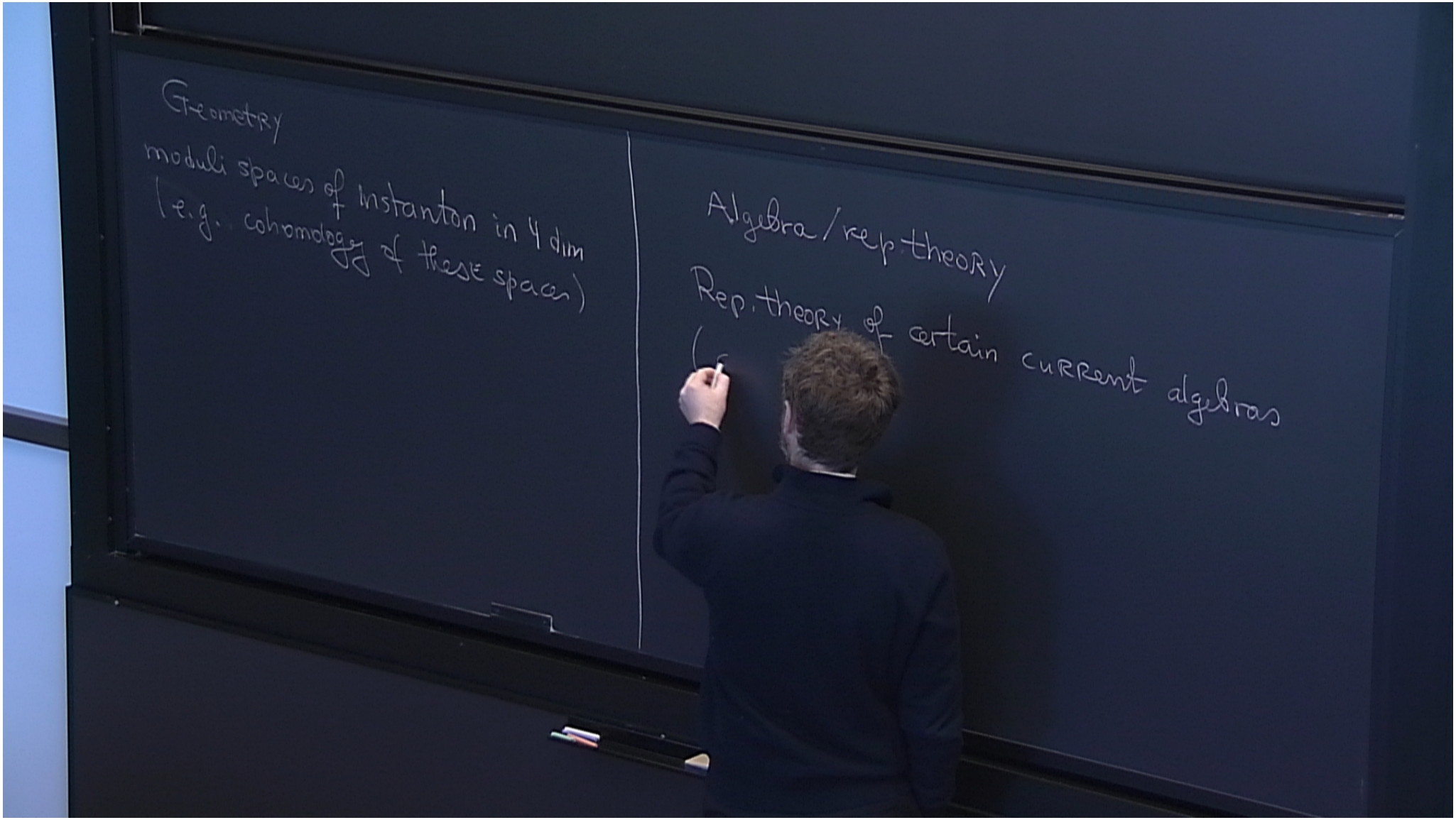
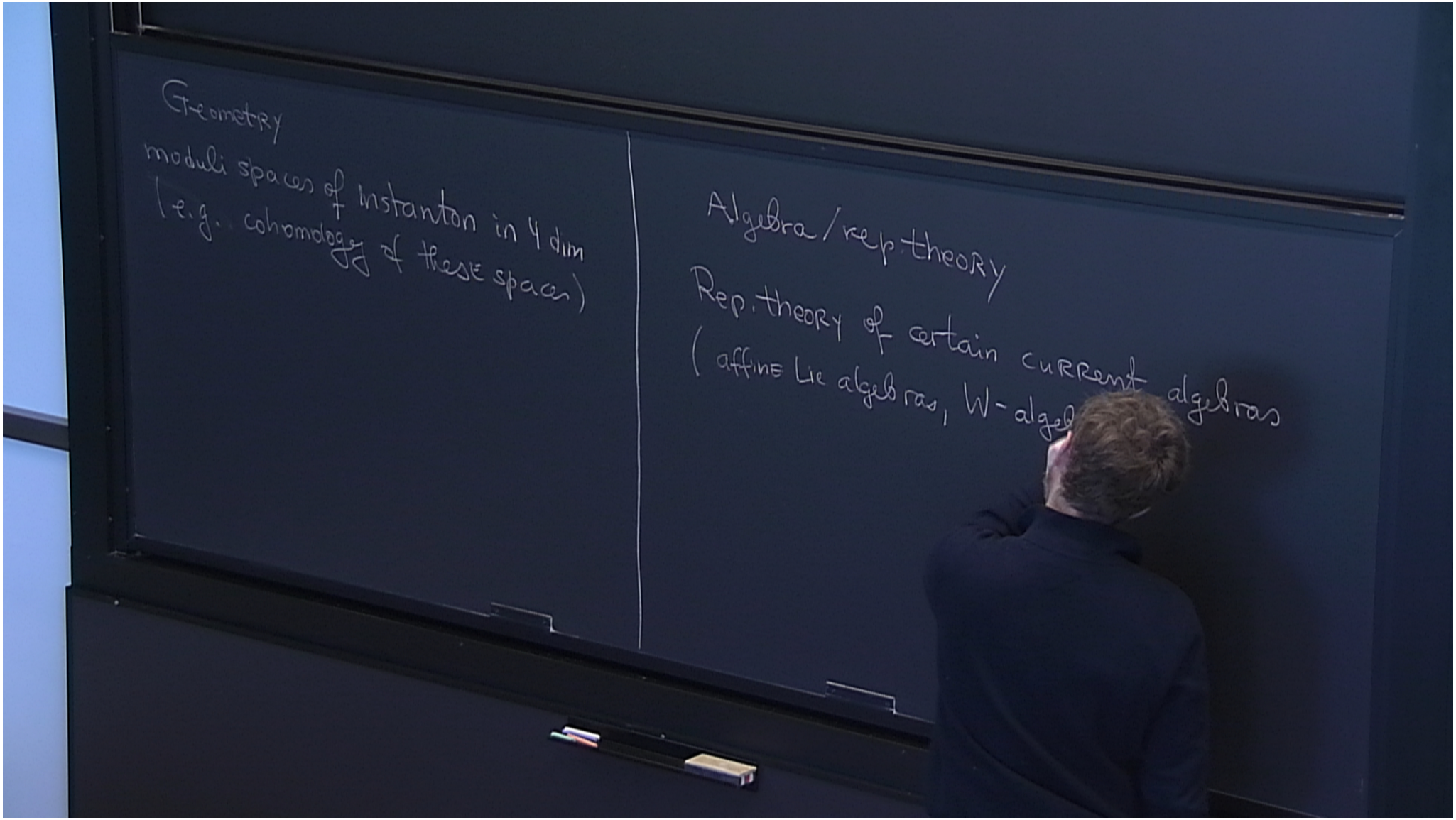
Title: Instanton moduli spaces, intersection cohomology and W-algebras or is there a mathematically rigorous formulation of the AGT conjecture?

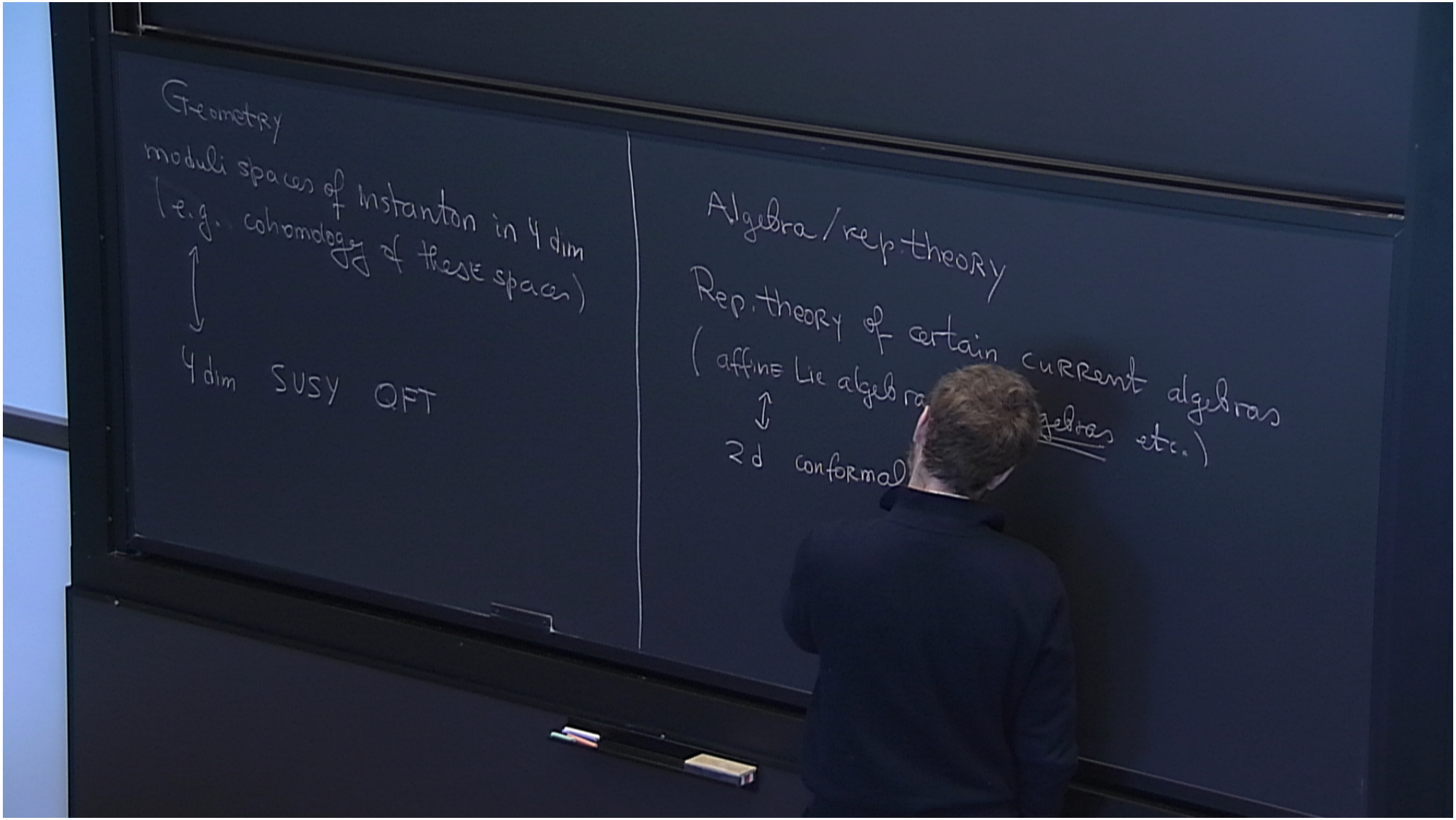
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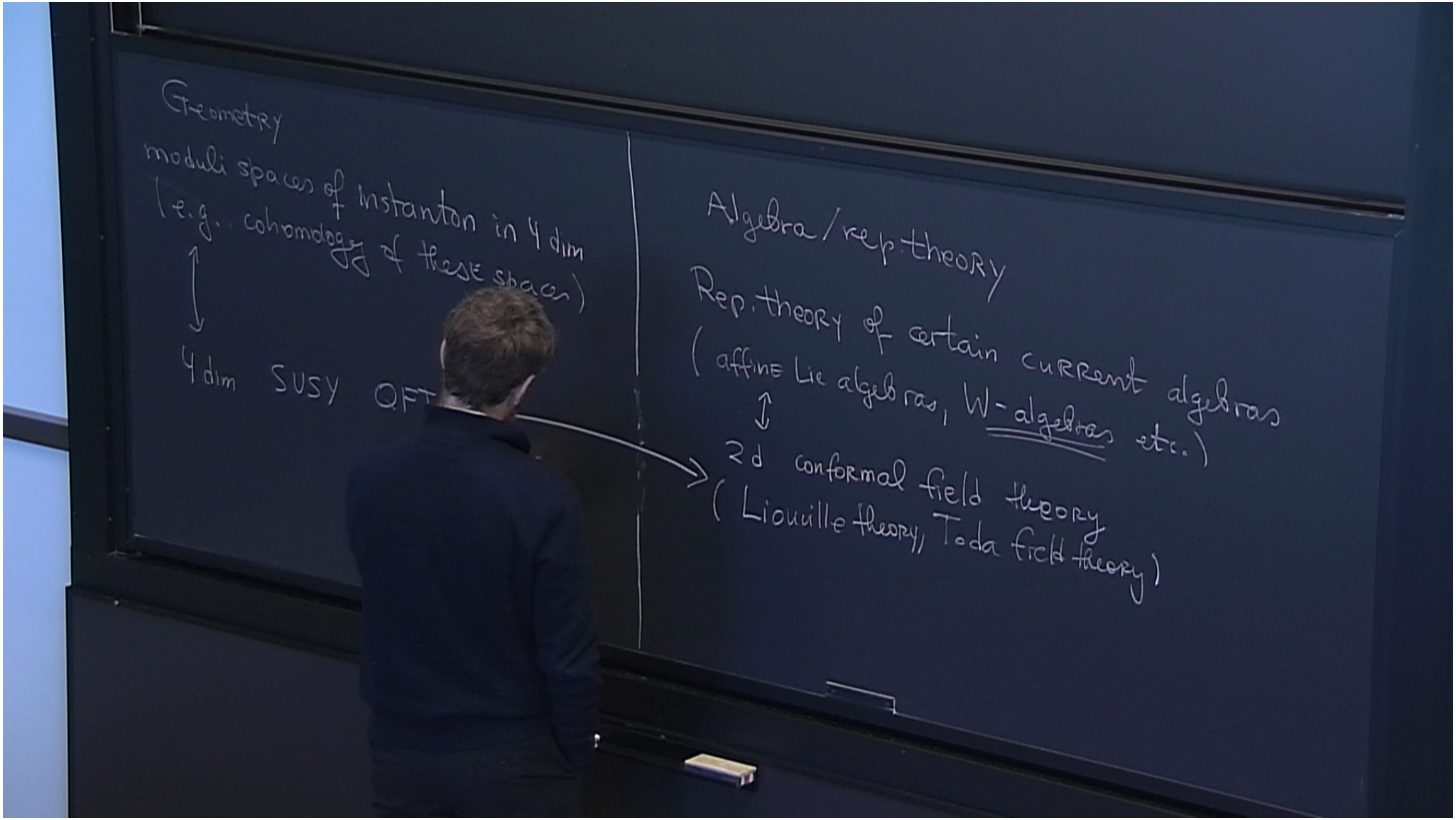
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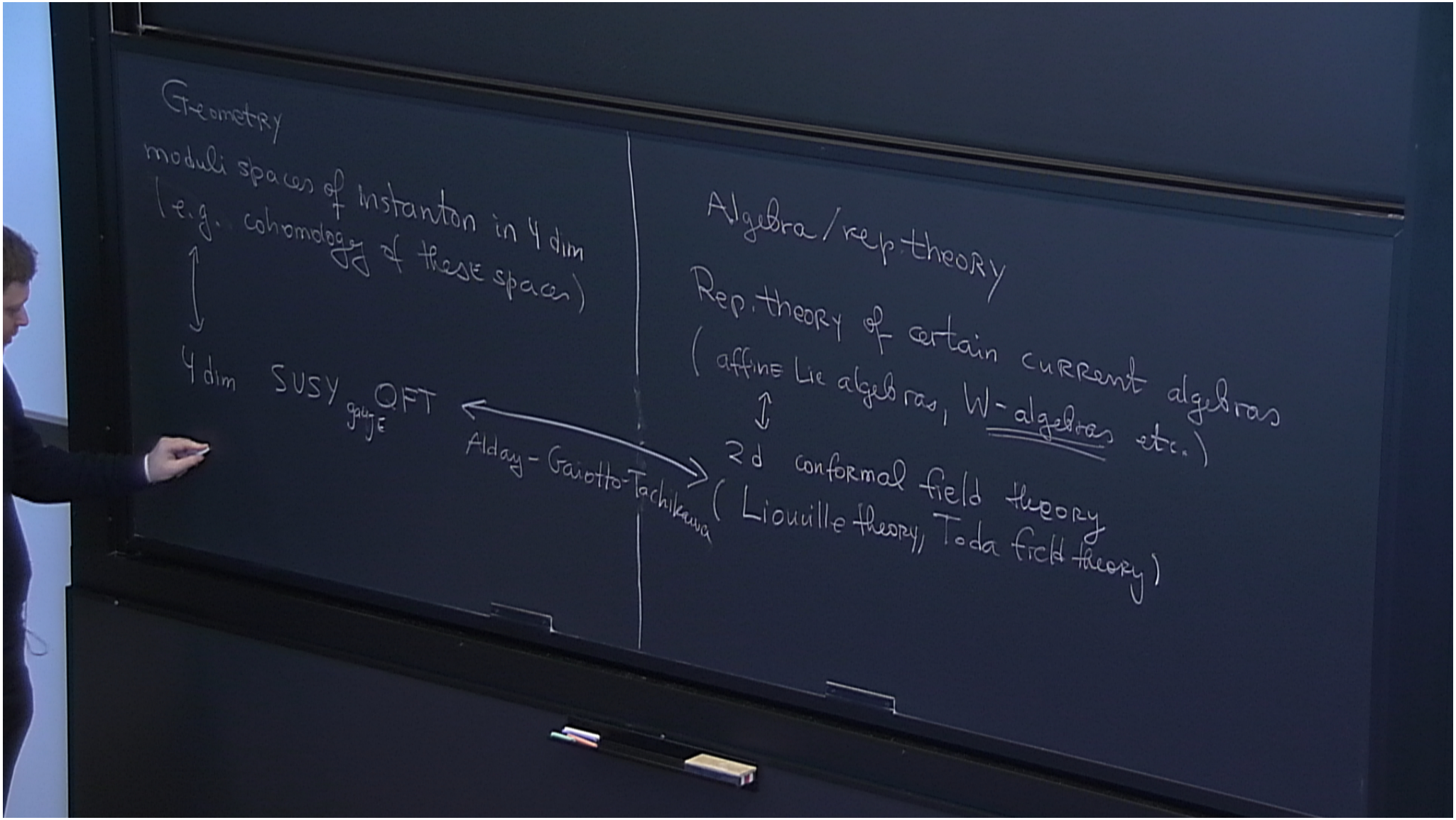
Abstract:

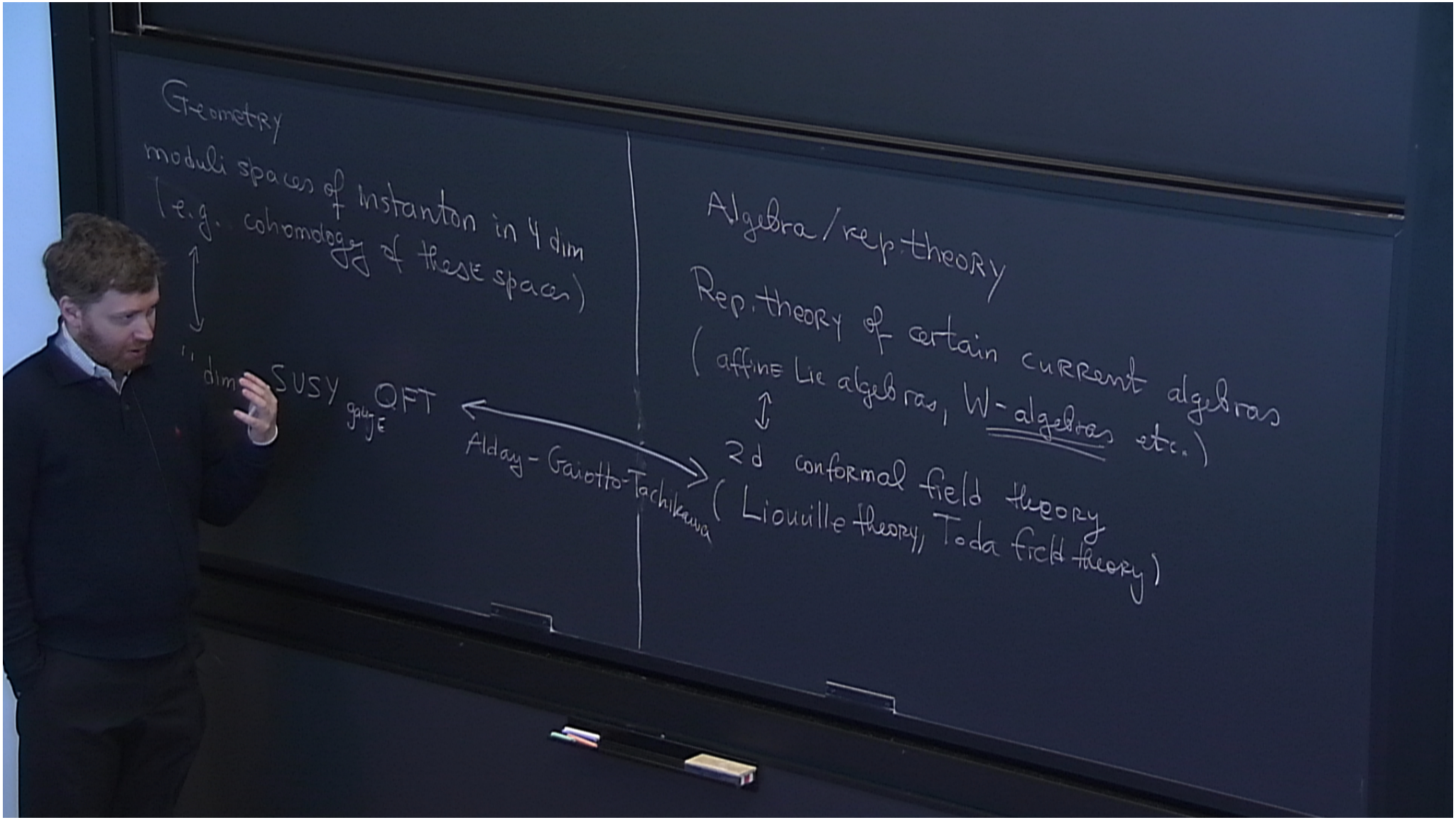

Geometry
moduli spaces of instanton in 4 dim
(e.g. cohomology of these spaces)
Algebra / rep. theory
Rep. theory of certain current algebras

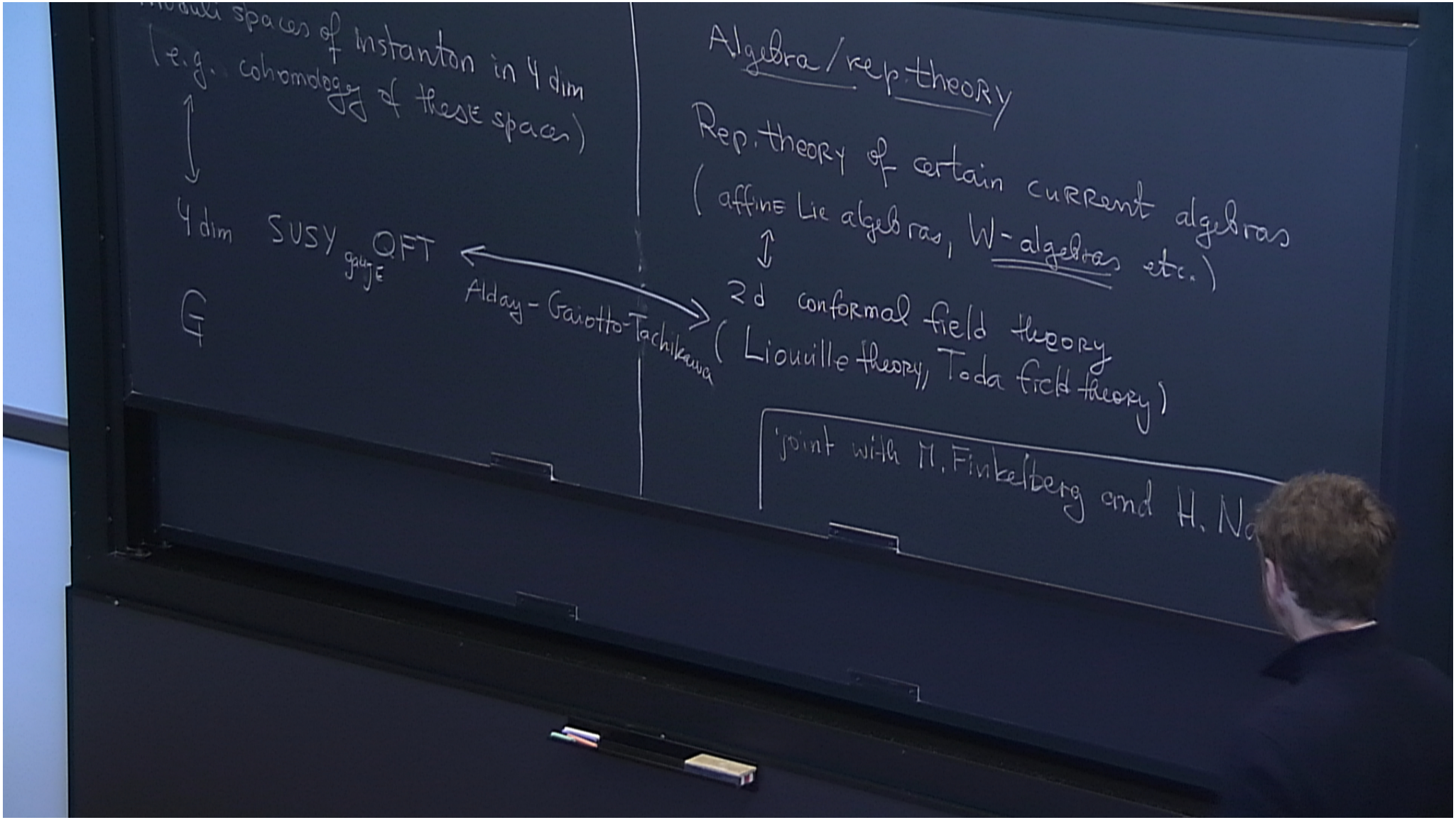

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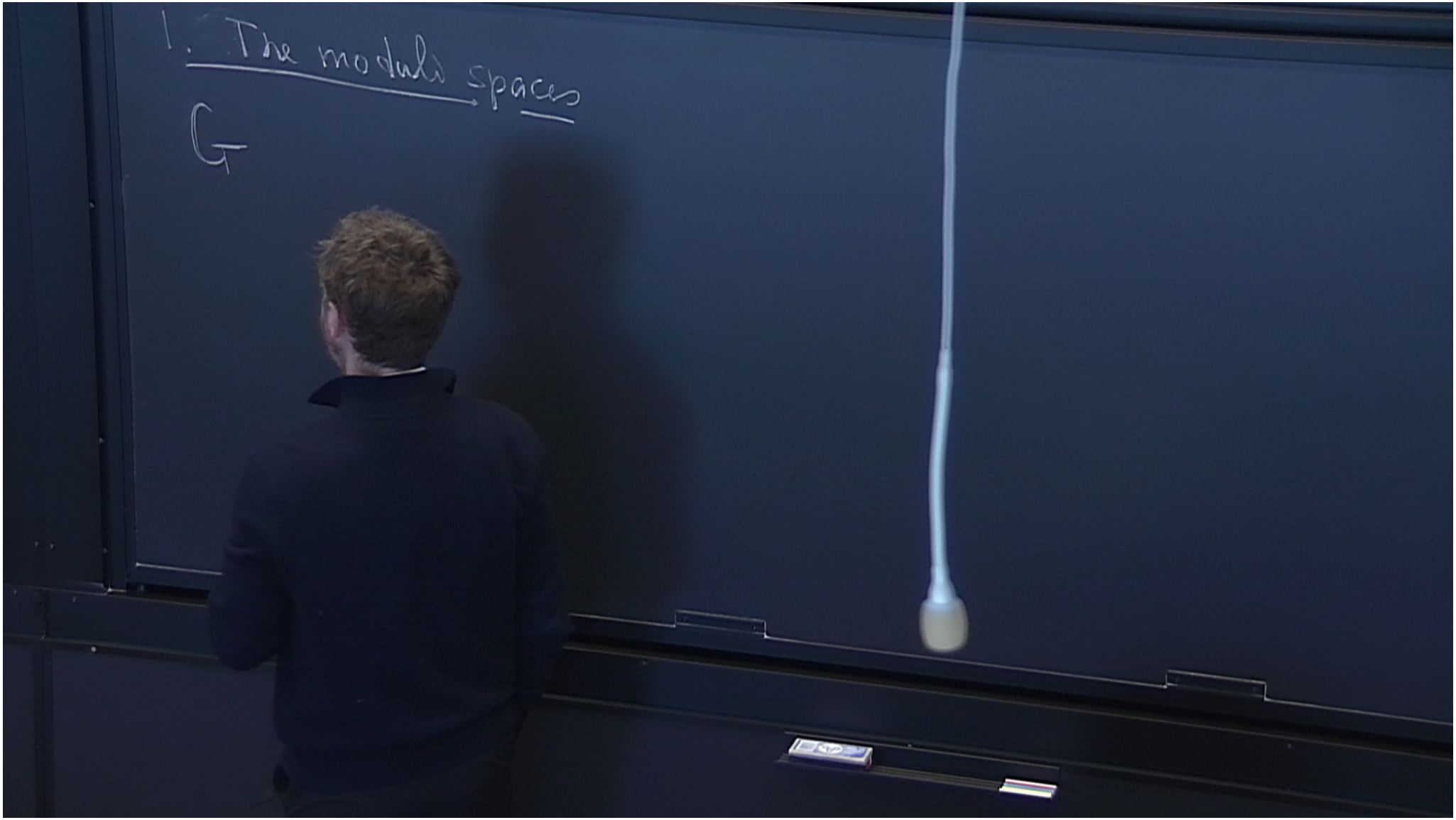

