Title: Exact results in supersymmetric gauge theories in various dimensions

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Abstract: One of the central challenges in theoretical physics is to develop non-perturbative methods to describe quantitatively the dynamics of strongly coupled quantum fields. Much progress in this direction has been made for theories with a higher degree of symmetry, such as conformal symmetry or supersymmetry. In recent years the method of localisation has allowed to obtain a great deal of exact results for supersymmetric gauge theories in various dimensions which has led to the discovery of new surprising correspondences such as the celebrated Alday-Gaiotto-Tachikawa correspondence. I will review some recent results which indicate that partition functions of supersymmetric theories formulated on compact manifolds can be expressed in terms of a small set of fundamental building blocks.

Exact results in supersymmetric gauge theories in various dimensions

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PI Colloquium, March 2014

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Sectors of supersymmetric gauge theories have been mapped to exactly solvable spin models and integrable systems. In particular the study of the 4d $\mathcal{N} = 4$ theory has been the main common playground of the integrability and gauge/string communities.

Results obtained in the framework of $\mathcal{N} = 2$ theories, such as topological strings and mirror symmetry, have been real breakthroughs in pure mathematics.

Localisation

One of the main tools to obtain exact results in QFTs is localisation.

The basic idea of localisation is the following. Take δ, a Grasmann-odd (non-anomalous) symmetry of the path integral, s.t. $\delta^2 = \mathcal{L}_B$ is Grasmann-even symmetry, and deform the action to

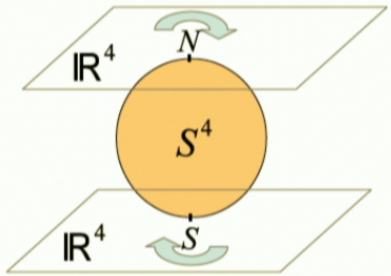
$$S_0(\phi) \to S_t(\phi) = S(\phi) + t\delta V \,,$$

where V is a Grasmann-odd operator with $\delta^2 V = \mathcal{L}_B V = 0$. It easy to see that:

$$\frac{d}{dt} Z_t = -\int [D\phi] \delta V e^{-S(\phi) - t\delta V} = \int [D\phi] \delta \left(V e^{-S(\phi) - t\delta V} \right) = 0 \,.$$

In recent years localisation has been applied to the path integral of supersymmetric theories formulated on compact manifolds.

The first example is the localisation of 4d $\mathcal{N} = 2$ theories on S^4: [Pestun]



$$\Rightarrow \quad Z_{S^4} = \int [Da] \; Z_{cl} \; Z_{1loop} \left| \mathcal{Z}_{inst} \right|^2$$

The path integral, at generic values of the coupling, reduces to an integral over the Cartan of the gauge group:

- $Z_{cl} = e^{-\frac{4\pi^2 a^2}{g^2_{YM}}}$ is the value of the action at the localising locus
- Z_{1loop} are the quadratic fluctuations around the localising locus
- Instantons $\mathcal{Z}_{inst}$ at the north pole and anti-instantons $\bar{\mathcal{Z}}_{inst}$ at the south pole contribute to the localising locus. $\mathcal{Z}_{inst}$ is a specification of the Ω-background result. [Nekrasov]

The S^4 localisation was first applied to prove a conjecture about Wilson loops in $\mathcal{N} = 4$ theory, stating that:

[Erickson-Semenoff-Zarembo],[Drukker-Gross]

$$\langle W_R \rangle_{S^4}^{\mathcal{N}=4} = \frac{\int [Da]\ tr_R e^{2\pi i a}\ e^{-\frac{4\pi^2 a^2}{g_{YM}^2}}}{\int [Da] e^{-\frac{4\pi^2 a^2}{g_{YM}^2}}}\ .$$

This corresponds to a specialisation of the result for the $\mathcal{N} = 2$ case [Pestun].

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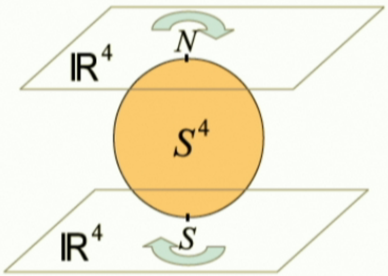

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M2-branes

The ABJM theory [Aharony-Bergman-Jafferis-Maldacena] was proposed to describe N coincident M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

It is a 3d supersymmetric quiver Chern-Simons theory with matter

$$U(N)_k \times U(N)_{-k}\,, \qquad \lambda = \frac{N}{k} = \text{`t Hooft coupling}$$

It has a gravity dual realisation as M theory on $AdS_4 \times S^7/\mathbb{Z}_k$ or type IIA on $AdS_4 \times \mathbb{CP}^3$.

