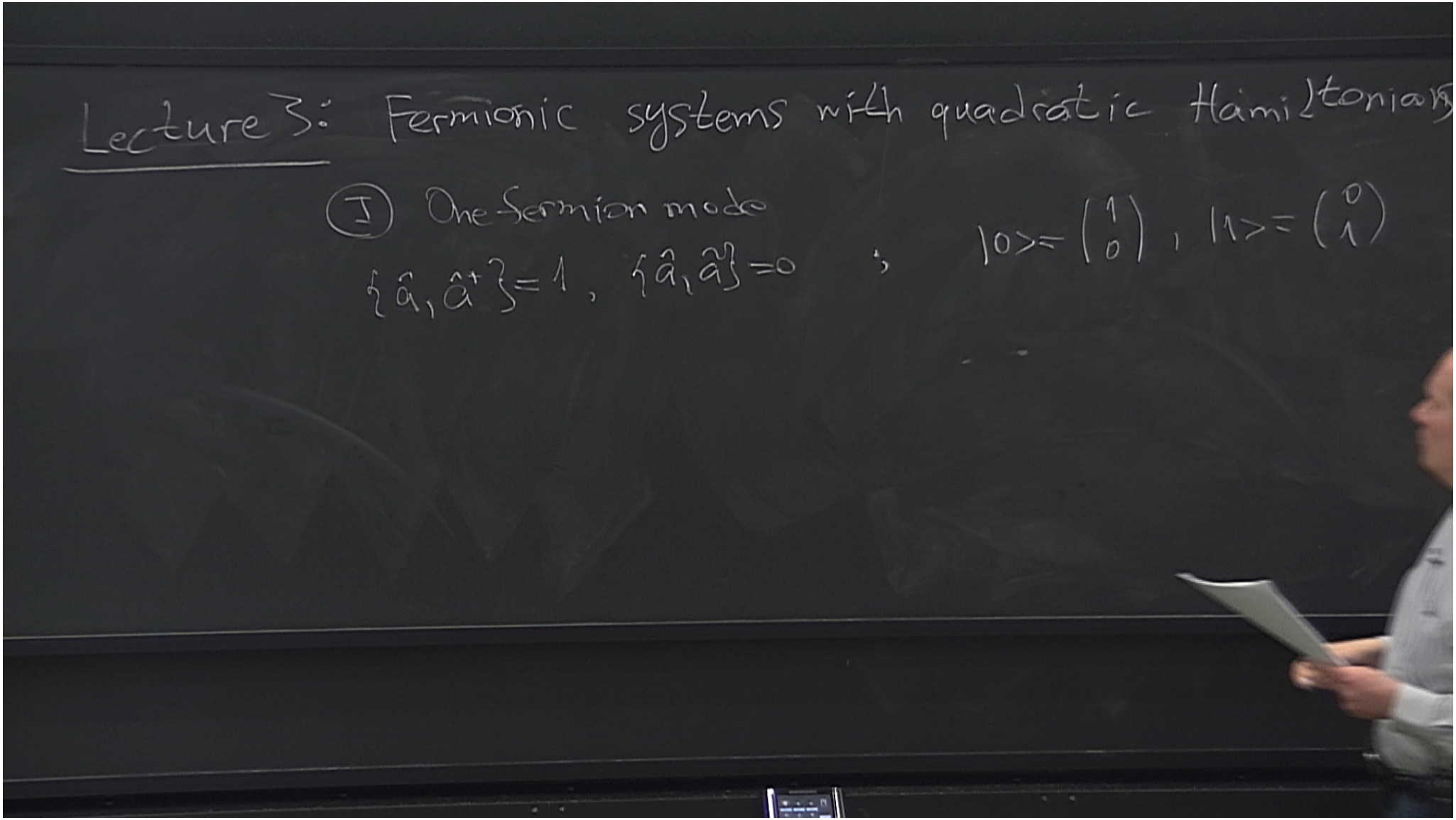
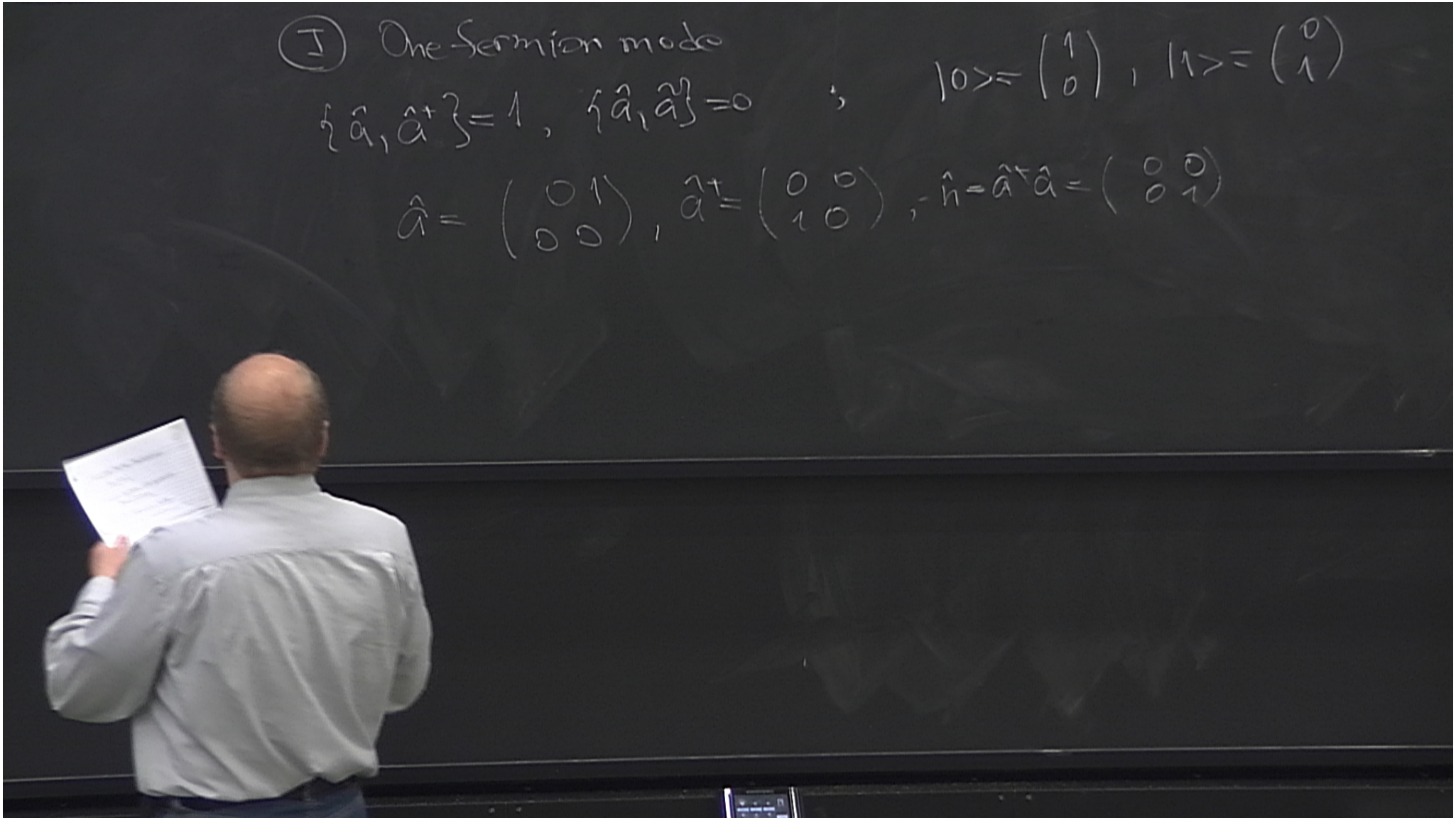
Title: 13/14 PSI - Explorations in Condensed Matter - Lecture 1

Date: Mar 17, 2014 10:15 AM

URL: http://pirsa.org/14040022

Abstract:


Lecture 3: Fermionic systems with quadratic Hamiltonians
(I) One-fermion mode
$\{\hat{a}, \hat{a}^\dagger\} = 1$, $\{\hat{a}, \hat{a}\} = 0$ ; $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$


(1) One-fermion mode
$\{\hat{a}, \hat{a}^\dagger\} = 1$, $\{\hat{a}, \hat{a}\} = 0$ ; $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\hat{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{a}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\hat{n} = \hat{a}^\dagger \hat{a} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

# Lecture 3: Fermionic systems with quadratic Hamiltonians

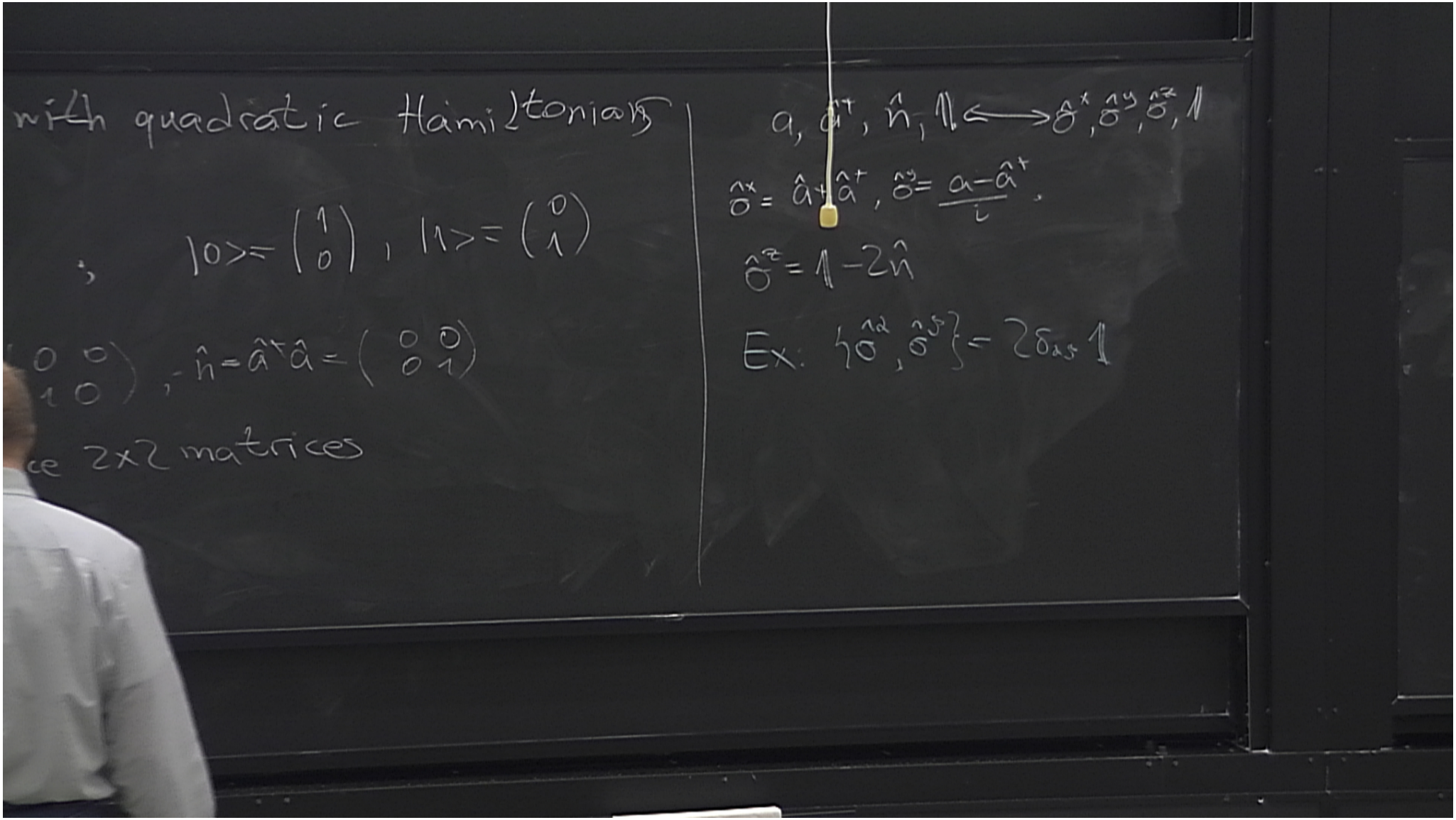
(I) One-fermion mode

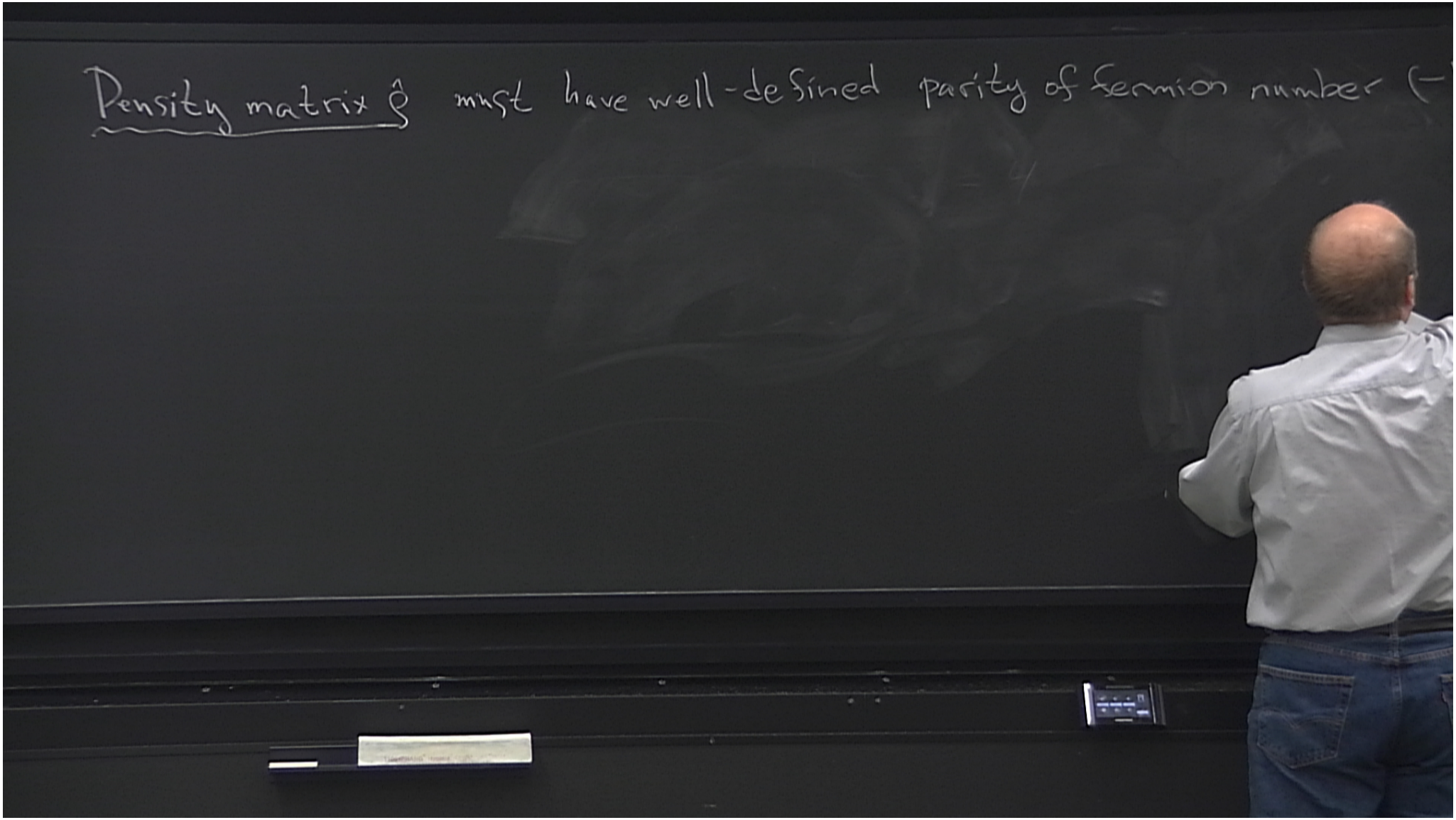
$\{\hat{a}, \hat{a}^\dagger\} = 1$, $\{\hat{a}, \hat{a}\} = 0$ ; $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