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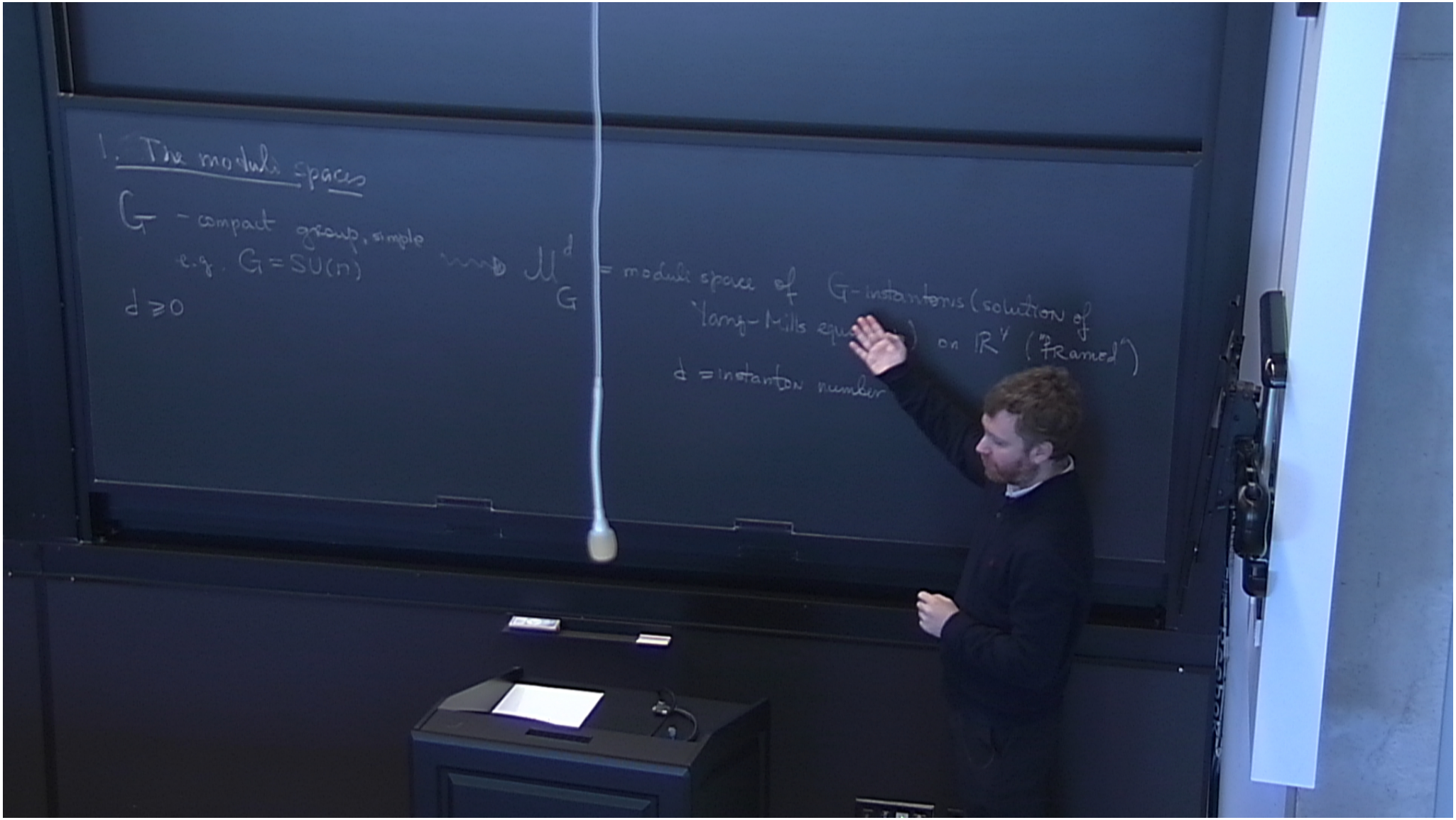

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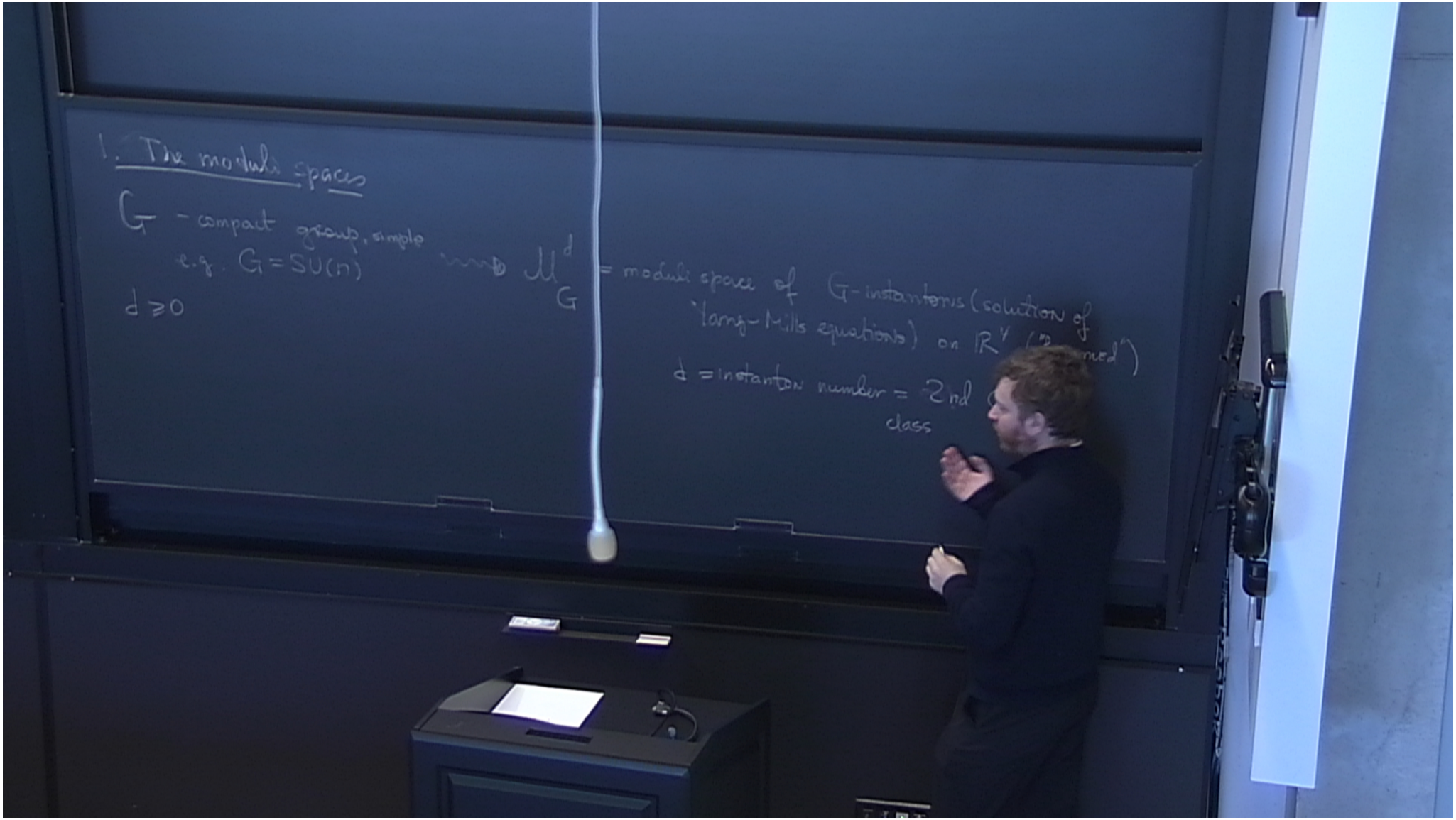

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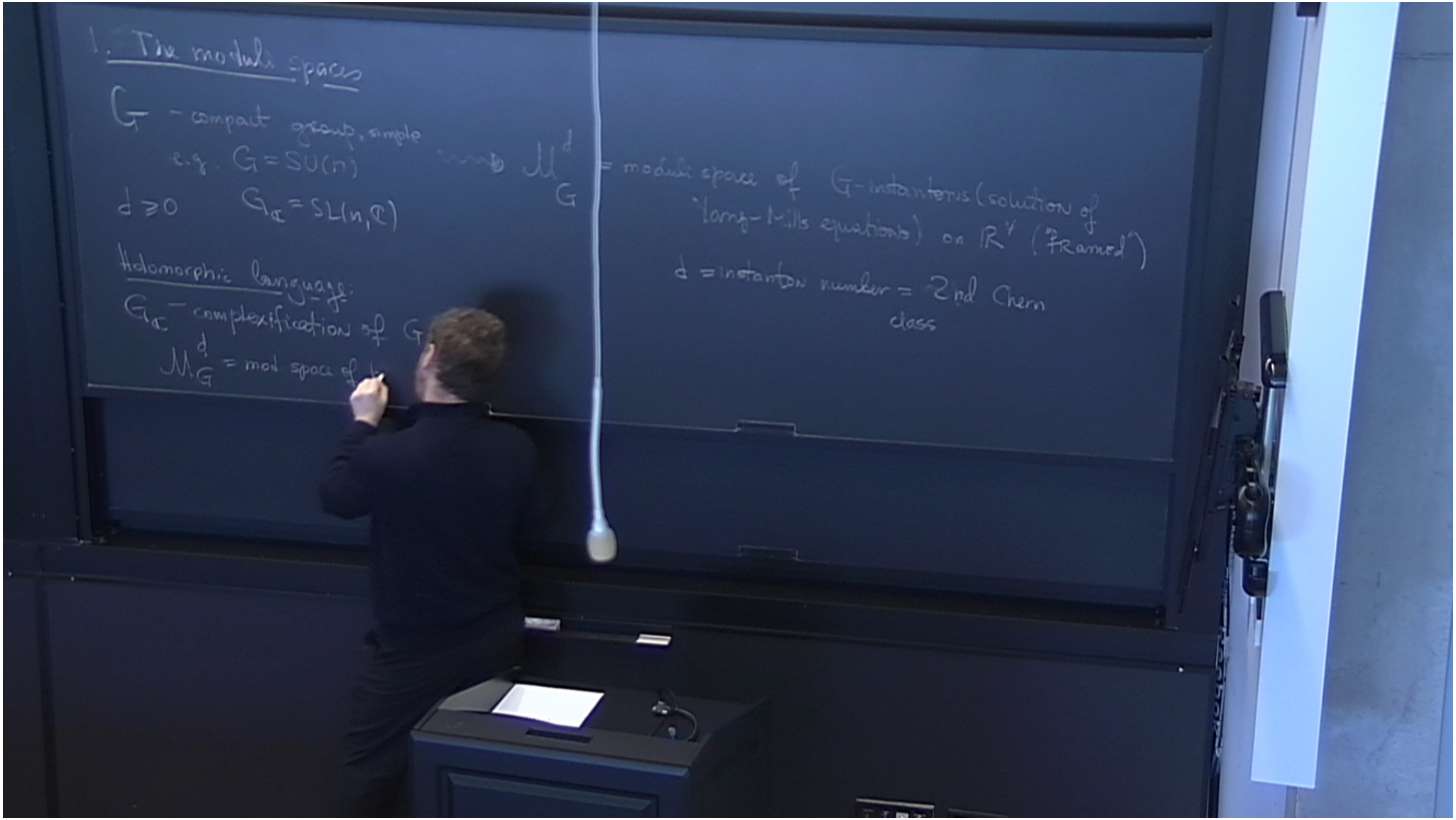

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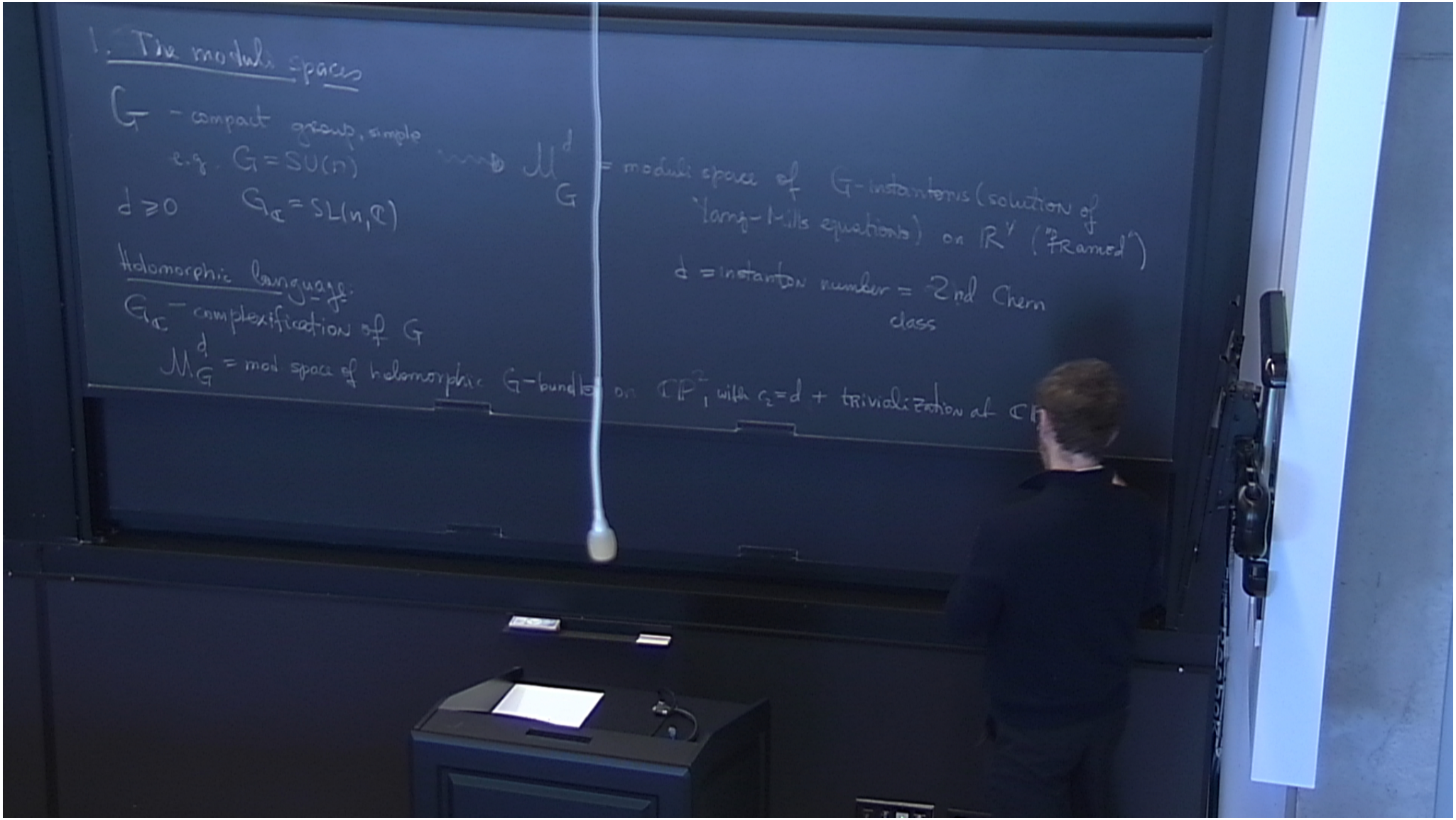

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(Liouville theory, Toda field theory)
joint with M. Finkelberg and H. Na

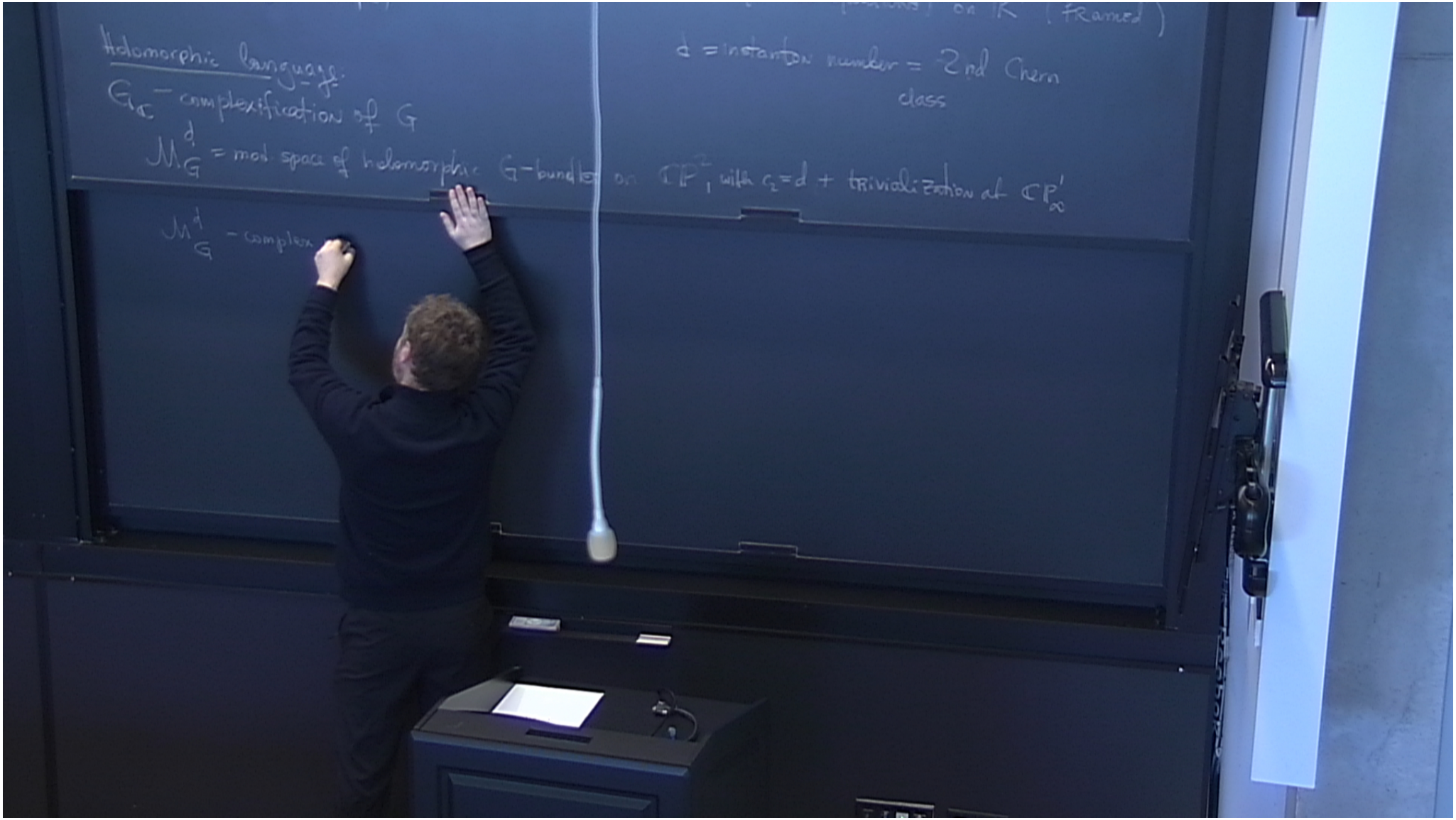

1. The moduli spaces
G

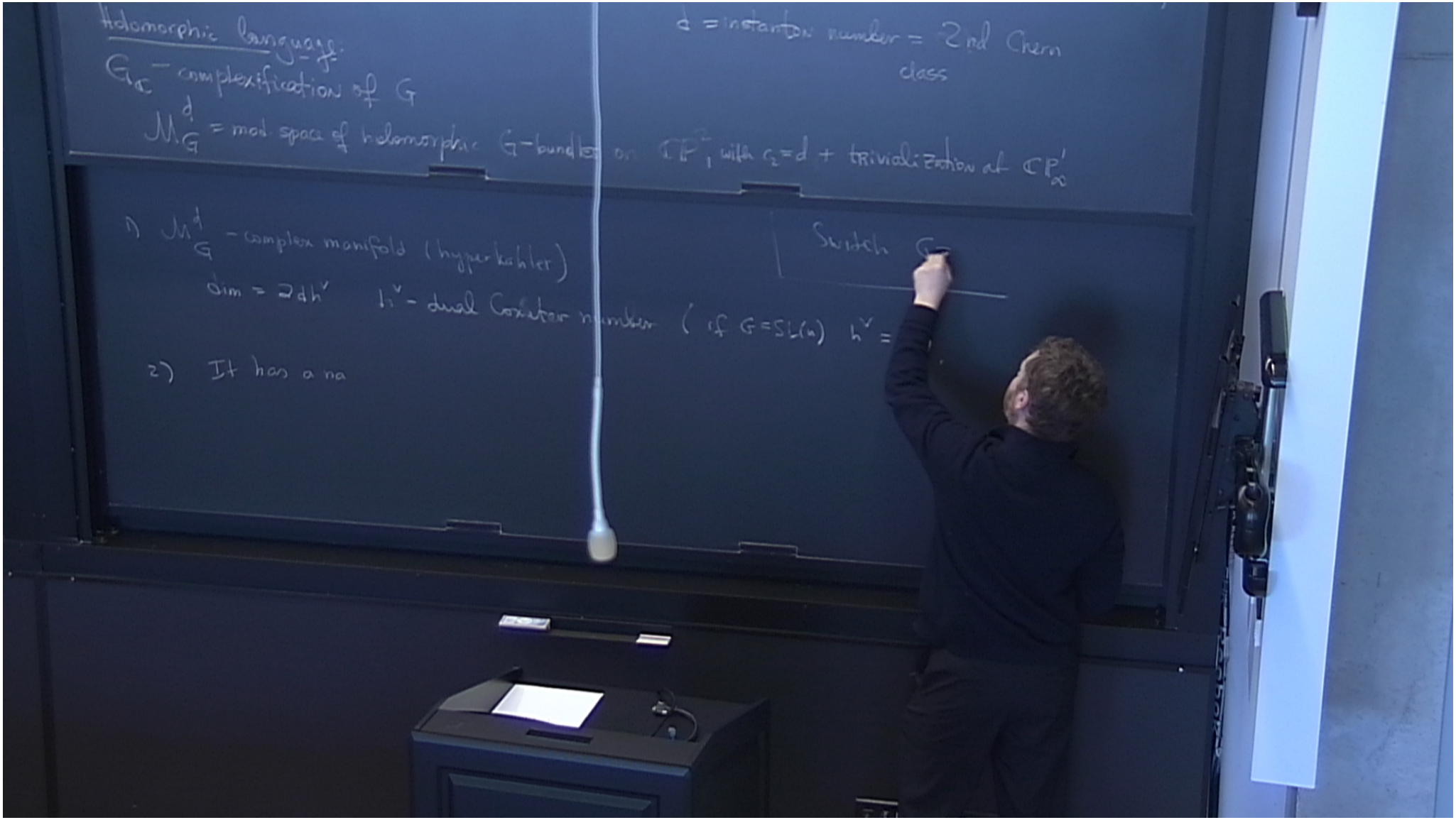

1. The moduli spaces
G - compact group, simple
e.g. G = SU(n)
d ≥ 0
$\mathcal{M}_G^d$ = moduli space of G-instantons (solution of
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d = instanton number