M2-branes

Via localisation ABJM theories reduce to matrix models:
[Kapustin-Willett-Yaakov]

$$Z_{ABJM} = \int \prod_{i=1,j}^{N} d\mu_i d\nu_j \frac{\prod_{i<j}\left(2\sin(\frac{\mu_1-\mu_j}{2})\right)^2\left(2\sinh(\frac{\nu_1-\nu_j}{2})\right)^2}{\prod_{i,j}\left(2\cos(\frac{\mu_1-\nu_j}{2})\right)^2} e^{\frac{k}{4\pi i}(\sum_j \nu_j^2-\sum_i \mu_i^2)} .$$

- The ‘t Hooft expansion (large N, fixed λ) yields: [Drukker-Mariño-Putrov]

  $$\log Z_{ABJM} = \sum_{g=0} F_g(\lambda) N^{2-2g} \;\rightarrow\; \mathrm{F}_0(\lambda) \text{ is the interpolating function!}$$

- The “M-theory” expansion (large N, fixed k) [Herzog-Klebanov-Pufu-Tesileanu] which can be studied with the Fermi gas approach [Mariño-Putrov], shows membrane instantons/D2-branes must be included to cancel singularities in the resummation of word-sheet instantons corrections!

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M5-branes

The theory of N coincident M5 branes from the supergravity dual is expected to scale as N^3 at large N.

This feature should be visible in its 5d reduction, the 5d maximally supersymmetric Yang-Mills (SYM) theory, which was argued to capture non-trivial information of the compactified 6d theory.

[Douglas],[Lambert-Papageorgakis-SchmitSommerfeld]

Duality test

The moduli space of $\mathcal{N} = 4$ theories in 3d consists of a Coulomb and a Higgs branch corresponding to fluctuations of massless vector multiplets and hypermultiplets. Mirror symmetry acts on this moduli space exchanging the Higgs and Coulomb branches of mirror pairs [Intrilligator-Seiberg].

3d mirror symmetry applies strictly only to the IR limit where the coupling runs to infinity. A perturbative comparison of quantities on the two sides of the duality is therefore not possible.

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Thanks to localisation path integrals of $\mathcal{N}=4$ theories on S^3, become simple matrix integrals which depend on mass and FI parameters. [Kapustin-Willett-Yaakov]

Hence, if theories A and B are supped to be a mirror pair, we can check that their partition functions are equal:

$$Z^A_{S^3} = Z^B_{S^3} \, .$$

AGT correspondence

The Alday-Gaiotto-Tachikawa correspondence relates:

- 4d "class S" $\mathcal{N}=2$ gauge theories $\mathcal{T}_{g,n}$, obtained wrapping M5 on $C_{g,n}$ [Gaiotto]. These theories enjoy S-duality corresponding to different pant-decompositions of $C_{g,n}$.
- Liouville theory on $C_{g,n}$. It is a non-rational 2d CFT, characterised by 3-point functions and spectrum. Consistency requires crossing symmetry of correlators.

$$\langle \prod_i^n V_{\alpha_i} \rangle_{C_{g,n}} = \int D\alpha \; C \cdots C \, |\mathcal{F}_\alpha^{\alpha_i}|^2 = \int [Da] \; Z_{1loop} \left| \mathcal{Z}_{cl} \mathcal{Z}_{inst} \right|^2 = Z_{S^4}[\mathcal{T}_{g,n}]$$

2dCFT	4d gauge theory
Virasoro conf block : $\mathcal{F}_\alpha^{\alpha_i}$	$\mathcal{Z}_{\rm inst}$
3point functions : $C(\alpha_1, \alpha_2, \alpha_3)$	$Z_{\rm 1loop}$
cross ratio z	$e^{2\pi i \tau}$
external momenta α_i	masses m_i
internal momentum α	coulomb branch a
crossing symmetry	S – duality

→Use Verlinde loop operators to study Wilson-'t Hooft loops.

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde], [Drukker-Gomis-Okuda-Teschner].

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3d-3d correspondence

The 3d-3d correspondence relates: [Dimofte-Gukov],[Dimofte-Gukov-GaiottoI,II]

- 3d "class R" $\mathcal{N}=2$ (IR SCFTs) gauge theories $\mathcal{T}(M_3)$, obtained wrapping M5 on a 3-manifold M_3. Different Lagrangian description of the same IR theory are related by 3d mirror symmetry.
- Analytically continued Chern-Simons theory on M_3. $SL(2,\mathbb{C})$ connections can be constructed "patching" $SL(2,\mathbb{C})$ connections on ideal tetrahedra. Consistency requires that different tetrahedra-decompositions of M_3 give the same result.

