Title: 13/14 PSI - Explorations in Condensed Matter - Lecture 6

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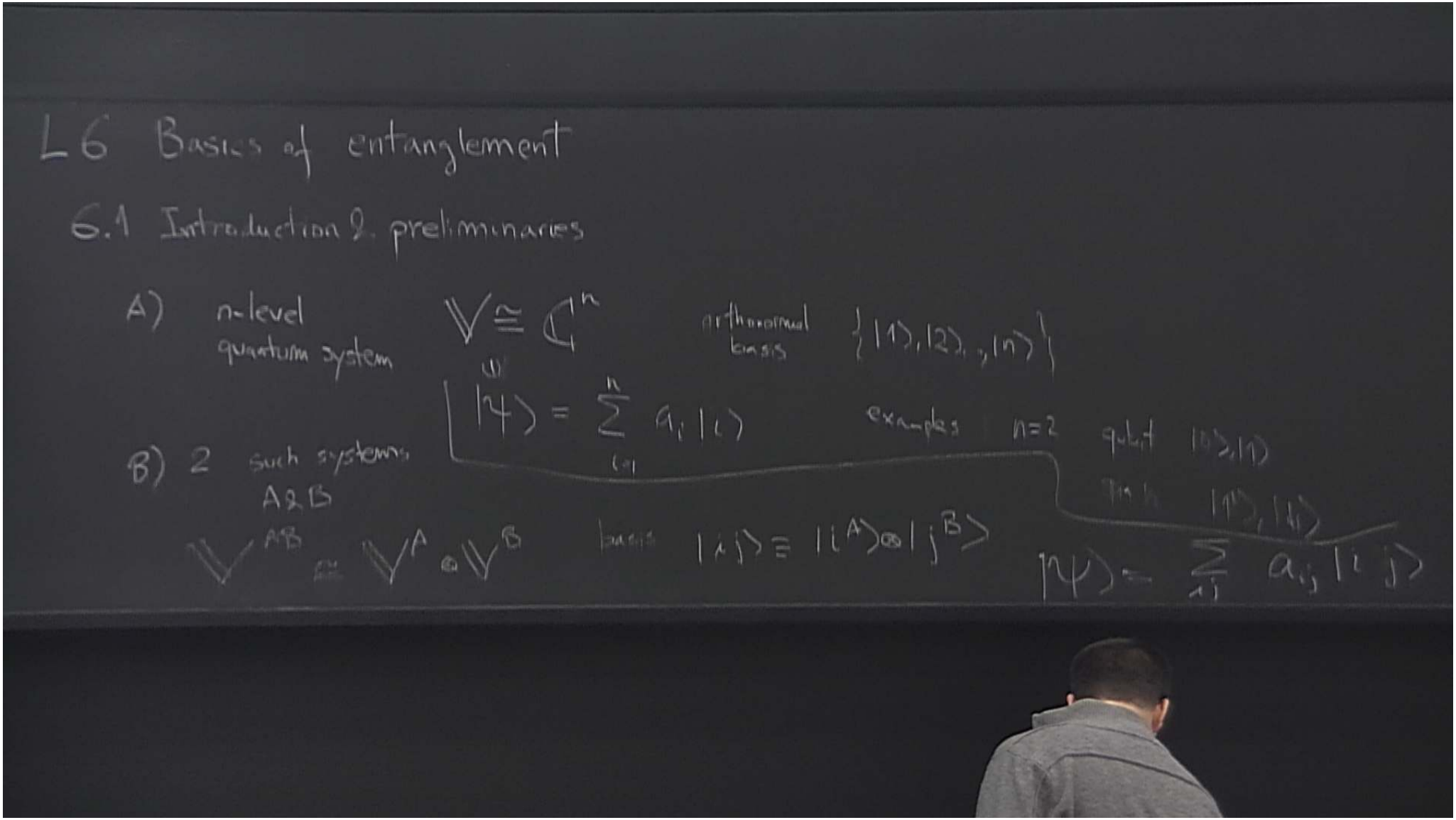
URL: http://pirsa.org/14040029

Abstract:

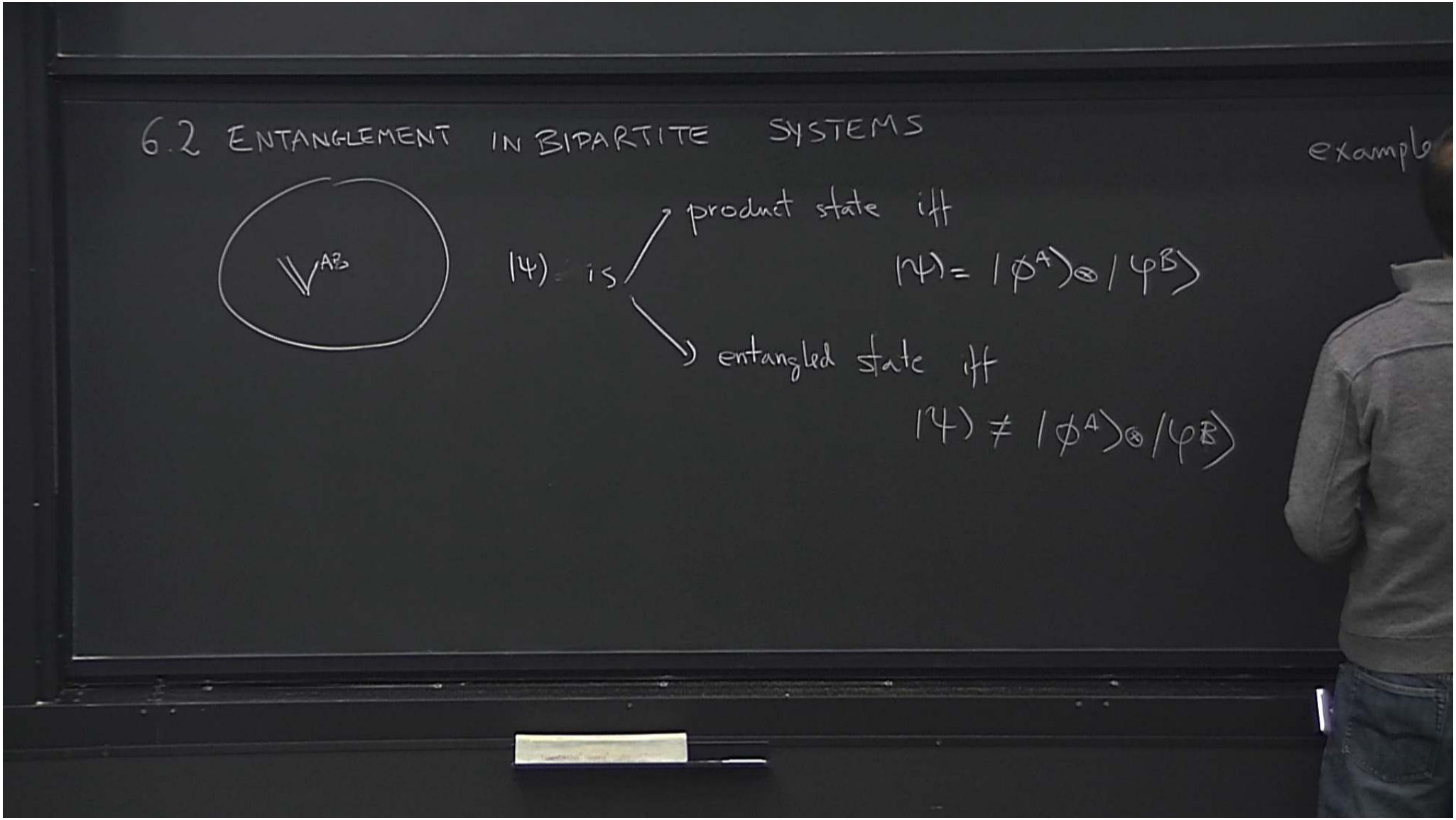

L6 Basics of entanglement
6.1 Introduction & preliminaries
A) n-level quantum system
$V \cong \mathbb{C}^n$
orthonormal basis $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$
$|\psi\rangle = \sum_{i=1}^{n} a_i |i\rangle$
examples $n=2$ qubit $|0\rangle, |1\rangle$
$n=3$ $|1\rangle, |2\rangle$
B) 2 such systems A & B
$V^{AB} \cong V^A \otimes V^B$
basis $|i,j\rangle \equiv |i^A\rangle \otimes |j^B\rangle$
$|\psi\rangle = \sum_{ij} a_{ij} |i,j\rangle$

c) REDUCED DENSITY MATRIX

$$\hat{\rho}^A \equiv \mathrm{tr}_B \, |\psi\rangle\langle\psi| = \sum_m \langle m^B| \, |\psi\rangle\langle\psi| \, |m^B\rangle = \sum (\hat{\rho}^A)_{ik} \, |i^A\rangle\langle k^A|$$

$$(\hat{\rho}^A)_{ik} = \sum_m a_{im} a^*_{km} = (A A^\dagger)_{ik}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots \\ a_{21} & \\ \vdots & \end{pmatrix}$$


6.2 ENTANGLEMENT IN BIPARTITE SYSTEMS
V^{AB}
$|\psi\rangle$ is
product state iff
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
entangled state iff
$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$
example

examples: $|1^A\rangle \otimes |1^B\rangle \equiv |11\rangle$

$|11\rangle + |22\rangle$

$|11\rangle + |12\rangle = |1\rangle \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle$

$|\phi_n^+\rangle \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |kk\rangle \quad \in \mathbb{C}^n \otimes \mathbb{C}^n$

maximally entangled state

(B) Schmidt decomposition

Singular value decomposition (SVD)

$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$

$|\phi_n^+\rangle \equiv$

$$M = U S V^+$$

$m \times n \quad m \times m \quad m \times n \quad n \times n$

$$= \begin{bmatrix} & \\ & \end{bmatrix}_m \begin{bmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{bmatrix}_{m \times n} \begin{bmatrix} & \\ & \\ & \end{bmatrix}_n$$