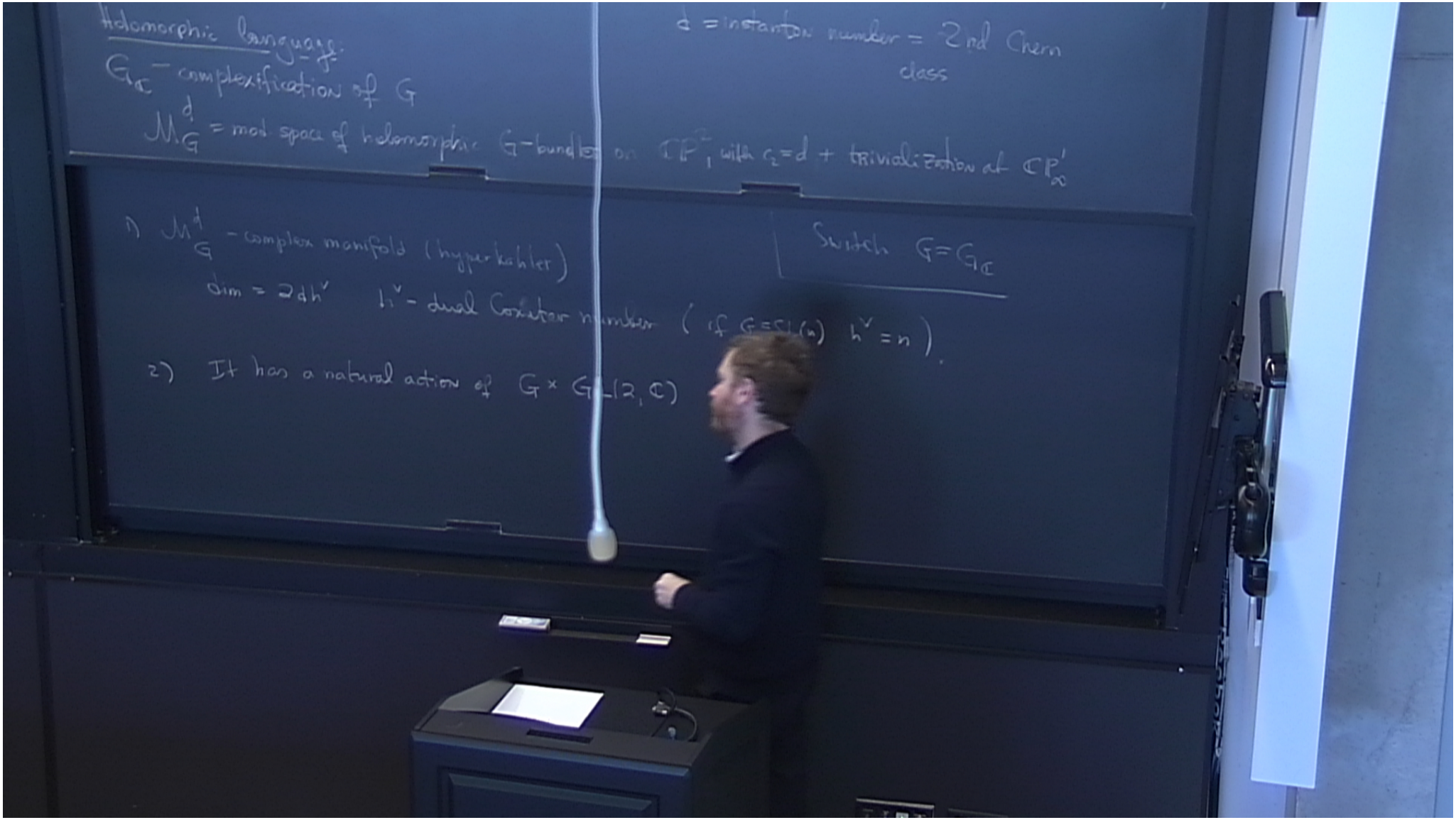

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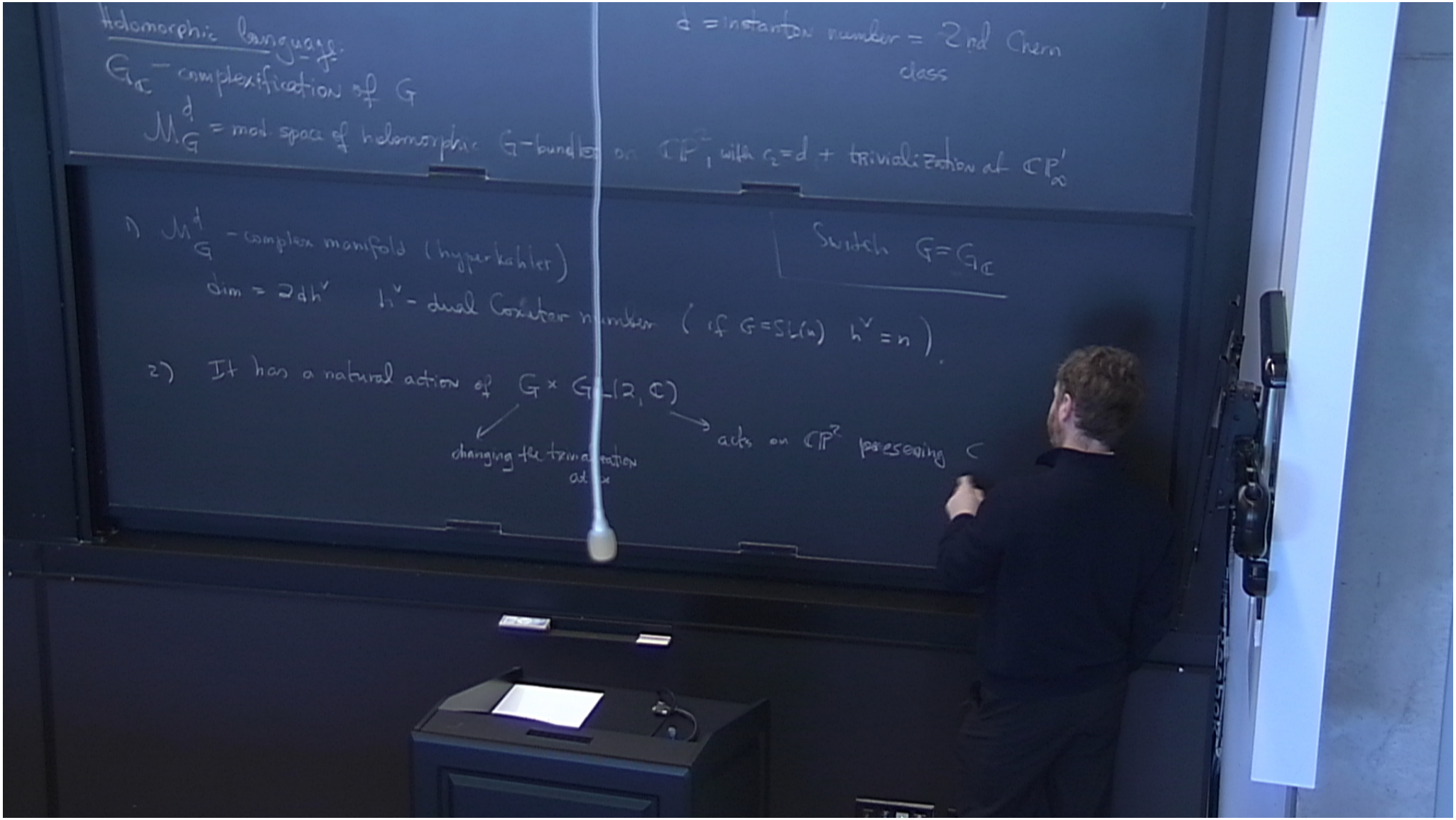

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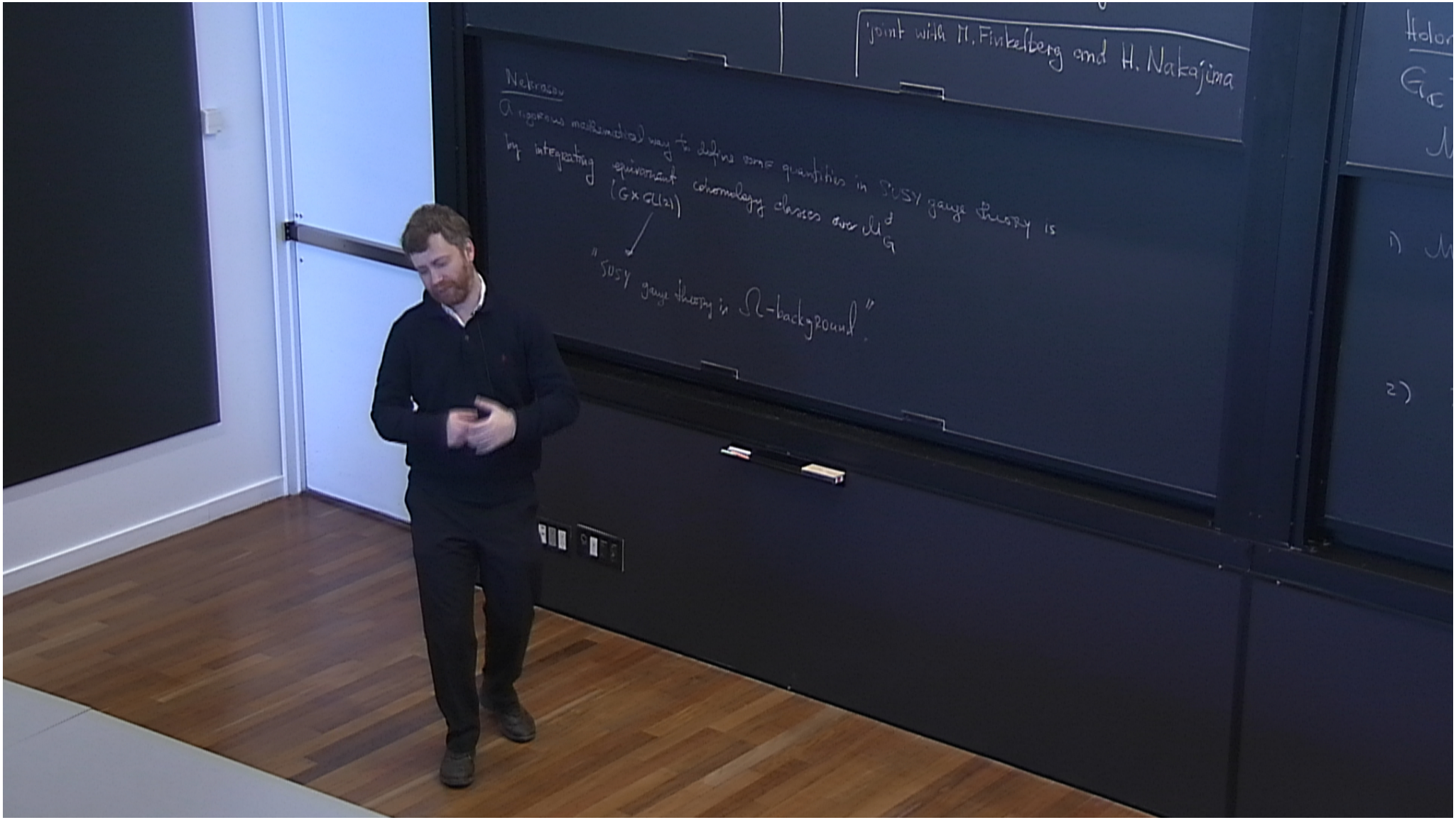

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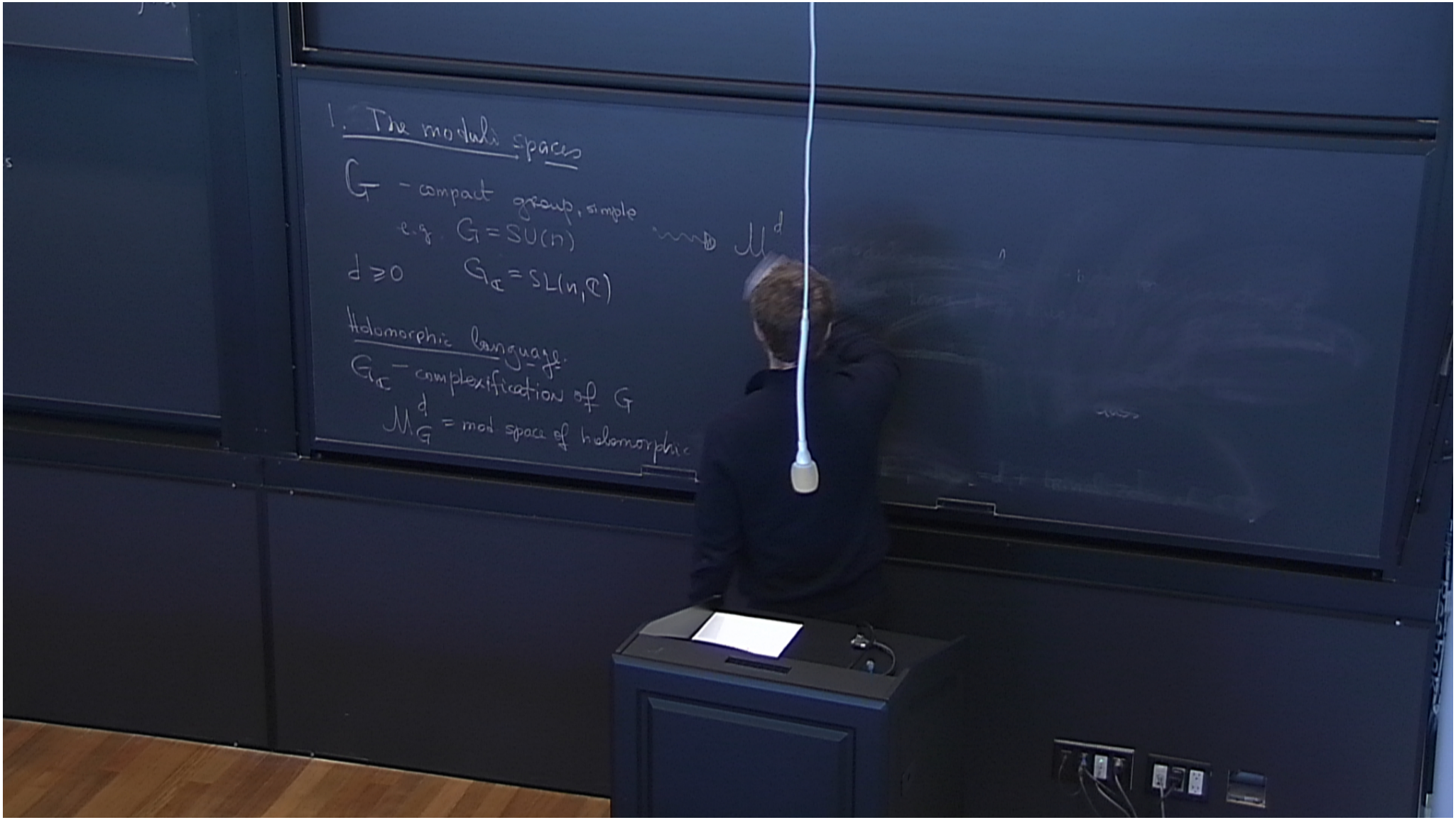

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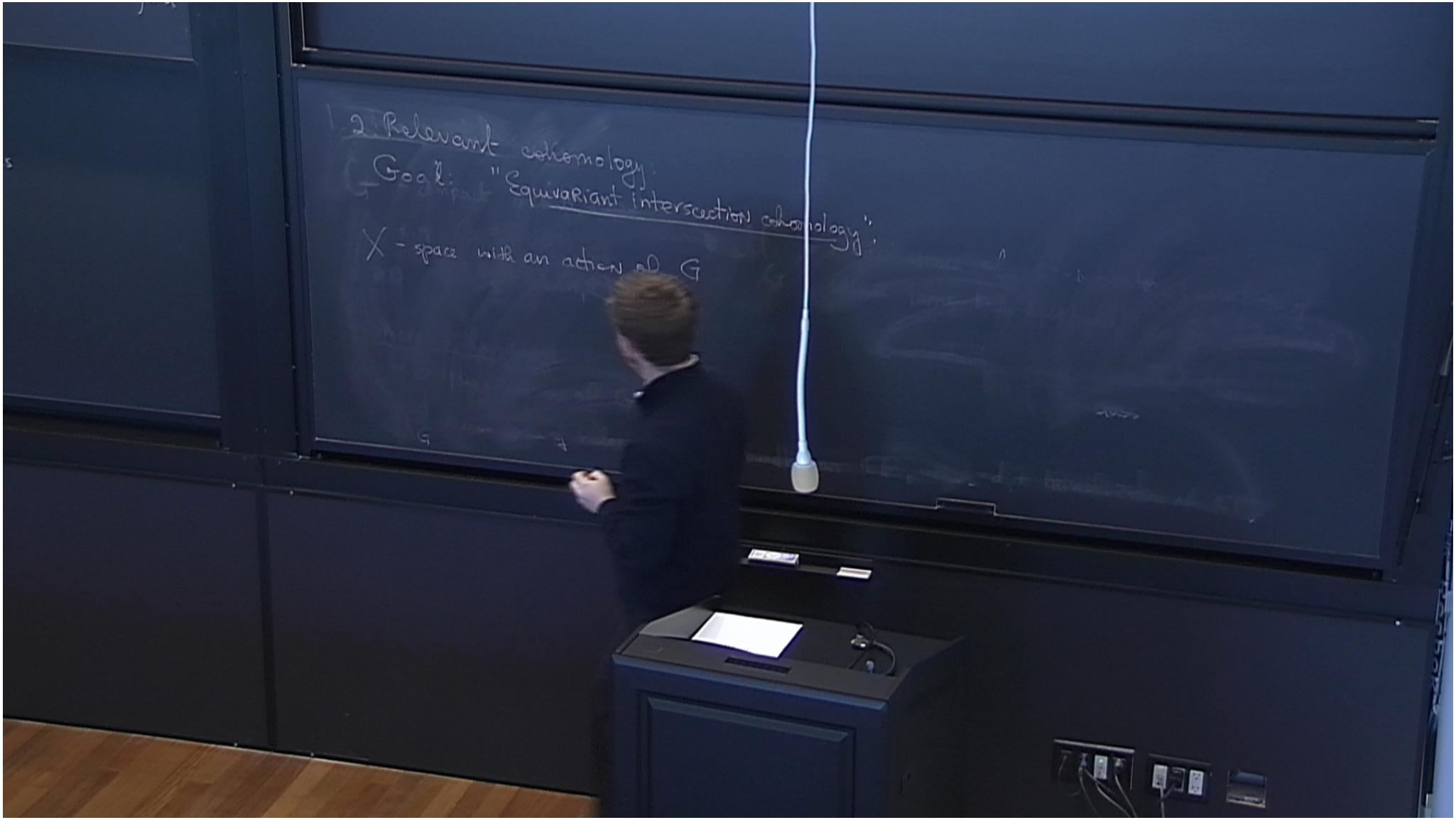

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Switch G

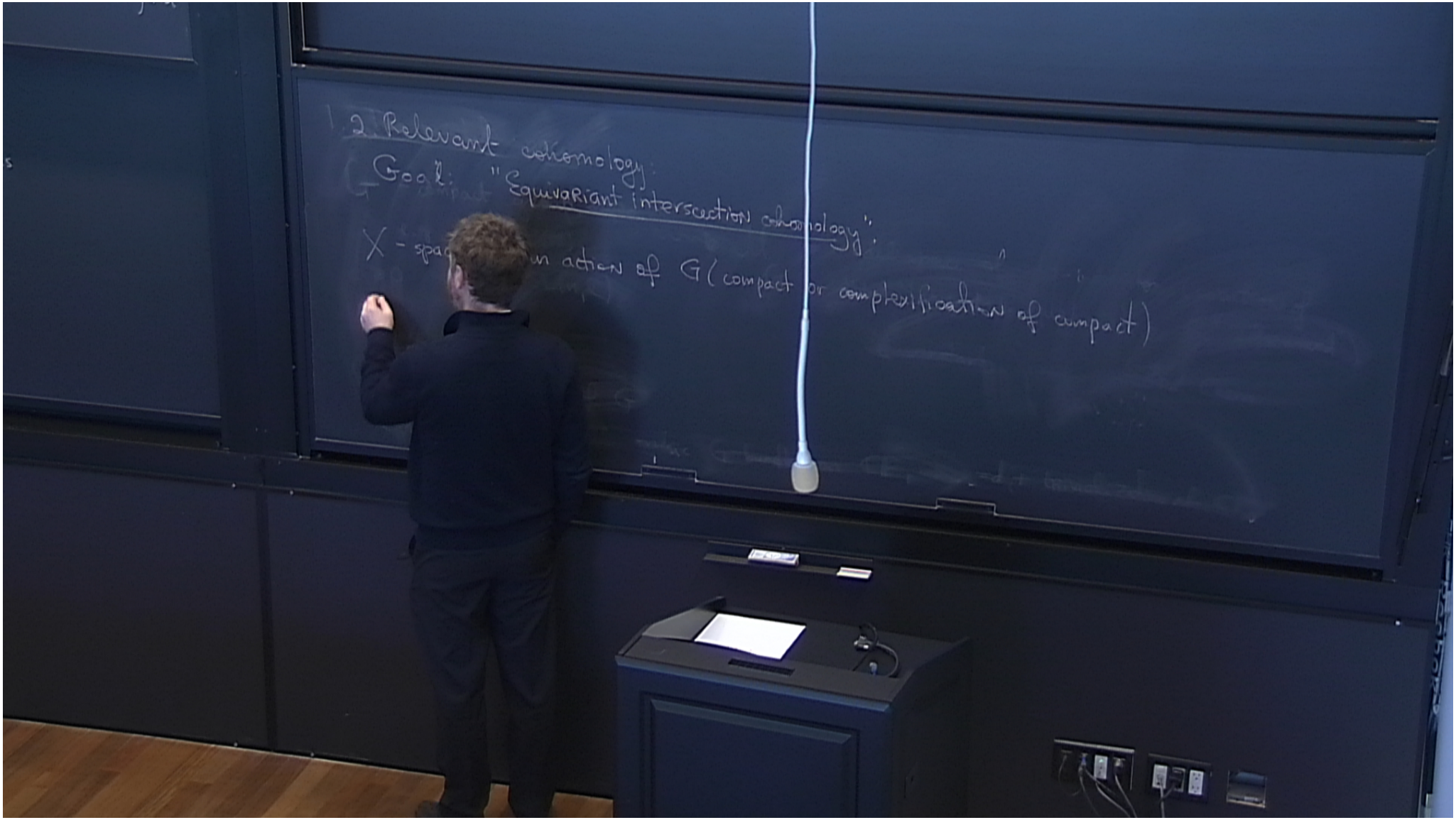

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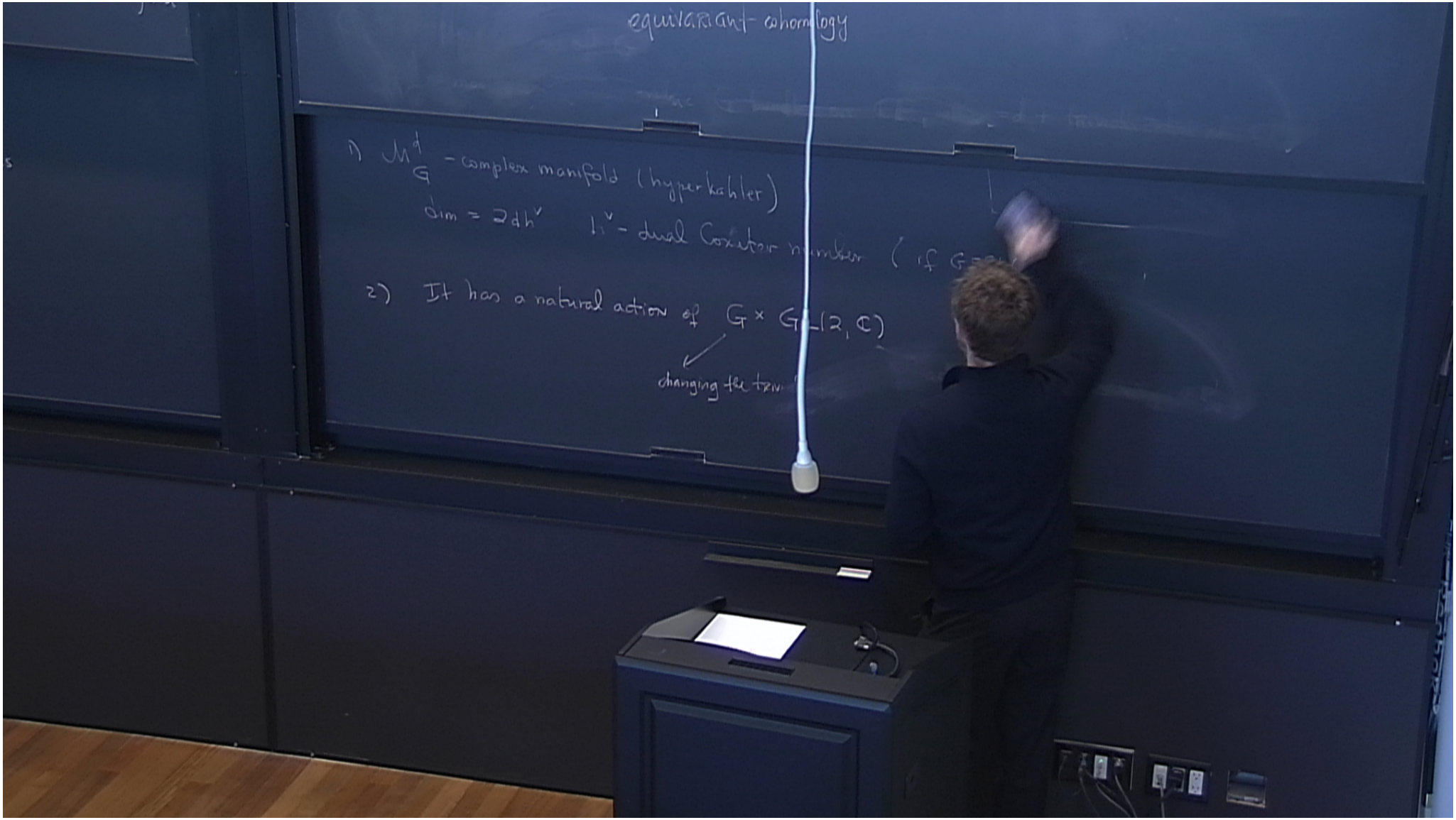

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changing the trivialization at ∞
acts on $\mathbb{CP}^2$ preserving

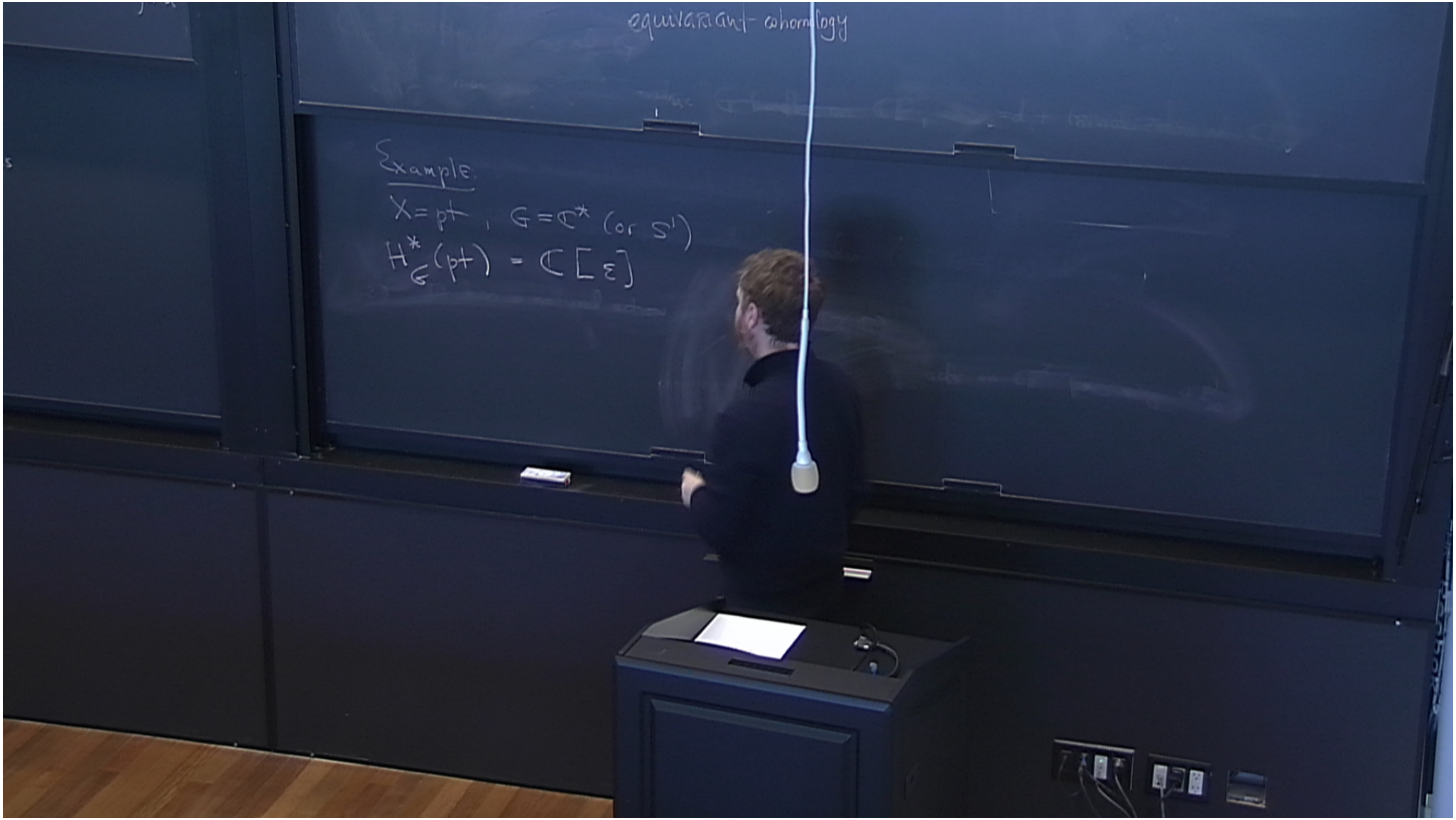

joint with M. Finkelberg and H. Nakajima
Nekrasov
A rigorous mathematical way to define some quantities in SUSY gauge theory is
by integrating equivariant cohomology classes over $\mathcal{M}_G^d$
$(G \times GL(2))$
"SUSY gauge theory is Ω-background"