Partition functions on S^3 and $S^2 \times S^1$ of class R theories are mapped to partition functions of Chern-Simons theories on M_3 with different analytic continuations.

This list is not complete... For example very recently a 2d-4d correspondence has been put forward, relating 4-manifolds M_4 to 2d $\mathcal{N} = 2, 0$ theories $\mathcal{T}(M_4)$. [Gadde-Gukov-Putrov]

These correspondences are very exciting and besides unveiling new mathematical structures of gauge theories they actually provide new efficient computational tools.

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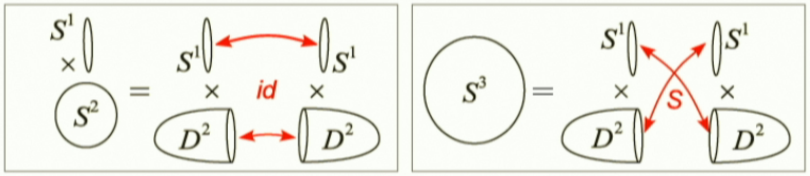
Now I'd like to discuss another aspect of partition functions of supersymmetric theories on compact manifolds: their decomposition in terms of a set of fundamental building blocks.

3d factorisation

S^3 and $S^2 \times S^1$ can be obtained by gluing solid tori $D^2 \times S^1$ with appropriate identification of the boundary tori:



Is there a QFT analogue of this decomposition?

The factorisation of 3d partition functions:

- works also on lens spaces S^3/Z_p, in this case:

$$\tau \to \tilde{\tau} = \frac{\tau}{p\tau - 1},$$

  as expected from Heegaard splitting [Nieri-SP, to appear]. What about more general Seifert manifolds?
- can be proved using an alternative localisation scheme (Higgs branch) [Benini-Pealers],[Fujitsuka-Honda-Yoshida]
- can be understood as an effective projection, since we can stretch the geometry, without changing the partition function, to the configuration of cigars connected by an infinitely long tube: [Alday-Martelli-Richmond-Sparks]

$$Z_{M_g} = \langle 1|1\rangle_g = \sum_\alpha \langle 1|\alpha\rangle\langle\alpha|1\rangle_g = \sum_\alpha \left|\left|\mathcal{B}_\alpha^{3d}\right|\right|_g^2 .$$

→ see also 3d tt* geometries [Cecotti-Gaiotto-Vafa].

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$\rightarrow$ see also 3d tt* geometries [Cecotti-Gaiotto-Vafa].

3d blocks have interesting properties:

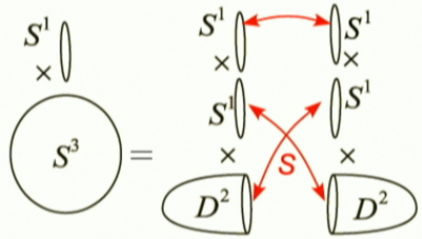
- they are solution of difference equations/XXZ Baxter equation
- varying x, they undergo Stokes jumps

$$\mathcal{B}_\alpha^{3d}(x,q) \to M_\alpha^\beta \mathcal{B}_\beta^{3d}(x,q)\,, \quad \mathcal{B}_\alpha^{3d}(\tilde{x},\tilde{q}) \to (M^{-1T})_\alpha^\beta \mathcal{B}_\beta^{3d}(\tilde{x},\tilde{q})\,,$$

  so blocks are covariant under symmetries of partition functions.
- via the 3d-3d correspondence they belong to a basis of wave-functions in analytically continued Chern-Simons theory. The index α then labels flat connections.
- Monodromies of blocks on curves in the parameter space correspond to the insertion of defects operators at the boundary torus. [Gadde-Gukov-Putrov]

4d factorisation

Very recent results for the $S^3 \times S^1$ partition function/Superconformal Index. Also in this case the block decomposition reflects the geometry: [Pealers], [Yoshida]



$$\Rightarrow \quad Z_{S^3 \times S^1} = \sum_\alpha \left|\left|\mathcal{B}^{4d}_\alpha\right|\right|^2_g$$

- 4d blocks $\mathcal{B}^{4d}_\alpha$ are $D^2 \times T^2$ partition functions.
- 4d blocks are solution of difference equations, probably connected to XYZ Baxter equation
- Partition functions on these backgrounds are metric independent [Closset-Dumitrescu-Festuccia-Komargodski], so the stretching/projection argument should apply also to this case.