$m \quad n$

Example $m \leq n$

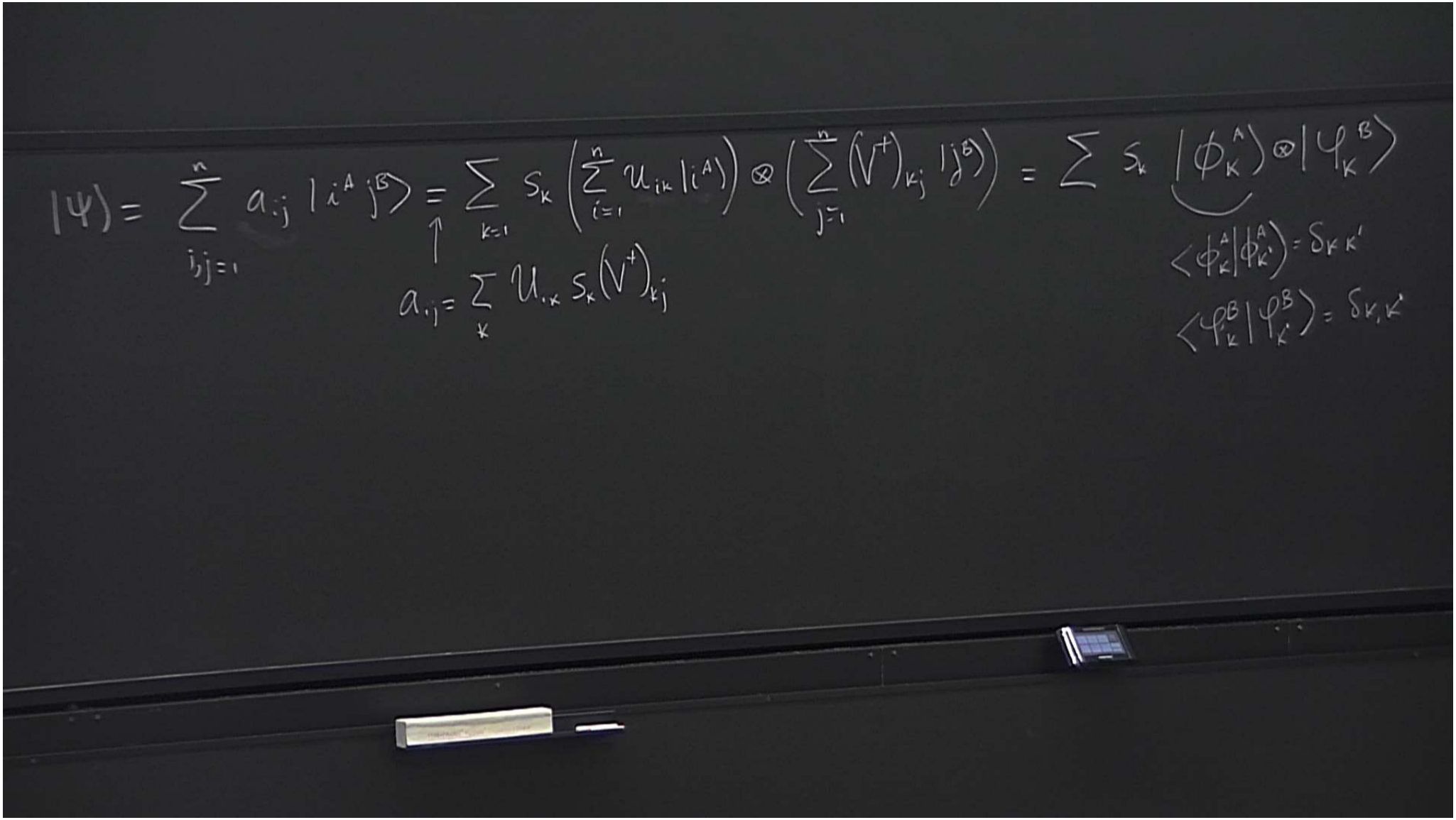
$$UU^+ = U^+U = \mathbb{1}_m$$

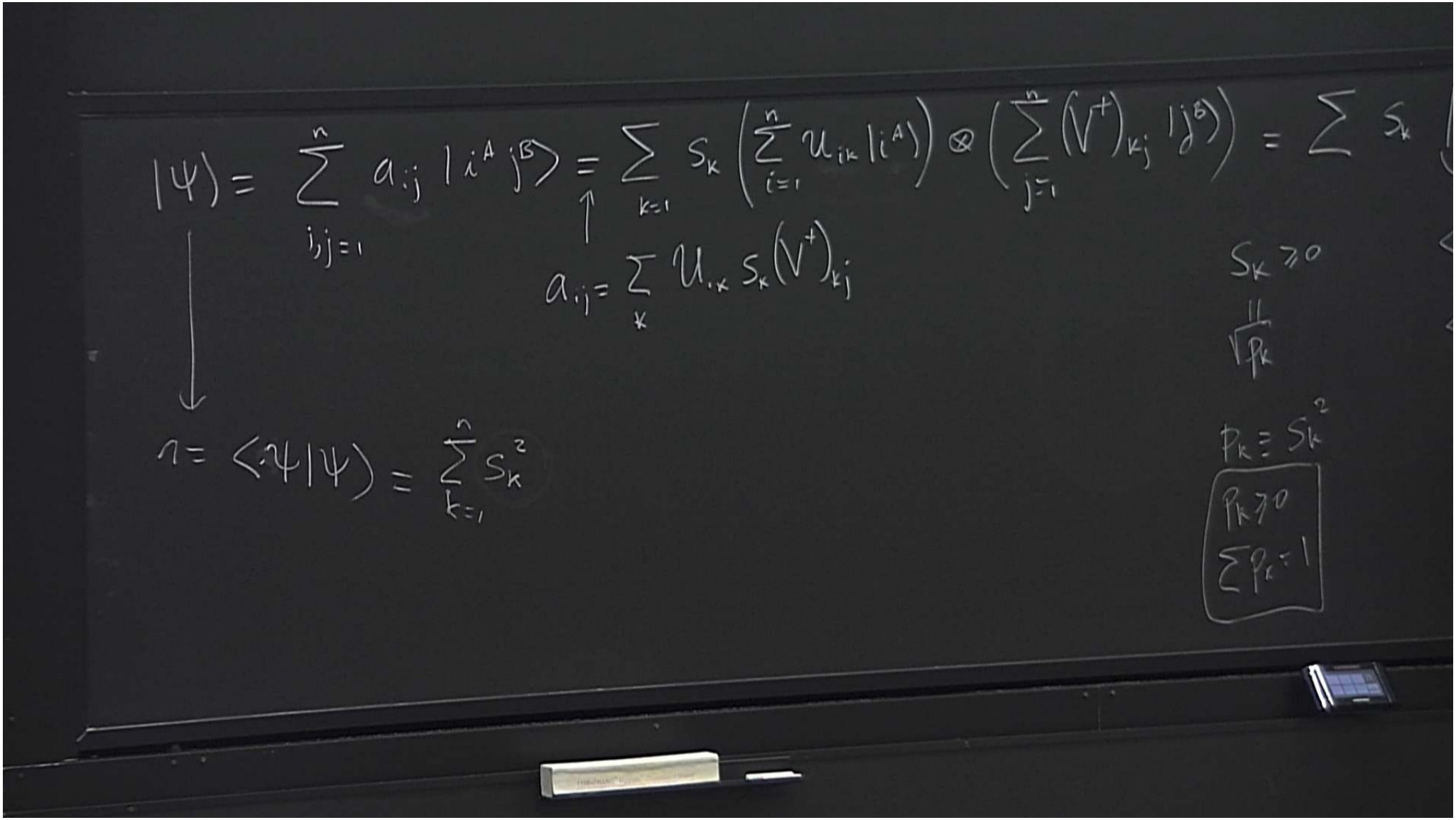
$$VV^+ = V^+V = \mathbb{1}_n$$

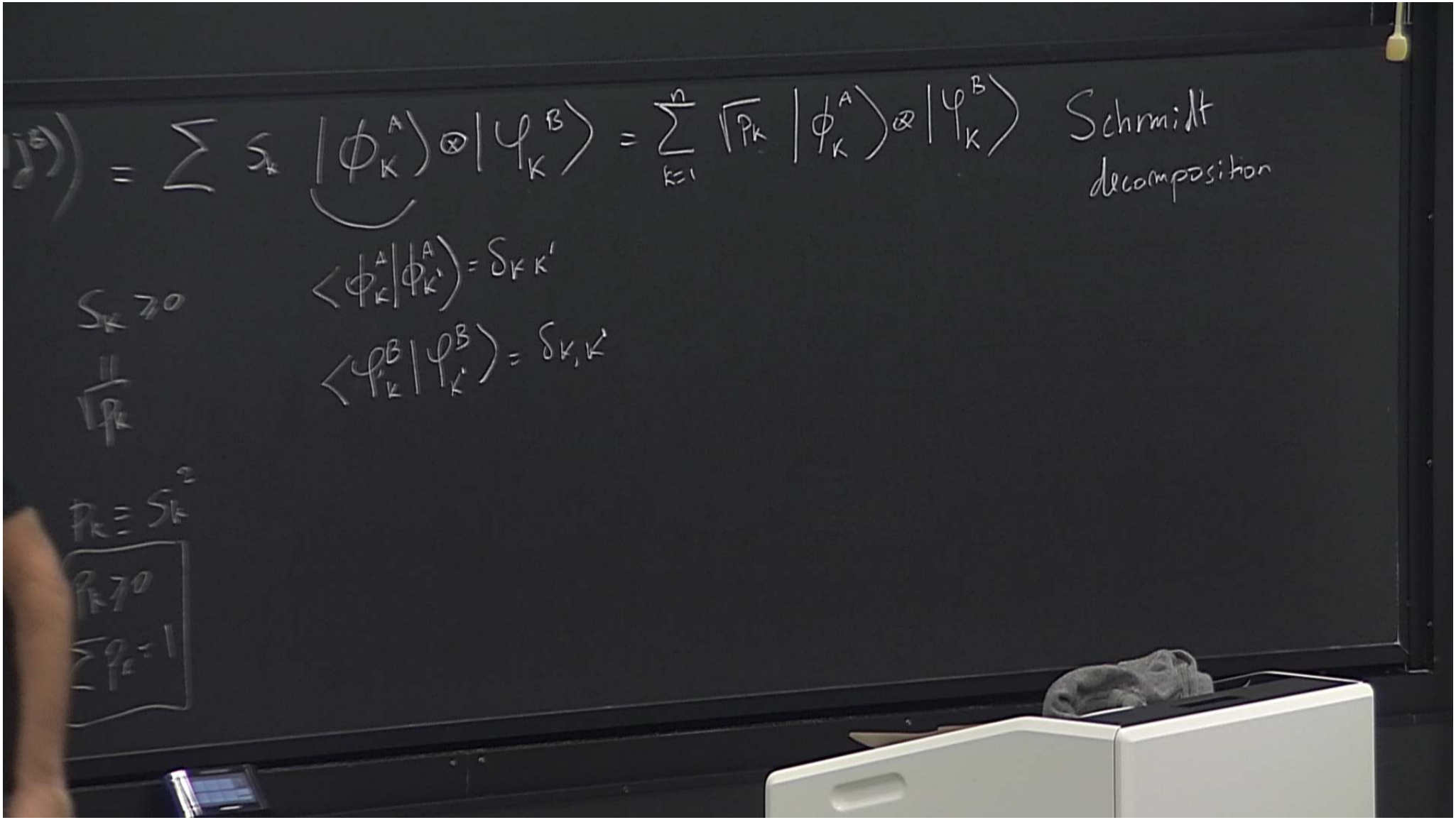
$$S = \begin{bmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \\ & & & s_n \end{bmatrix}$$