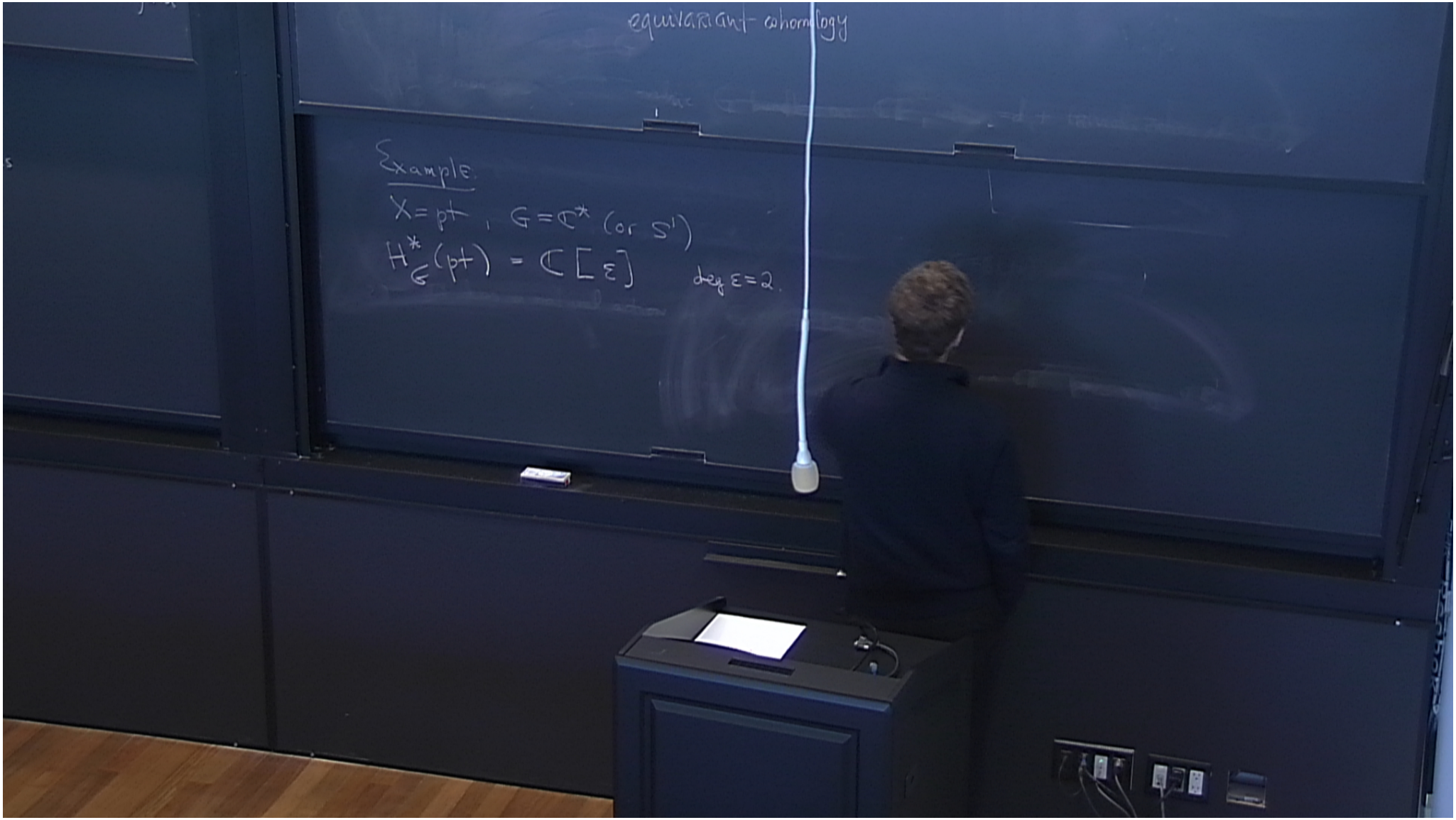

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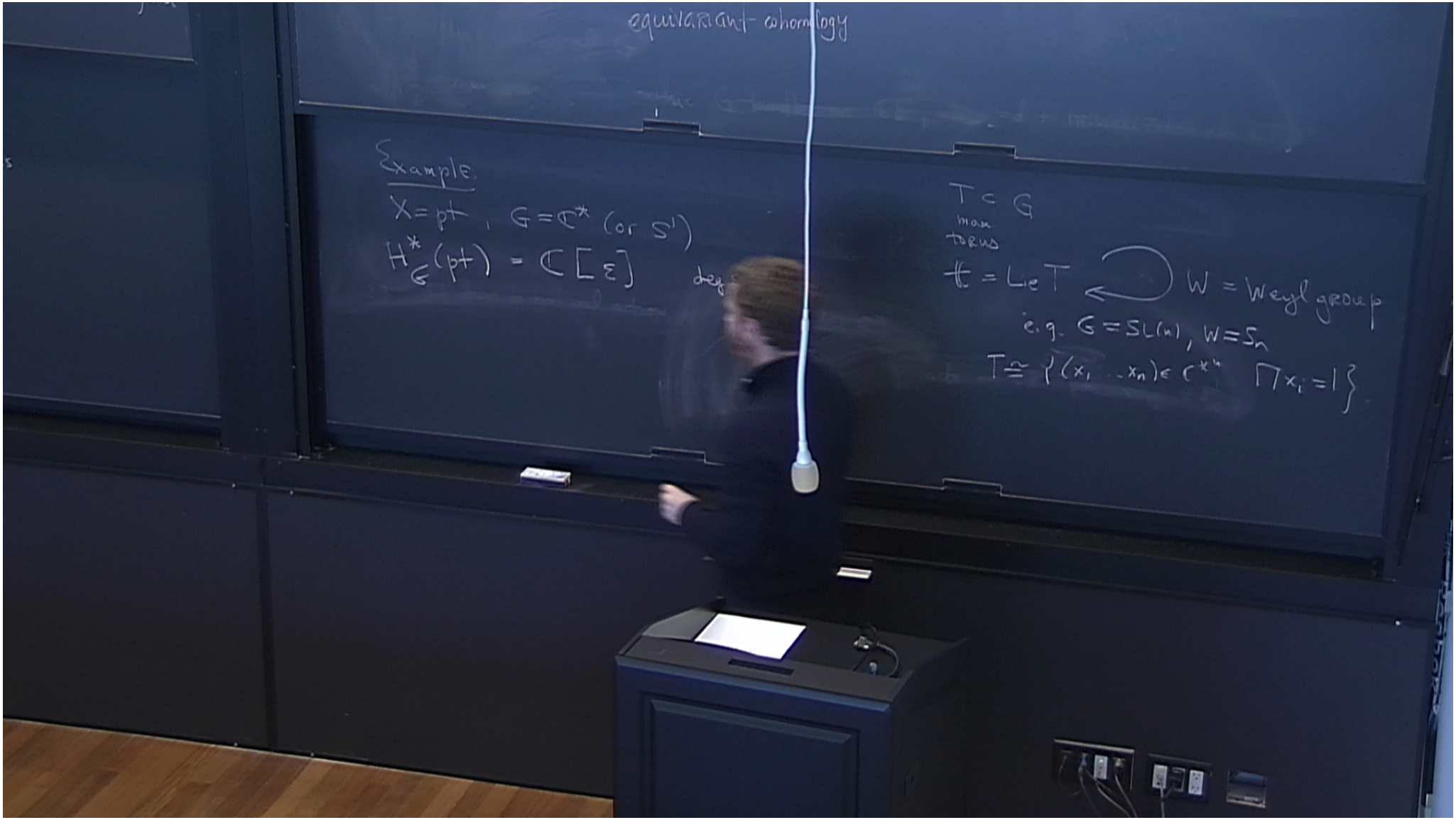

2 Relevant cohomology
Goal: "Equivariant intersection cohomology".
X - space with an action of G

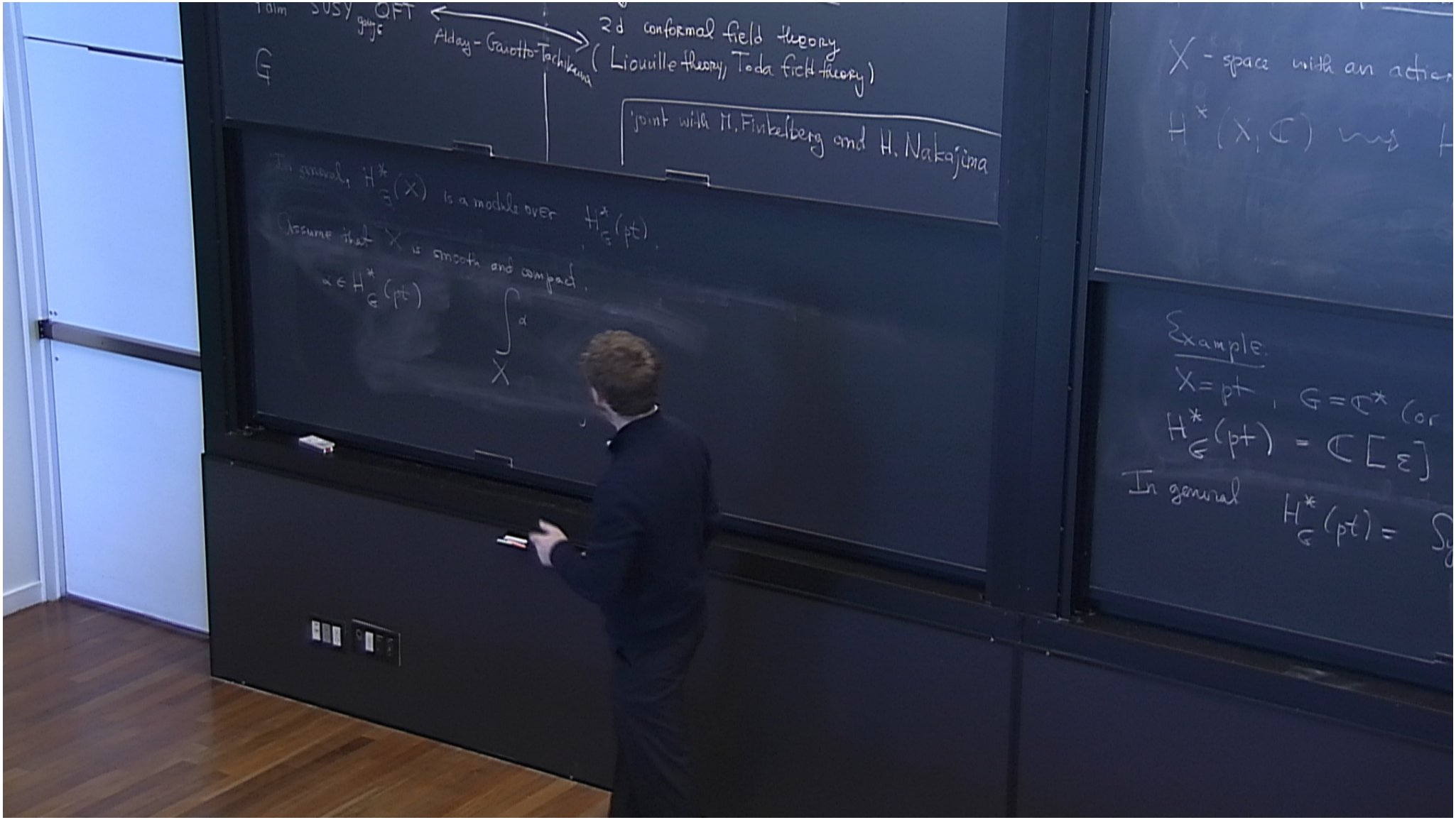

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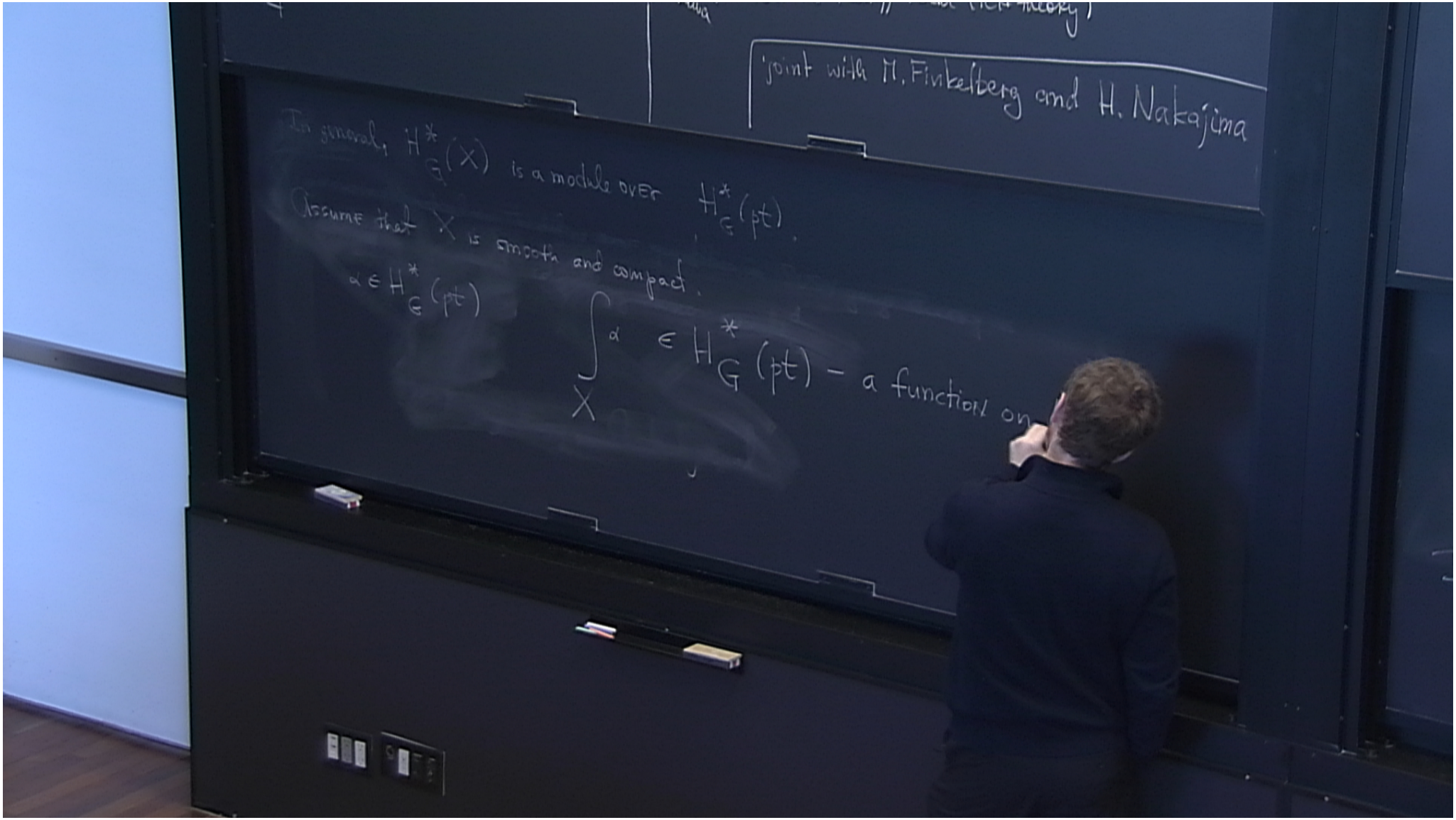

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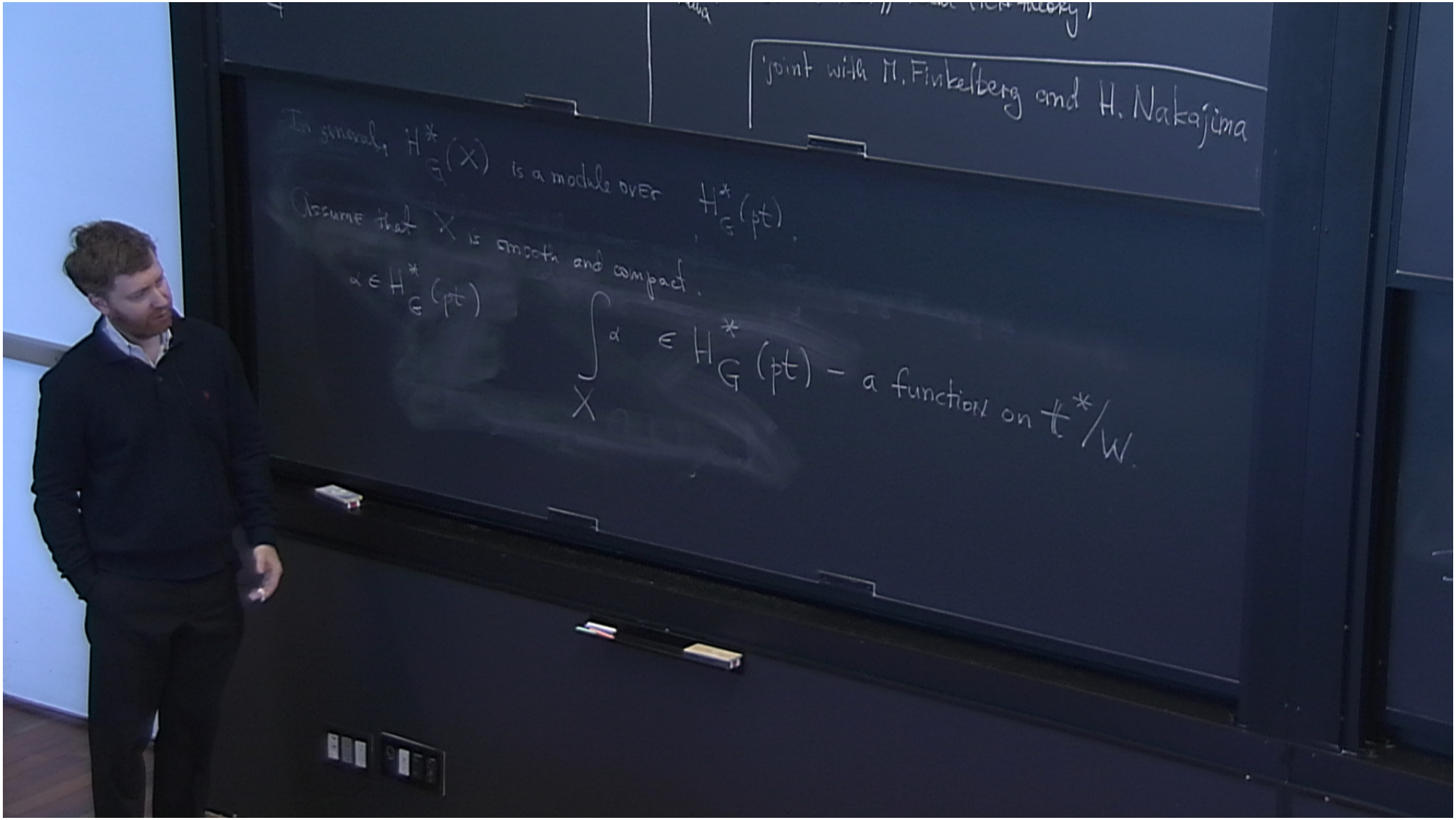

equivariant cohomology
Example
$X = pt$, $G = \mathbb{C}^*$ (or S^1)
$H^*_G(pt) = \mathbb{C}[\varepsilon]$

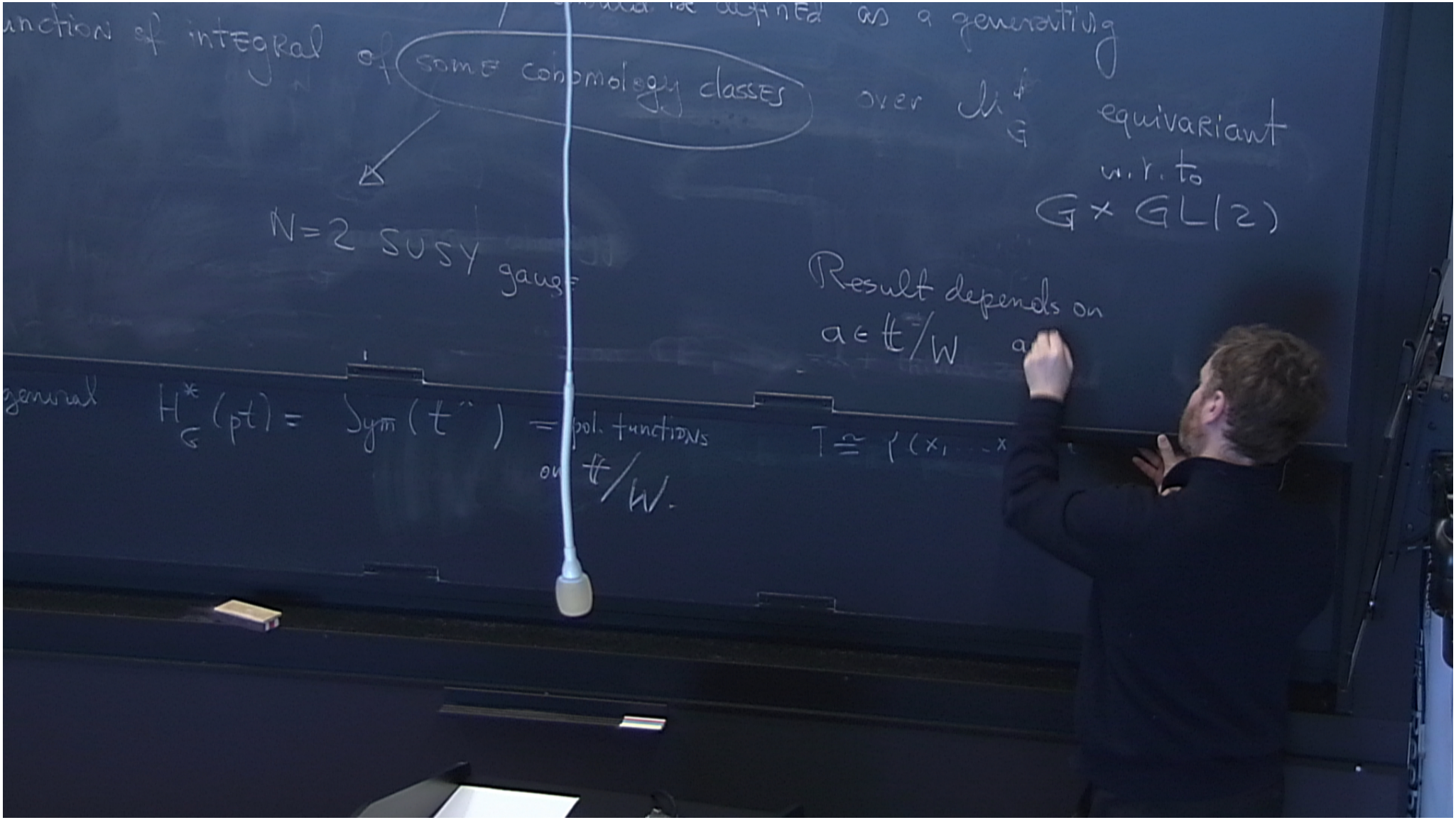

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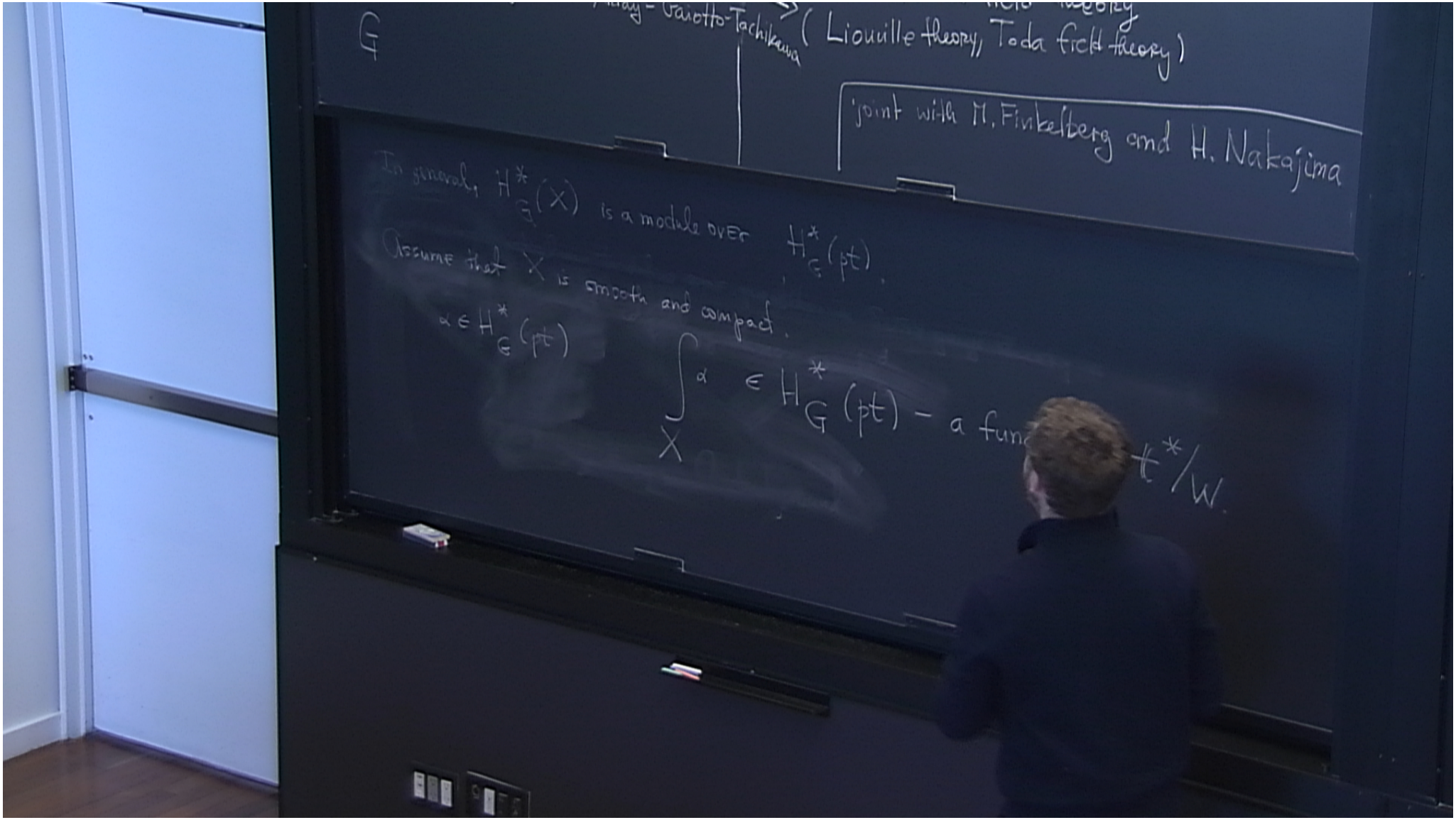

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X = pt , G = ℂ* (or S¹)
H*_G(pt) = ℂ[ε]
deg
T ⊂ G
max torus
𝔱 = Lie T
W = Weyl group
e.g. G = SL(n), W = Sn
T ≅ {(x1 ... xn) ∈ ℂ^n ∏ xi = 1}

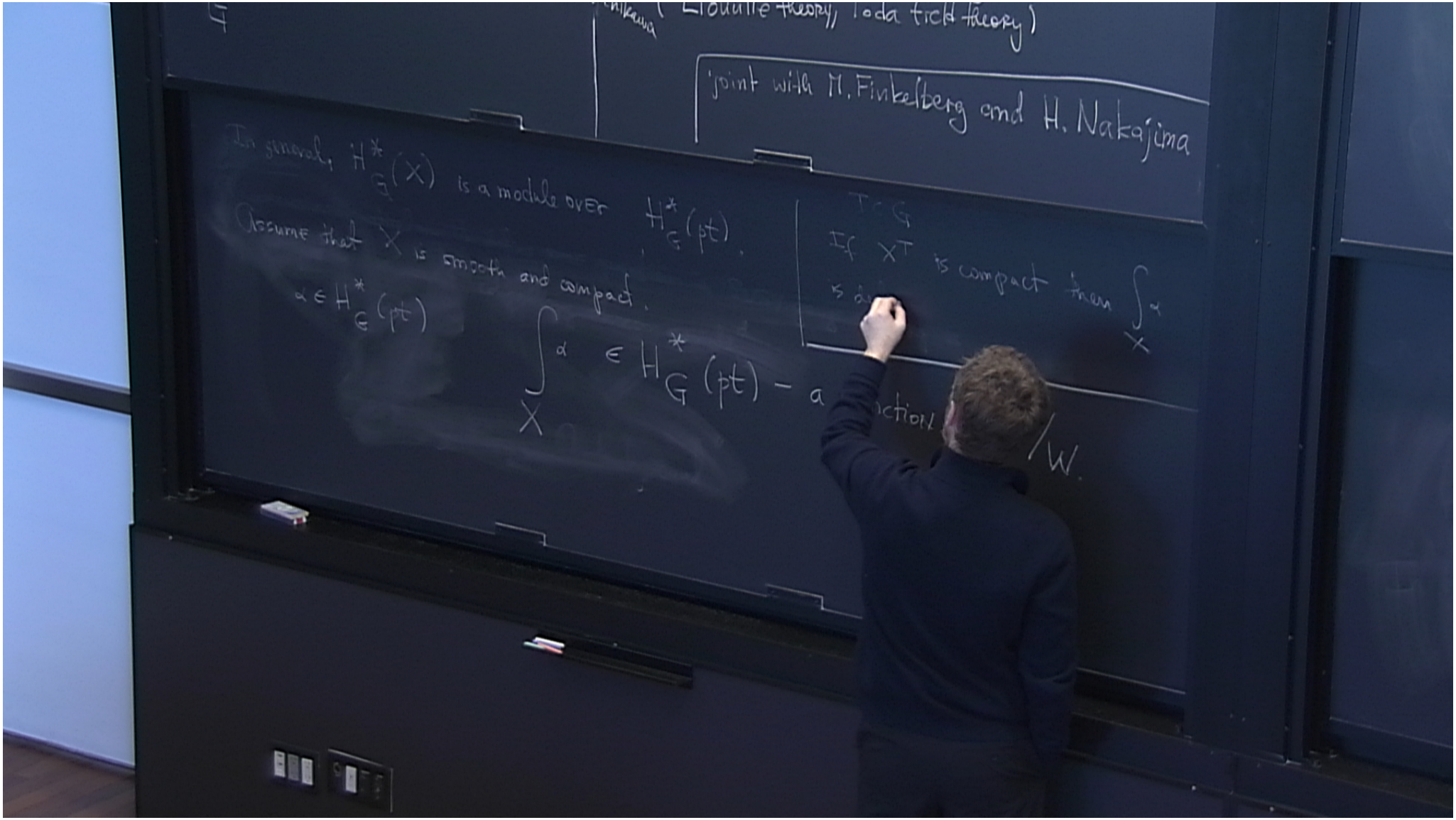

2d conformal field theory
(Liouville theory, Toda field theory)
joint with M. Finkelberg and H. Nakajima
Example.
In general