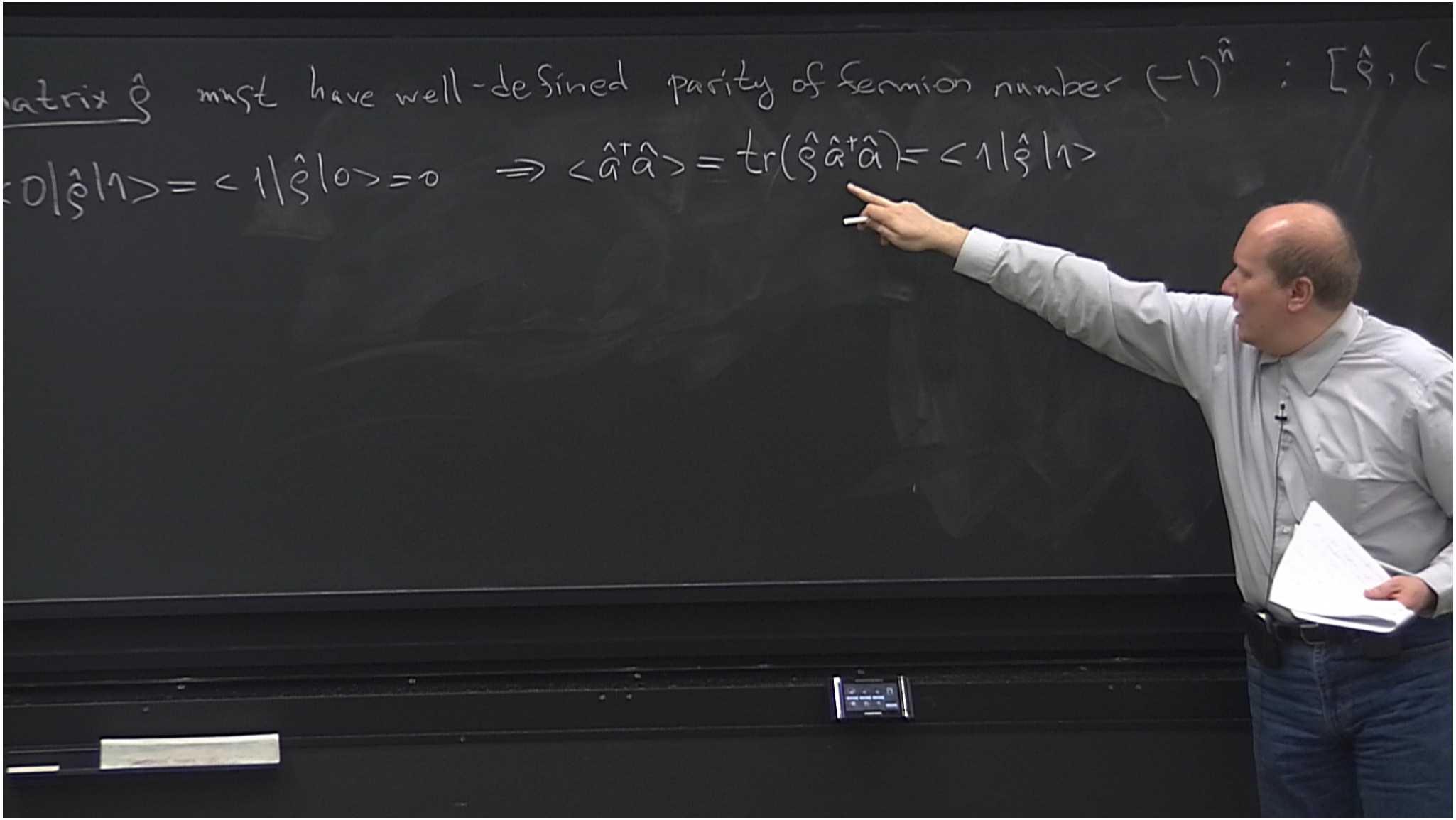
$\hat{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{a}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\hat{n} = \hat{a}^\dagger \hat{a} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

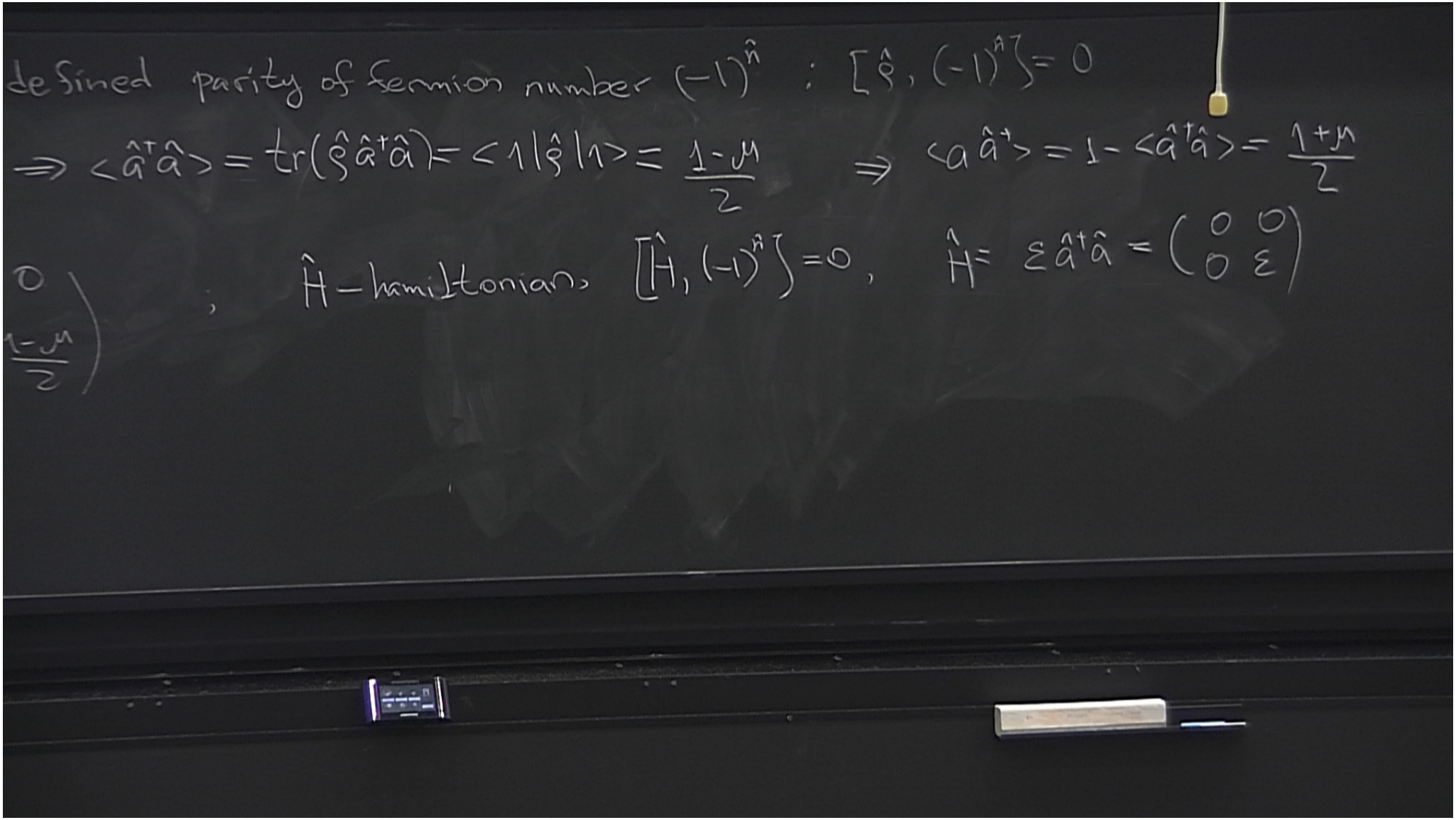
$\hat{a}, \hat{a}^\dagger, \hat{n}, \mathbb{1}$ – form basis in space 2×2 matrices

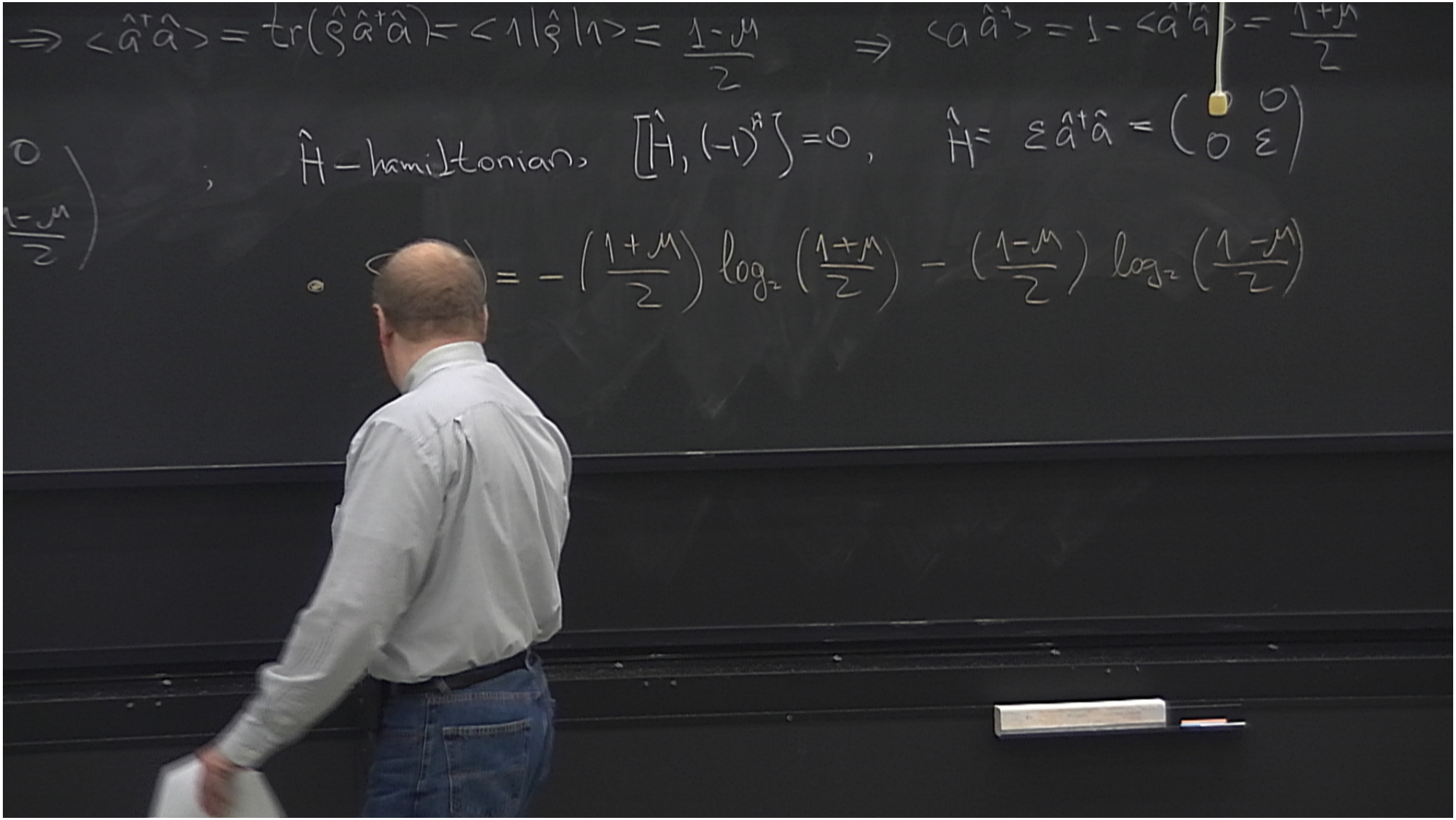


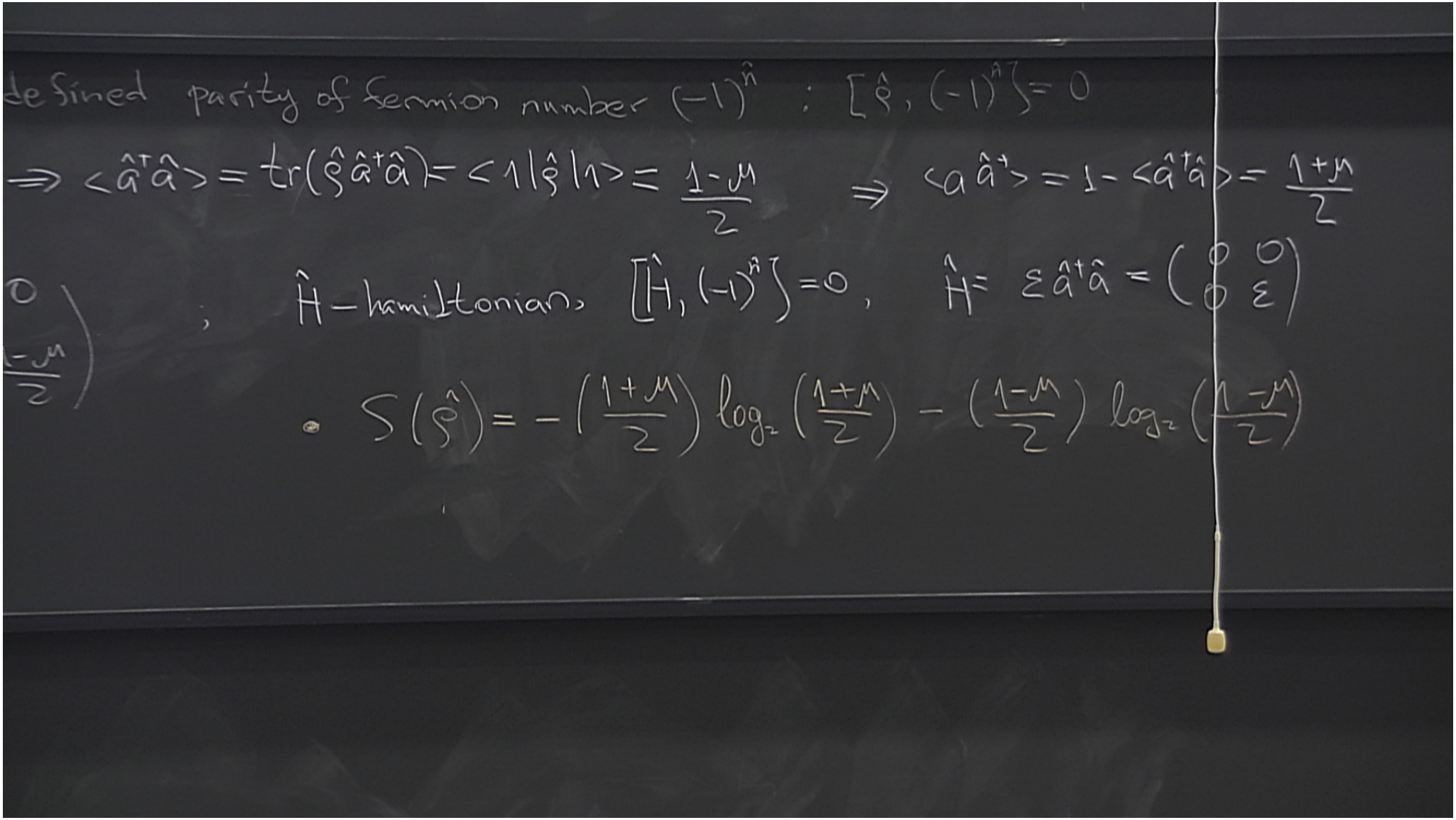

with quadratic Hamiltonians
$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\hat{n} = \hat{a}^\dagger \hat{a} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
ce 2×2 matrices
$a, a^\dagger, \hat{n}, \mathbb{1} \longleftrightarrow \hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z, \mathbb{1}$
$\hat{\sigma}^x = \hat{a} + \hat{a}^\dagger$, $\hat{\sigma}^y = \frac{a - \hat{a}^\dagger}{i}$.
$\hat{\sigma}^z = \mathbb{1} - 2\hat{n}$
Ex: $\{\hat{\sigma}^x, \hat{\sigma}^y\} = 2\delta_{xy}\mathbb{1}$