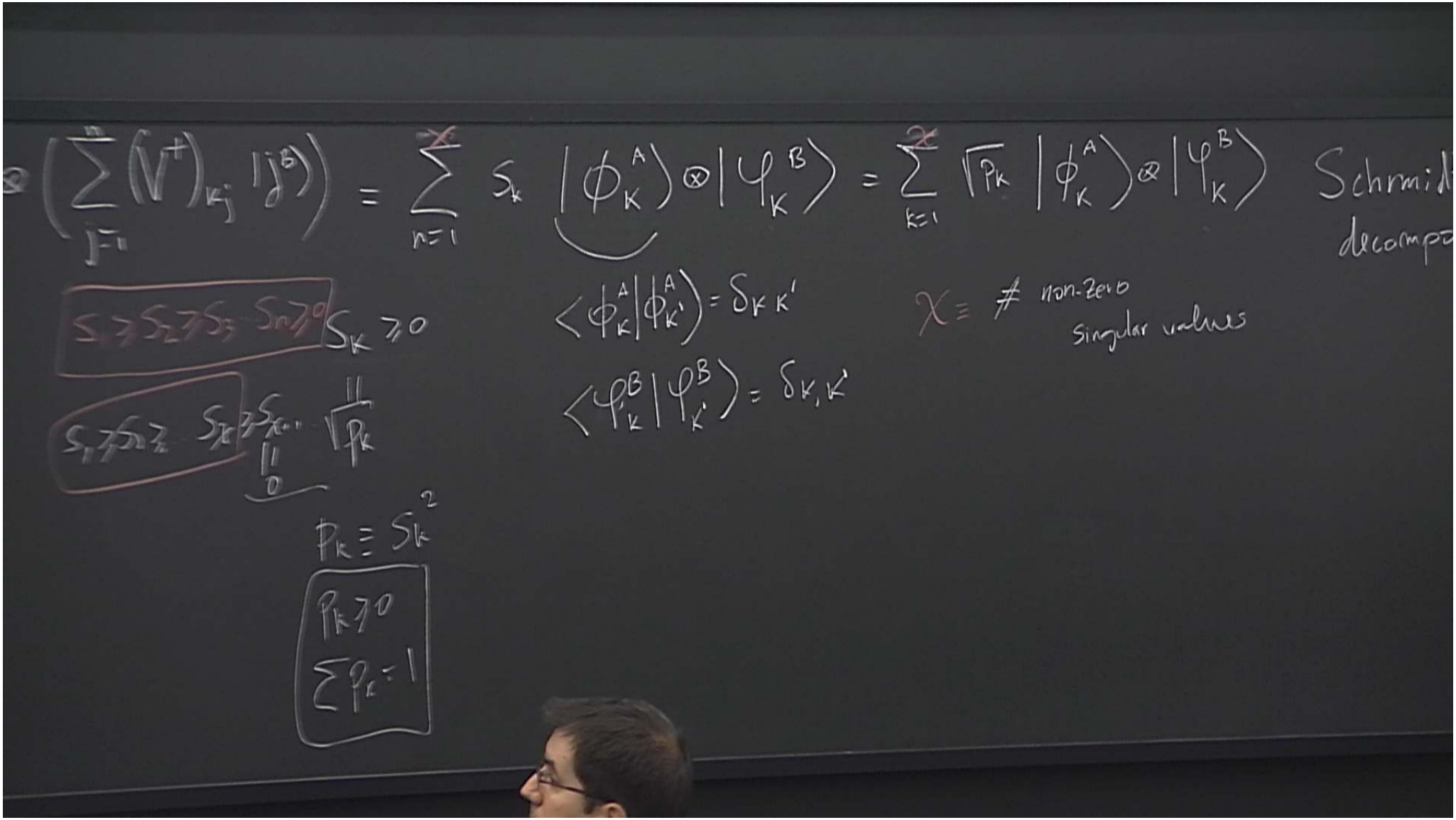
$s_i \geq 0$

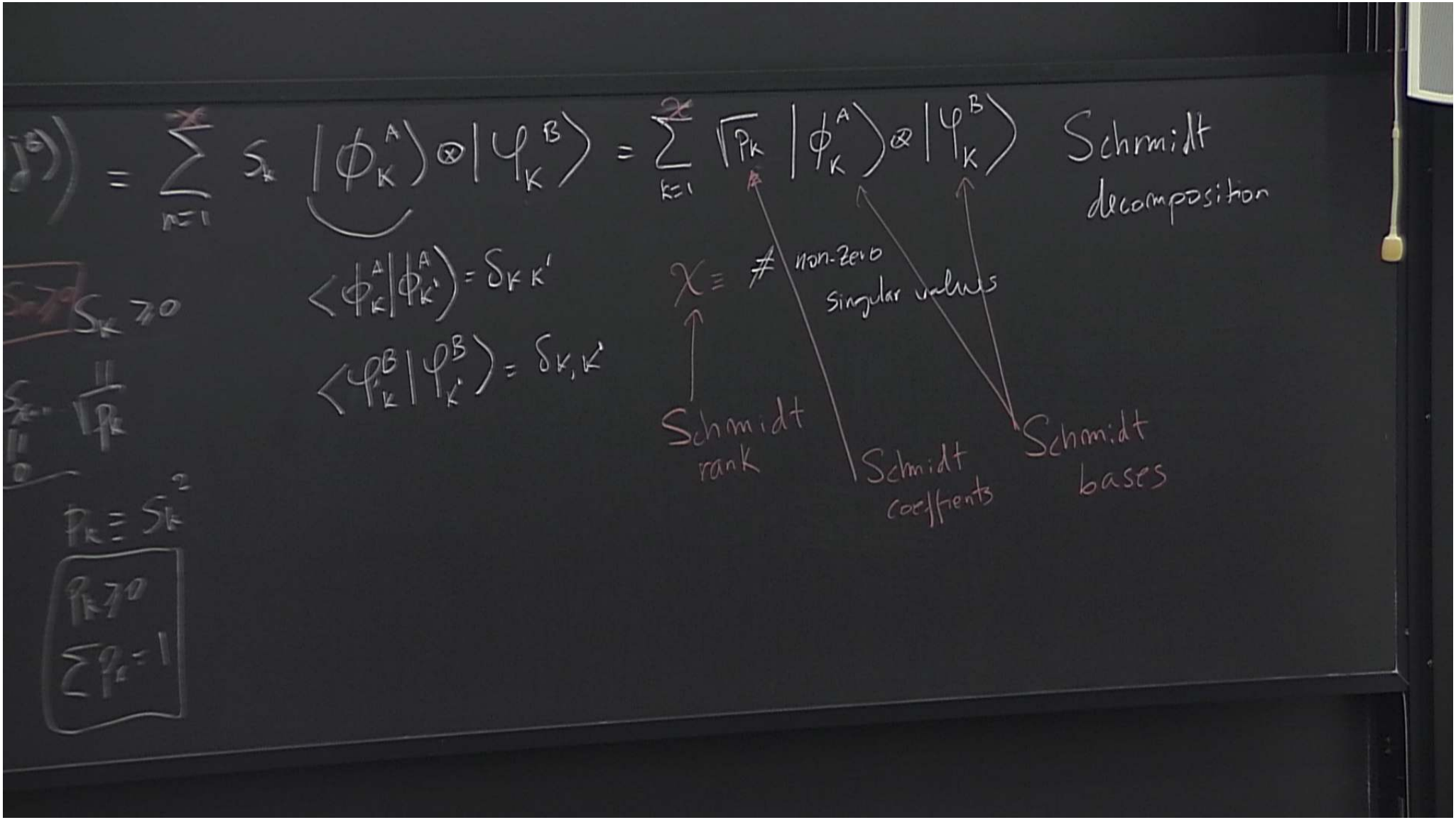
singular values of M


$|\psi\rangle = \sum_{i,j=1}^{n} a_{ij} |i^A j^B\rangle = \sum_{k=1} s_k \left(\sum_{i=1}^{n} u_{ik} |i^A\rangle\right) \otimes \left(\sum_{j=1}^{n} (V^\dagger)_{kj} |j^B\rangle\right) = \sum s_k |\phi_k^A\rangle \otimes |\varphi_k^B\rangle$
$a_{ij} = \sum_k u_{ik} s_k (V^\dagger)_{kj}$
$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{kk'}$
$\langle \varphi_k^B | \varphi_{k'}^B \rangle = \delta_{kk'}$