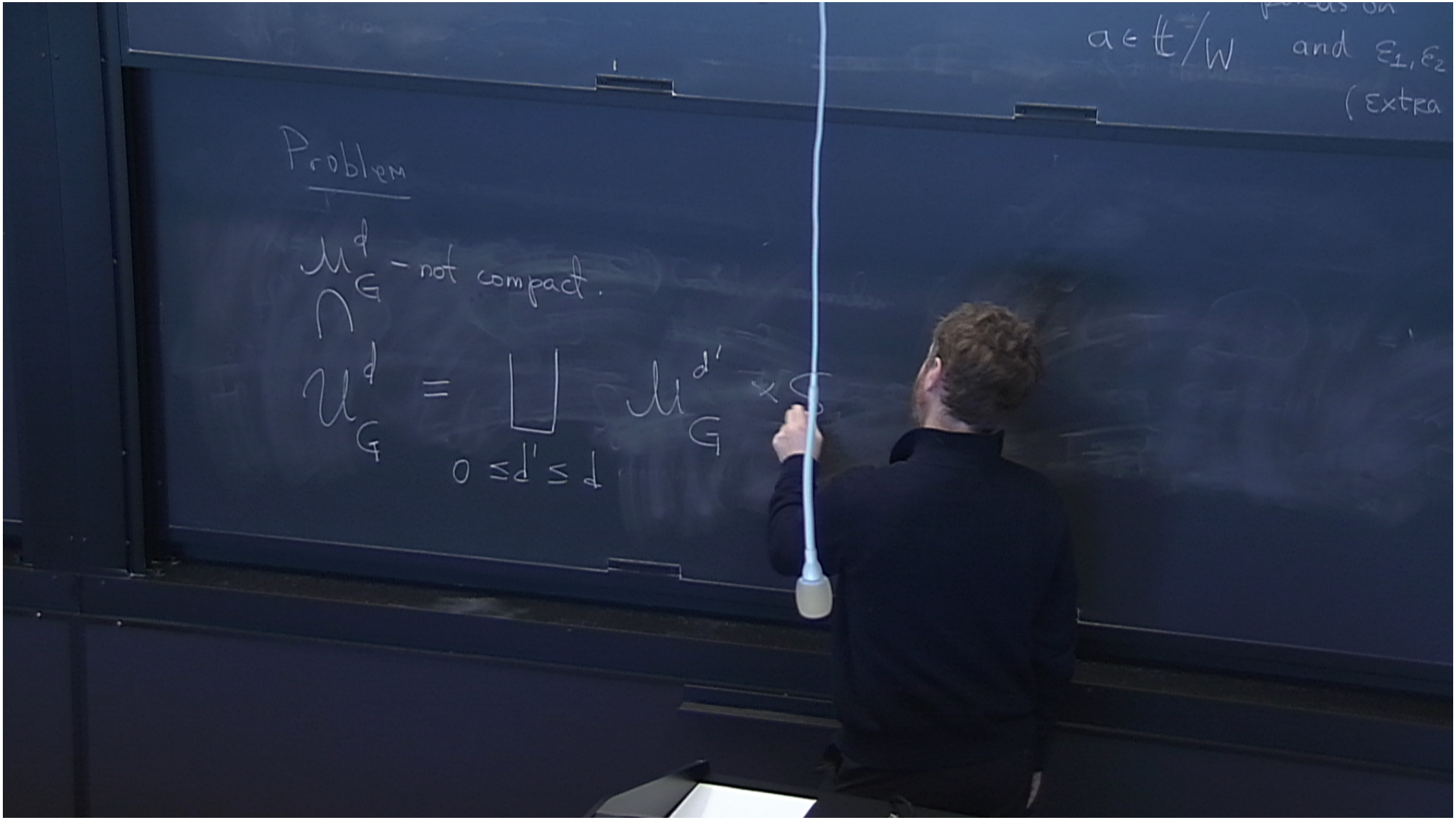

joint with M. Finkelberg and H. Nakajima
In general, $H^*_G(X)$ is a module over $H^*_G(pt)$.
Assume that X is smooth and compact.
$\alpha \in H^*_G(pt)$
$\int_X \alpha \in H^*_G(pt)$ – a function on

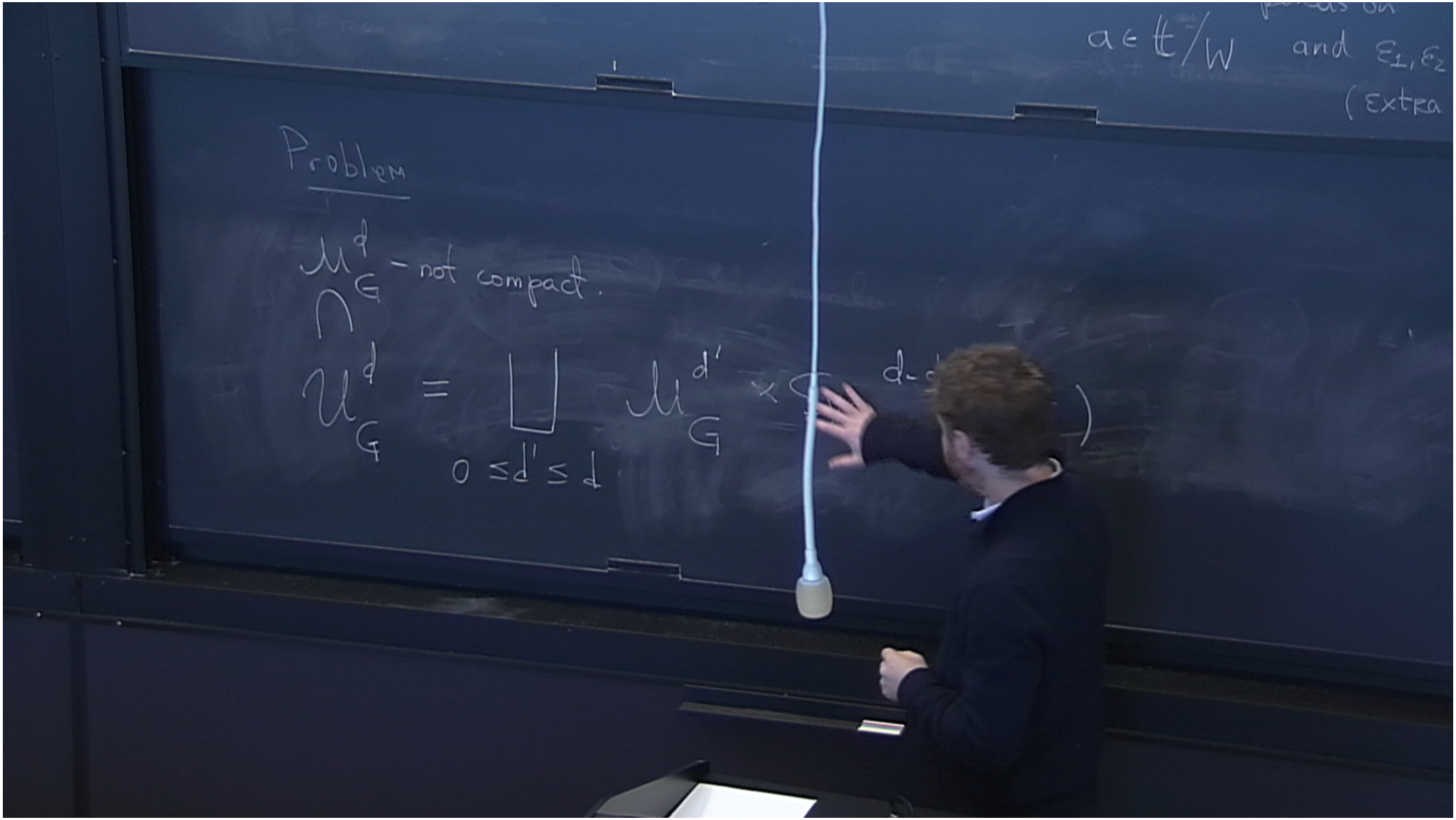

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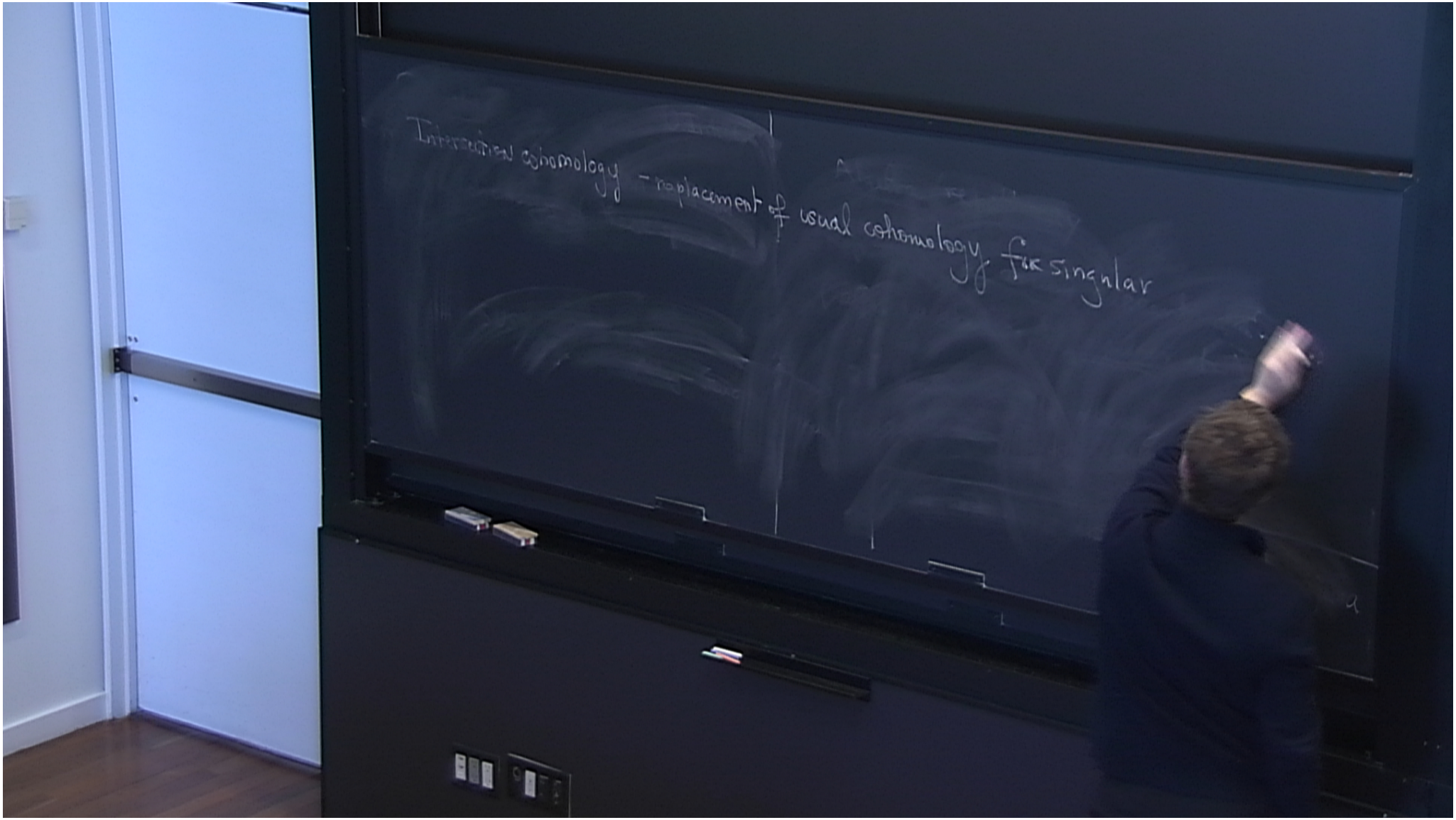

function of integral of some cohomology classes over $\mathcal{M}^{\ddagger}_{G}$ equivariant w.r.to $G \times GL(2)$
N=2 SUSY gauge
Result depends on
$a \in t/W$
general $H^{*}_{G}(pt) = Sym(t^{\vee}) =$ pol. functions on t/W.
$T \cong$

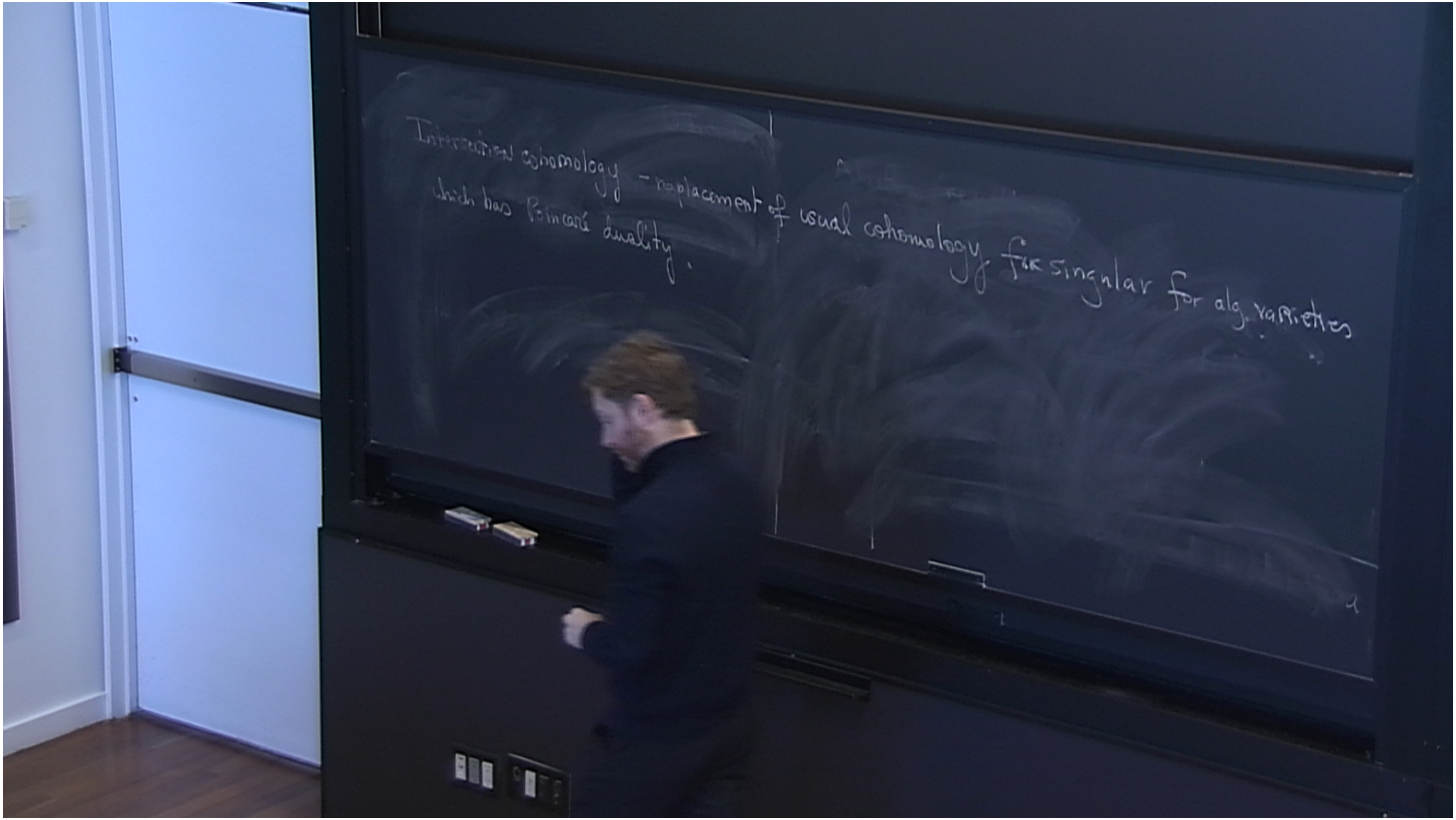

G
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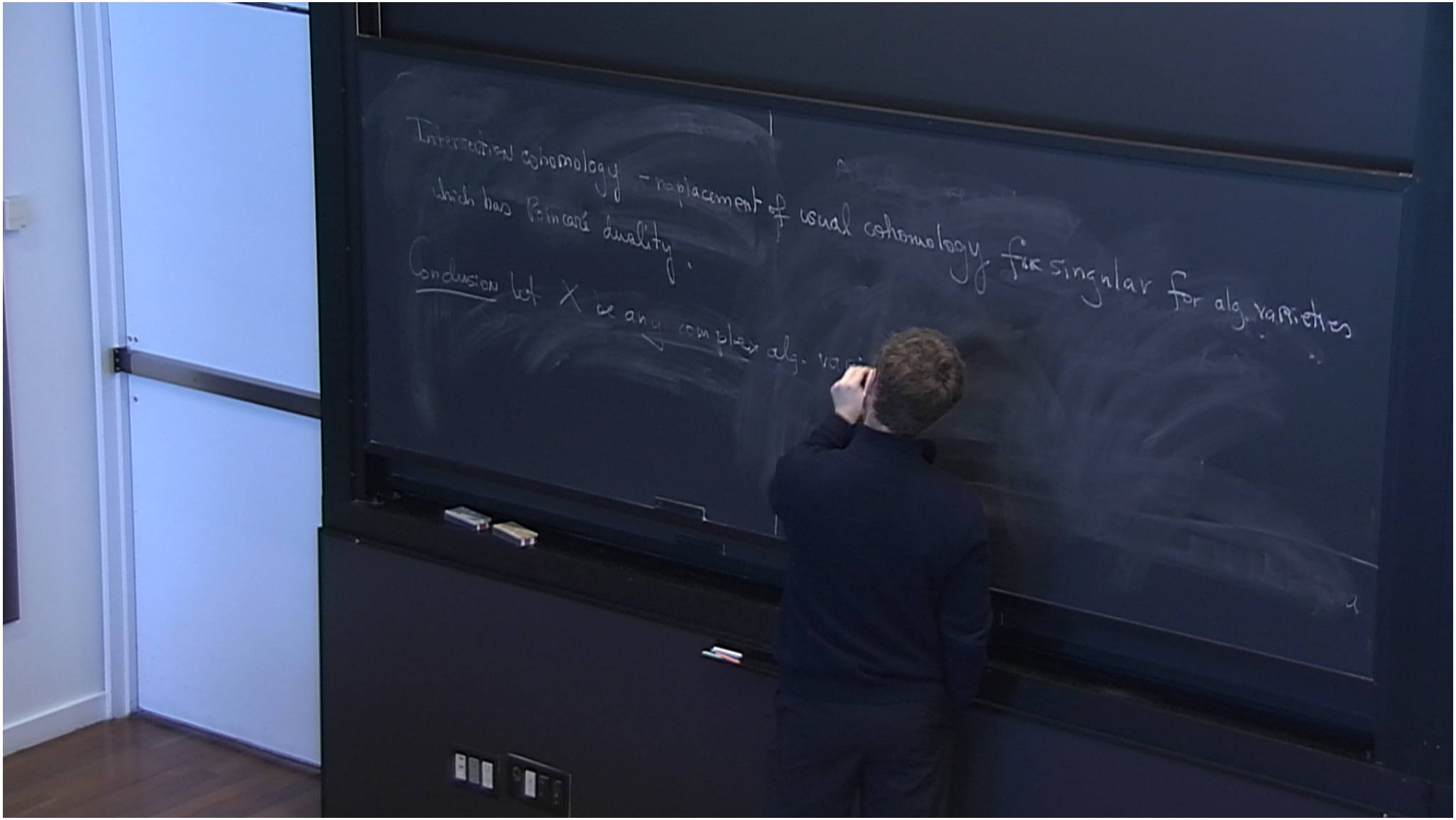

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$\int_X \alpha \in H^*_G(pt)$ – a
If X^T is compact then $\int_X \alpha$
/W.

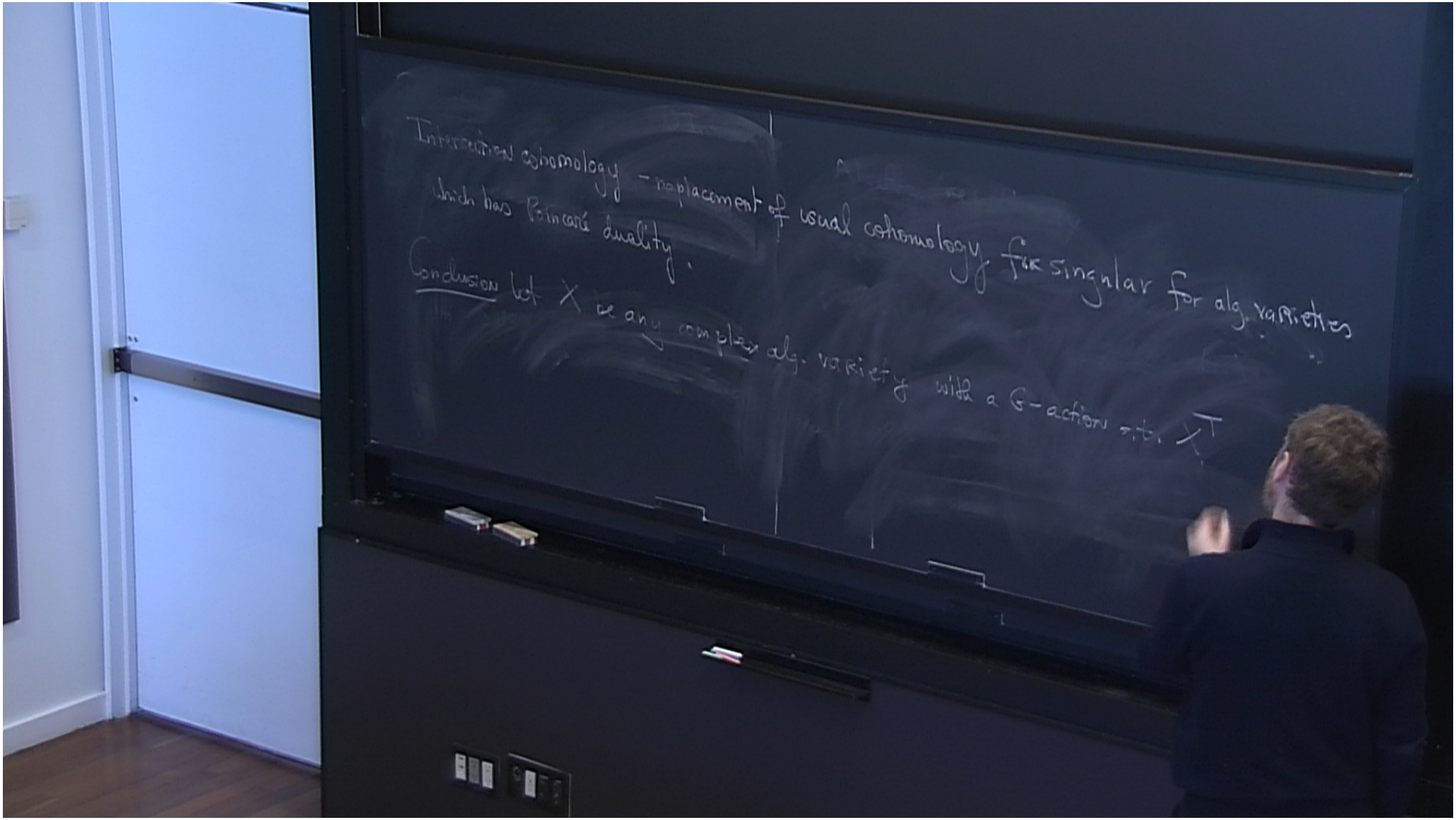

a ∈ t/W
and ε1, ε2
(Extra
Problem
$\mathcal{M}_G^d$ - not compact.
$\cap$
$\mathcal{U}_G^d = \bigsqcup_{0 \le d' \le d} \mathcal{M}_G^{d'} \times S$

