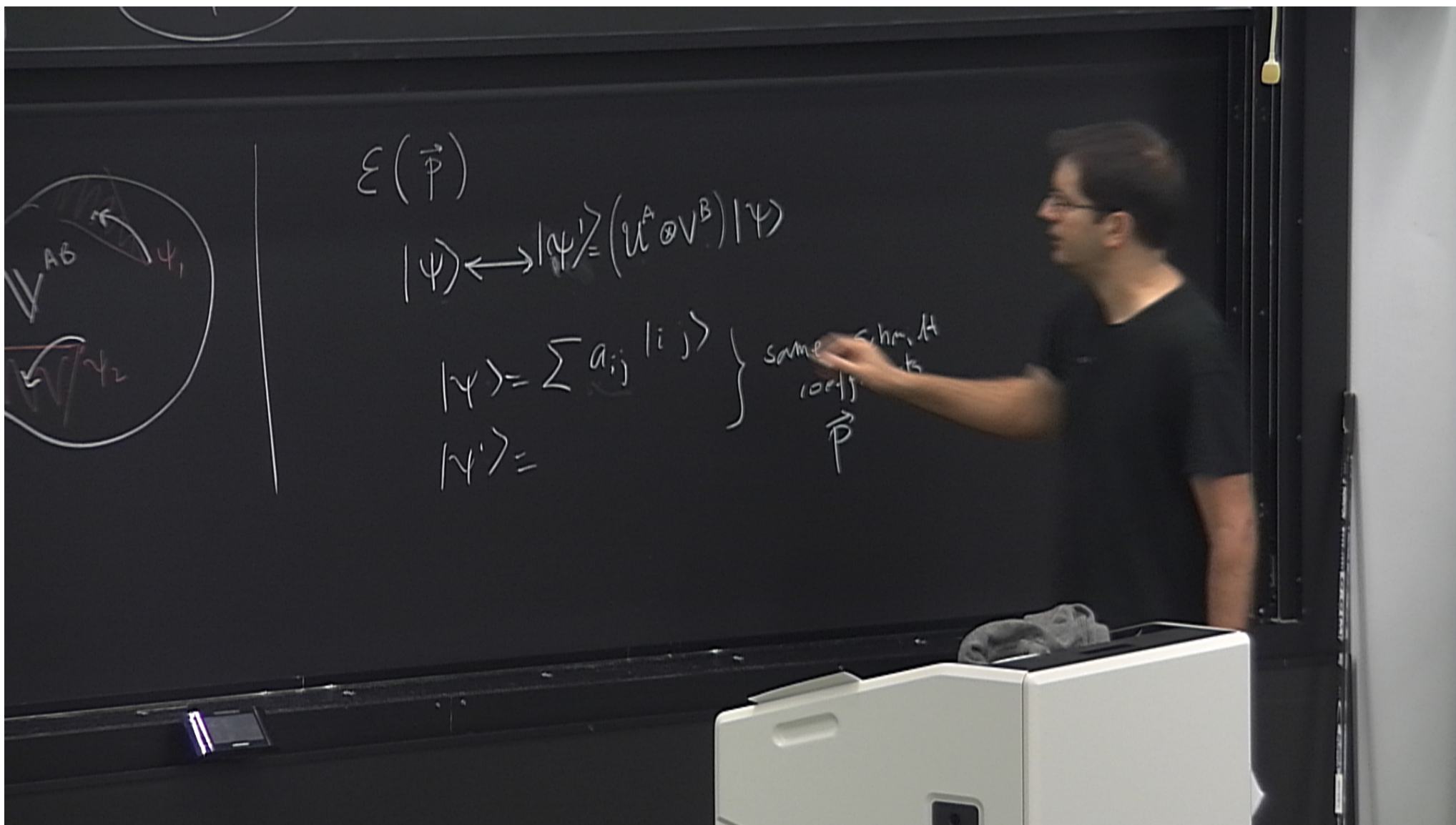


$$\mathcal{E}(\vec{P})$$

$$|\psi\rangle \leftrightarrow |\psi'\rangle = (U^A \otimes V^B) |\psi\rangle$$







maximally entangled state

(B) Entanglement entropies $\left\{ \begin{array}{l} \text{von Neumann entr.} \\ \text{Renyi entropies} \end{array} \right.$

Shannon entropy

m

n

n

maximally entangled state

(B) Entanglement entropies $\left\{ \begin{array}{l} \text{von Neumann entr.} \\ \text{Reny: entropies} \end{array} \right.$

Shannon entropy

$$S(\vec{p}) \equiv - \sum_k p_k \log_2(p_k)$$

Examples.
 $|\psi\rangle = |\psi^A\rangle \otimes |\psi^B\rangle$ $\hat{S}^A =$

Examples.

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle \quad \hat{S}^A = |\phi^A\rangle\langle\phi^A|$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

$$S(\hat{S}^A) = -1 \underbrace{\log_2(1)}_0 - 0 \underbrace{\log_2 0}_0 - \dots = 0$$

Examples.

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle \quad \hat{S}^A = |\phi^A\rangle\langle\phi^A|$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

$$S(\hat{S}^A) = -1 \underbrace{\log_2(1)}_0 - 0 \underbrace{\log_2 0}_0 - \dots = 0$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle) \quad \hat{S}^A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{p} = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right)$$

$$S(\hat{S}^A) = -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2} = -\log_2 \frac{1}{2} = \log 2 = 1$$

Examples.

$$0 \text{ ebit} \quad |\psi\rangle = |\phi^A\rangle \otimes |\psi^B\rangle \quad \hat{\rho}^A = |\phi^A\rangle\langle\phi^A|$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

$$S(\hat{\rho}^A) = -1 \log_2(1) - \underbrace{0 \log_2 0}_{=0} - \dots = 0$$

$$1 \text{ ebit} \quad |\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |00\rangle) \quad \hat{\rho}^A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{p} = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right)$$

$$S(\hat{\rho}^A) = -\log_2 \frac{1}{2} = \log_2 2 = 1$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |k\rangle$$

$$\hat{\rho}^A = \frac{1}{n} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\vec{p} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$$

$$S(\hat{\rho}^A)$$

Examples.

$$0 \text{ ebit} \quad |\psi\rangle = |\phi^A\rangle \otimes |\psi^B\rangle \quad \hat{\rho}^A = |\phi^A\rangle\langle\phi^A|$$

$$\vec{p} = (1, 0, 0, \dots, 0)$$

$$S(\hat{\rho}^A) = -1 \underbrace{\log_2(1)}_0 - 0 \underbrace{\log_2(0)}_0 - \dots = 0$$

$$1 \text{ ebit} \quad |\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |22\rangle) \quad \hat{\rho}^A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{p} = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right)$$

$$S(\hat{\rho}^A) = -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2} = -\log_2 \frac{1}{2} = \log_2 2 = 1$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |kk\rangle \quad \hat{\rho}^A = \frac{1}{n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \vec{p} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$$

$$S(\hat{\rho}^A) = \left(-\frac{1}{n} \log_2 \left(\frac{1}{n}\right)\right) n = \log_2(n)$$