$|\psi\rangle = \sum_{i,j=1}^{n} a_{ij} |i^A j^B\rangle = \sum_{k=1} s_k \left(\sum_{i=1}^{n} u_{ik} |i^A\rangle\right) \otimes \left(\sum_{j=1}^{n} (V^\dagger)_{kj} |j^B\rangle\right) = \sum s_k$
$a_{ij} = \sum_k u_{ik} s_k (V^\dagger)_{kj}$
$s_k \geq 0$
$\sqrt{p_k}$
$p_k \equiv s_k^2$
$p_k \geq 0$
$\sum p_k = 1$
$1 = \langle\psi|\psi\rangle = \sum_{k=1}^{n} s_k^2$


$= \sum s_k |\phi_k^A\rangle \otimes |\varphi_k^B\rangle = \sum_{k=1}^{n} \sqrt{p_k} |\phi_k^A\rangle \otimes |\varphi_k^B\rangle$
Schmidt decomposition
$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{k k'}$
$\langle \varphi_k^B | \varphi_{k'}^B \rangle = \delta_{k,k'}$
$s_k \geq 0$
$\| \sqrt{p_k}$
$p_k = s_k^2$
$p_k \geq 0$
$\sum p_k = 1$


Schmidt
decomp
$s_1 \geq s_2 \geq s_3 \geq \cdots s_n \geq 0$
$s_k \geq 0$
$s_1 \geq s_2 \geq \cdots s_k \geq s_{k+1} = 0$
$\sqrt{p_k}$
$p_k \equiv s_k^2$
$p_k \geq 0$
$\sum p_k = 1$
$\chi \equiv \#$ non-zero
singular values


Schmidt decomposition
non-zero singular values
Schmidt rank
Schmidt coefficients
Schmidt bases

$|\psi\rangle = \sum_{ij} a_{ij} |i\,j\rangle$

$|11\rangle + |12\rangle = |1\rangle \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$|\phi_n^+\rangle \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |k\,k\rangle \quad \in \mathbb{C}^n \otimes \mathbb{C}^n$

maximally entangled state

Examples $|\psi\rangle = |\phi\rangle \otimes |\varphi\rangle$ $\chi = 1$ $\Longleftrightarrow$ product state

Schmidt rank

$\vec{p} = (1, 0, 0 \ldots 0)$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$ $\chi = 2 \Rightarrow$ entangled

$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$

$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |kk\rangle$ $\chi = n$

$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$

6.3 MEASURES OF ENTANGLEMENT

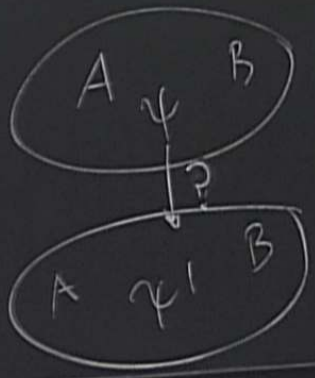
$|\psi\rangle$ → product state

$|\psi\rangle$ → entangled state

$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$

(A) Convertibility under LOCC (local operation & classical communication)