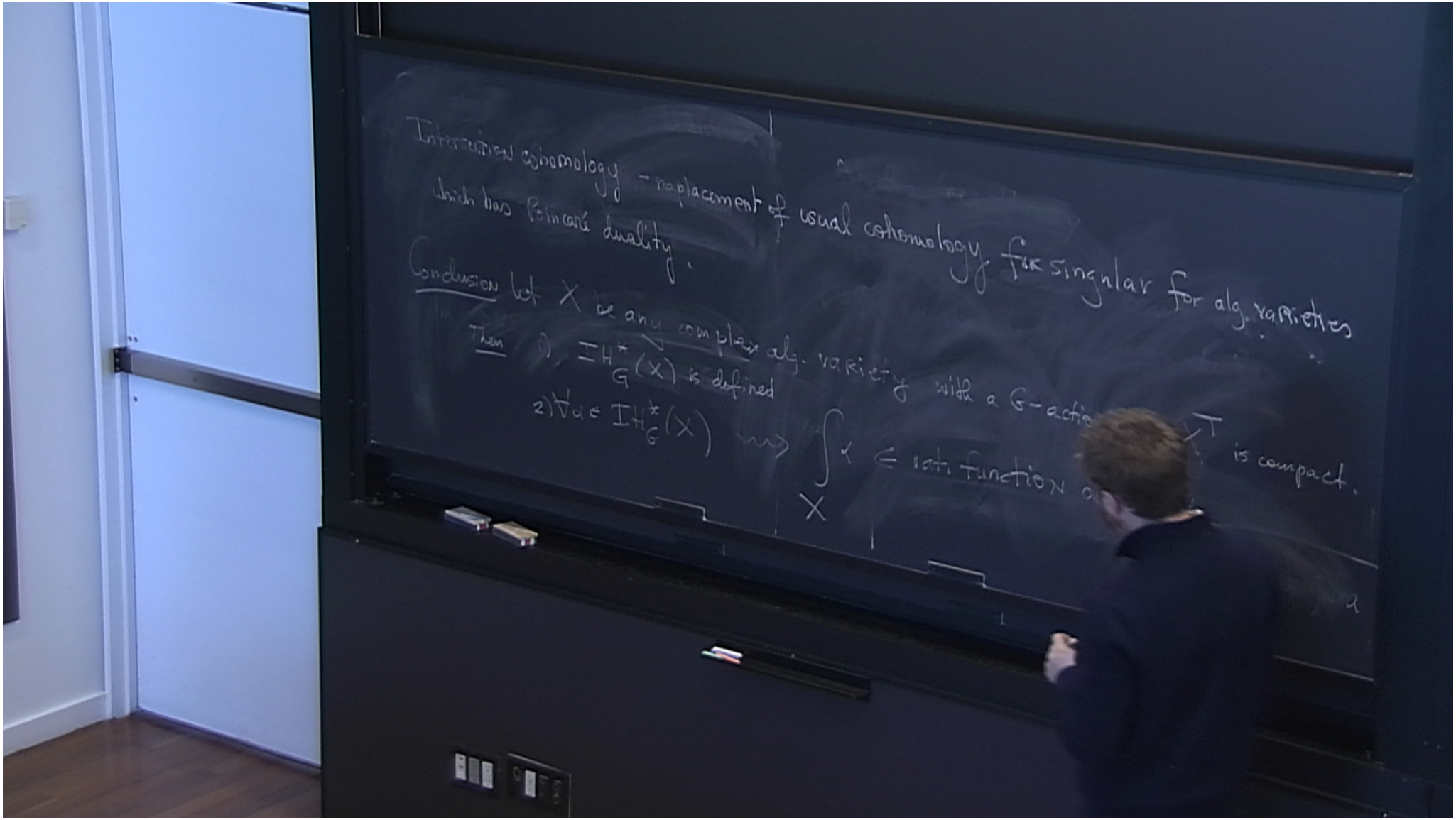

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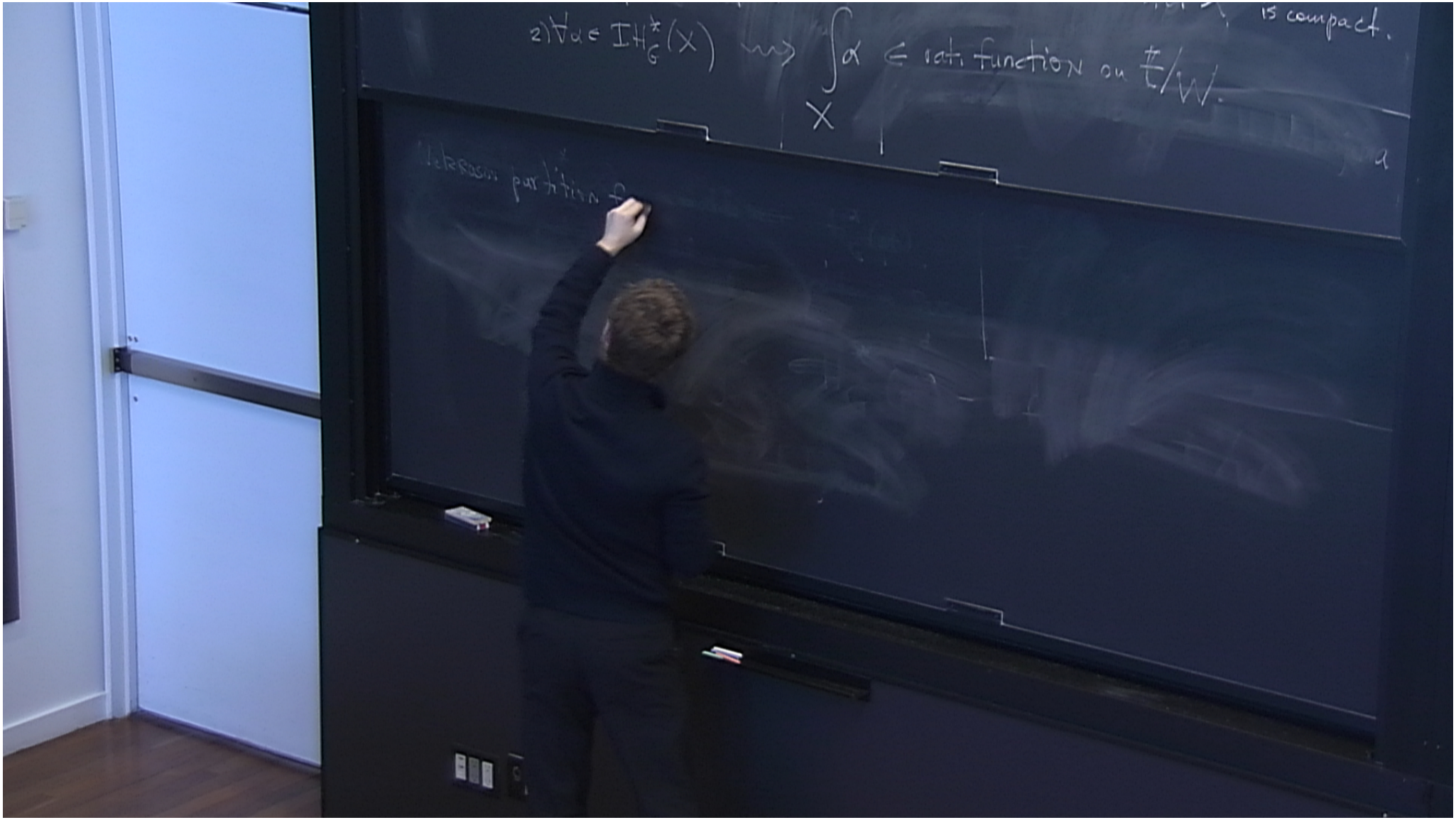

Intersection cohomology - replacement of usual cohomology for singular

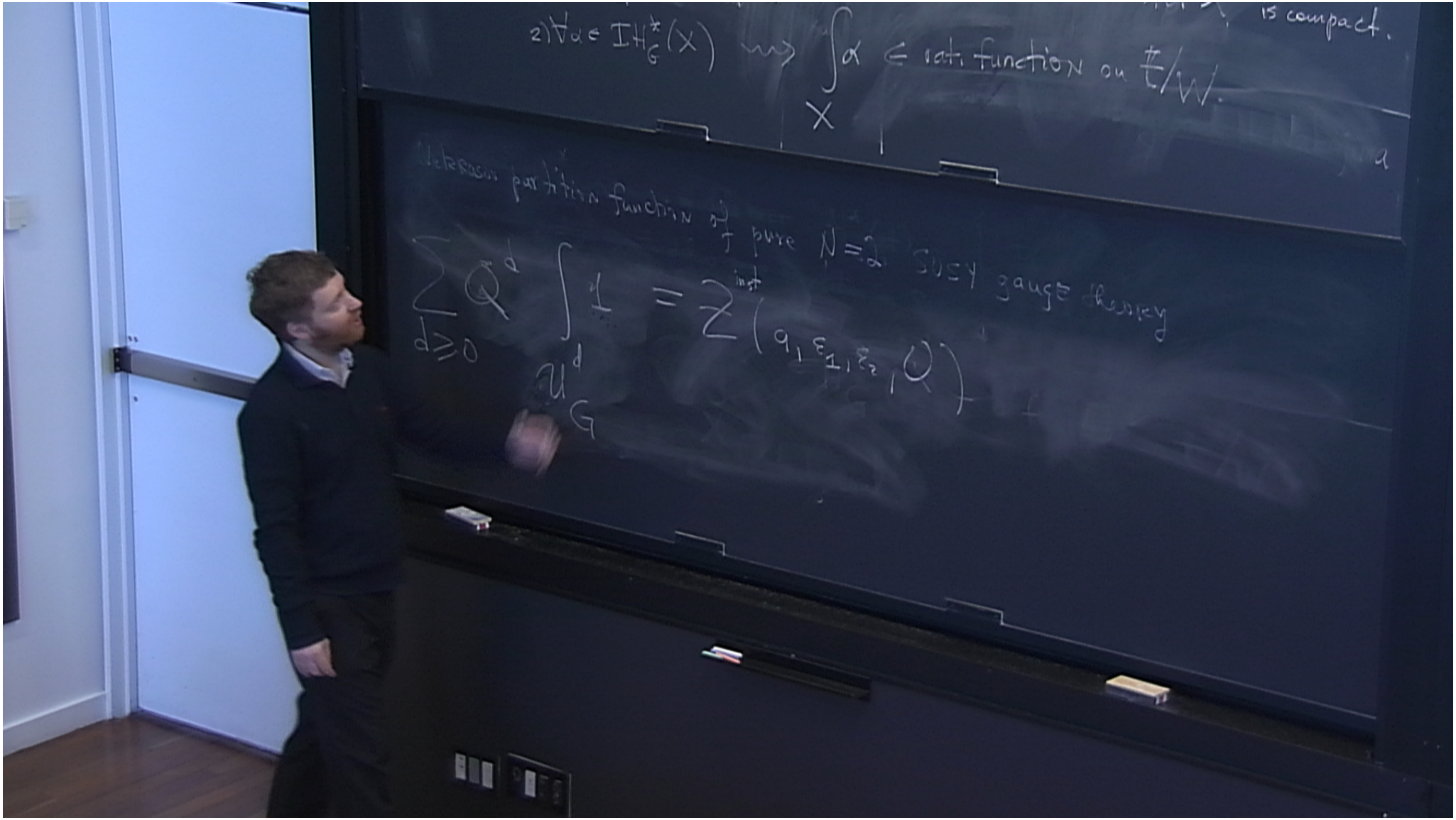

Intersection cohomology - replacement of usual cohomology for singular for alg. varieties
which has Poincaré duality.

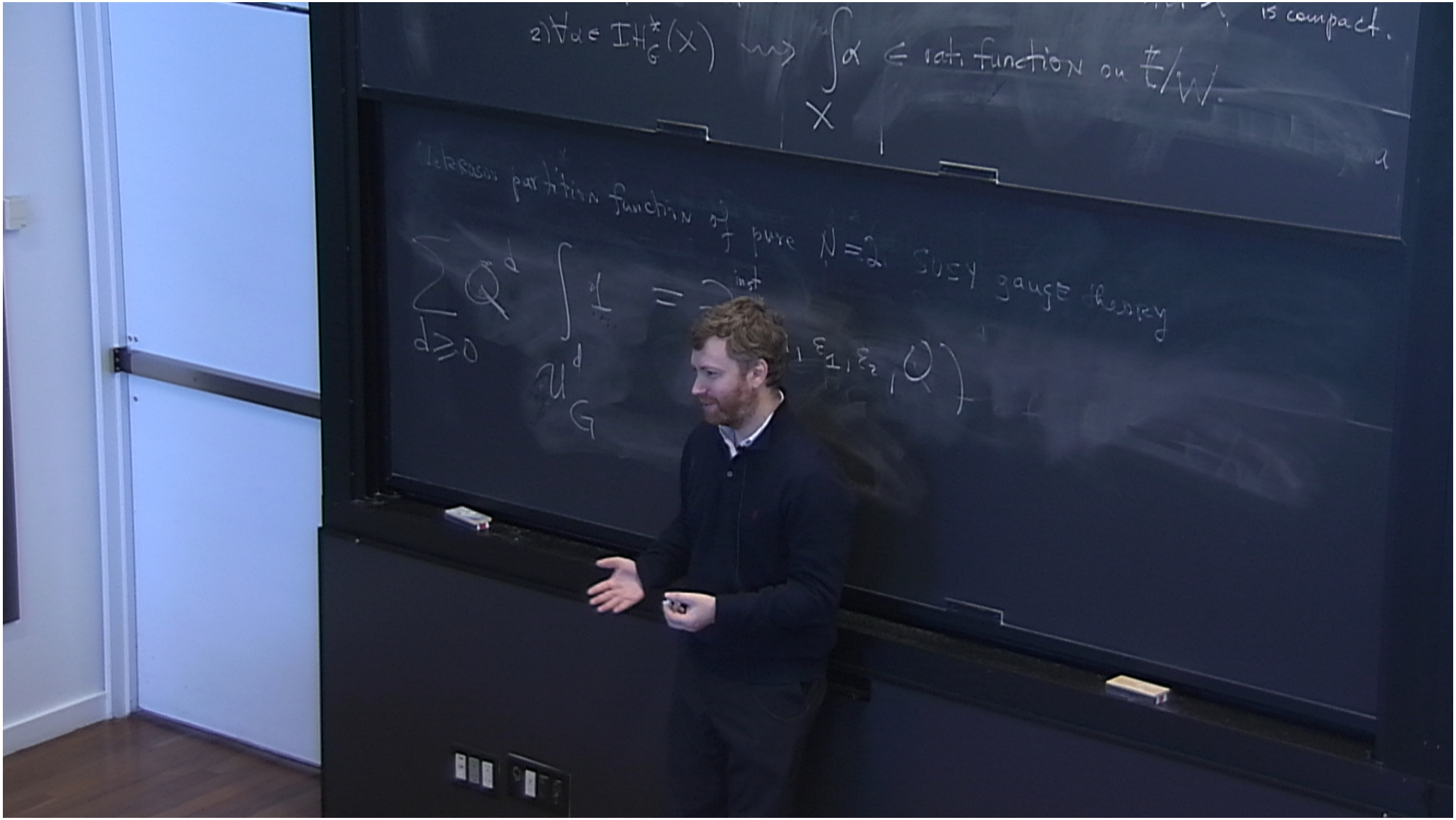

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Conclusion let X be any complex alg.

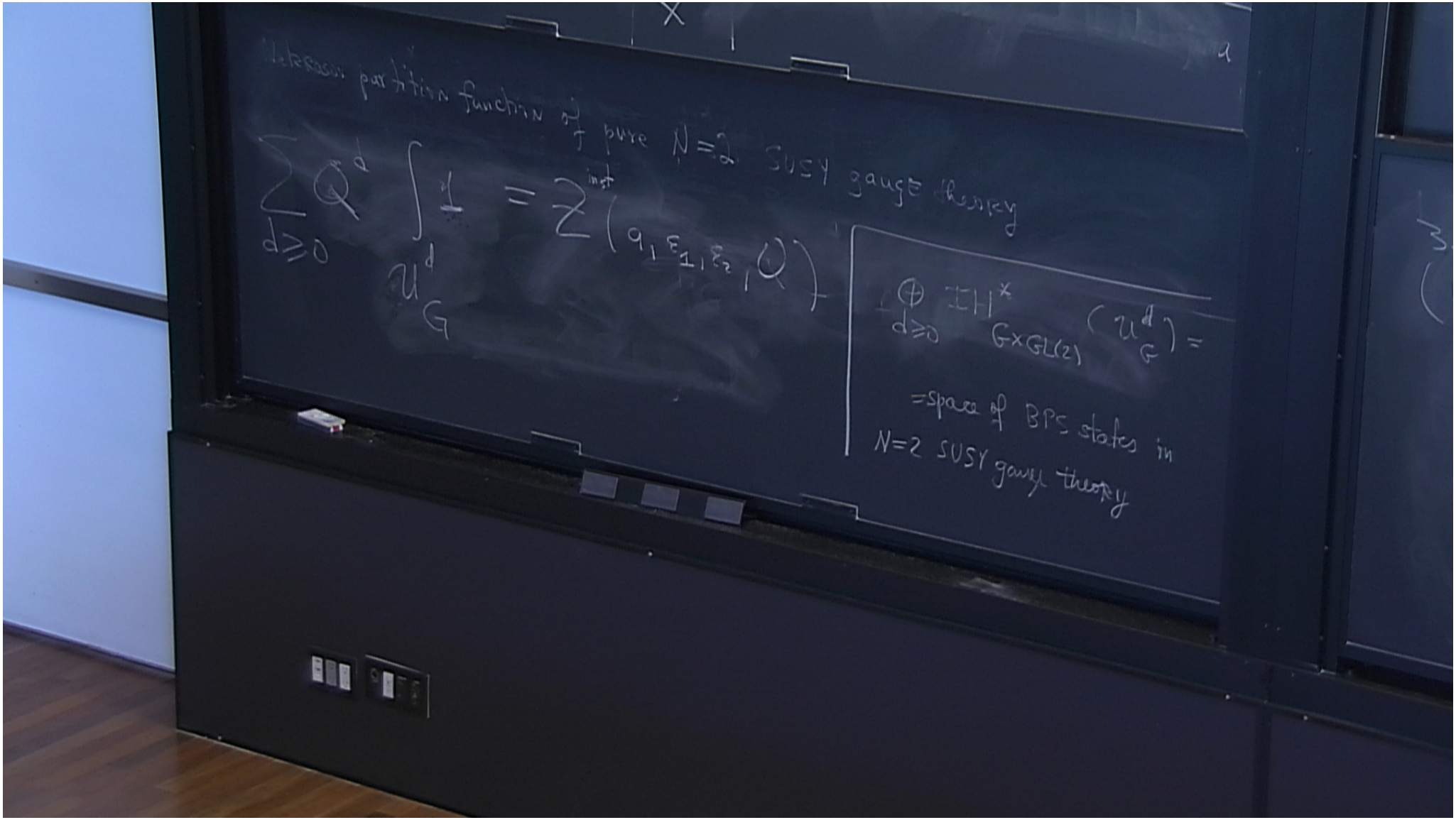

Intersection cohomology - replacement of usual cohomology for singular for alg. varieties
which has Poincaré duality.
Conclusion let X be any complex alg. variety with a G-action s.t.

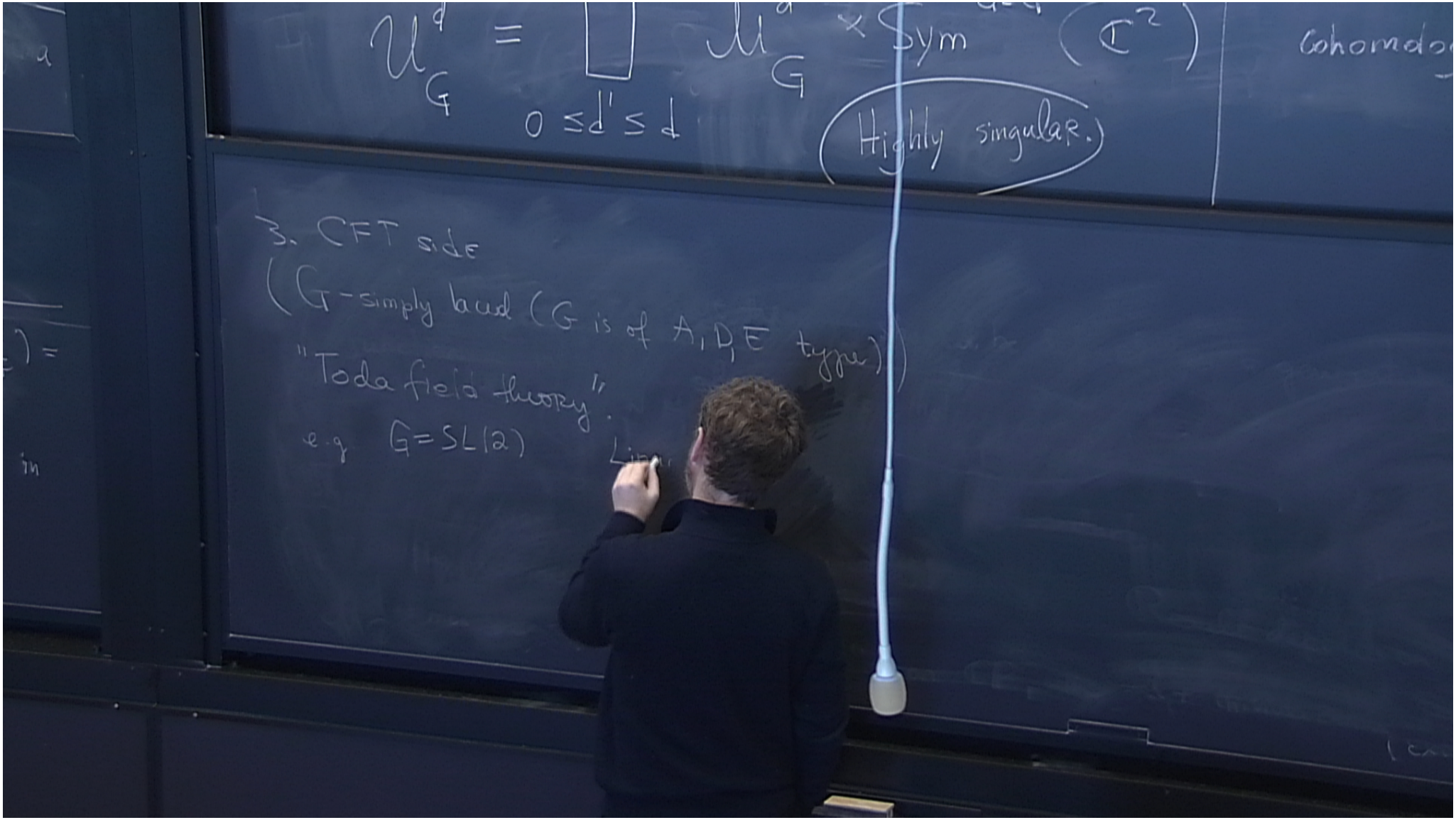

Intersection cohomology - replacement of usual cohomology for singular for alg. varieties
which has Poincaré duality.
Conclusion let X be an
Then 1) $IH^*_G(X)$ is defined
2) $\forall \alpha \in IH^*_G(X)$
is compact.

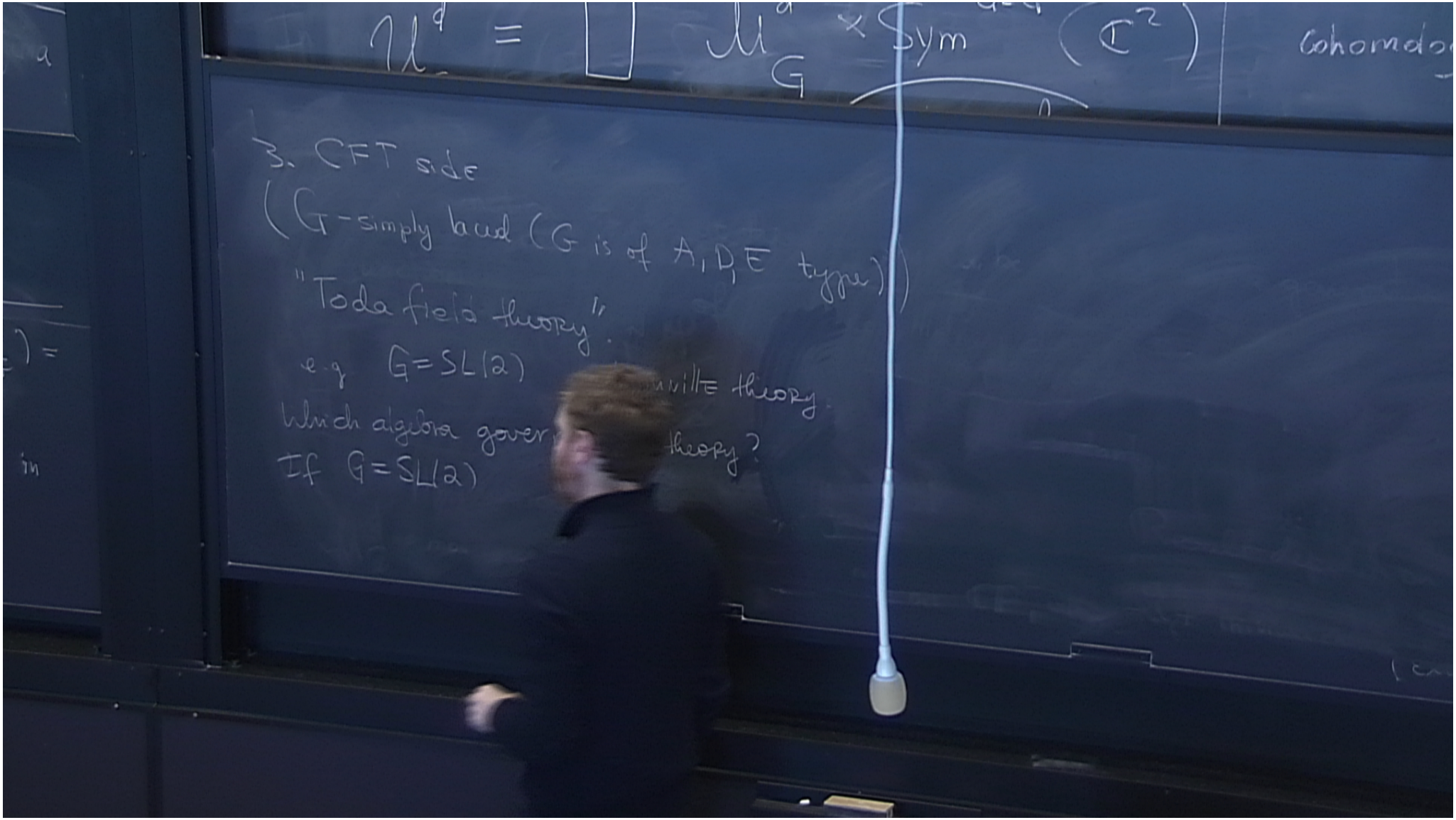

2) ∀α ∈ $IH^*_G(X)$ ⇝ $\int_X \alpha$ ∈ rat. function on $\mathfrak{t}/W$.
is compact.

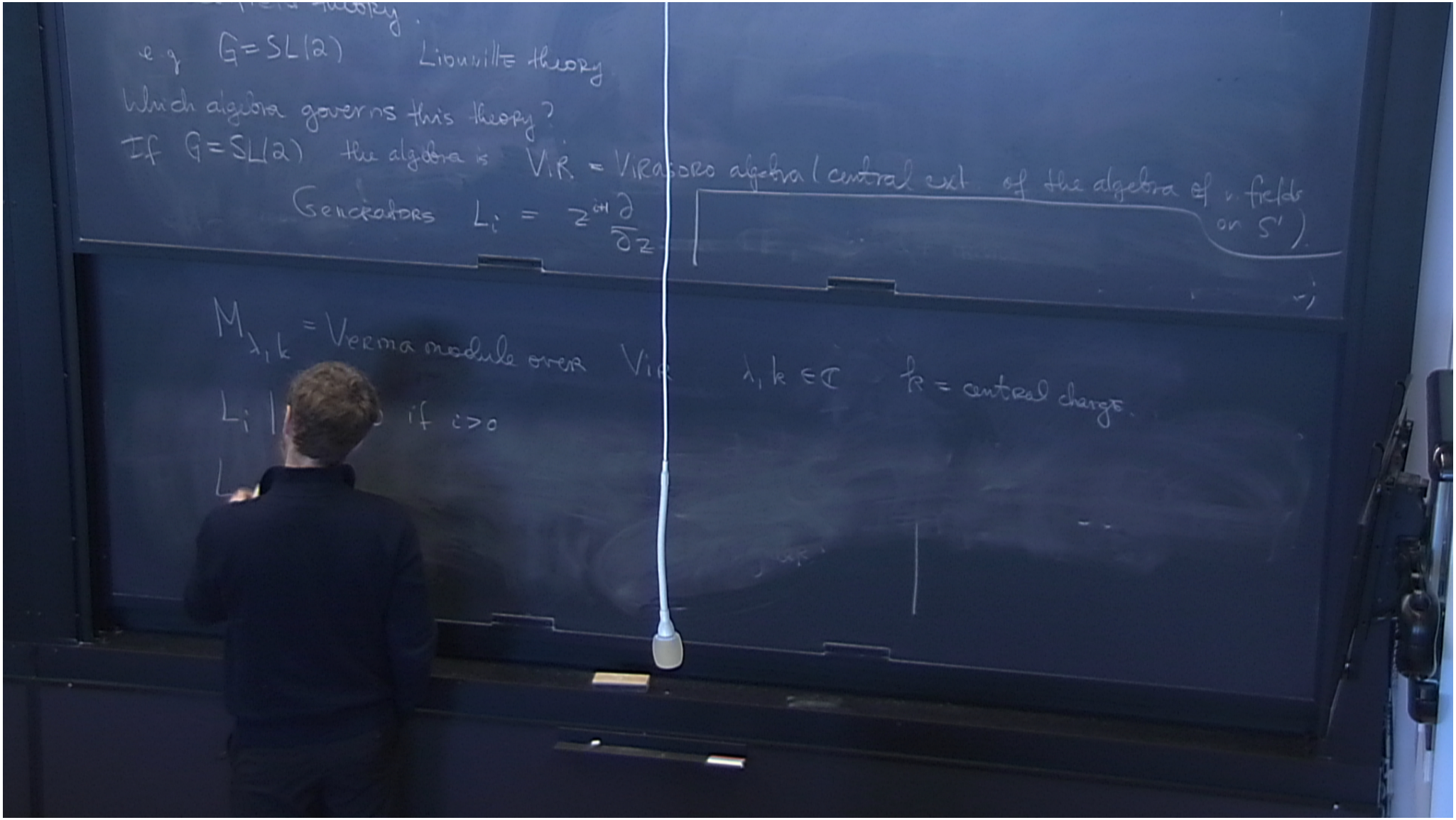

2) $\forall a \in \mathbb{I}H^*_G(X) \rightsquigarrow \int_X \alpha \in$ rat. function on $\mathfrak{t}/W$.
is compact.
Nekrasov partition function $\mathcal{Z}$ pure $N=2$ SUSY gauge theory
$\sum_{d \geq 0} Q^d \int_{\mathcal{U}^d_G} 1 = \mathcal{Z}^{inst}(q, \varepsilon_1, \varepsilon_2, a)$


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Nekrasov partition function Z_T pure N=2 susy gauge theory
Σ_{d≥0} Q^d ∫_{U^d_G} 1 = Z^inst(ε_1, ε_2, Q)


Instanton partition function of pure N=2 SUSY gauge theory
$\sum_{d\geq 0} Q^d \int_{\mathcal{U}_G^d} 1 = Z^{inst}(q, \varepsilon_1, \varepsilon_2, Q)$
$\bigoplus_{d\geq 0} H^*_{G\times GL(2)}(\mathcal{U}_G^d) =$
=space of BPS states in N=2 SUSY gauge theory


$\mathcal{U}_G^q = \square$
$0 \le d' \le d$
$\mathcal{U}_G^q \times$ Sym $(\mathbb{C}^2)$
Highly singular.
Cohomolo
3. CFT side
(G-simply laced (G is of A, D, E type))
"Toda field theory".
e.g. G=SL(2)


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(G - simply laced (G is of A, D, E type))
"Toda field theory".
e.g. G = SL(2)
Which algebra gover
If G = SL(2)
theory?


e.g. G=SL(2) Liouville theory
Which algebra governs this theory?
If G=SL(2) the algebra is Vir = Virasoro algebra (central ext of the algebra of v. fields on S¹)
Generators $L_i = z^{i+1}\frac{\partial}{\partial z}$
$M_{\lambda,k}$ = Verma module over Vir
$\lambda, k \in \mathbb{C}$
k = central charge.
L_i | if $i>0$

e.g. $G=SL(2)$ Liouville theory

Which algebra governs this theory?