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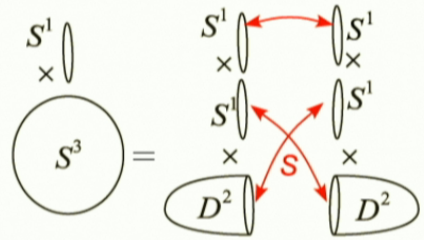

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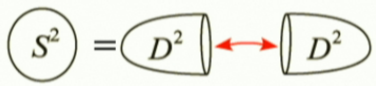
2d factorisation

Partition functions of $\mathcal{N} = (2, 2)$ theories on S^2 have been computed via Coulomb and Higgs-branch localisation. [Droud-Gomis-Le Floch-Lee], [Benini-Cremonesi]

It was also conjectured that for a GLSM, $Z_{S^2} = e^{-\mathcal{K}}$, where $\mathcal{K}$ is the exact Kähler potential. [Jockers-Kumar-Lapan-Marrison-Romo]

In the Higgs branch partition functions take a factorised form:



$$\Rightarrow \quad Z_{S^2} = \sum_{\alpha} \left|\left|\mathcal{B}^{2d}_{\alpha}\right|\right|^2$$

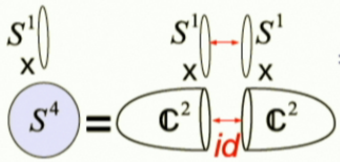
- 2d blocks $\mathcal{B}^{2d}_{\alpha}$ are cigars D^2 partition functions.
- Since Z_{S^2} is unaffected by the stretching [Gomis-Lee]

$$Z_{S^2} = {}_R\langle\bar{1}|1\rangle_R = -\log\mathcal{K}\,,$$

  the last step is the tt* definition of $\mathcal{K}$. [Cecotti-Vafa]
- Hemisphere partition functions was also computed via localisation and argued to capture the central charge of the D-brane (hemisphere boundary condition). [Honda-Okuda], [Hori-Romo], [Sugishita-Terashima]

5d factorisation

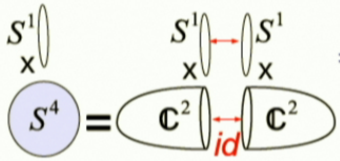
- 5d $\mathcal{N} = 1$ theories on $S^4 \times S^1$ [Kim-Kim-Lee],[Terashima],[Iqbal-Vafa]



$$\Rightarrow Z_{S^4 \times S^1} = \int d\sigma \; Z_{1loop} |\mathcal{Z}^{5d}_{inst}|^2 = \int d\sigma \; \prod_{k=1}^{2} (\mathcal{B}^{5d})_k$$

5d factorisation

- 5d $\mathcal{N}=1$ theories on $S^4 \times S^1$ [Kim-Kim-Lee],[Terashima],[Iqbal-Vafa]



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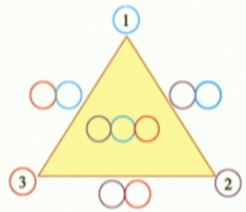
- 5d $\mathcal{N}=1$ theories on S^5 (toric Sasaki-Einstein manifold): [Kallen-Zabzine],[Hosomichi-Seong-Terashima],[Imamura],[Kim-Kim-Kim],[Lockart-Vafa]

S^5:



$$\Rightarrow Z_{S^5} = \int d\sigma\ Z_{cl} Z_{1loop} \prod_{k=1}^{3} (\mathcal{Z}^{5d}_{inst})_k = \int d\sigma\ \prod_{k=1}^{3} (\mathcal{B}^{5d})_k$$

$\rightarrow$5d blocks are $\mathbb{C}^2 \times S^1$ partition functions: $\mathcal{B}^{5d} := \mathcal{Z}^{5d}_{cl} \mathcal{Z}^{5d}_{1loop} \mathcal{Z}^{5d}_{inst}$.
[Nieri-SP-Passerini-Torrielli]

Very recently it has been found shown that partition functions of 5d $\mathcal{N} = 1$ theories on generic toric Sasaki-Einstein manifolds, T^3 fibrations over an n-gon, can be expressed in terms of 5d blocks.
[Qiu-Tizzano-Winding-Zabzine]

They first use localisation to compute the perturbative partition function:

$$Z_n^{pert} = \int d\sigma \; Z_{cl} \; Z_{1loop} \, .$$

Then they prove that

$$Z_{cl} Z_{1loop} = \prod_{k=1}^{n} \, (\mathcal{Z}_{cl}^{5d})_k (\mathcal{Z}_{1loop}^{5d})_k \, .$$

Finally the complete each sector with the non-perturbative term $(\mathcal{Z}_{inst}^{5d})_k$ and obtain:

$$Z_n = \int d\sigma \, \prod_{k=1}^{n} \, (\mathcal{B}^{5d})_k \, .$$

→A very large class of new results thanks to the factorisation paradigm!