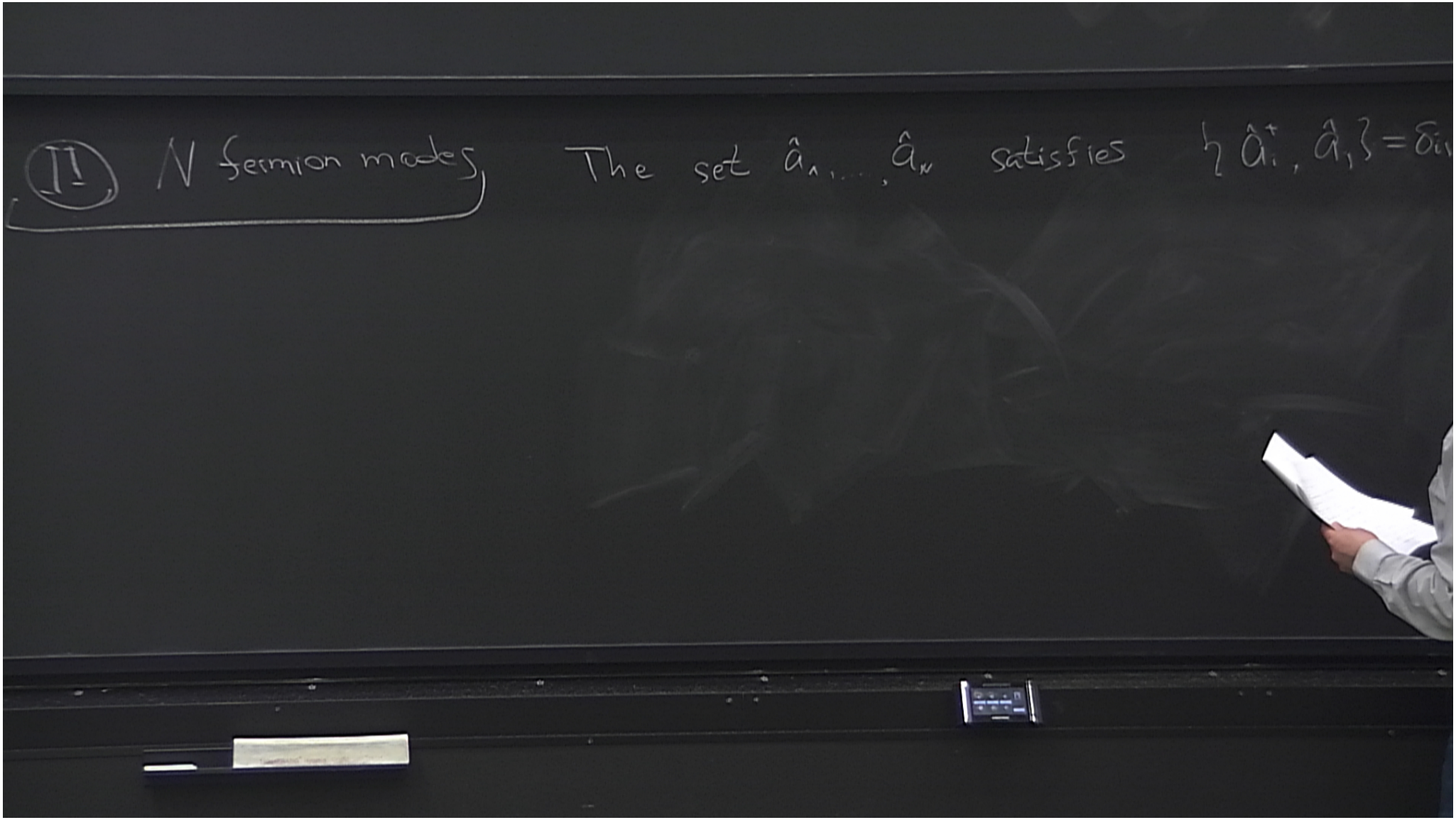

Density matrix $\hat{\rho}$ must have well-defined parity of fermion number


matrix $\hat{\rho}$ must have well-defined parity of fermion number $(-1)^{\hat{n}}$ : $[\hat{\rho}, (-$
$\langle 0|\hat{\rho}|1\rangle = \langle 1|\hat{\rho}|0\rangle = 0 \Rightarrow \langle \hat{a}^\dagger \hat{a}\rangle = \mathrm{tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \langle 1|\hat{\rho}|1\rangle$


defined parity of fermion number $(-1)^{\hat{n}}$ : $[\hat{\rho}, (-1)^{\hat{n}}] = 0$
$\Rightarrow \langle \hat{a}^\dagger \hat{a} \rangle = \mathrm{tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \langle 1|\hat{\rho}|1\rangle = \frac{1-\mu}{2}$
$\Rightarrow \langle a\hat{a}^\dagger \rangle = 1 - \langle \hat{a}^\dagger\hat{a}\rangle = \frac{1+\mu}{2}$
$\hat{H}$ - hamiltonian, $[\hat{H}, (-1)^{\hat{n}}] = 0$, $\hat{H} = \varepsilon \hat{a}^\dagger\hat{a} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix}$
$\begin{pmatrix} 0 \\ \frac{1-\mu}{2} \end{pmatrix}$


$\Rightarrow \langle \hat{a}^\dagger \hat{a} \rangle = \mathrm{tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \langle 1|\hat{\rho}|1\rangle = \frac{1-\mu}{2}$
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$= -\left(\frac{1+\mu}{2}\right)\log_2\left(\frac{1+\mu}{2}\right) - \left(\frac{1-\mu}{2}\right)\log_2\left(\frac{1-\mu}{2}\right)$


defined parity of fermion number $(-1)^{\hat{n}}$ : $[\hat{\rho}, (-1)^{\hat{n}}] = 0$
$\Rightarrow \langle \hat{a}^\dagger \hat{a} \rangle = tr(\hat{\rho}\hat{a}^\dagger\hat{a}) = \langle 1|\hat{\rho}|1\rangle = \frac{1-\mu}{2}$
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$S(\hat{\rho}) = -\left(\frac{1+\mu}{2}\right)\log_2\left(\frac{1+\mu}{2}\right) - \left(\frac{1-\mu}{2}\right)\log_2\left(\frac{1-\mu}{2}\right)$


II. N fermion modes. The set $\hat{a}_1, \ldots, \hat{a}_N$ satisfies $\{\hat{a}_i^\dagger, \hat{a}_j\} = \delta_{ij}$

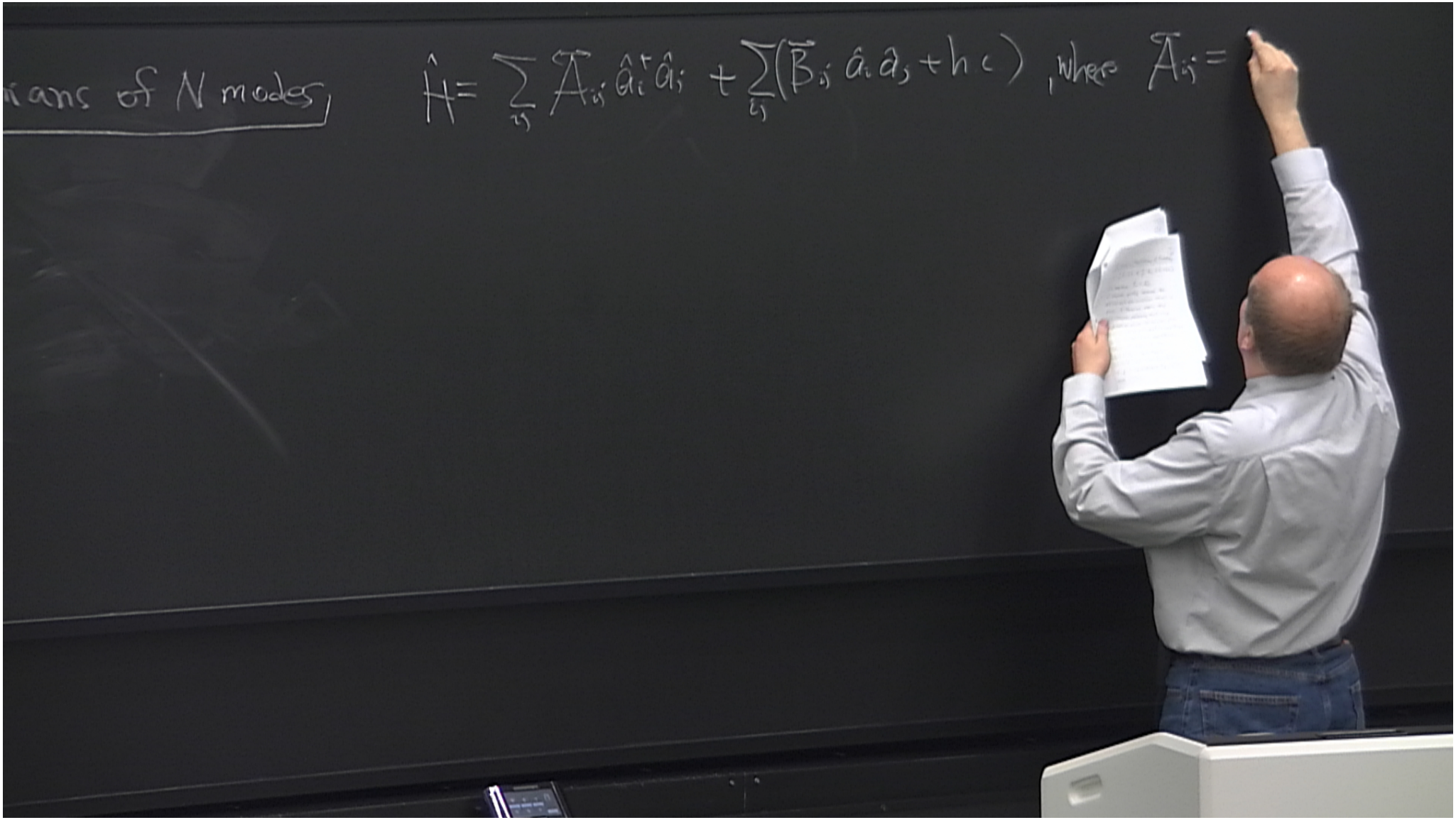
The set $\hat{a}_1, \ldots, \hat{a}_N$ satisfies $\{\hat{a}_i^\dagger, \hat{a}_j\} = \delta_{ij}$. $\{\hat{a}_i, \hat{a}_j\} = 0$ $(i, j = 1, \ldots, N$

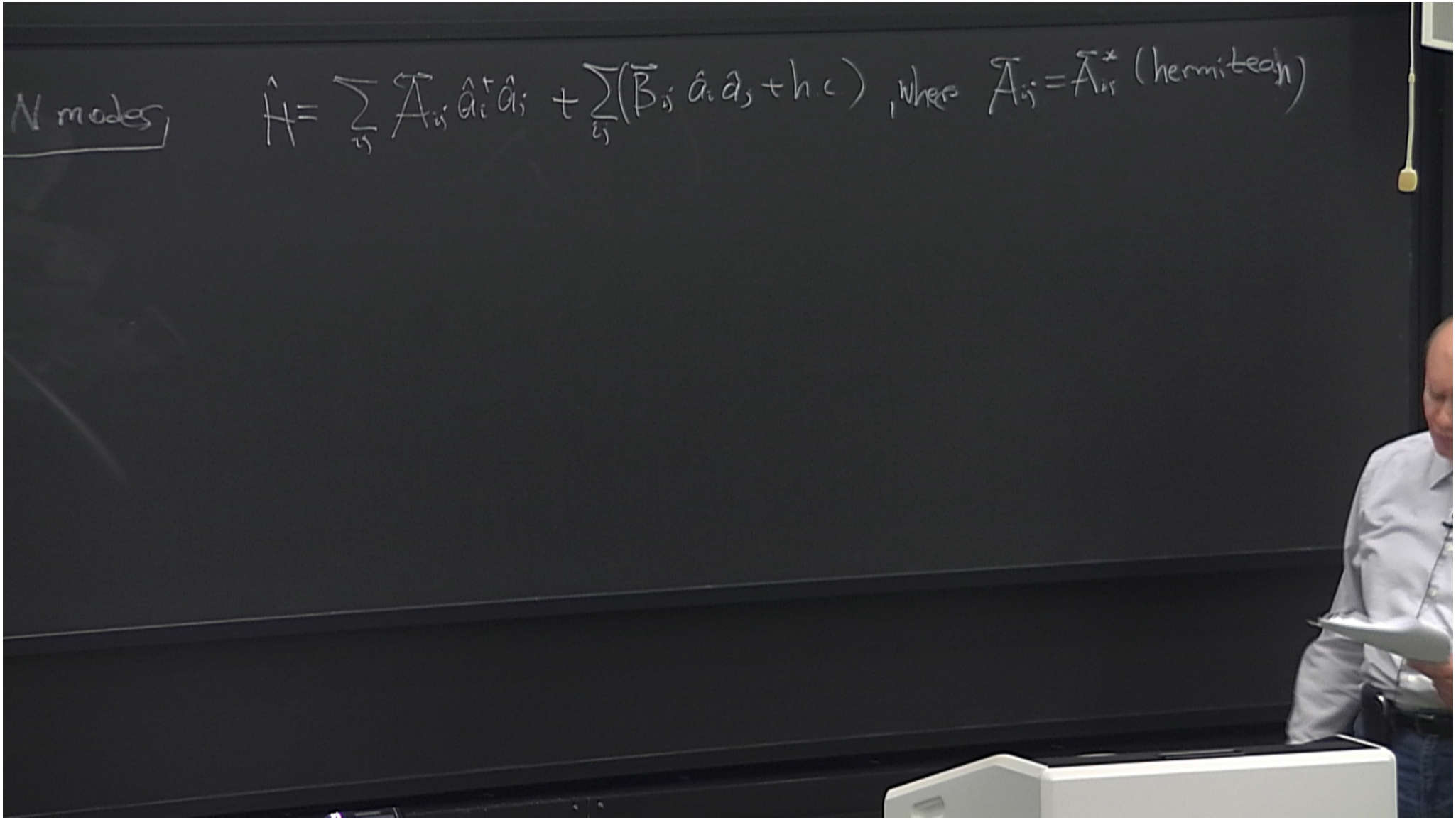
$\hat{n} = \sum_i \hat{n}_i = \sum_i \hat{a}_i^\dagger \hat{a}_i$ — total particle # operator