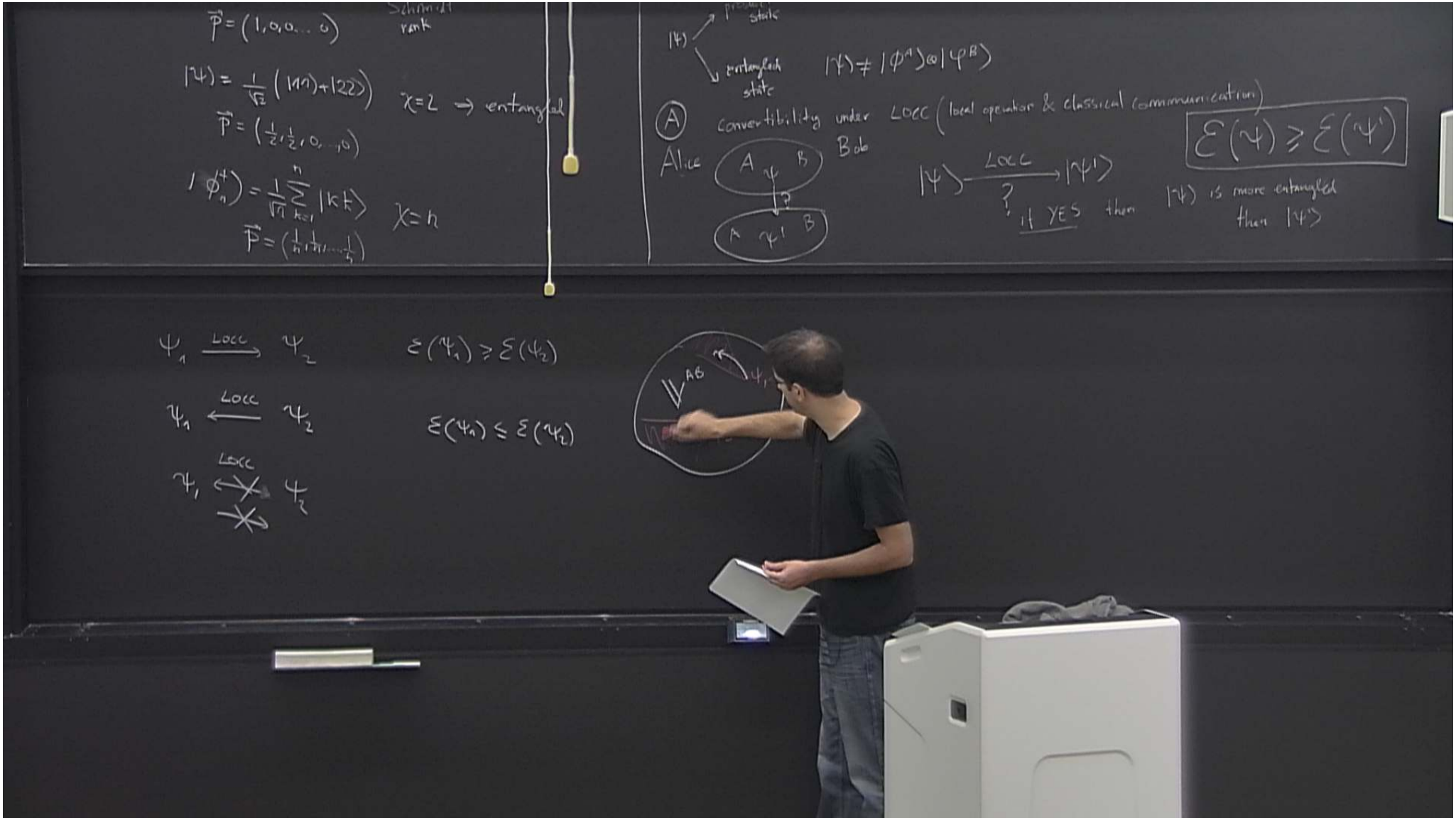
Alice — Bob

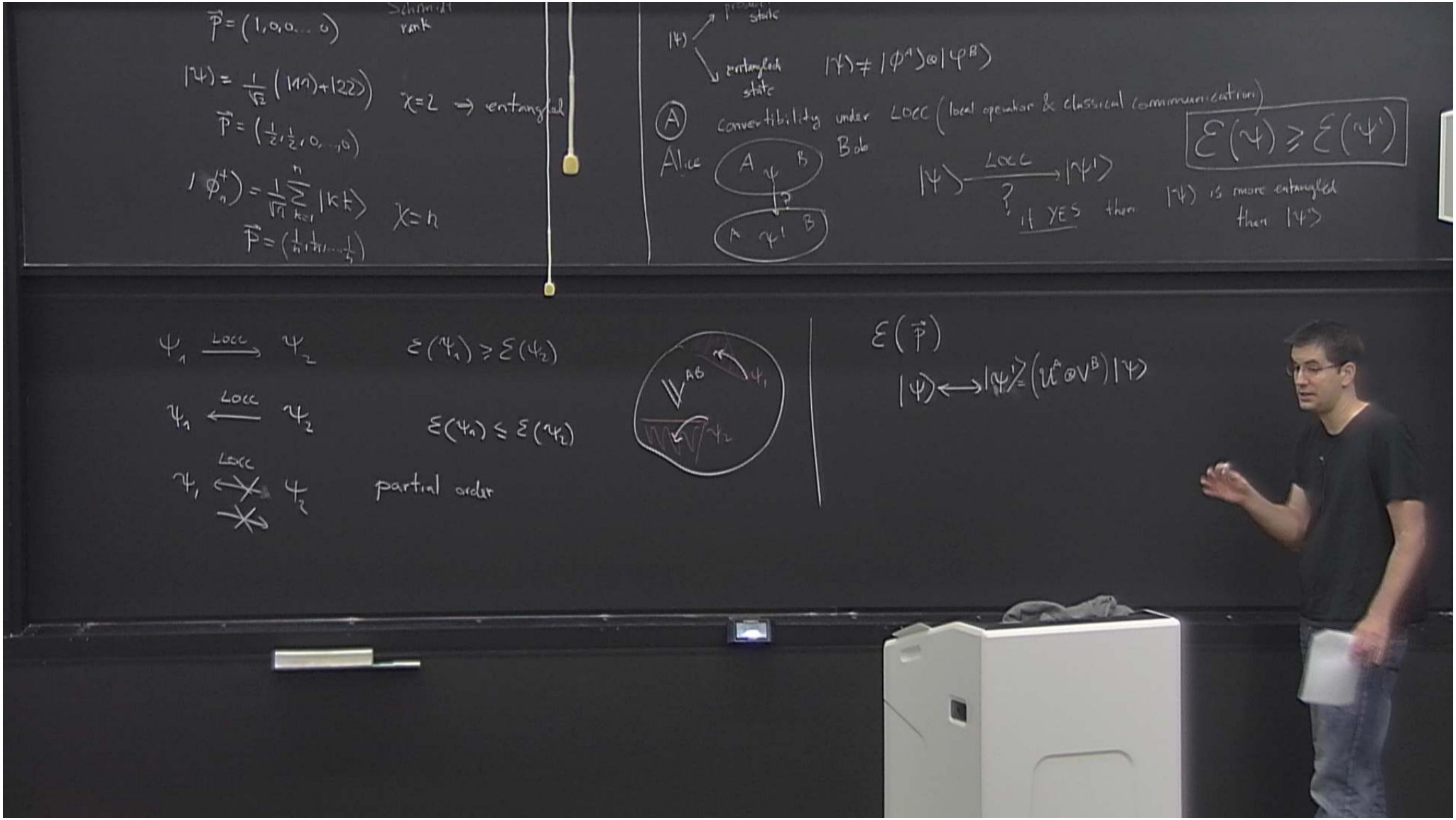


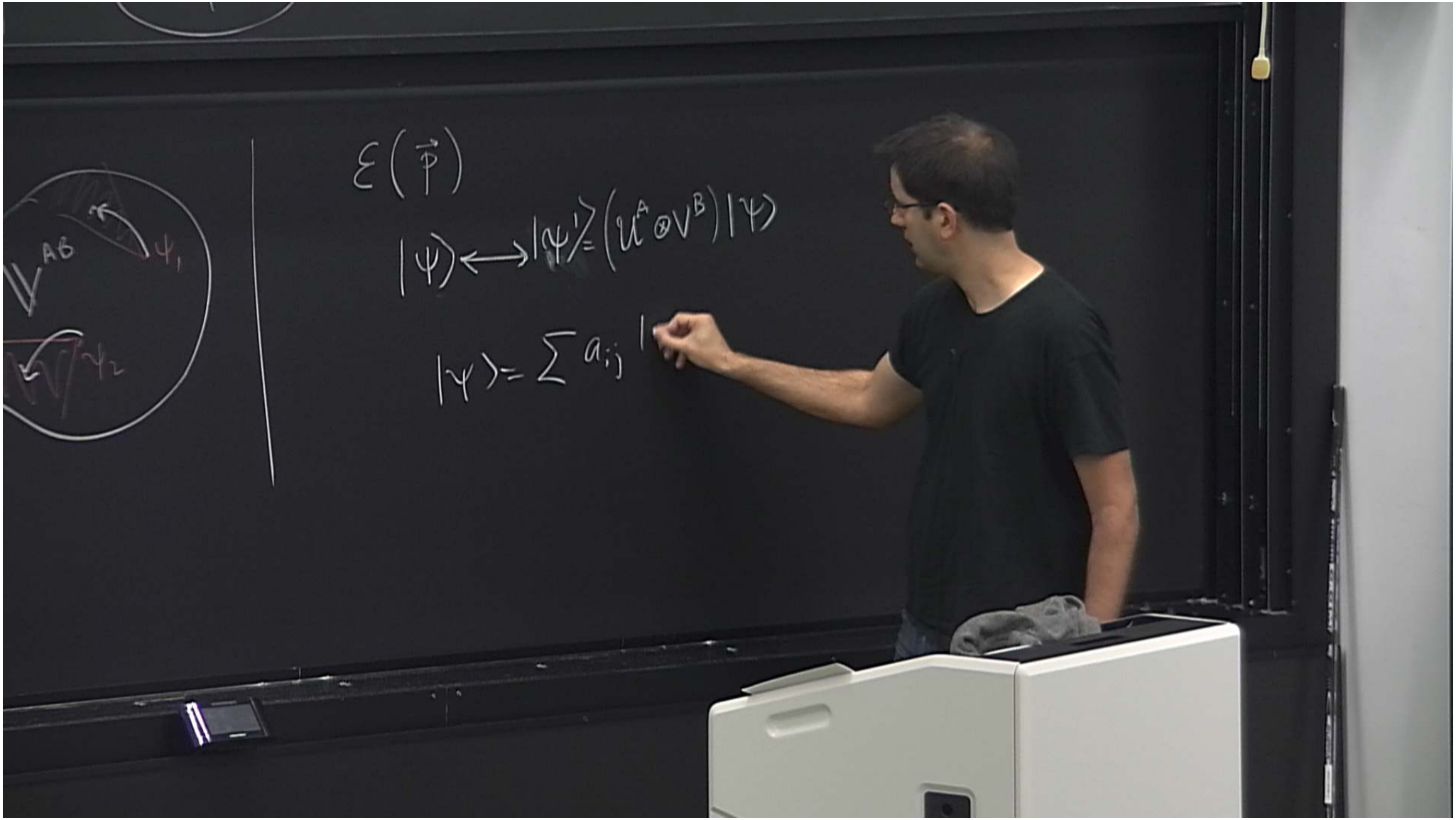
$|\psi\rangle \xrightarrow{LOCC} |\psi'\rangle$?

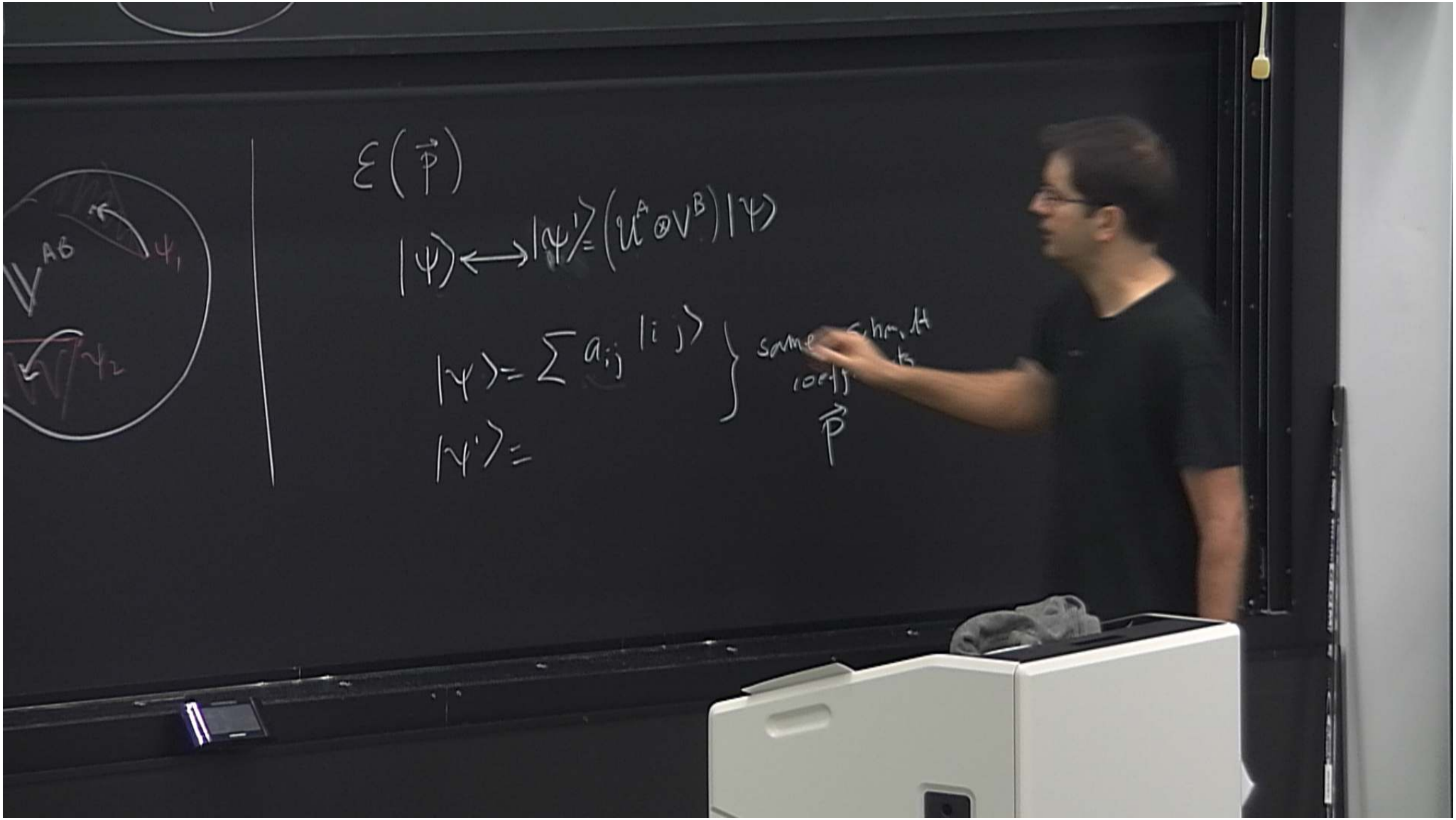
$\mathcal{E}(\psi) \geq \mathcal{E}(\psi')$