If $G=SL(2)$ the algebra is $V.R.$ = Virasoro algebra (central ext of the algebra of v. fields on S^1)

Generators $L_i = z^{i+1}\frac{\partial}{\partial z}$

$M_{\lambda,k}$ = Verma module over $V.R.$ $\lambda, k \in \mathbb{C}$ k = central charge.

$L_i|\lambda\rangle = 0$ if $i>0$

$L_0|\lambda\rangle = \lambda|\lambda\rangle$

e.g. $G = SL(2)$ Liouville theory

Which algebra governs this theory?

If $G = SL(2)$ the algebra is Vir = Virasoro algebra (central ext of the algebra of $\cdot$ fields on S^1)

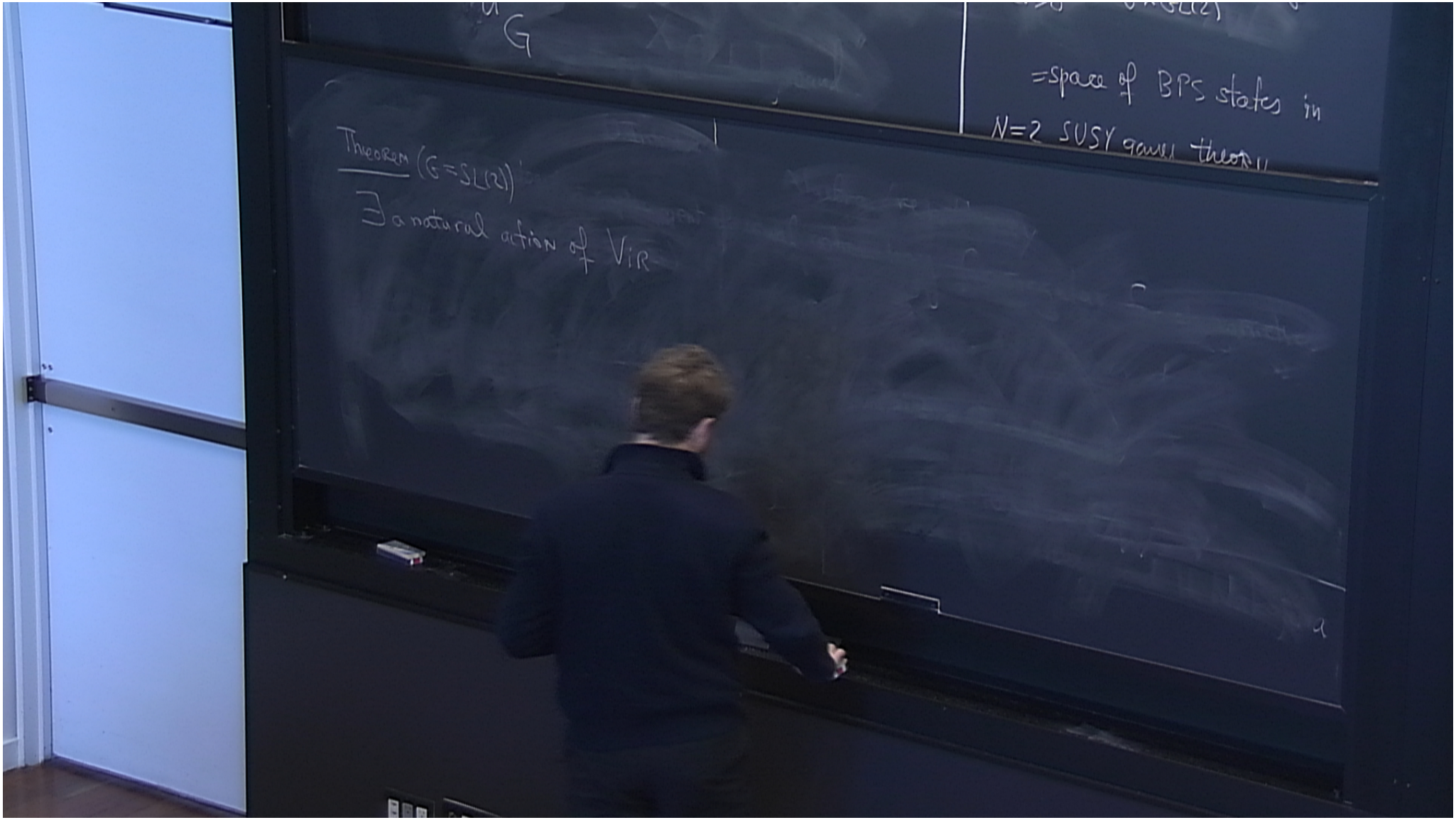
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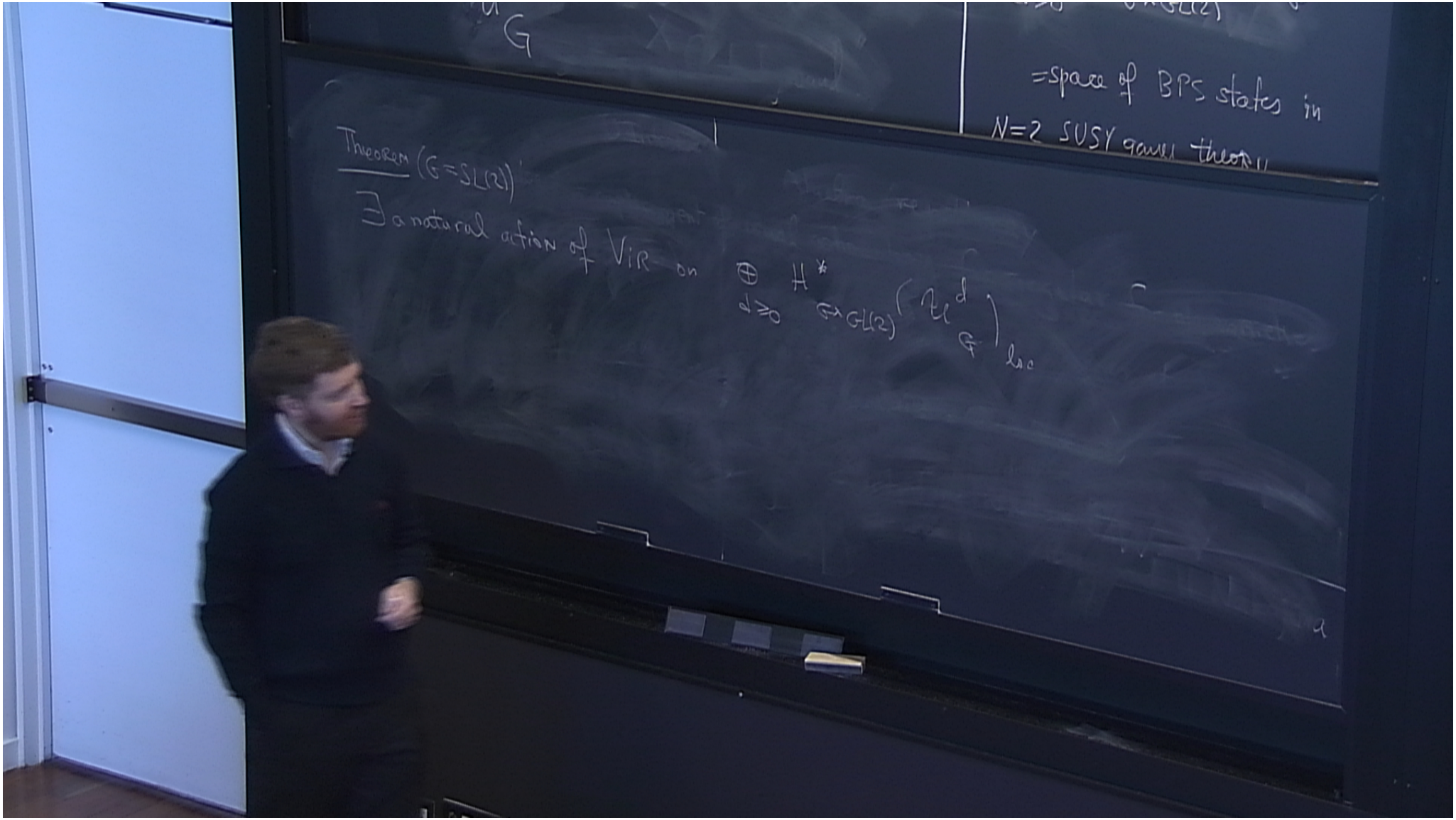
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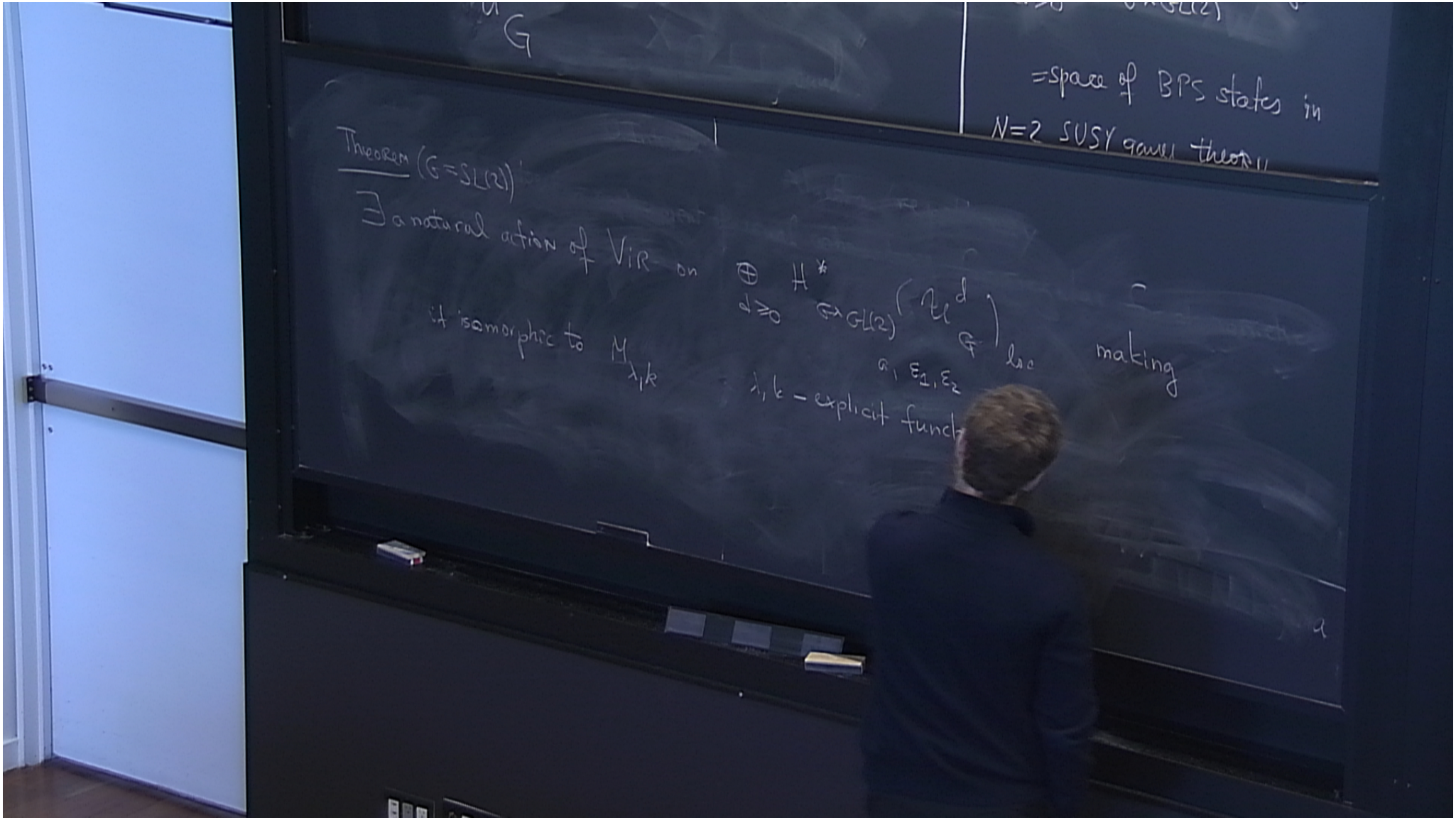
$L_i |\lambda\rangle = 0$ if $i > 0$

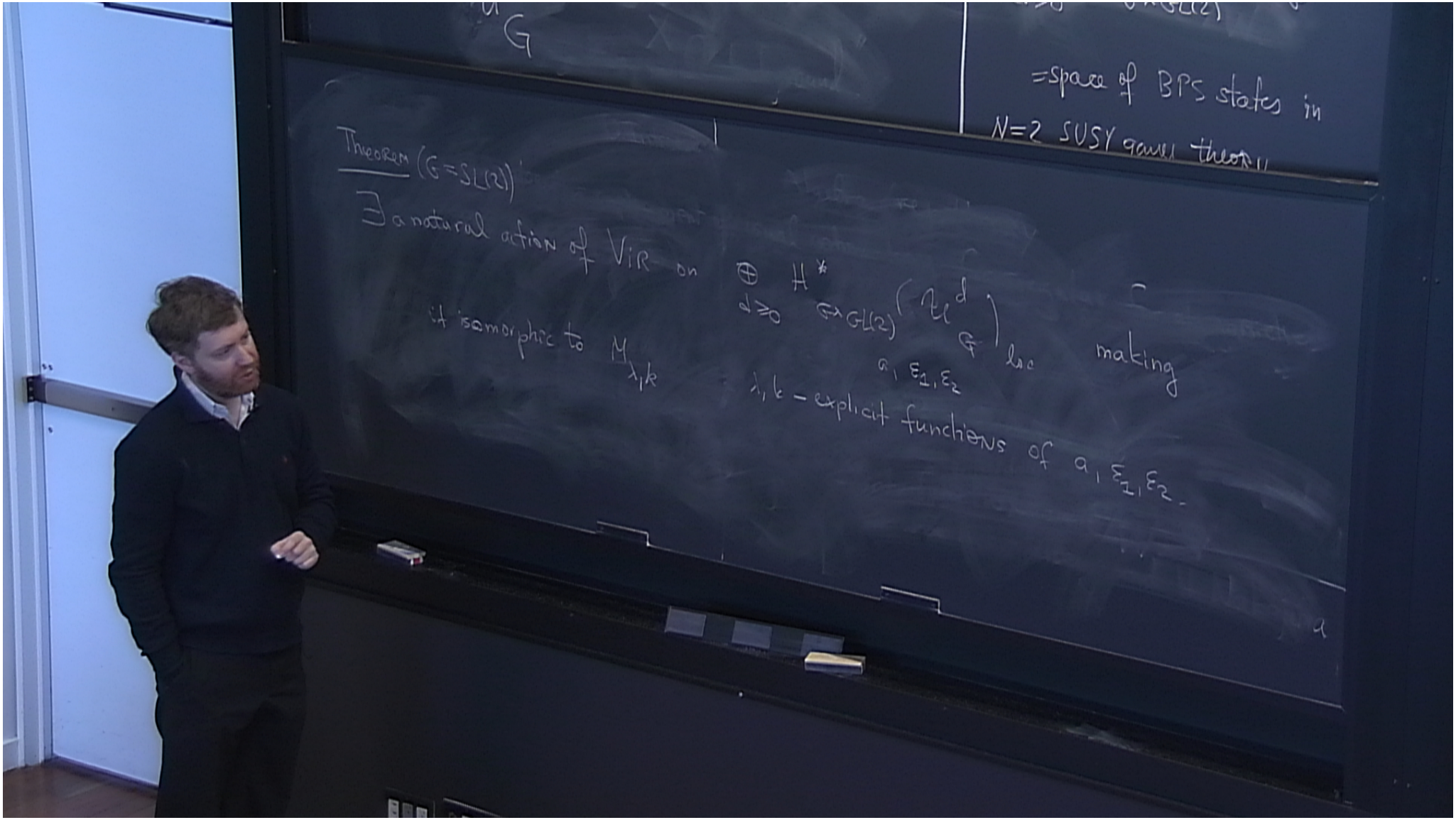
$L_0 |\lambda\rangle = \lambda \cdot |\lambda\rangle$

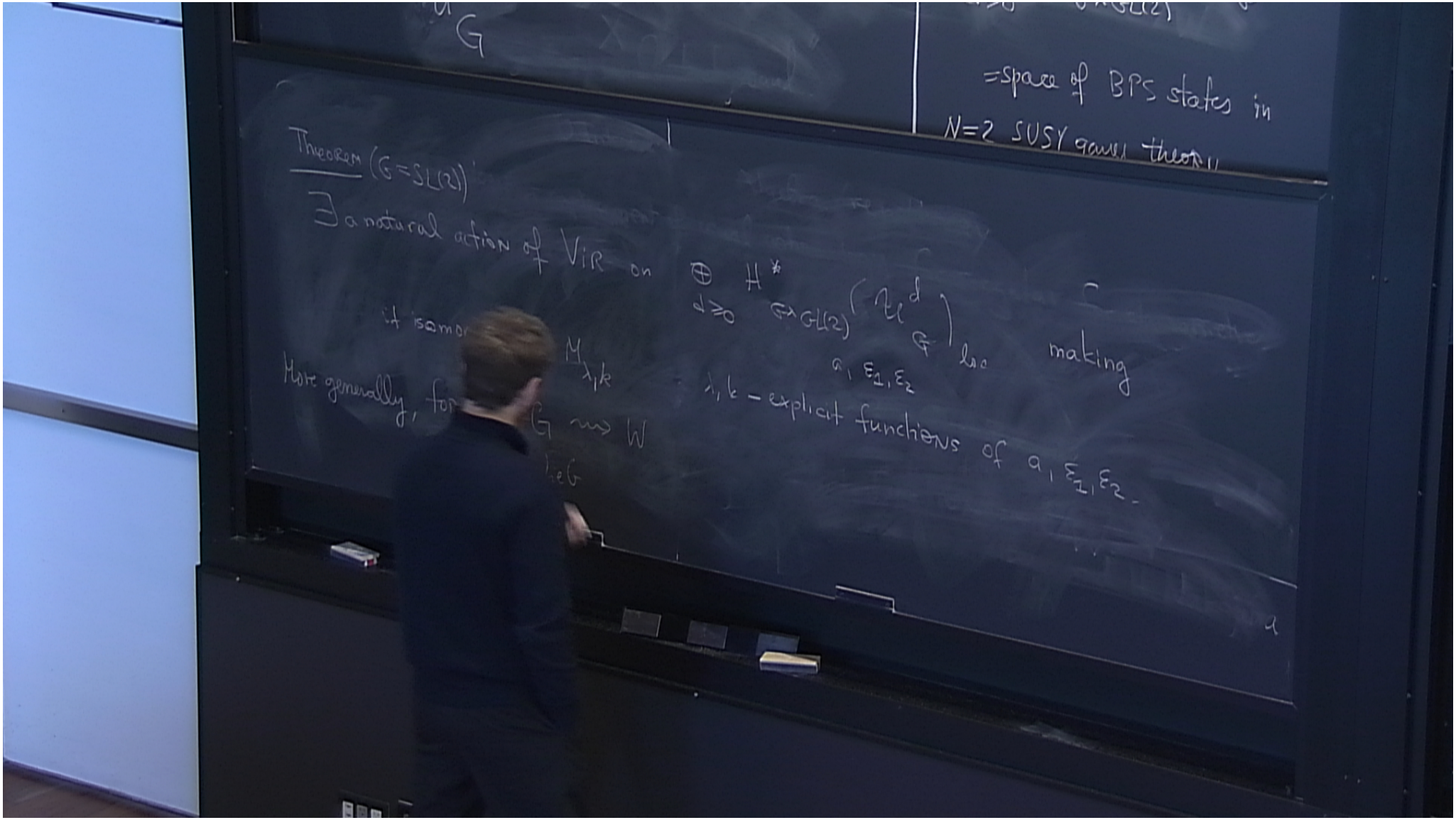
Want: Compare $M_{\lambda,k}$ with $\bigoplus_{d \geq 0}$

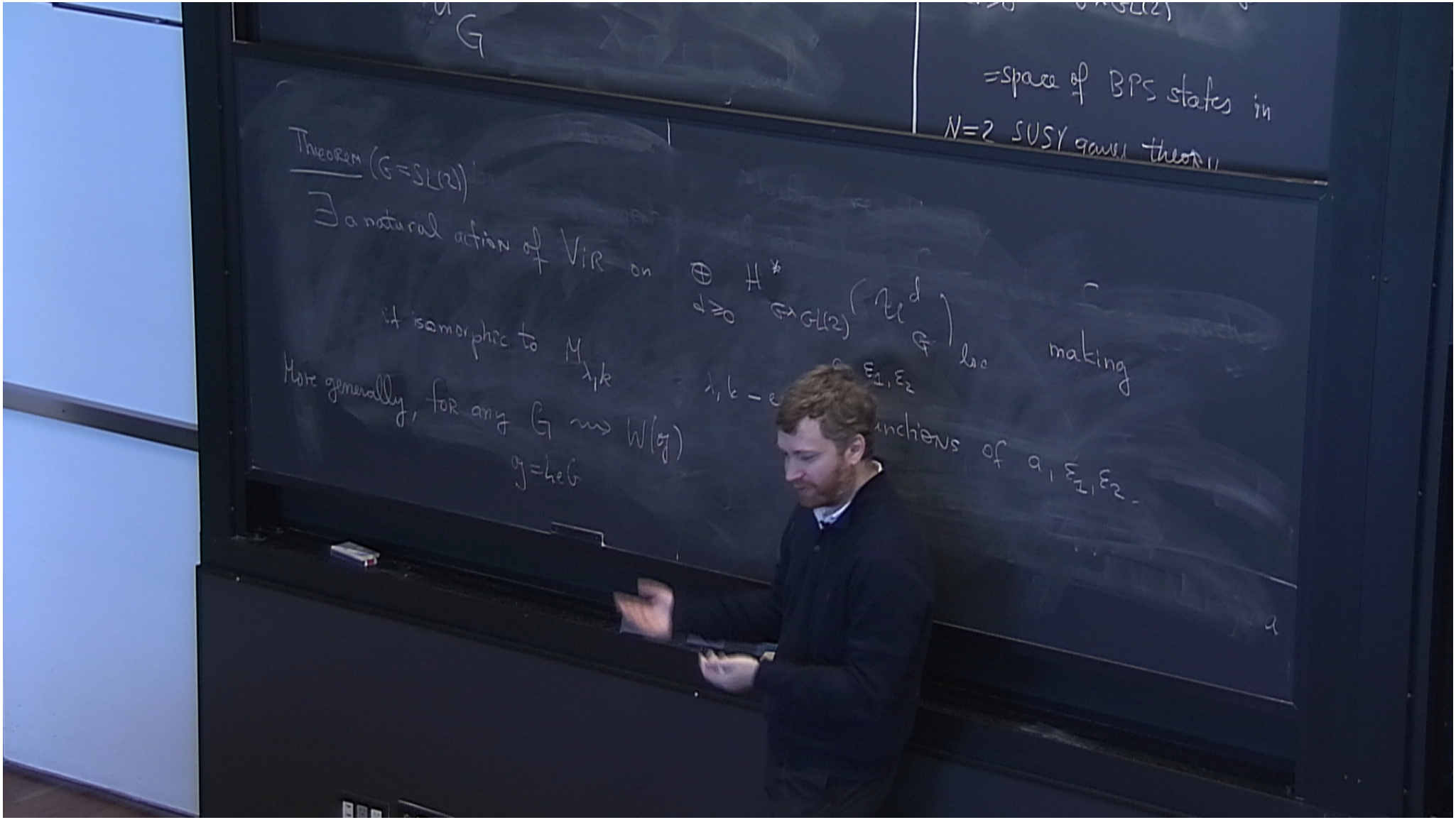

G
=space of BPS states in
N=2 SUSY gauge theory
THEOREM (G=SL(2))
∃ a natural action of Vir

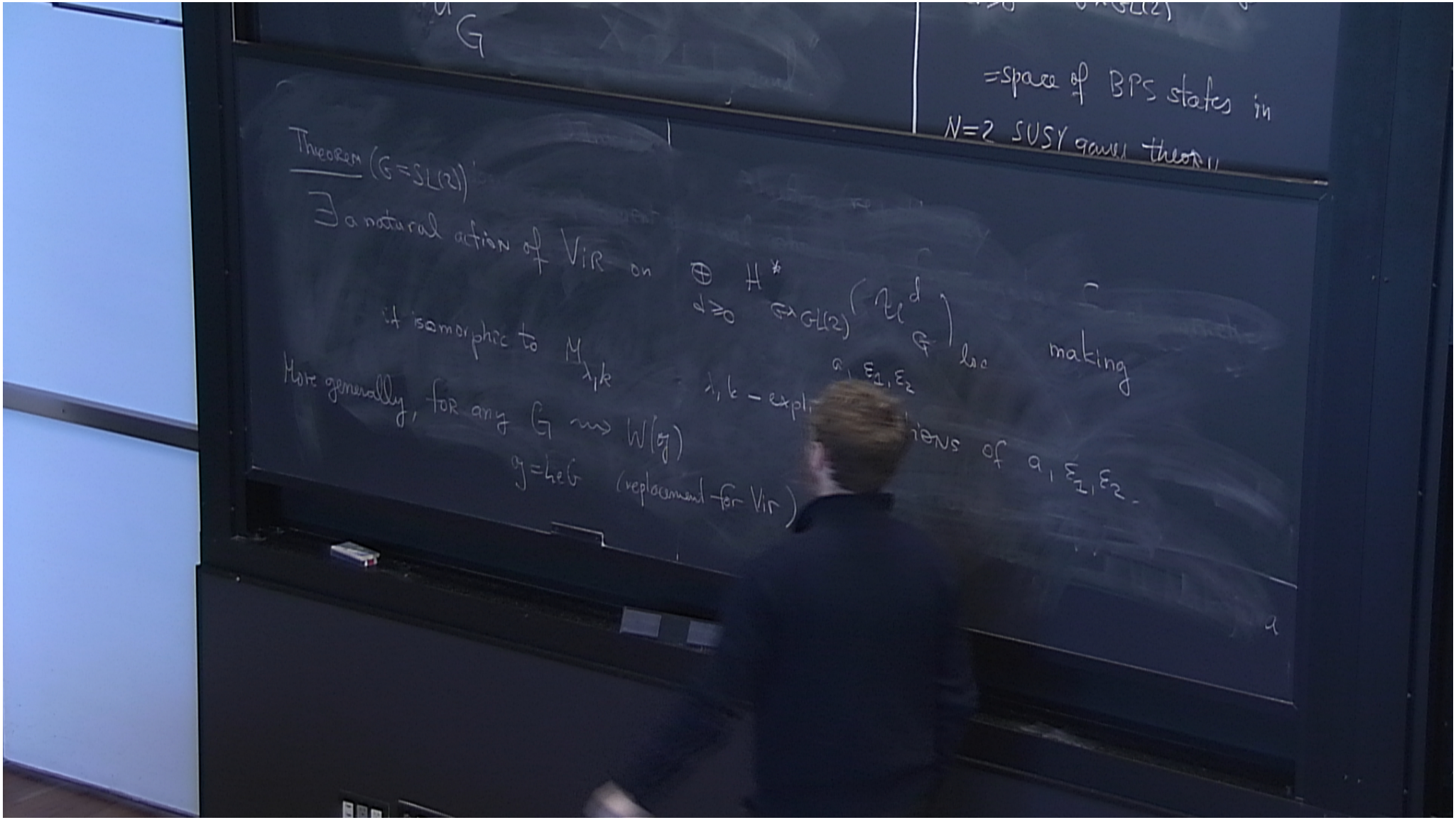

G
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N=2 SUSY gauge theory
THEOREM (G=SL(2))
∃ a natural action of Vir on ⊕ H*
d≥0 G^SL(2)