(II) N fermion modes The set $\hat{a}_1, \ldots, \hat{a}_N$ satisfies $\{\hat{a}_i^\dagger, \hat{a}_j\} =$

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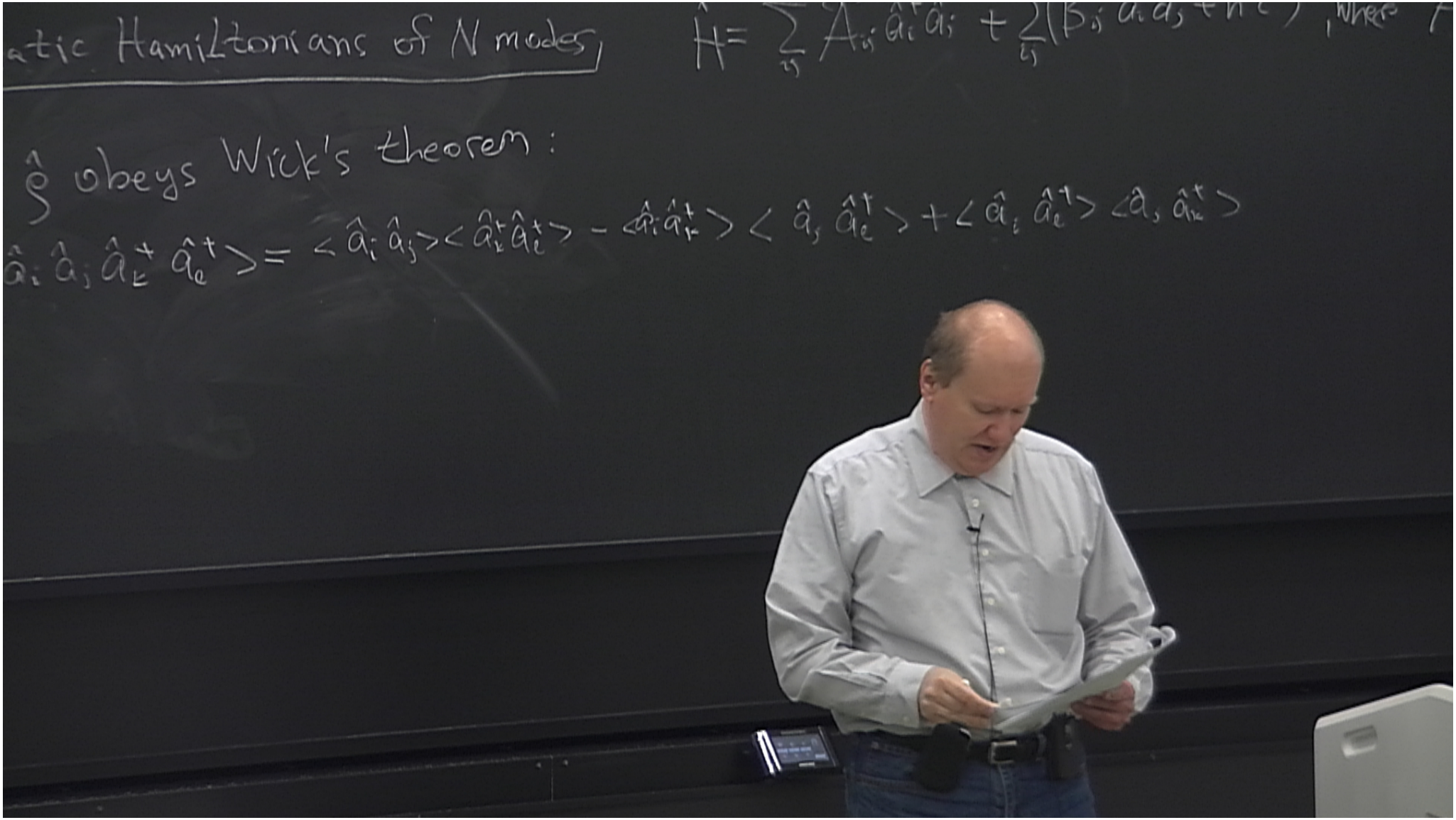
- Basis: $2^N$ possible states $|n_1, n_2, \ldots, n_N\rangle$, $n_i = 0, 1$
- $\hat{\rho}, \hat{H}$ must satisfy $[\hat{\rho}, (-1)^{\hat{n}}] = 0$, $[\hat{H}, (-1)^{\hat{n}}] = 0$

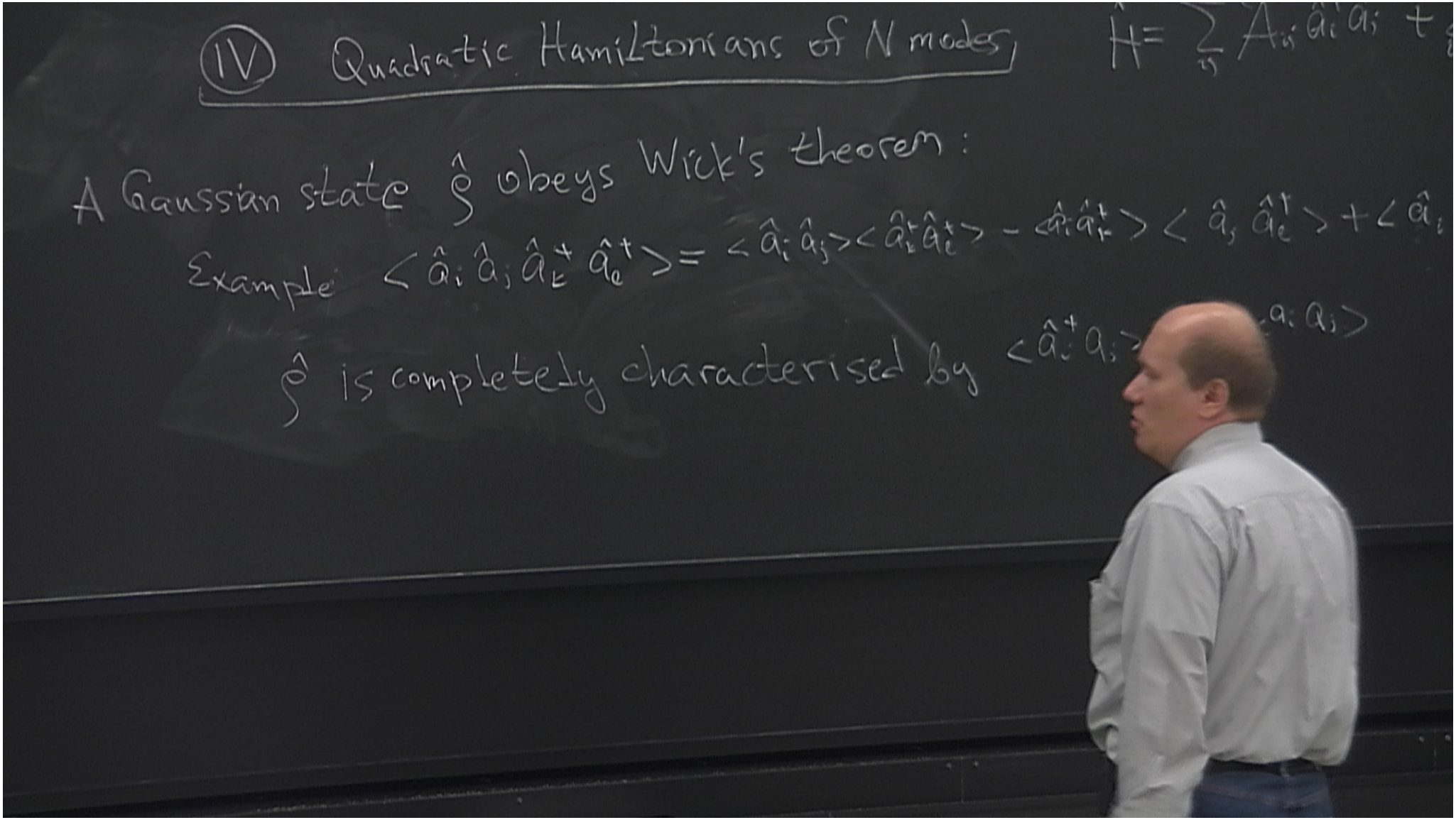

ians of N modes
$\hat{H} = \sum_{ij} \vec{A}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{ij} (\vec{B}_{ij} \hat{a}_i \hat{a}_j + h.c.)$, where $\vec{A}_{ij} =$

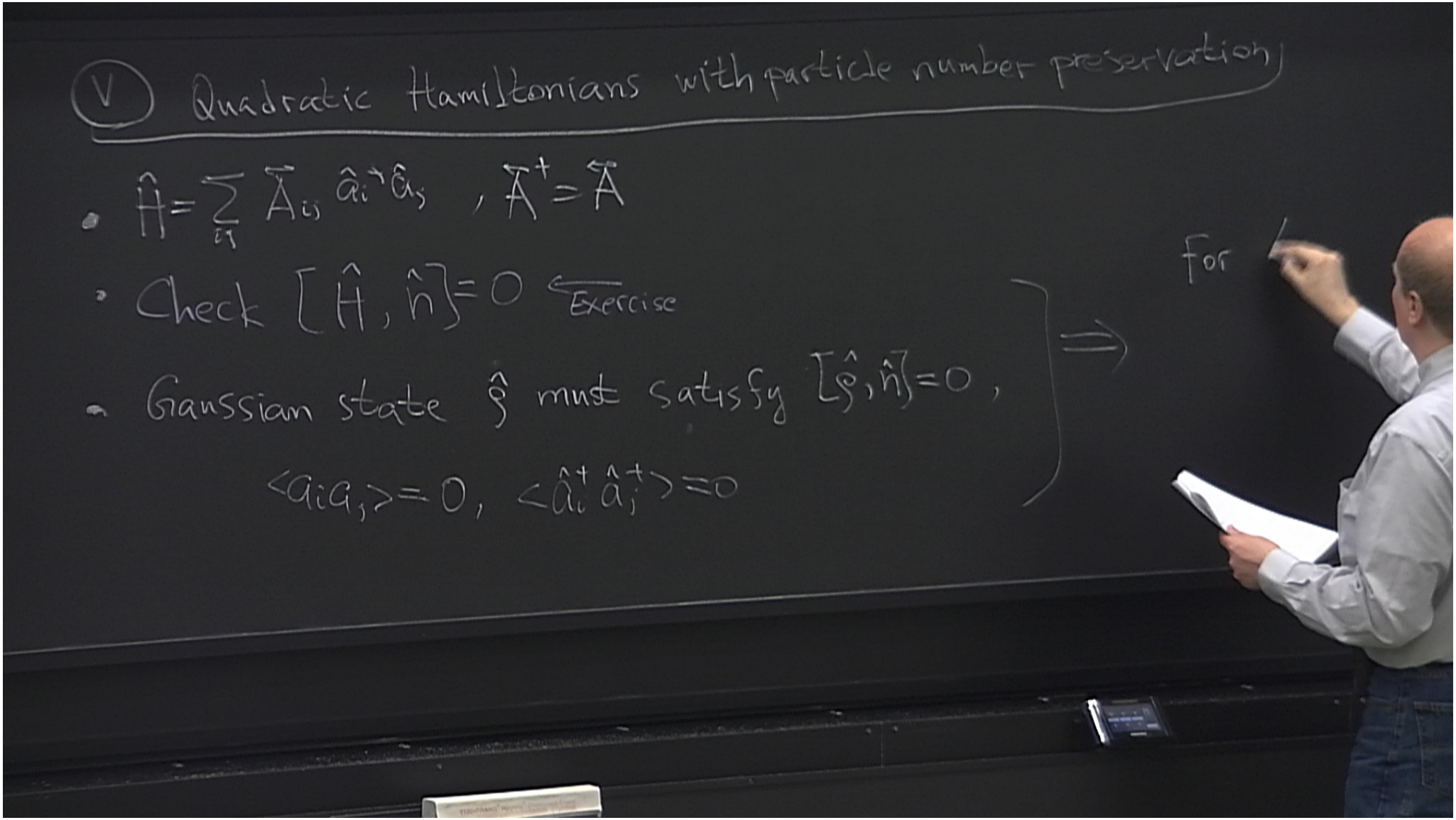

N modes
$\hat{H} = \sum_{ij} \vec{A}_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{ij} (\vec{B}_{ij} \hat{a}_i \hat{a}_j + h.c.)$, where $\vec{A}_{ij} = \vec{A}_{ji}^*$ (hermitean)

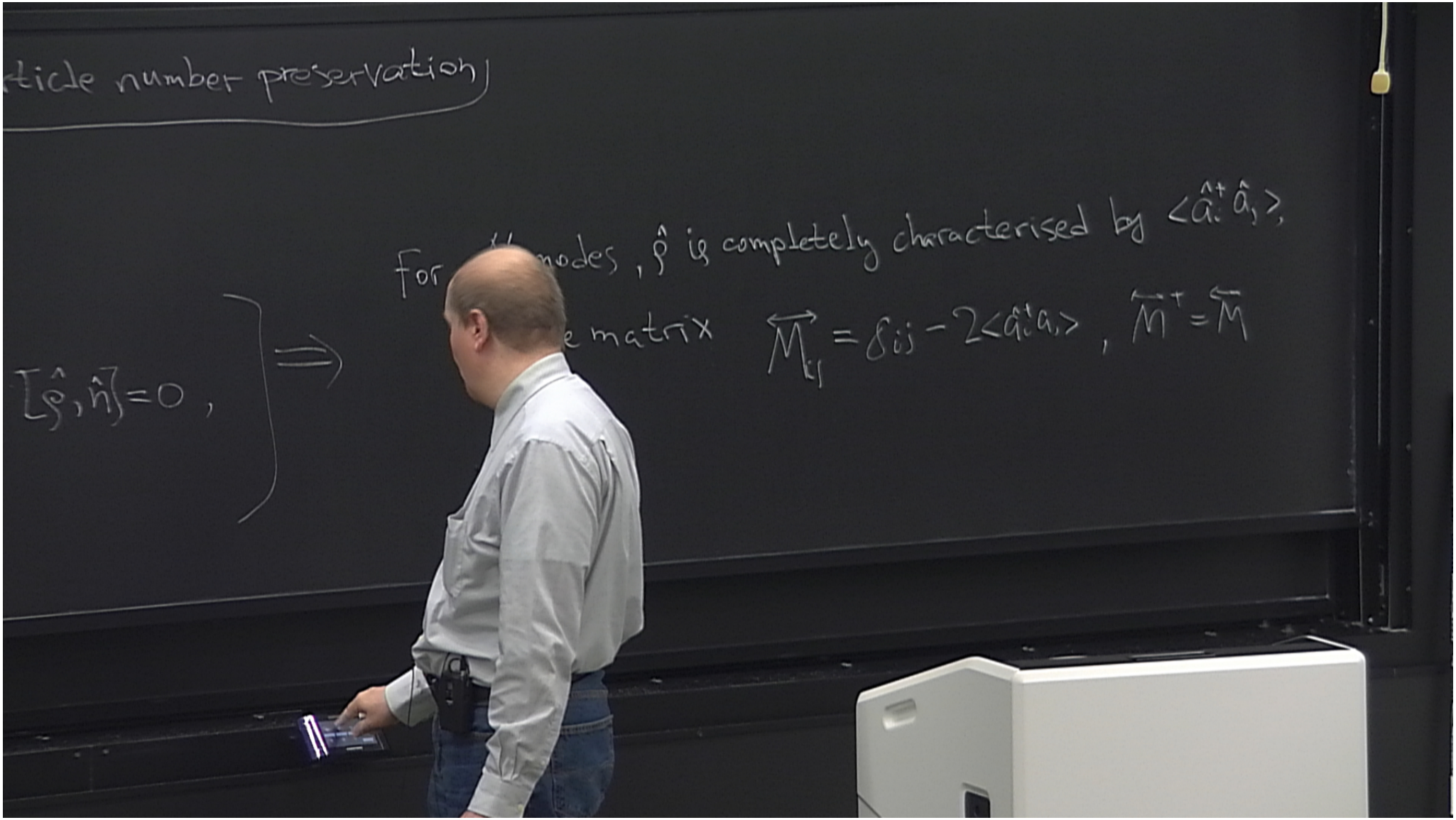
A Gaussian state $\hat{\rho}$ obeys Wick's theorem:

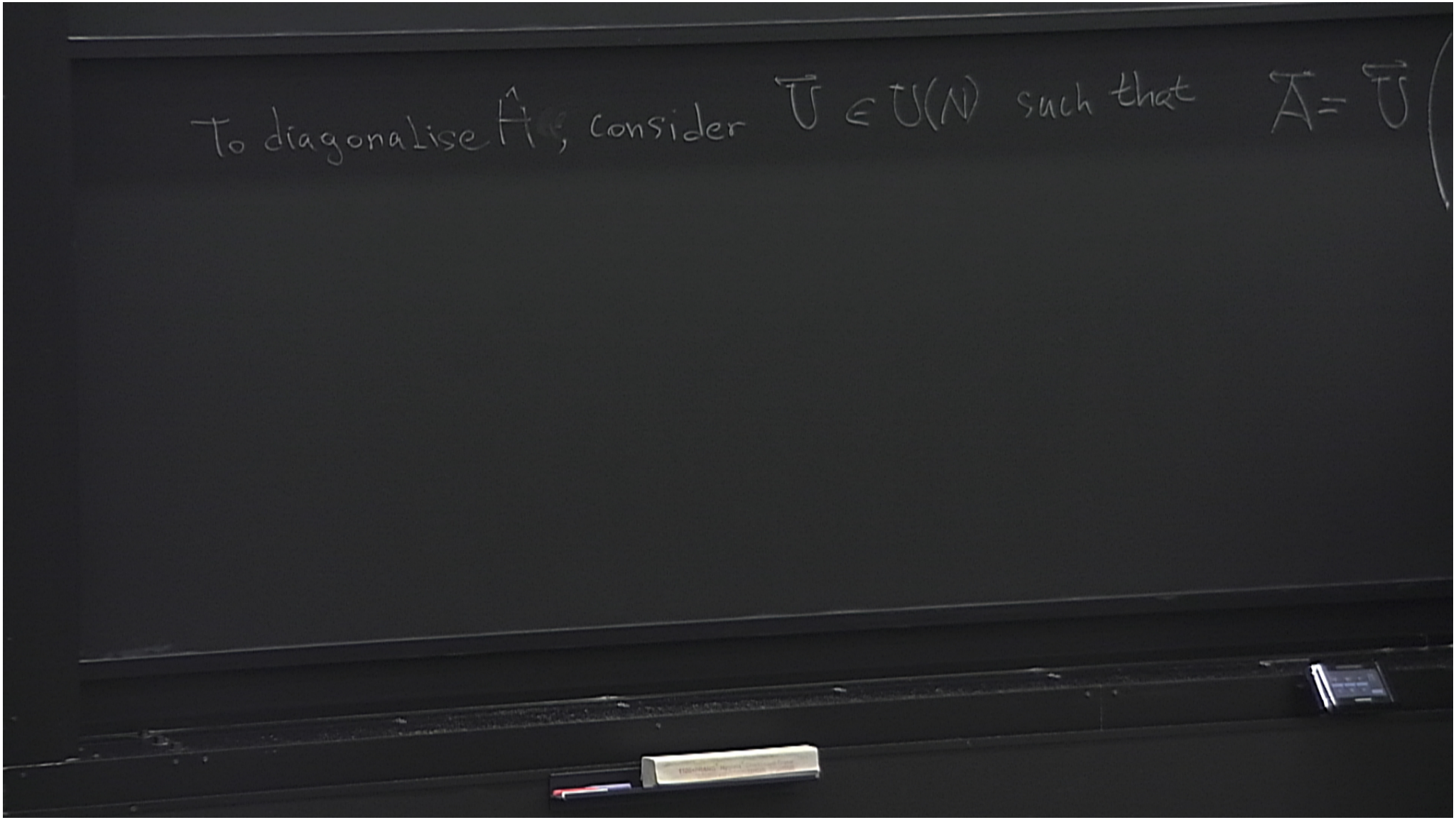
Example $\langle \hat{a}_i \hat{a}_j \hat{a}_k^\dagger \hat{a}_\ell^\dagger \rangle = \langle \hat{a}_i \hat{a}_j \rangle \langle \hat{a}_k^\dagger \hat{a}_\ell^\dagger \rangle - \langle \hat{a}_i \hat{a}_k^\dagger \rangle \langle \hat{a}_j \hat{a}_\ell^\dagger$

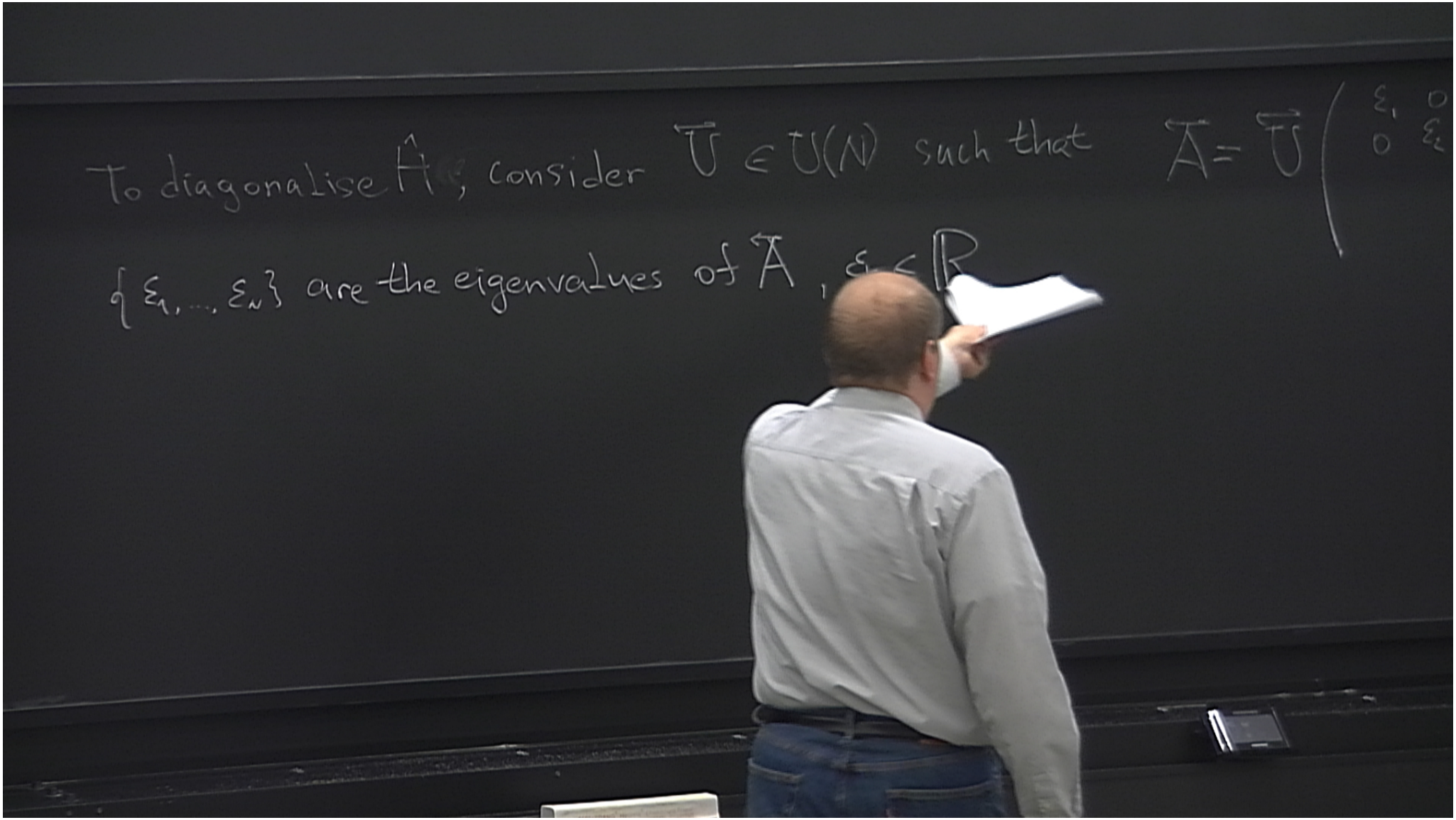

atic Hamiltonians of N modes
$\hat{H} = \sum_{ij} A_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2}\sum_{ij}(B_{ij} a_i a_j + h.c.)$, where
$\hat{\rho}$ obeys Wick's theorem:
$\langle \hat{a}_i \hat{a}_j \hat{a}_k^\dagger \hat{a}_\ell^\dagger \rangle = \langle \hat{a}_i \hat{a}_j \rangle \langle \hat{a}_k^\dagger \hat{a}_\ell^\dagger \rangle - \langle \hat{a}_i \hat{a}_k^\dagger \rangle \langle \hat{a}_j \hat{a}_\ell^\dagger \rangle + \langle \hat{a}_i \hat{a}_\ell^\dagger \rangle \langle \hat{a}_j \hat{a}_k^\dagger \rangle$


(IV) Quadratic Hamiltonians of N modes
A Gaussian state obeys Wick's theorem:
Example
is completely characterised by


(V) Quadratic Hamiltonians with particle number preservation
$\hat{H} = \sum_{ij} \vec{A}_{ij} \hat{a}_i^\dagger \hat{a}_j$ , $\vec{A}^\dagger = \vec{A}$
Check $[\hat{H}, \hat{n}] = 0$ ⟵ Exercise
Gaussian state $\hat{\rho}$ must satisfy $[\hat{\rho}, \hat{n}] = 0$ ,
$\langle a_i a_j \rangle = 0$, $\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \rangle = 0$
⟹
for


ticle number preservation
$[\hat{\rho}, \hat{n}] = 0$ ,
$\Rightarrow$
for modes , $\hat{\rho}$ is completely characterised by $\langle \hat{a}_i^\dagger \hat{a}_j \rangle$,
matrix $\overleftrightarrow{M}_{ij} = \delta_{ij} - 2\langle \hat{a}_i^\dagger a_j \rangle$ , $\overleftrightarrow{M}^\dagger = \overleftrightarrow{M}$

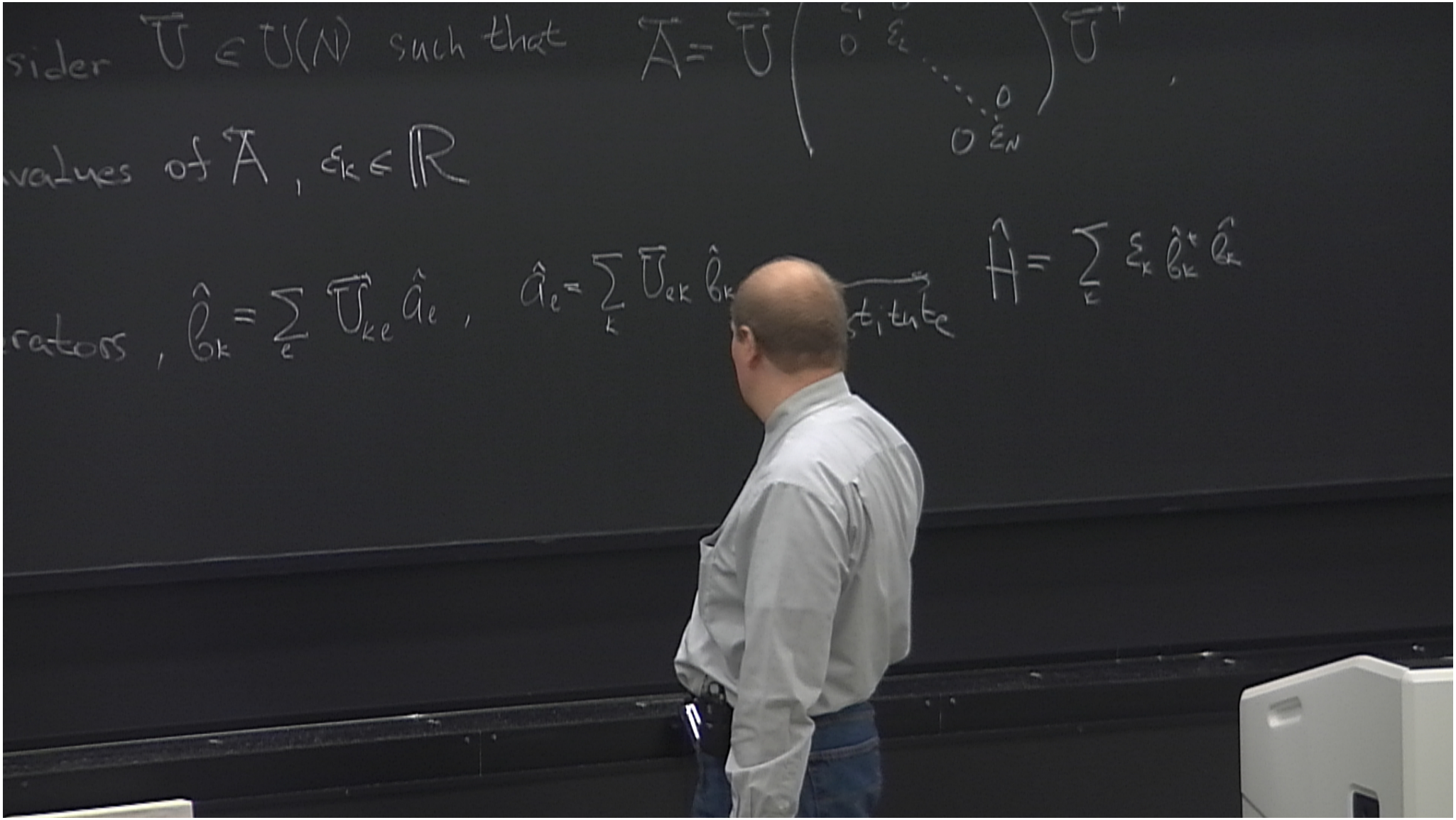

To diagonalise $\hat{A}$, consider $\bar{U} \in U(N)$ such that $\bar{A} = \bar{U}$