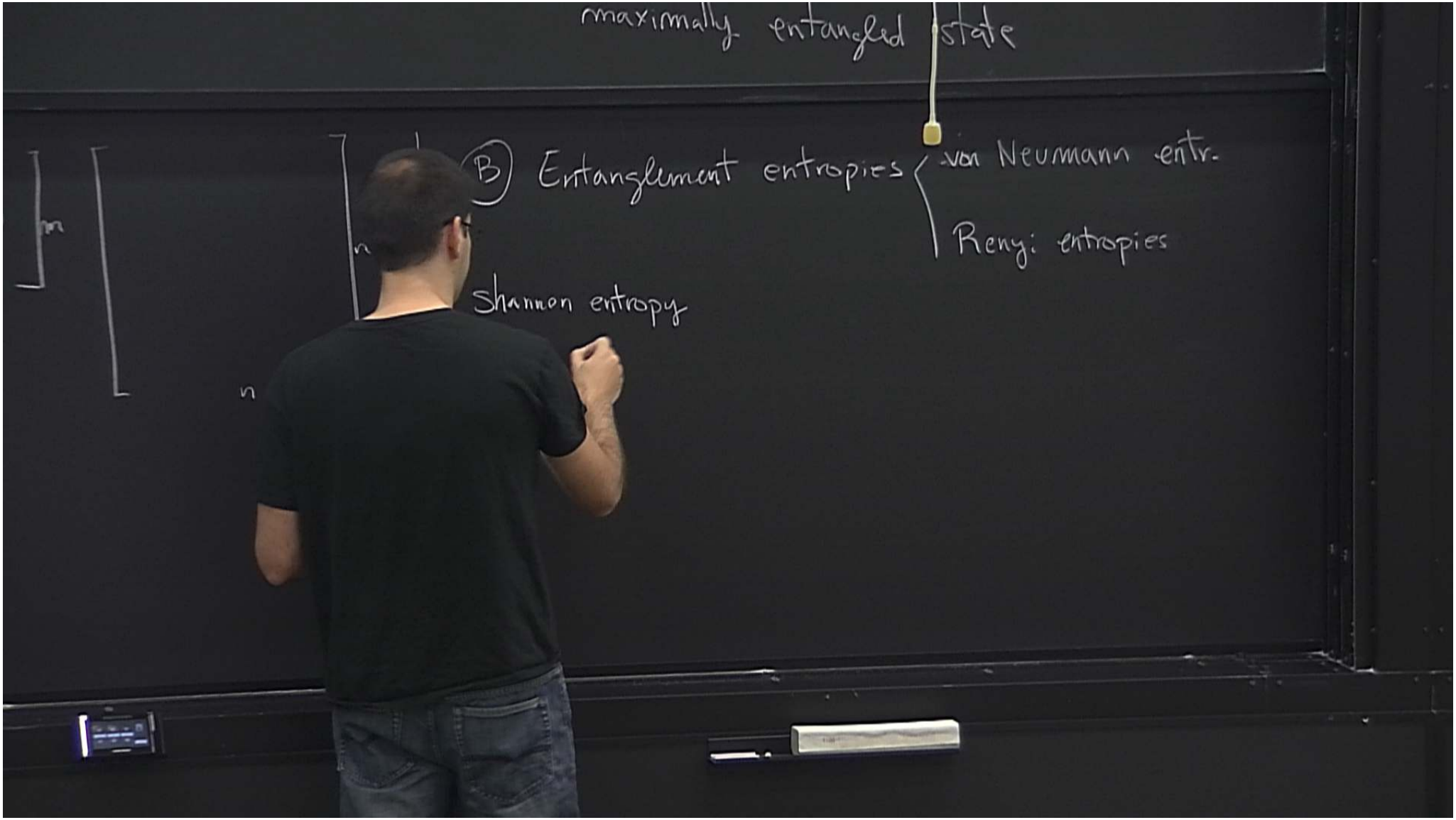
if YES then $|\psi\rangle$ is more entangled than $|\psi'\rangle$

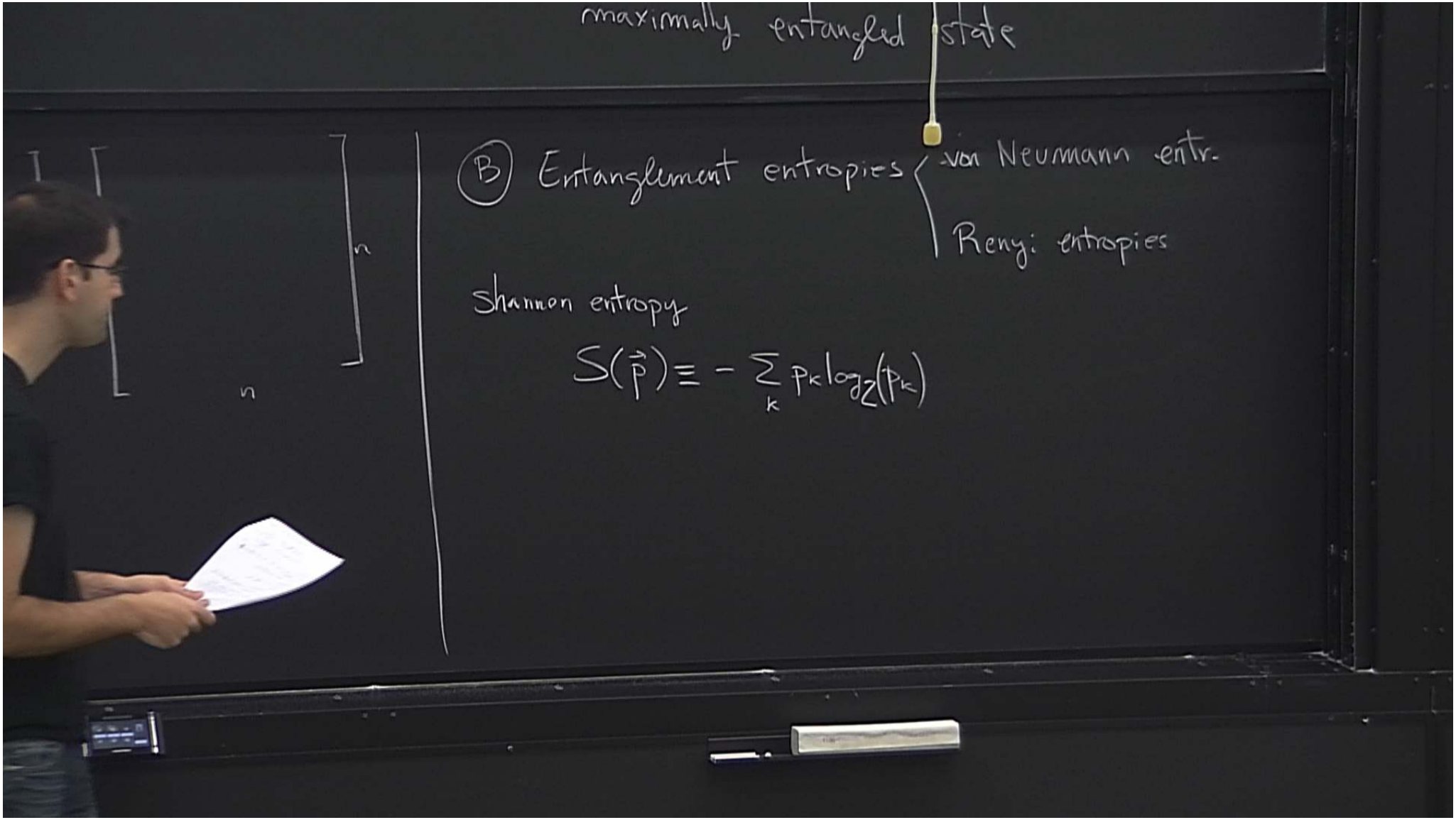

$\vec{p} = (1, 0, 0 \ldots 0)$
Schmidt rank
$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$
$\chi = 2 \Rightarrow$ entangled
$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$
$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |k\,k\rangle$
$\chi = n$
$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$
$|\psi\rangle$
product state
entangled state
$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$
A
Alice
convertibility under LOCC (local operation & classical communication)
Bob
A ψ B
A ψ' B
$|\psi\rangle \xrightarrow{LOCC} |\psi'\rangle$
?
if YES then
$\mathcal{E}(\psi) \geq \mathcal{E}(\psi')$
$|\psi\rangle$ is more entangled than $|\psi'\rangle$
$\psi_1 \xrightarrow{LOCC} \psi_2$
$\mathcal{E}(\psi_1) \geq \mathcal{E}(\psi_2)$
$\psi_1 \xleftarrow{LOCC} \psi_2$
$\mathcal{E}(\psi_1) \leq \mathcal{E}(\psi_2)$
LOCC
ψ_1 ψ_2
AB
ψ_1