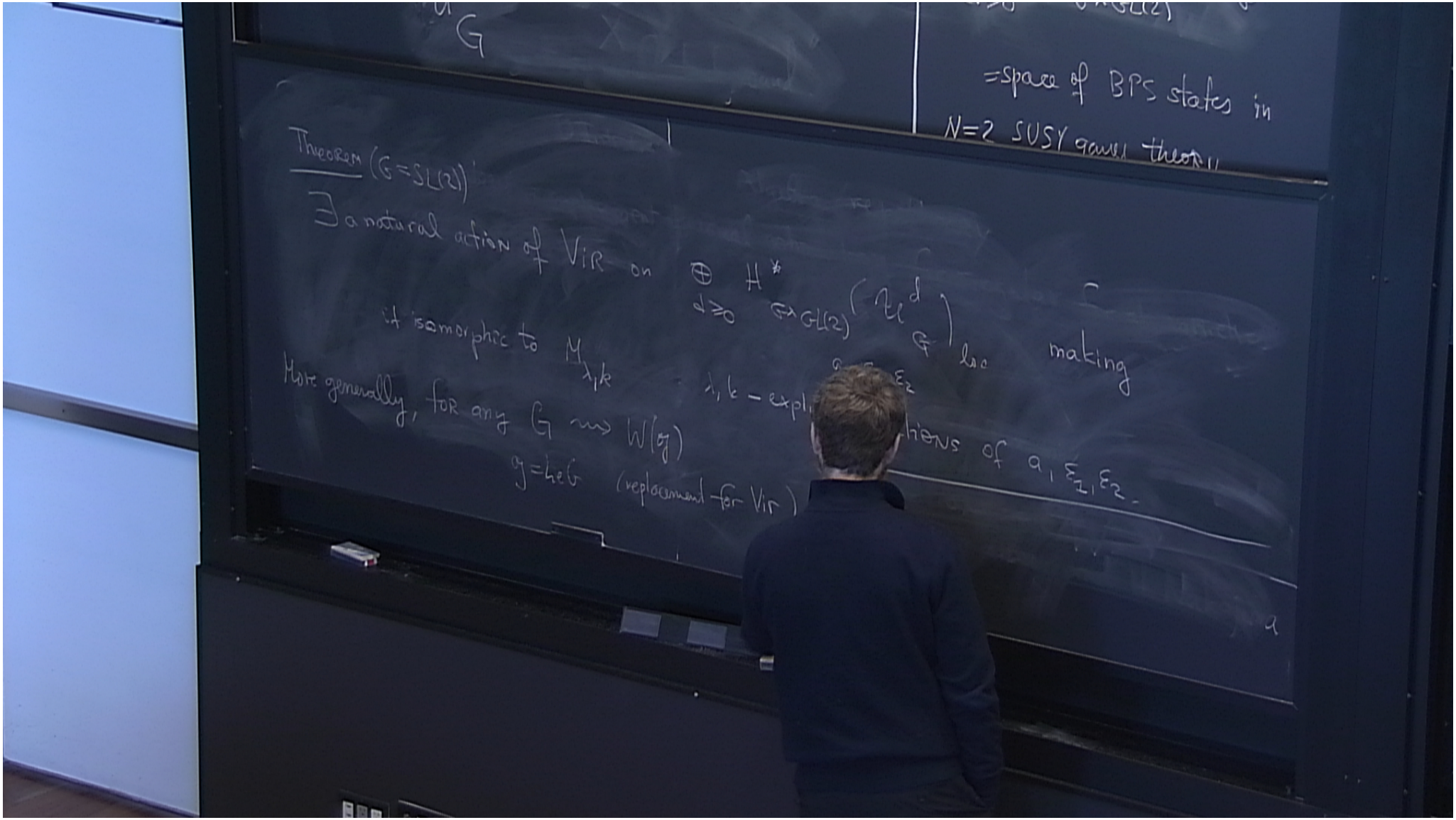

G
=space of BPS states in
N=2 SUSY gauge theor
THEOREM (G=SL(2))
∃ a natural action of Vir on ⊕ H*
d≥0
it isomorphic to M
λ,k
G↷GL(2)
ℓ
d
G
loc
making
a, ε1, ε2
λ, k – explicit

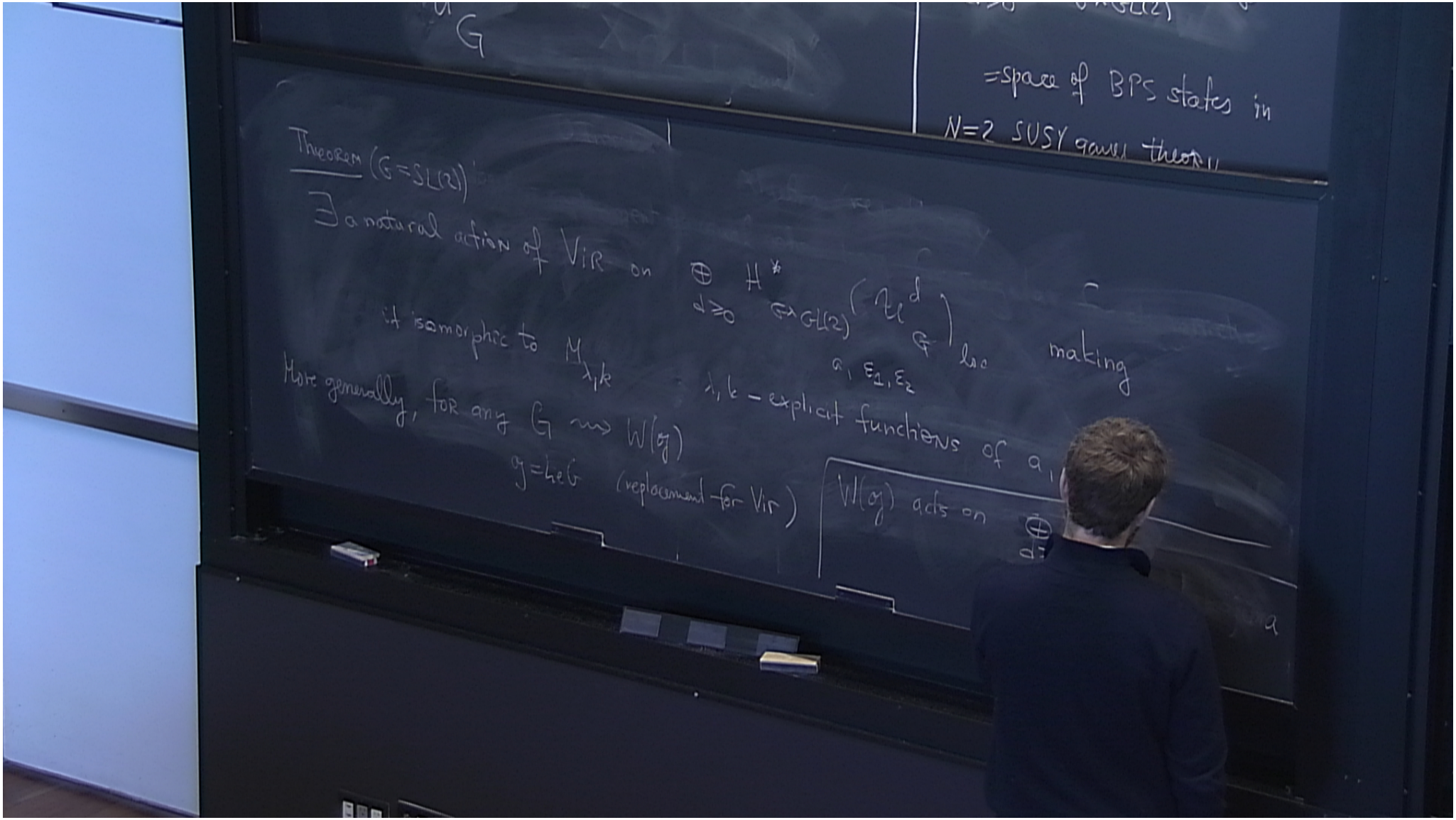

G
=space of BPS states in
N=2 SUSY gauge theory
THEOREM (G=SL(2))
∃ a natural action of Vir on ⊕ H*
d≥0 G^SL(2)(ℂ^d, G) loc making
it isomorphic to M_{λ,k}
a, ε1, ε2
λ, k – explicit functions of a, ε1, ε2.

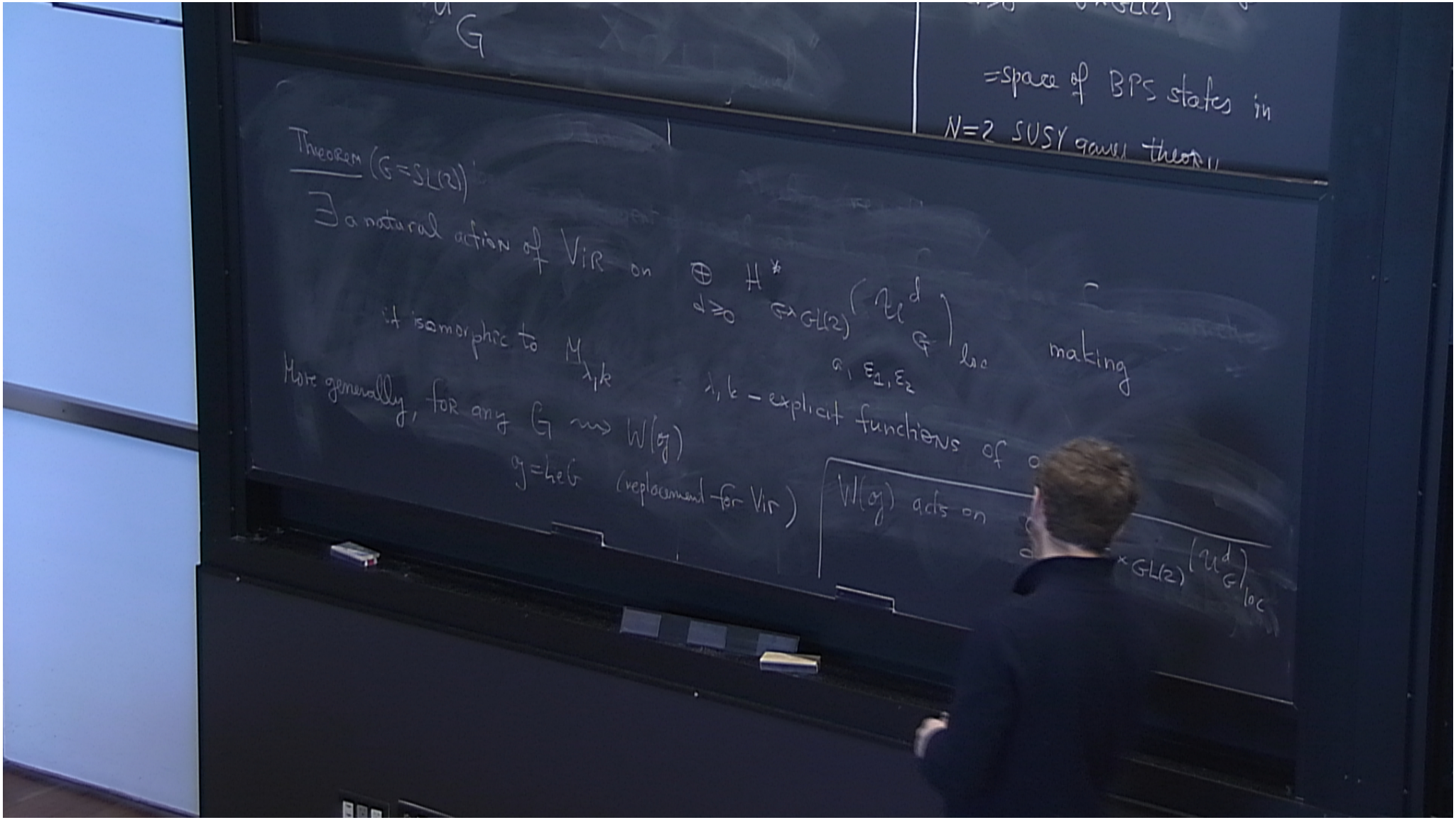

G
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Theorem (G=SL(2))
∃ a natural action of Vir on ⊕ H*
d≥0 G×GL(2)
making
a, ε1, ε2
λ, k – explicit functions of a, ε1, ε2.

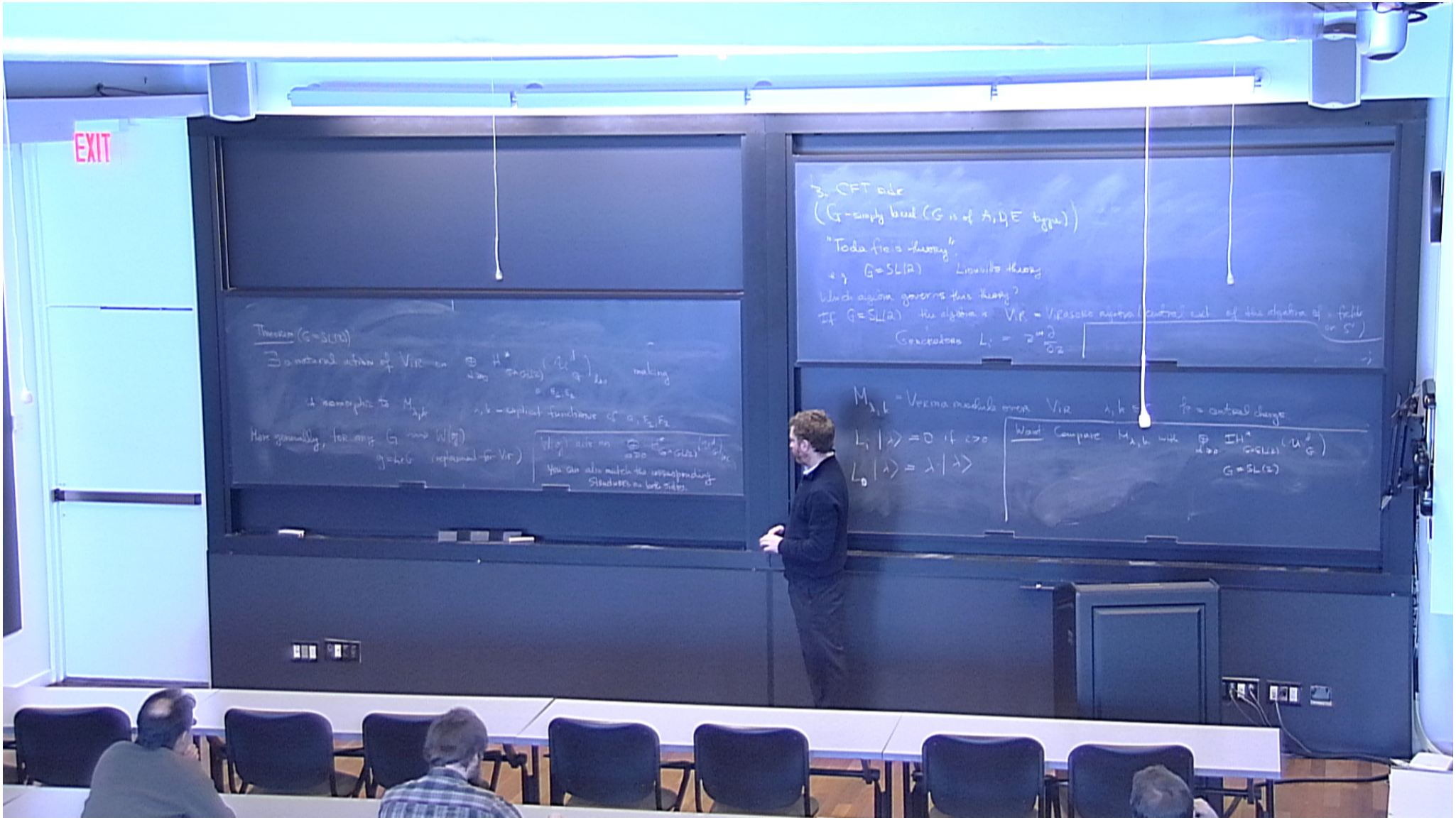

G
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Theorem (G=SL(2))
∃ a natural action of Vir on ⊕ H*
d≥0
isomorphic to M
λ,k
More generally, for any G ⇝ W(g)
g=LieG
making
functions of a, ε1, ε2.

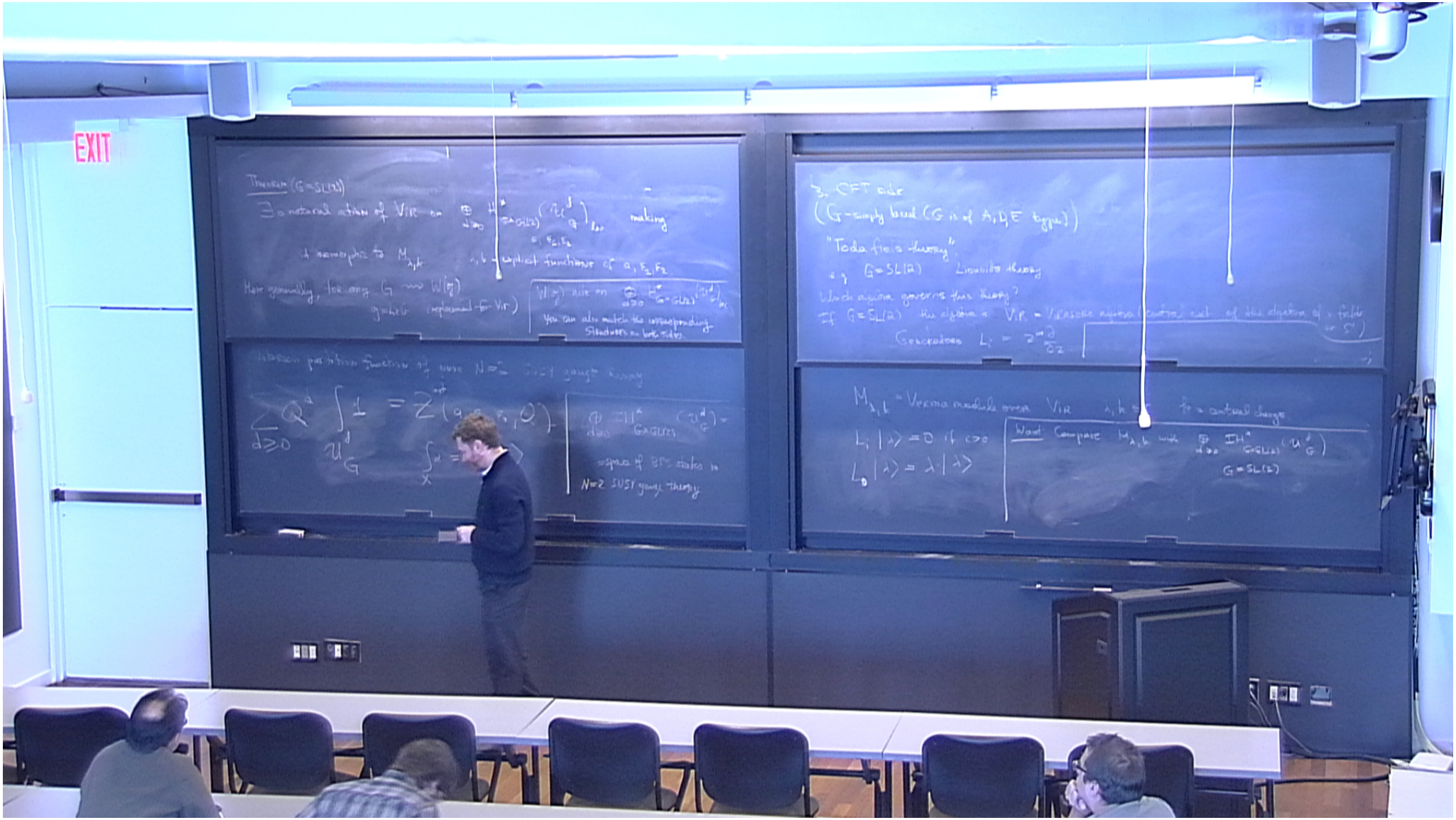

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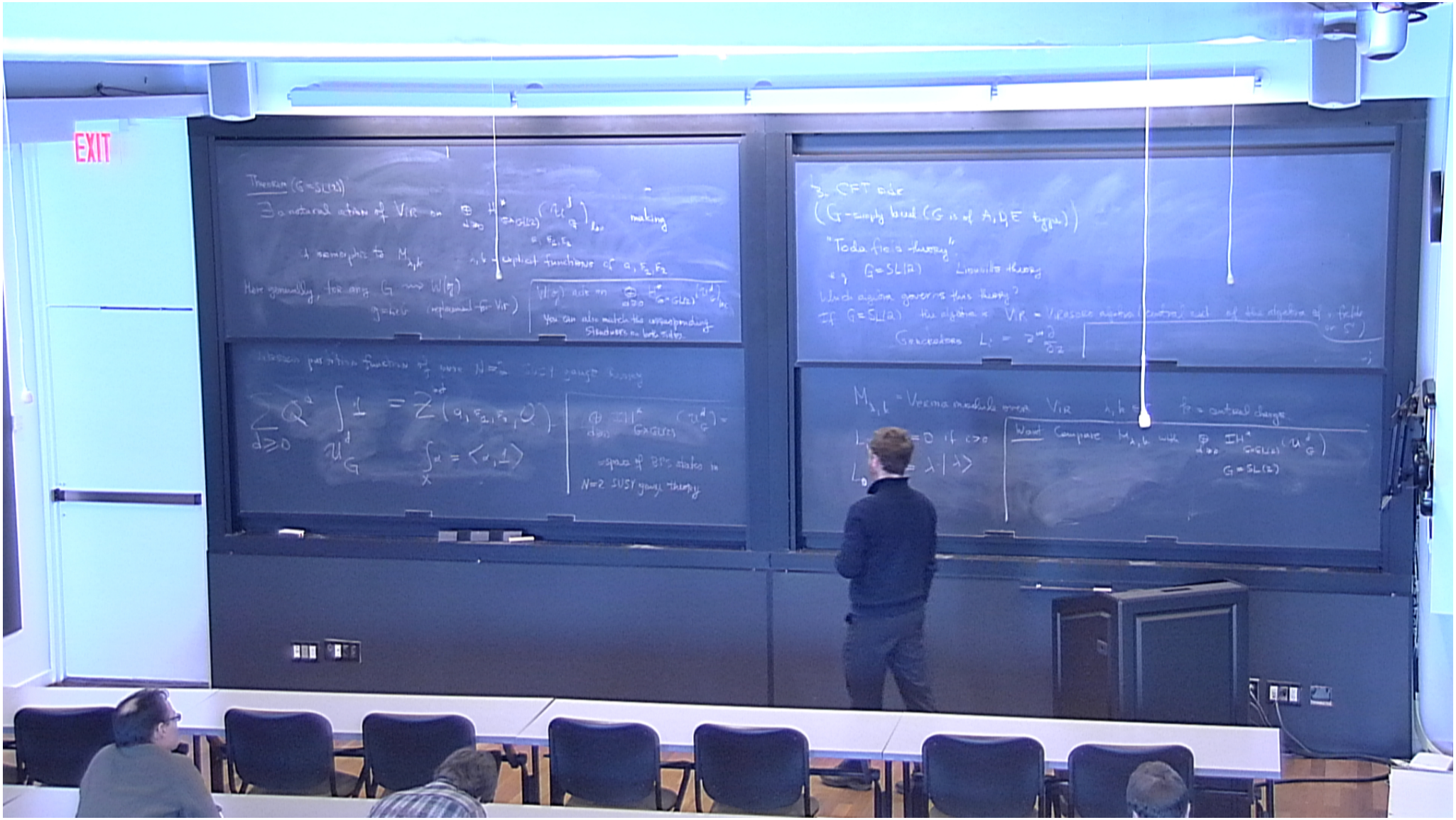

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making
ε2
ε1, ε2.

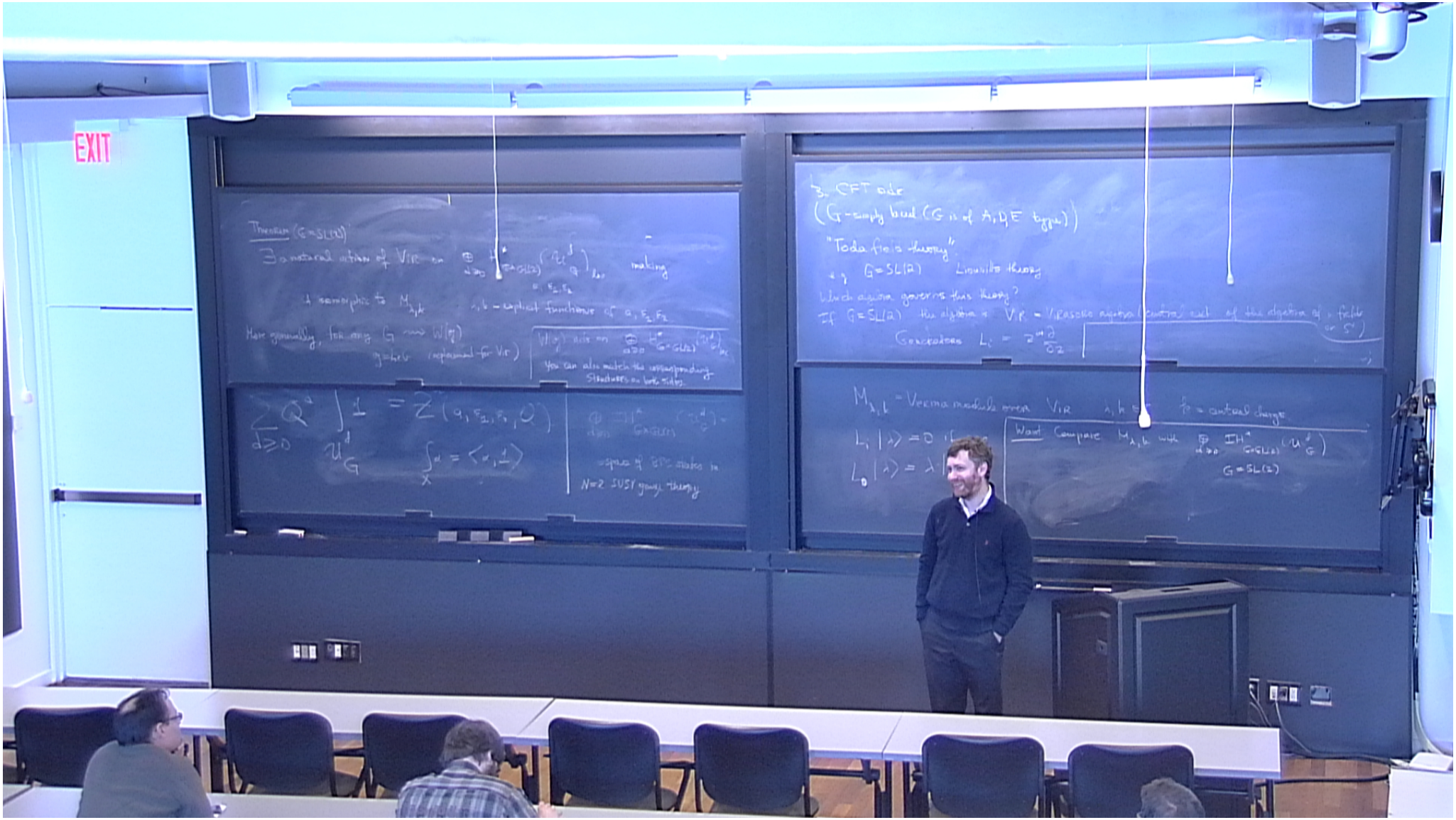

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More generally, for any G ⇝ W(g)
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λ, k – explicit functions of a,
W(g) acts on ⊕
a


G
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N=2 SUSY gauge theory
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it is isomorphic to M
λ,k
Moreover generally, for any G ⇝ W(g)
g=LieG (replacement for Vir)
λ, k – explicit functions of
making
W(g) acts on


EXIT
3. CFT side
"Toda field theory"
G=SL(2) Liouville theory
Theorem (G=SL(2))
Want Compare


EXIT


EXIT


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3. CFT side
"Toda field theory"
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