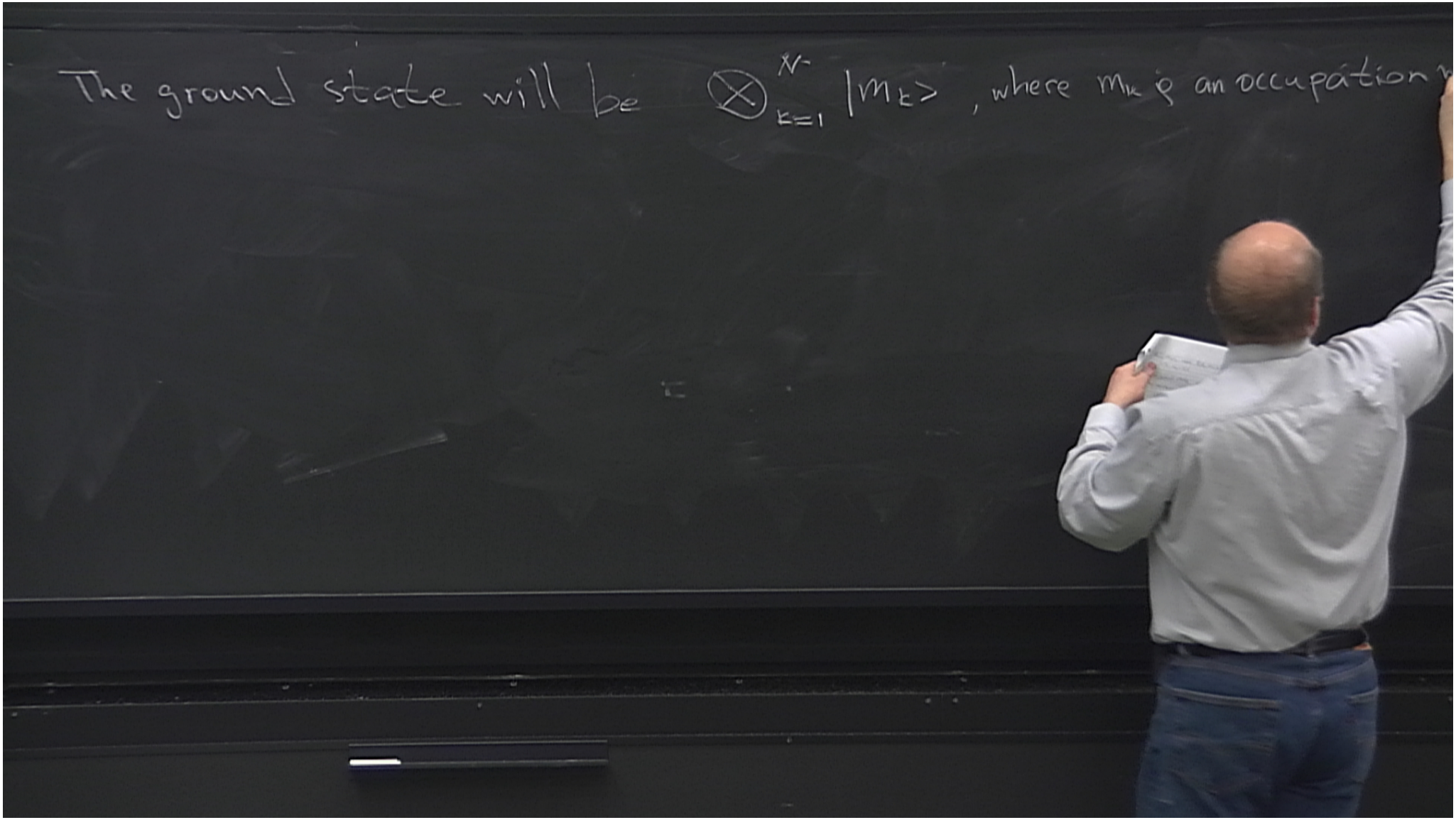

To diagonalise $\hat{A}$, consider $\vec{U} \in U(N)$ such that $\vec{A} = \vec{U}$
$\{\varepsilon_1, \ldots, \varepsilon_N\}$ are the eigenvalues of $\vec{A}$, $\varepsilon \in \mathbb{R}$

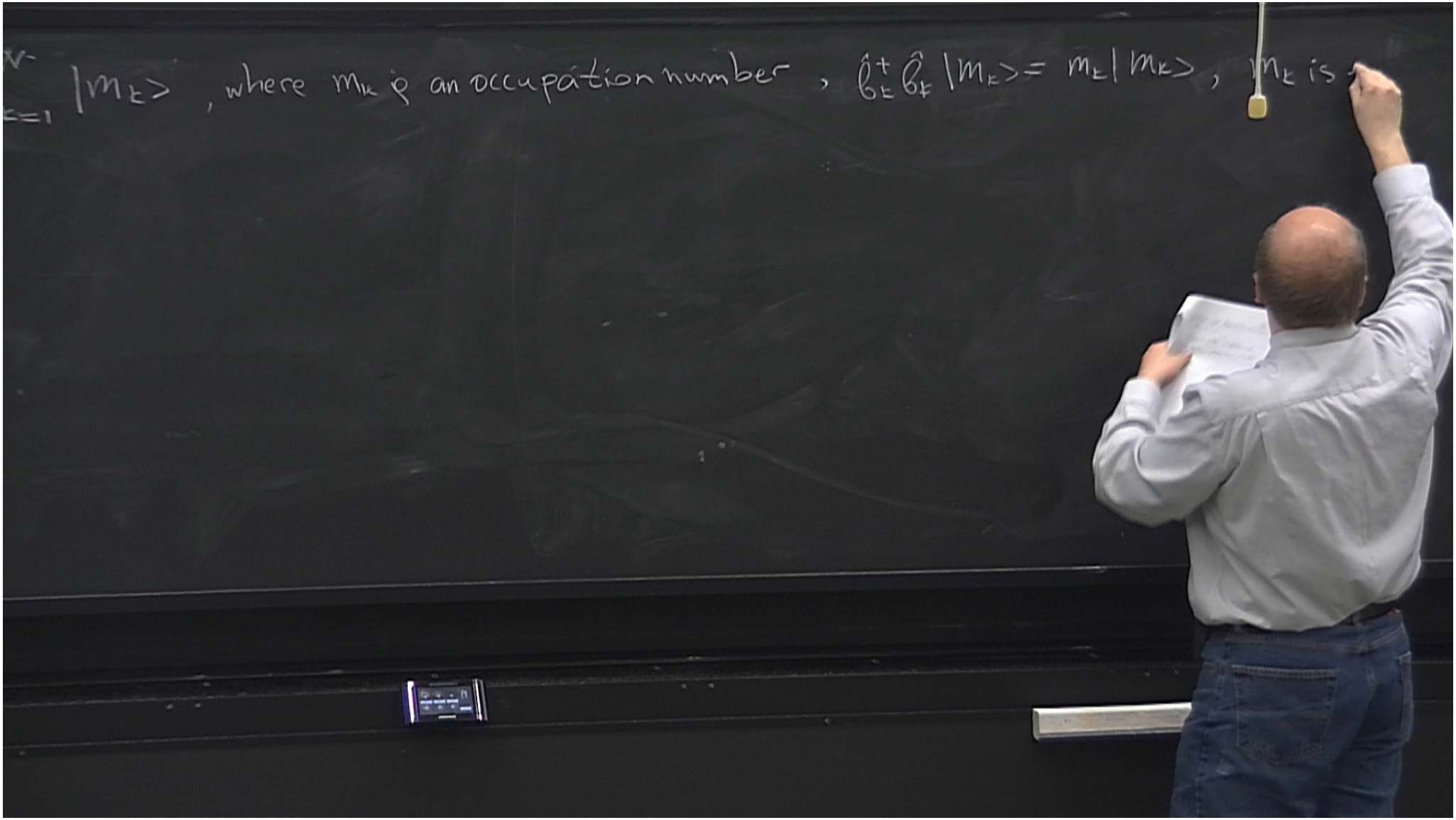
To diagonalise $\hat{A}$, consider $\vec{U} \in U(N)$ such that $\vec{A} = \vec{U} \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$

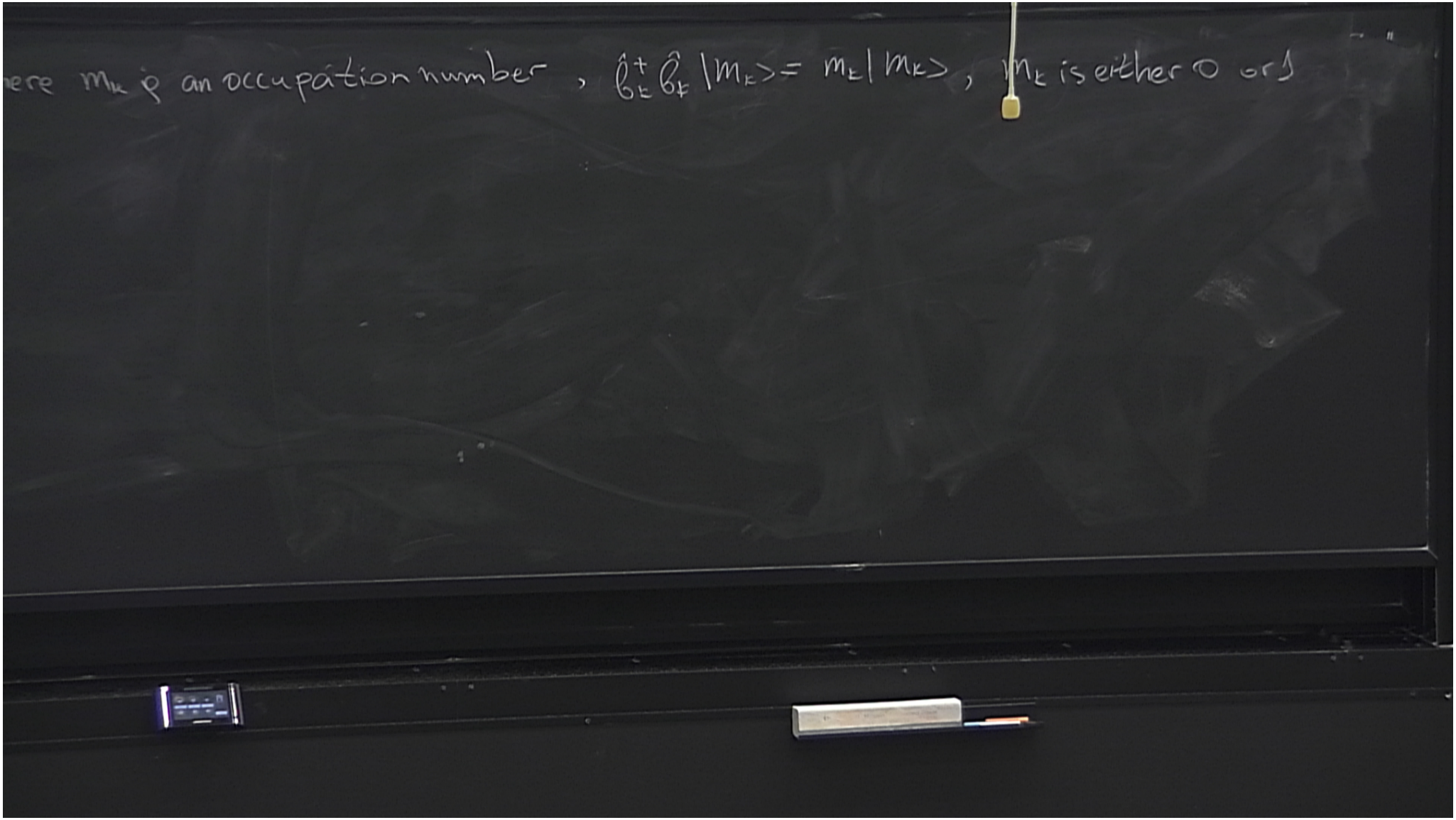
$\{\varepsilon_1, \ldots, \varepsilon_N\}$ are the eigenvalues of $\vec{A}$, $\varepsilon_k \in \mathbb{R}$

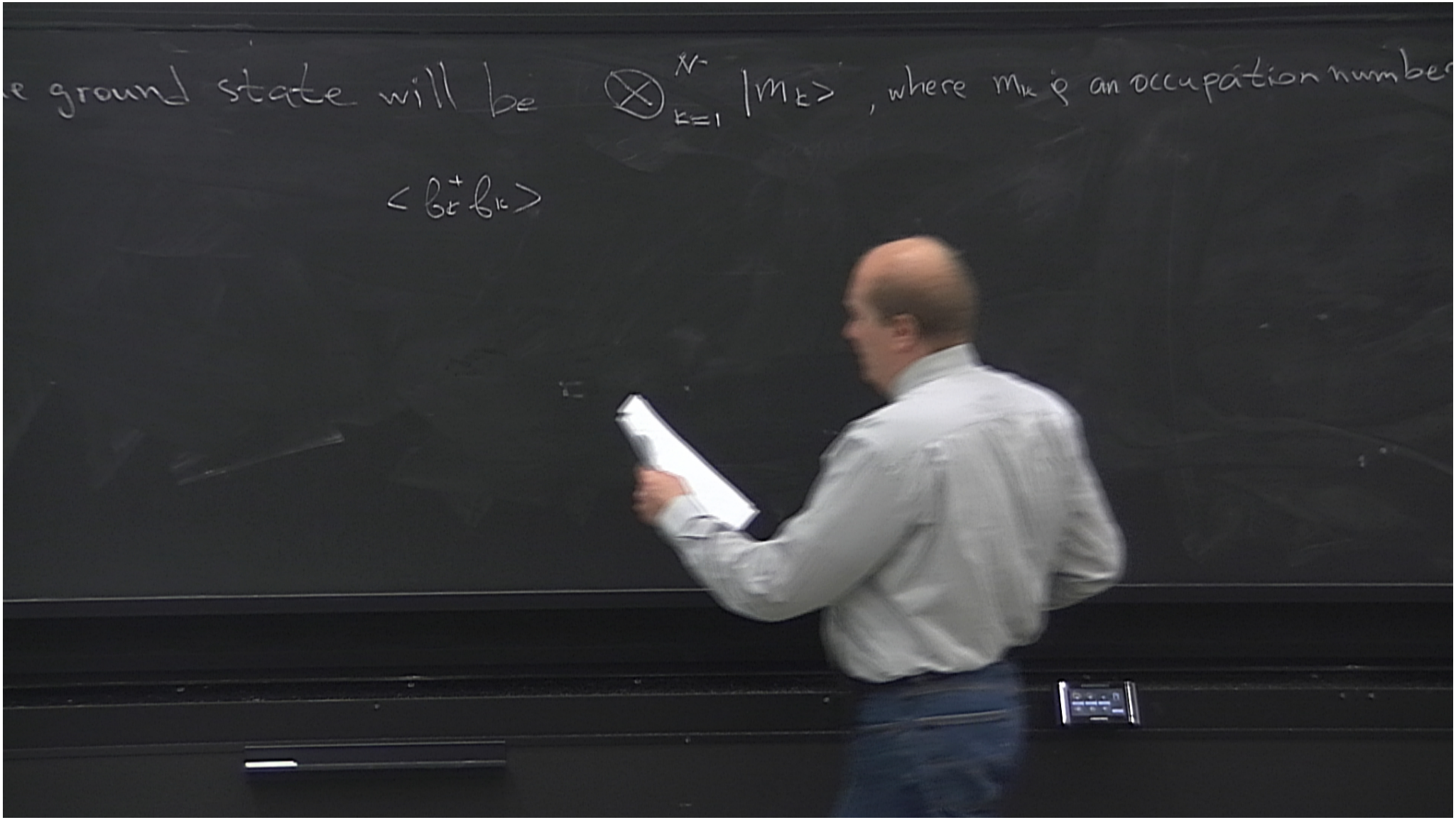
New set of fermionic operators, $\hat{\beta}_k = \sum_\ell \vec{U}_{k\ell} \hat{a}_\ell$, $\hat{a}_\ell = \sum_k \vec{U}_{\ell k} \hat{\beta}_k$ $\xrightarrow{\text{substitute}}$


sider $U \in U(N)$ such that $\hat{A} = U$
values of $\hat{A}$, $\varepsilon_k \in \mathbb{R}$
$\hat{b}_k = \sum_e U_{ke} \hat{a}_e$,
$\hat{a}_e = \sum_k U_{ek} \hat{b}_k$
$\hat{H} = \sum_k \varepsilon_k \hat{b}_k^\dagger \hat{b}_k$