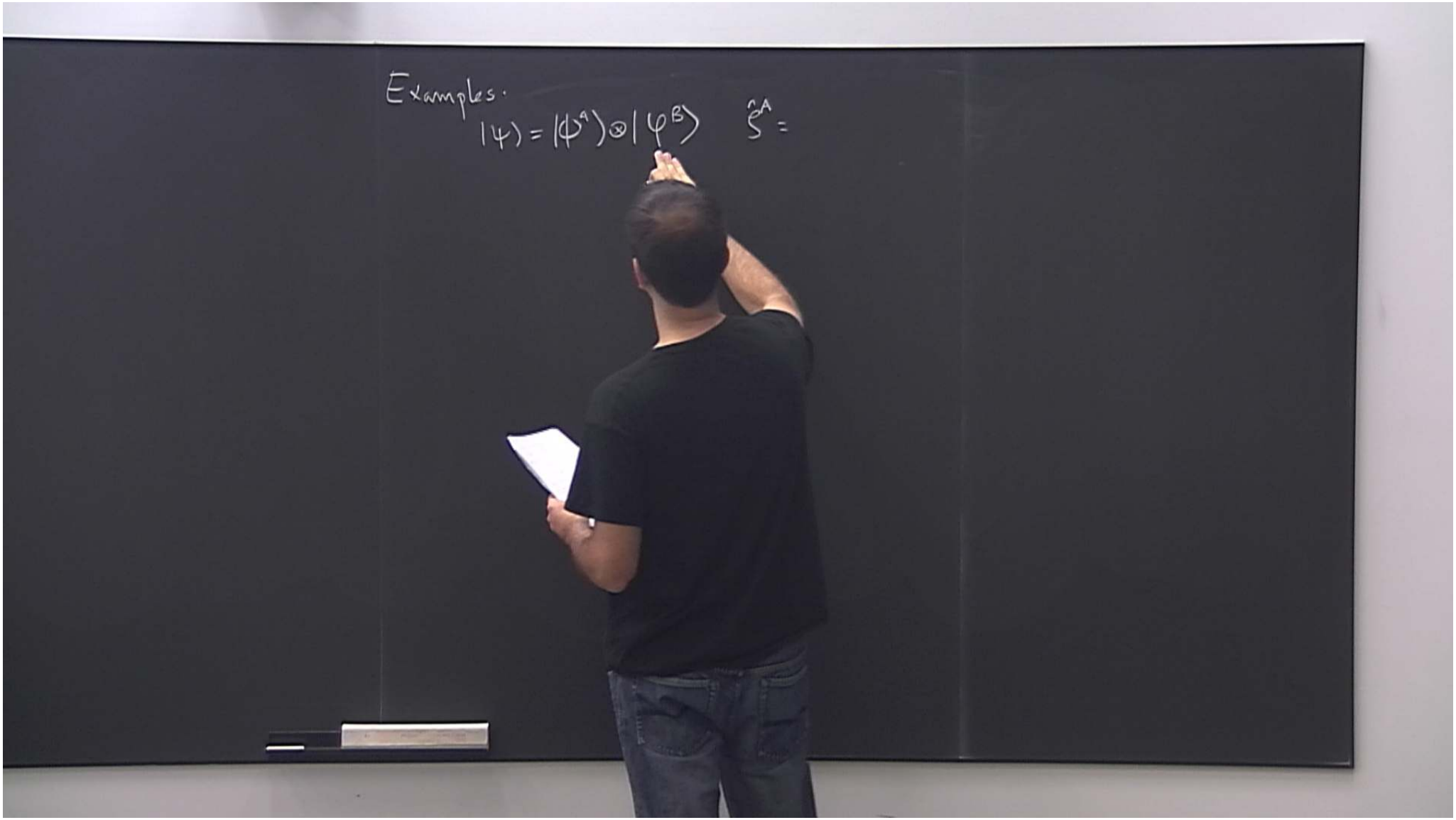

$\vec{p} = (1, 0, 0 \ldots 0)$
Schmidt rank
$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$
$\chi = 2 \Rightarrow$ entangled
$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$
$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |kk\rangle$
$\chi = n$
$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})$
$|\psi\rangle$
product state
entangled state
$|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$
A
Alice
Convertibility under LOCC (local operator & classical communication)
Bob
A ψ B
A ψ' B
$|\psi\rangle \xrightarrow{LOCC} |\psi'\rangle$
?
if YES then
$\mathcal{E}(\psi) \geq \mathcal{E}(\psi')$
$|\psi\rangle$ is more entangled than $|\psi'\rangle$
$\psi_1 \xrightarrow{LOCC} \psi_2$
$\mathcal{E}(\psi_1) \geq \mathcal{E}(\psi_2)$
$\psi_1 \xleftarrow{LOCC} \psi_2$
$\mathcal{E}(\psi_1) \leq \mathcal{E}(\psi_2)$
$\psi_1 \not\xleftrightarrow{LOCC} \psi_2$
partial order
V^{AB}
ψ_1
ψ_2
$\mathcal{E}(\vec{p})$
$|\psi\rangle \longleftrightarrow |\psi'\rangle = (U^A \otimes V^B)|\psi\rangle$

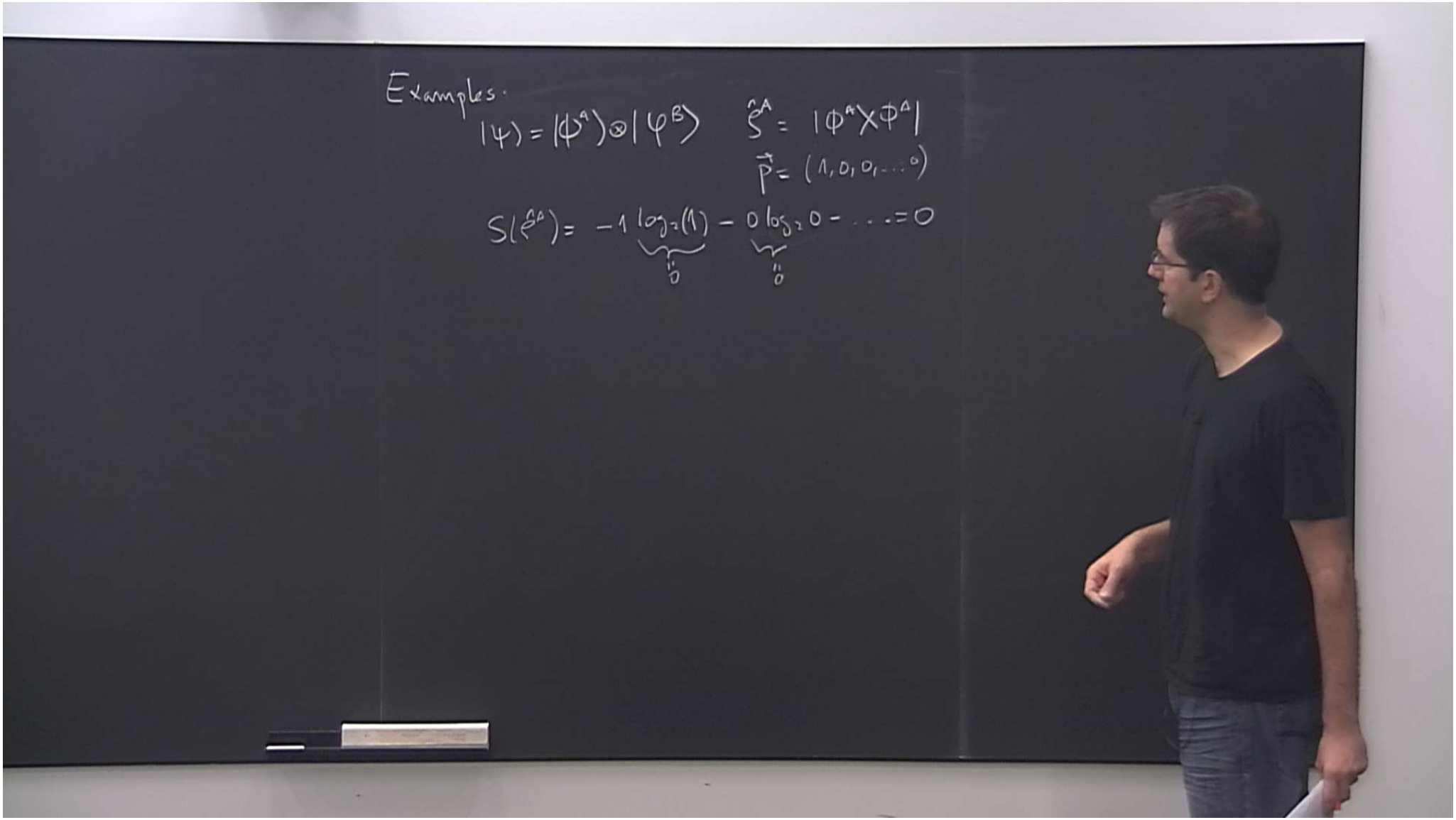

$\mathcal{E}(\vec{p})$
$|\psi\rangle \longleftrightarrow |\psi'\rangle = (U^A \otimes V^B)|\psi\rangle$
$|\psi\rangle = \sum a_{ij}$
AB
ψ_1
ψ_2

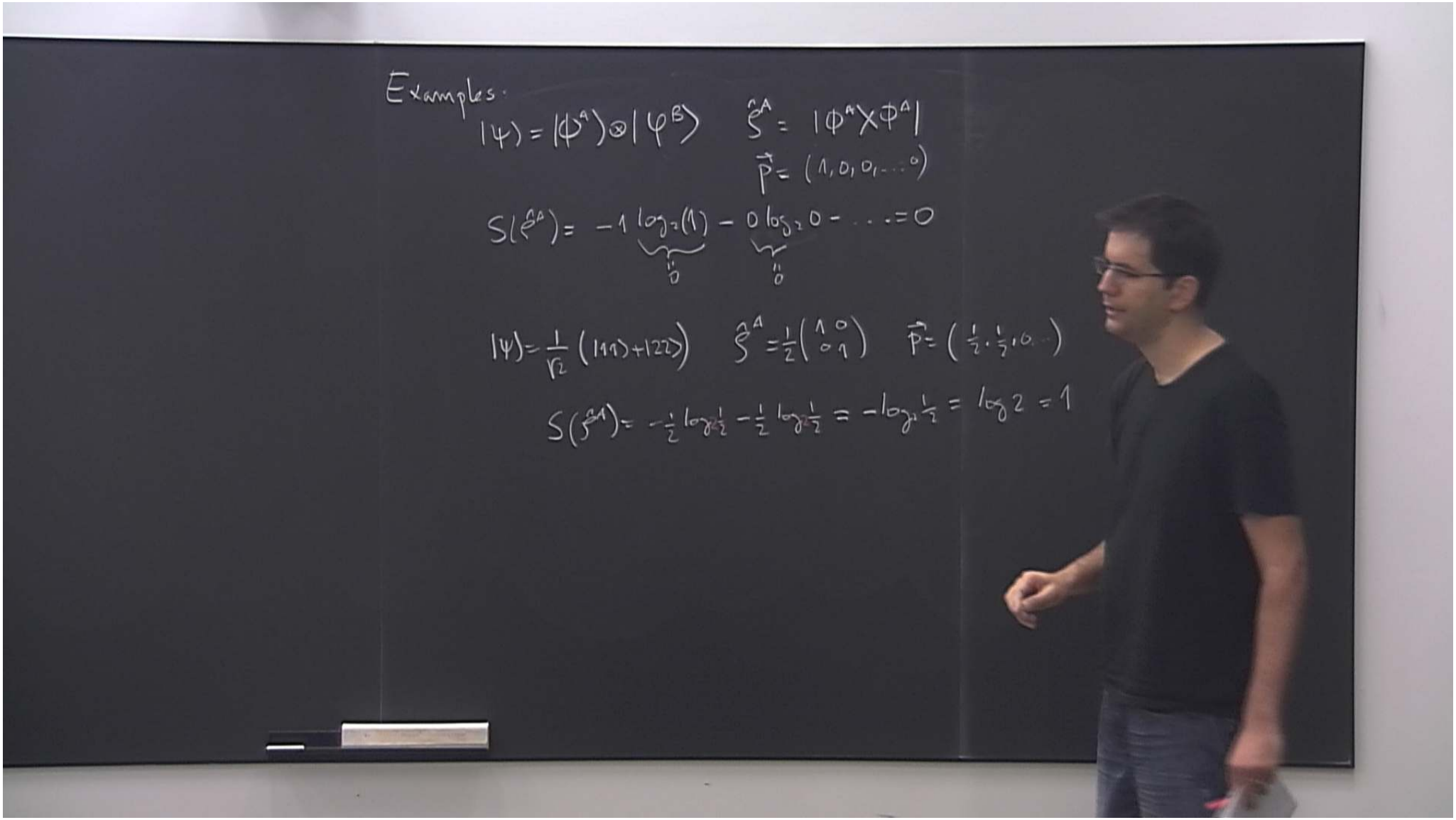

$\mathcal{E}(\vec{p})$
$|\psi\rangle \longleftrightarrow |\psi'\rangle = (U^A \otimes V^B)|\psi\rangle$
$|\psi\rangle = \sum a_{ij} |i\,j\rangle$
same
$\vec{p}$
AB
ψ_1
ψ_2

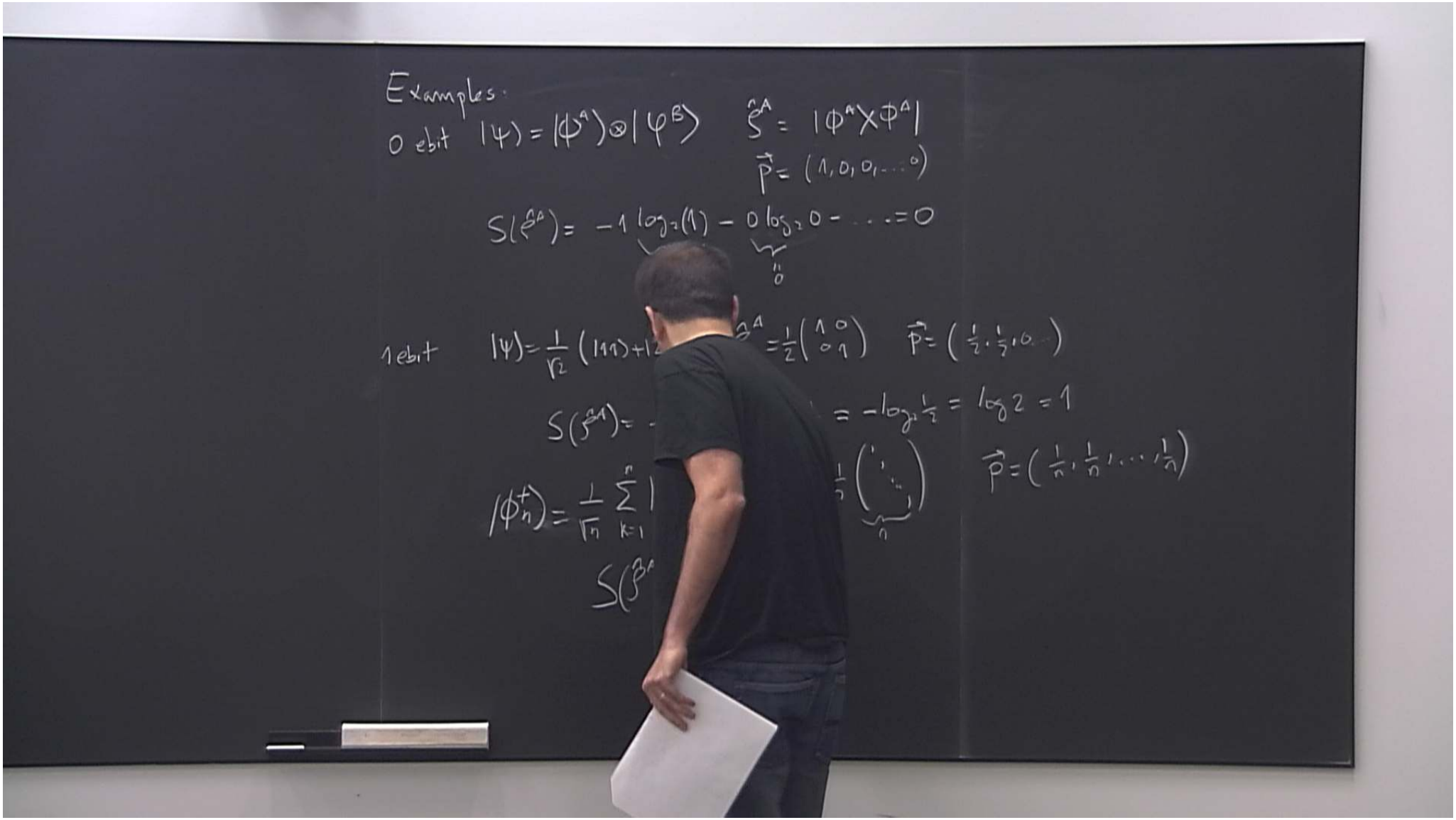

maximally entangled state
B) Entanglement entropies
von Neumann entr.
Renyi entropies
Shannon entropy

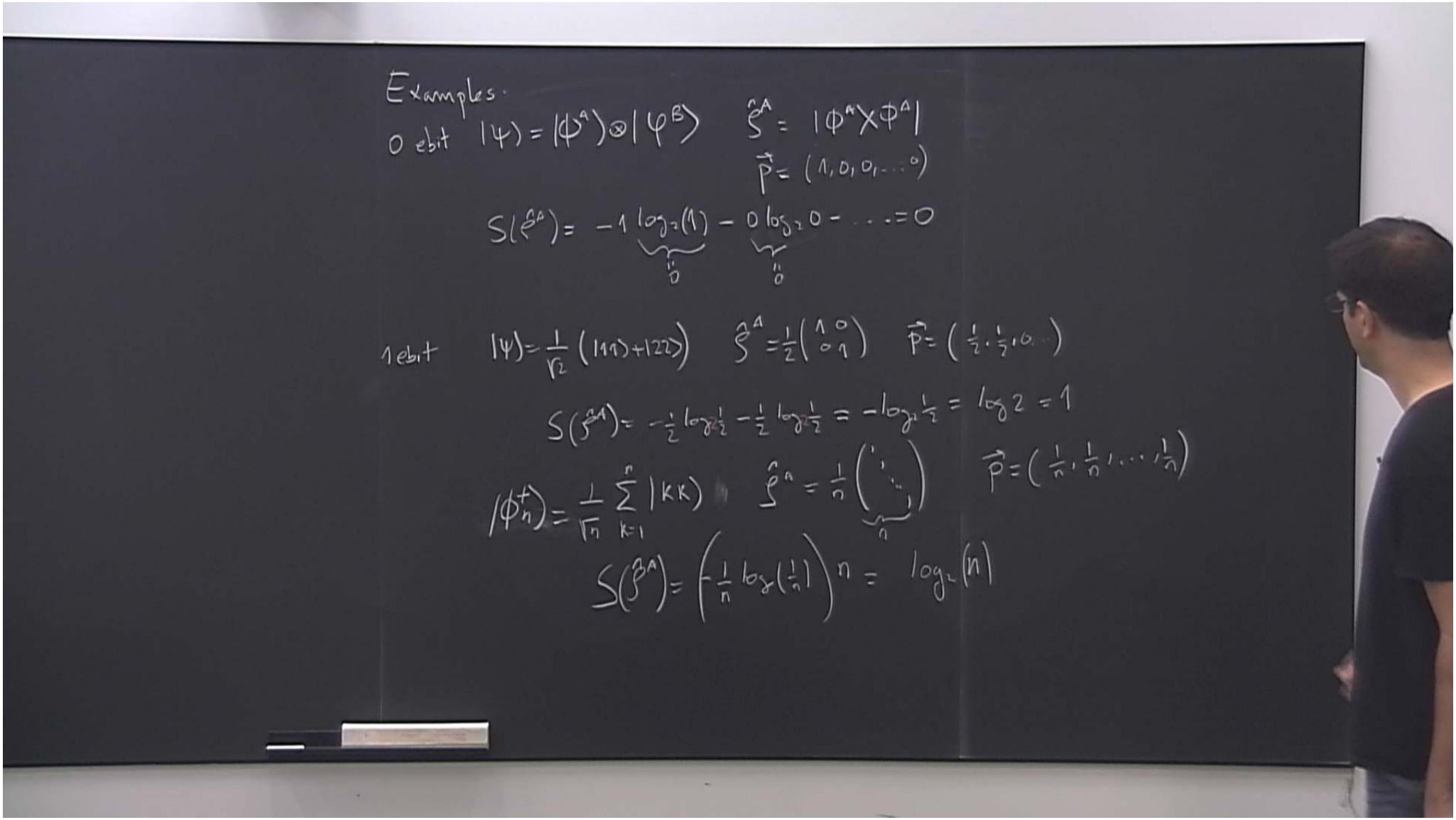

maximally entangled state
n
n
(B) Entanglement entropies
von Neumann entr.
Renyi entropies
Shannon entropy
$S(\vec{p}) \equiv -\sum_k p_k \log_2(p_k)$


Examples.
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
$\hat{S}^A =$


Examples.
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
$\hat{\rho}^A = |\phi^A\rangle\langle\phi^A|$
$\vec{p} = (1, 0, 0, \ldots, 0)$
$S(\hat{\rho}^A) = -1 \underbrace{\log_2(1)}_{=0} - 0 \underbrace{\log_2 0}_{=0} - \ldots = 0$


Examples.
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
$\hat{\rho}^A = |\phi^A\rangle\langle\phi^A|$
$\vec{p} = (1, 0, 0, \ldots, 0)$
$S(\hat{\rho}^A) = -1 \log_2(1) - 0 \log_2 0 - \ldots = 0$
$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$
$\hat{\rho}^A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots)$
$S(\hat{\rho}^A) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = -\log_2\frac{1}{2} = \log 2 = 1$


Examples:
0 ebit $|\psi\rangle = |\phi^a\rangle \otimes |\varphi^B\rangle$
$\hat{S}^A = |\phi^a\rangle\langle\phi^a|$
$\vec{p} = (1, 0, 0, \ldots, 0)$
$S(\rho^A) = -1 \log_2(1) - 0 \log_2 0 - \ldots = 0$
1 ebit
$|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |2$
$\rho^A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots)$
$S(\rho^A) = -$
$= -\log_2 \frac{1}{2} = \log 2 = 1$
$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n}$
$\frac{1}{n}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$
$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$
$S(\rho^A$


Examples.
0 ebit $|\psi\rangle = |\phi^a\rangle \otimes |\varphi^B\rangle$ $\hat{\rho}^A = |\phi^a\rangle\langle\phi^a|$
$\vec{p} = (1, 0, 0, \dots 0)$
$S(\hat{\rho}^A) = -1 \log_2(1) - 0 \log_2 0 - \dots = 0$
1 ebit $|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$ $\hat{\rho}^A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0 \dots)$
$S(\hat{\rho}^A) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = -\log\frac{1}{2} = \log 2 = 1$
$|\phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |kk\rangle$ $\hat{\rho}^n = \frac{1}{n}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ $\vec{p} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$
$S(\hat{\rho}^A) = \left(-\frac{1}{n}\log(\frac{1}{n})\right) n = \log_2(n)$