The ground state will be $\bigotimes_{k=1}^{N} |m_k\rangle$, where $m_k$ is an occupation

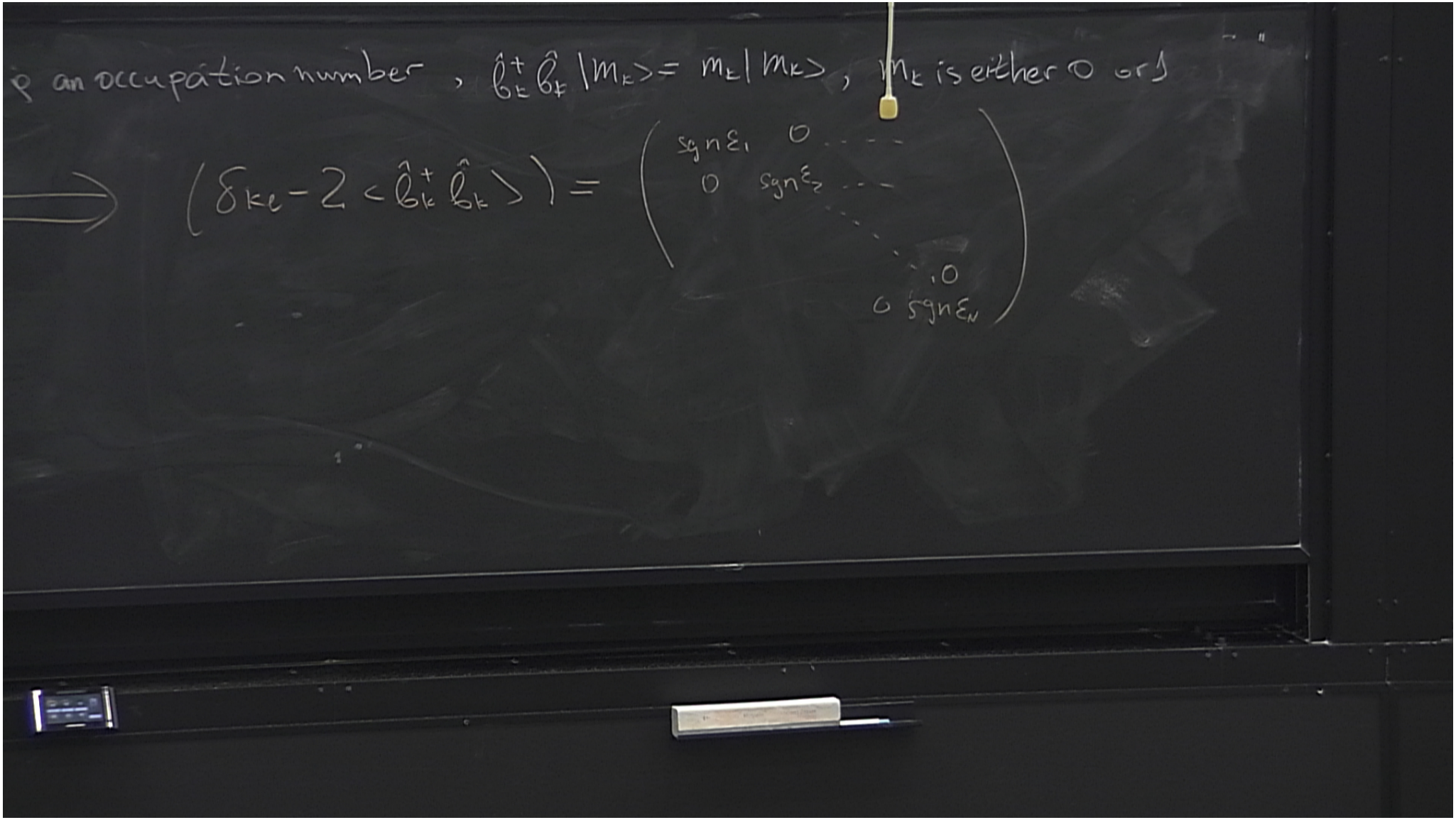

$|m_k\rangle$, where $m_k$ is an occupation number, $\hat{b}_k^\dagger \hat{b}_k |m_k\rangle = m_k |m_k\rangle$, $m_k$ is

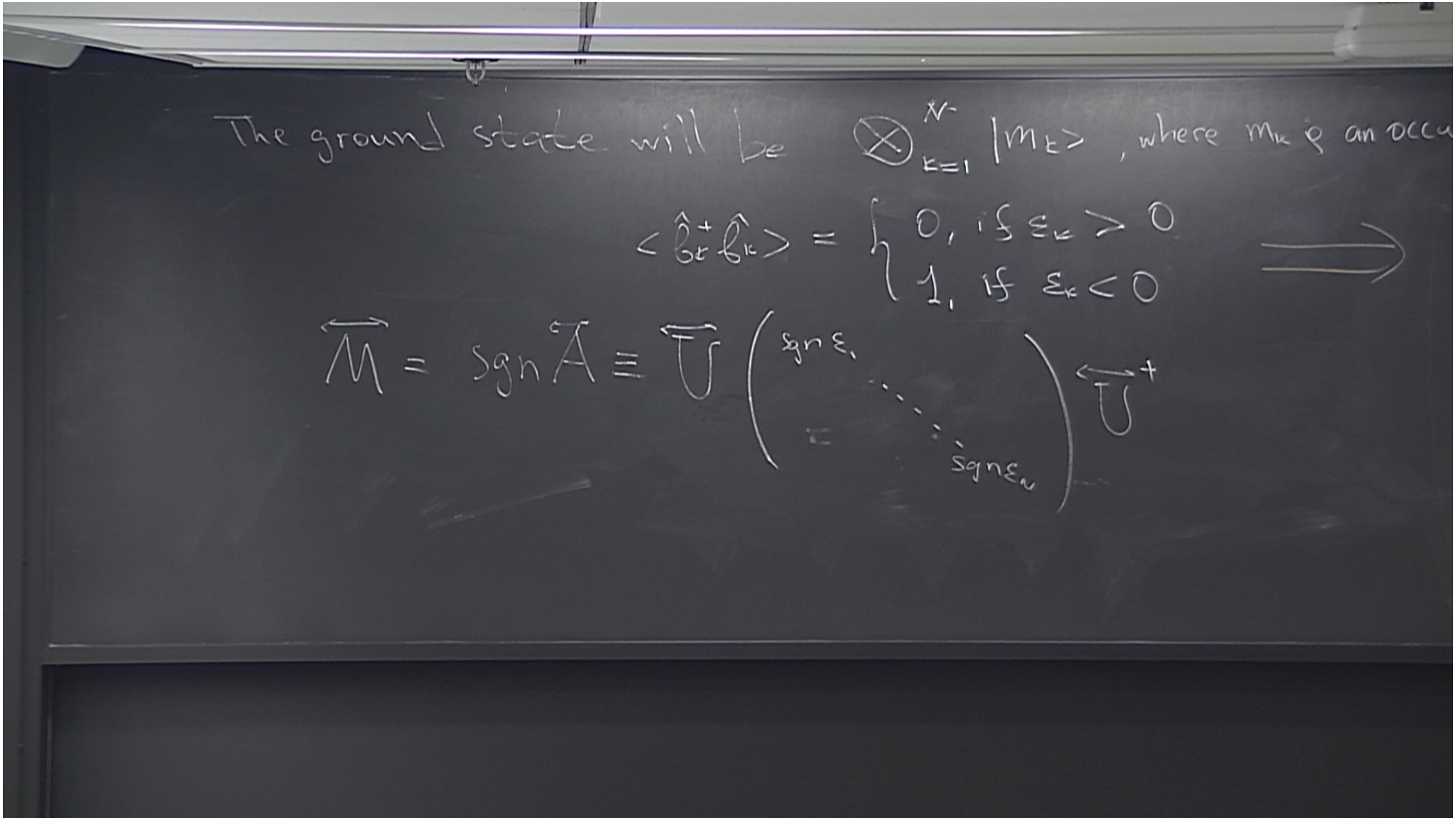

here $m_k$ is an occupation number , $\hat{b}_k^\dagger \hat{b}_k |m_k\rangle = m_k |m_k\rangle$ , $m_k$ is either 0 or 1


e ground state will be $\bigotimes_{k=1}^{N} |m_k\rangle$, where $m_k$ is an occupation numbe
$\langle b_k^\dagger b_k \rangle$

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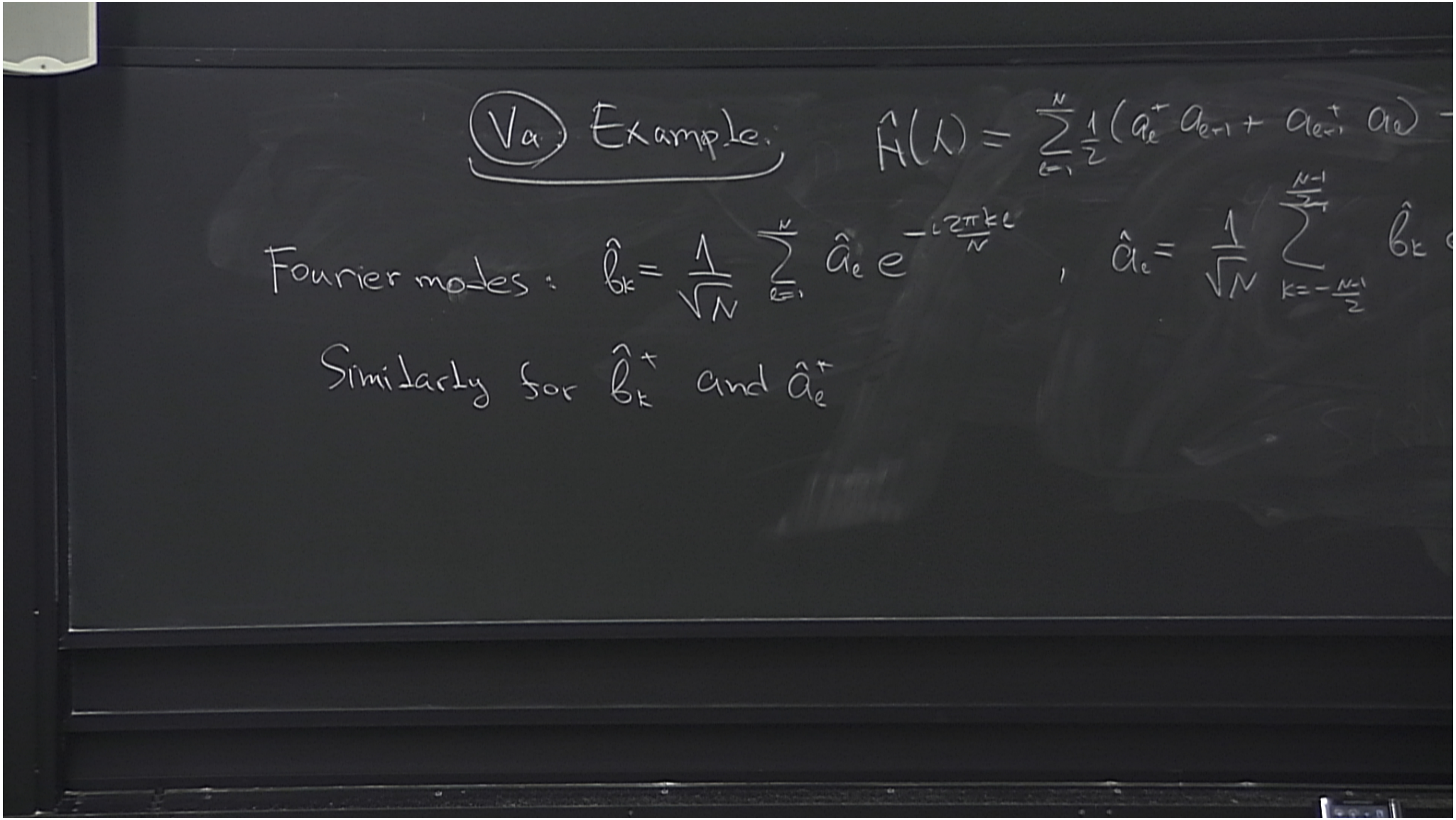
$$\langle b_k^\dagger b_k \rangle = \begin{cases} 0, & \text{if } \varepsilon_k > 0 \\ 1, & \text{if } \varepsilon_k < 0 \end{cases}$$


$\hat{b}_k^\dagger \hat{b}_k |m_k\rangle = m_k |m_k\rangle$, $m_k$ is either 0 or 1
$(\delta_{kl} - 2\langle \hat{b}_k^\dagger \hat{b}_k \rangle) =$
$\mathrm{sgn}\varepsilon_1$ 0 ...
0 $\mathrm{sgn}\varepsilon_2$ ...
0
0 $\mathrm{sgn}\varepsilon_N$


The ground state will be $\bigotimes_{k=1}^{N} |m_k\rangle$, where $m_k$ ...
$\langle \hat{b}_k^+ \hat{b}_k \rangle = \begin{cases} 0, \text{ if } \varepsilon_k > 0 \\ 1, \text{ if } \varepsilon_k < 0 \end{cases} \Longrightarrow$
$\overleftrightarrow{M} = \operatorname{sgn} \overleftrightarrow{A} \equiv \overleftrightarrow{U} \begin{pmatrix} \operatorname{sgn}\varepsilon_1 & & \\ & \ddots & \\ & & \operatorname{sgn}\varepsilon_N \end{pmatrix} \overleftrightarrow{U}^+$

Va) Example: $\hat{H}(\lambda) = \sum_{\ell=1}^{N} \frac{1}{2}\left(a_\ell^\dagger a_{\ell+1} + a_{\ell+1}^\dagger a_\ell\right) - 2\lambda \sum_{\ell=1}^{N} \hat{a}_\ell^\dagger \hat{a}_\ell$

Fourier modes: $\hat{b}_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} \hat{a}_\ell e^{-i\frac{2\pi k \ell}{N}}$, $\hat{a}_\ell = \frac{1}{\sqrt{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{b}_k e^{i\frac{2\pi}{N} k\ell}$

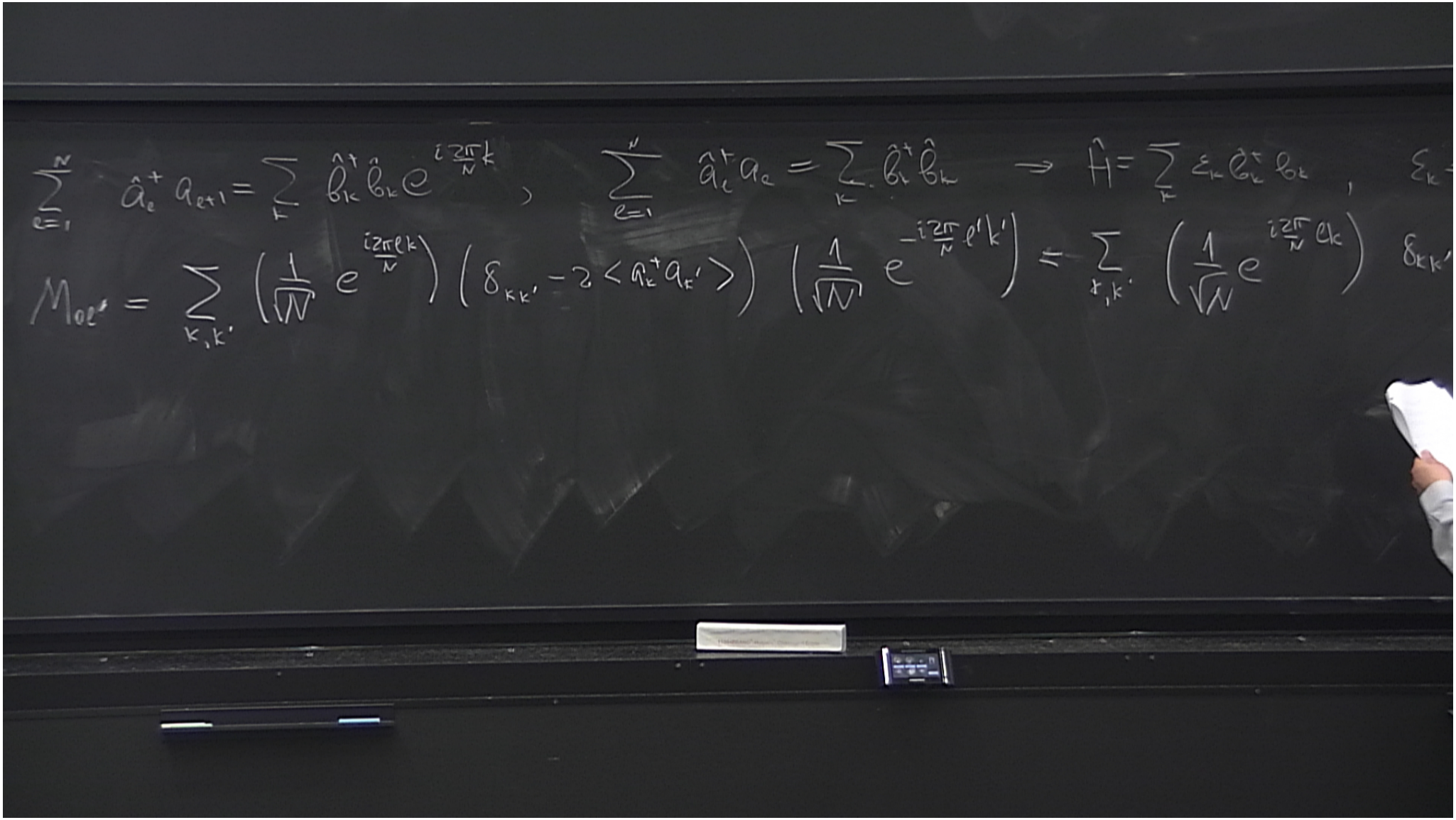

Va) Example:
$\hat{H}(\Lambda) = \sum_{\ell=1}^{N} \frac{1}{2}(a_\ell^+ a_{\ell-1} + a_{\ell-1}^+ a_\ell) -$
Fourier modes: $\hat{b}_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} \hat{a}_\ell e^{-i\frac{2\pi k \ell}{N}}$, $\hat{a}_\ell = \frac{1}{\sqrt{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{b}_k$
Similarly for $\hat{b}_k^+$ and $\hat{a}_\ell^+$

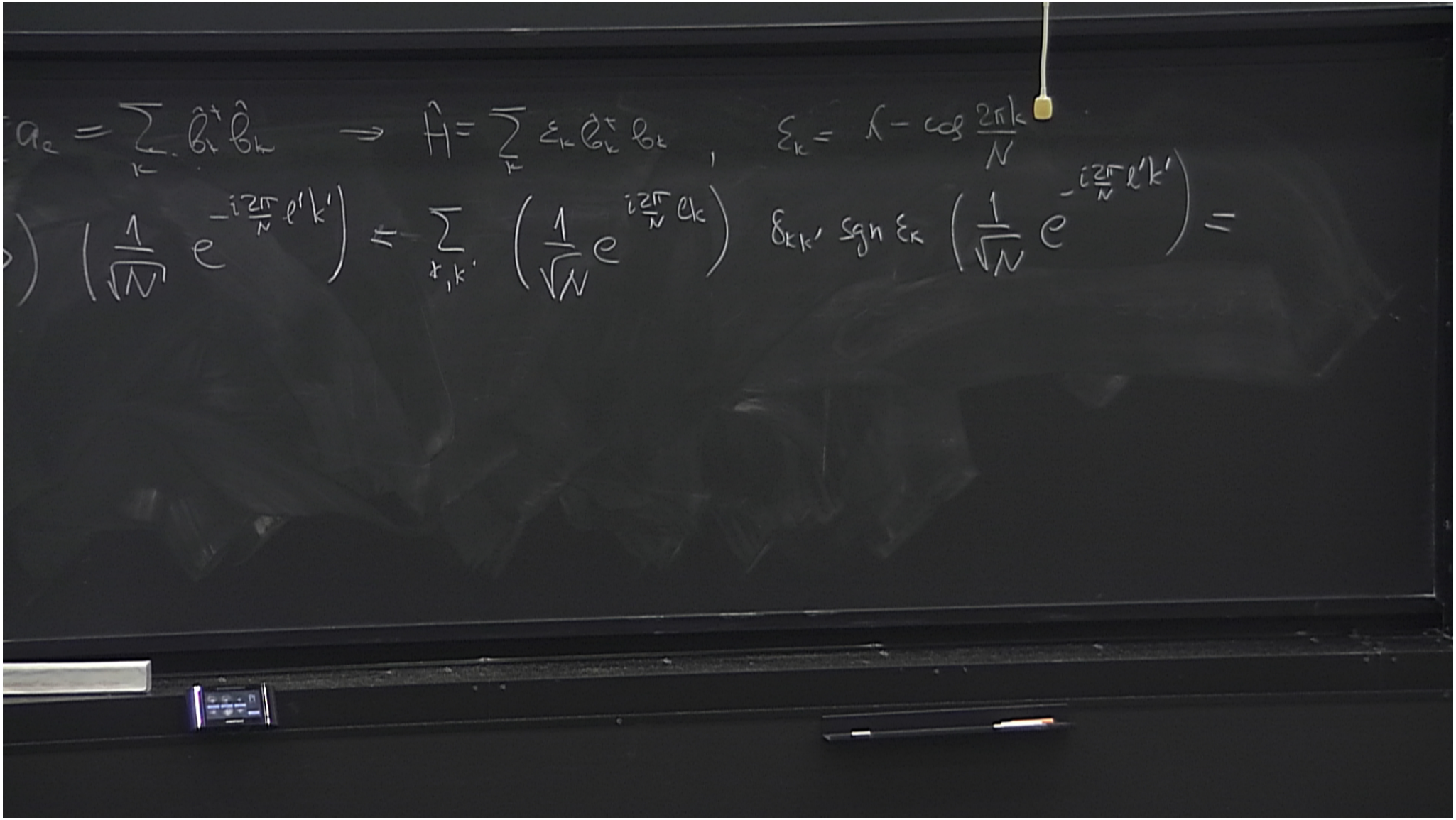
Va) Example: $\hat{H}(\lambda) = \sum_{\ell=1}^{N} \frac{1}{2}\left(\hat{a}_\ell^\dagger \hat{a}_{\ell+1} + \hat{a}_{\ell+1}^\dagger \hat{a}_\ell\right) - 2\lambda \sum_{\ell=1}^{N} \hat{a}_\ell^\dagger \hat{a}_\ell$

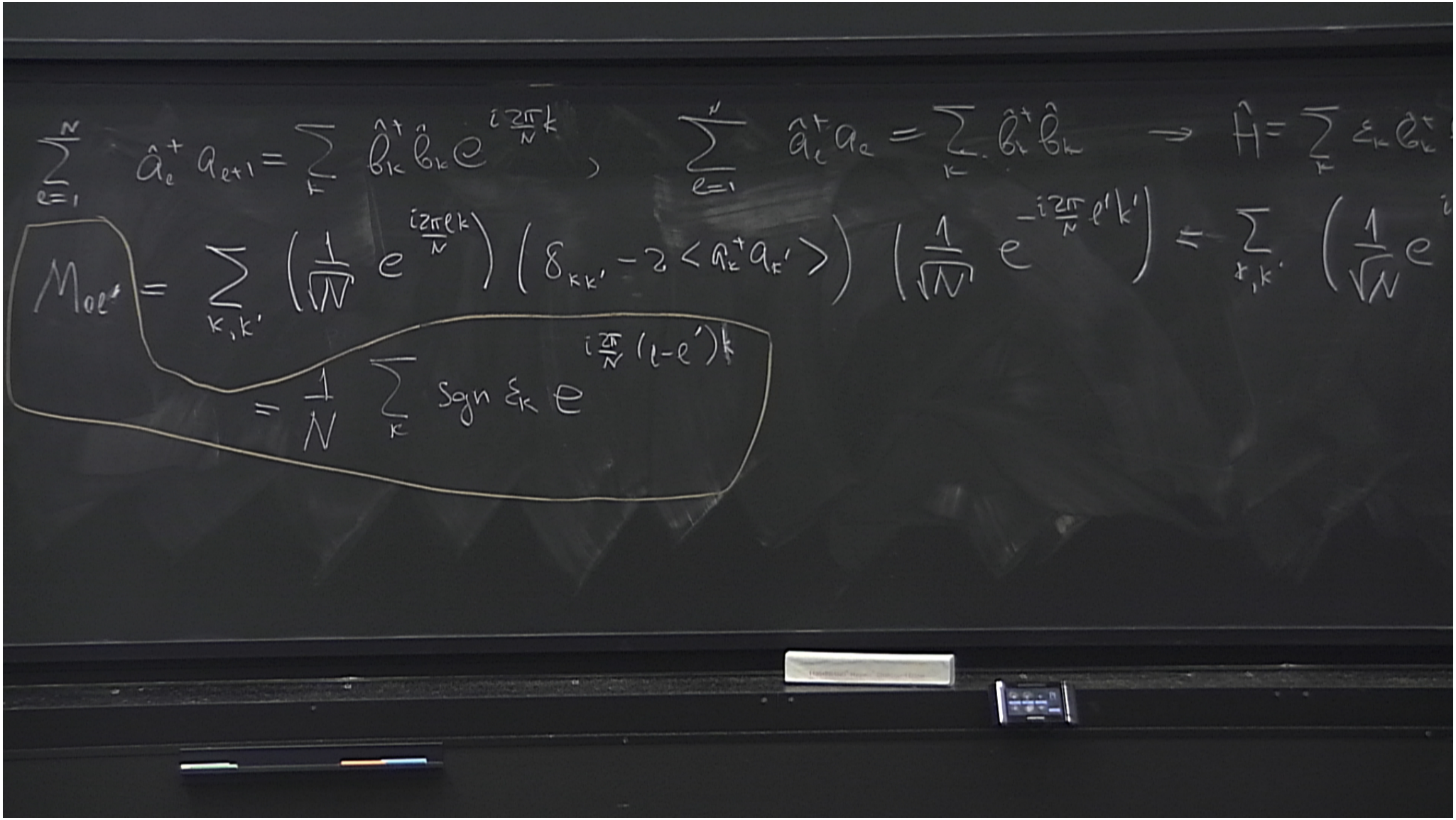
Fourier modes: $\hat{b}_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} \hat{a}_\ell e^{-i\frac{2\pi k \ell}{N}}$, $\hat{a}_\ell = \frac{1}{\sqrt{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \hat{b}_k e^{i\frac{2\pi}{N}k\ell}$ ←

Similarly for $\hat{b}_k^\dagger$ and $\hat{a}_\ell^\dagger$, Note: $U_{\ell k} = \frac{1}{\sqrt{N}} e^{i\frac{2\pi \ell k}{N}}$, $U$ is u

$$\sum_{\ell=1}^{N} \hat{a}_\ell^\dagger a_{\ell+1} = \sum_k \hat{b}_k^\dagger \hat{b}_k e^{i\frac{2\pi}{N}k} \quad , \quad \sum_{\ell=1}^{N} \hat{a}_\ell^\dagger a_\ell = \sum_k \hat{b}_k^\dagger \hat{b}_k$$


$\sum_{e=1}^{N} \hat{a}_e^\dagger a_{e+1} = \sum_k \hat{b}_k^\dagger \hat{b}_k e^{i\frac{2\pi}{N}k}$ , $\sum_{e=1}^{N} \hat{a}_e^\dagger a_e = \sum_k \hat{b}_k^\dagger \hat{b}_k \Rightarrow \hat{H} = \sum_k \varepsilon_k \hat{b}_k^\dagger \hat{b}_k$ ,
$M_{ee'} = \sum_{k,k'} \left(\frac{1}{\sqrt{N}} e^{\frac{i2\pi ek}{N}}\right)\left(\delta_{kk'} - 2\langle a_k^\dagger a_{k'}\rangle\right)\left(\frac{1}{\sqrt{N}} e^{-\frac{i2\pi}{N}e'k'}\right) = \sum_{k,k'} \left(\frac{1}{\sqrt{N}} e^{\frac{i2\pi}{N}ek}\right) \delta_{kk'}$


$a_\ell = \sum_k \hat{b}_k^\dagger \hat{b}_k \Rightarrow \hat{H} = \sum_k \varepsilon_k \hat{b}_k^\dagger \hat{b}_k$ , $\varepsilon_k = 1 - \cos\frac{2\pi k}{N}$
$\left(\frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}\ell' k'}\right) = \sum_{k,k'} \left(\frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}\ell k}\right) \delta_{kk'} \operatorname{sgn}\varepsilon_k \left(\frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}\ell' k'}\right) =$


$\sum_{e=1}^{N} \hat{a}_e^+ a_{e+1} = \sum_k \hat{b}_k^+ \hat{b}_k e^{i\frac{2\pi}{N}k}$ , $\sum_{e=1}^{N} \hat{a}_e^+ a_e = \sum_k \hat{b}_k^+ \hat{b}_k \rightarrow \hat{H} = \sum_k \xi_k \hat{b}_k^+$
$M_{ee'} = \sum_{k,k'} \left(\frac{1}{\sqrt{N}} e^{i\frac{2\pi ek}{N}}\right)\left(\delta_{kk'} - 2\langle a_k^+ a_{k'}\rangle\right)\left(\frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}e'k'}\right) = \sum_{k,k'} \left(\frac{1}{\sqrt{N}} e$
$= \frac{1}{N} \sum_k \operatorname{sgn} \xi_k \, e^{i\frac{2\pi}{N}(e-e')k}$