Remember the AGT correspondence:

- $\mathcal{Z}^{4d}_{inst} \leftrightarrow$ Virasoro conformal blocks
- $Z_{1loop} \leftrightarrow$Liouville 3-point functions
- $Z_{S^4} \leftrightarrow$Liouville correlators

Is there an analogue in 5d?

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Is there an analogue in 5d?

Hint: $\mathcal{Z}^{5d}_{inst} \leftrightarrow$ q-deformed Virasoro chiral blocks. [Awata-Yamada],[many many others]

Conjecture: S^5 and $S^4 \times S^1$ partition functions are captured by q-deformed Liouville correlators. [Nieri-SP-Passerini]

But what is q-deformed Liouville?

We can try to define it in terms of spectrum and 3-point functions.

q-deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t and generators T_n with $n \in \mathbb{Z}$.

[Shiraishi-Kubo-Awata-Odake], [Lukyanov-Pugai], [Frenkel-Reshetikhin], [Jimbo-Miwa]

$$
\begin{aligned}
[T_n\,,\,T_m] \;&=\; -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) \\
&\quad -\frac{(1-q)(1-t^{-1})}{1-p}((q/t)^n - (q/t)^{-n})\delta_{m+n,0}
\end{aligned}
$$

where $f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right]$

consistency conditions

We require that n-point correlators of $\mathcal{V}ir_{q,t}$ primaries V_{α_j} could be reduced to a product of 3-point functions and chiral $\mathcal{V}ir_{q,t}$ blocks.

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– show that chiral blocks $\mathcal{F}_{\alpha}^{\alpha_i}$ are *covariant* under change of pant-decompositions (Moore-Seiberg groupoid)

– glue blocks to form invariant correlators. In Liouville this amounts to take the holomorphic/anti-holomorphic pairing of the blocks:

$$\langle V_{\alpha_1} \cdots V_{\alpha_k} \rangle = \int d\alpha \; C \cdots C \; \bar{\mathcal{F}}_{\alpha}^{\alpha_i} \mathcal{F}_{\alpha}^{\alpha_i} \, .$$

gluing rules

In the q-deformed set-up we believe that there is more freedom (probably $SL(3, Z)$ ways) to glue $\mathcal{V}ir_{q,t}$ blocks.

Since $\mathcal{V}ir_{q,t}$ blocks $\mathcal{F}_{\alpha}^{\alpha_i} = \mathcal{Z}_{inst}^{5d}$, we try to construct q-deformed correlators using the gluing rules of 5d blocks in $S^4 \times S^1$ and S^5 partition functions.

Liouville CFT correlators can be defined and computed in a purely axiomatic fashion, without using the Lagrangian.

- conformal blocks are determined by the Virasoro algebra
- 3-point function can be obtained using degenerate reps of the Virasoro algebra + crossing symmetry=bootstrap approach
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We try to define and compute q-deformed Liouville CFT correlators in a purely axiomatic fashion, without knowing the Lagrangian.

- chiral blocks are determined by the $\mathcal{V}ir_{qt}$ algebra
- we determine 3-point function using degenerate reps of the $\mathcal{V}ir_{qt}$ algebra + crossing symmetry+gluing prescription= q-deformed bootstrap approach

Around $z = 0$

$$I_1^{(s)} = {}_2\Phi_1(A, B; C; z)\,, \qquad I_2^{(s)} = \frac{\theta(q^2C^{-1}z^{-1}; q)}{\theta(qC^{-1}; q)\theta(qz^{-1}; q)}\, {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2C^{-1}; z)$$

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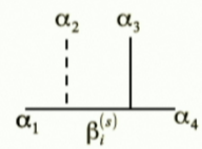
s-channel correlator:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(s)} K_{ij}^{(s)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_*^2 = \sum_i$$

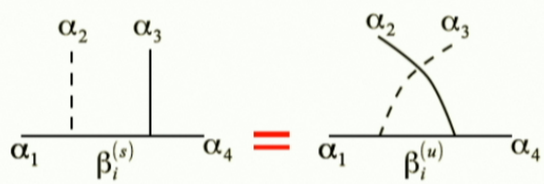


$K_{ij}^{(s)}$ is diagonal with elements related to 3-point functions:

$$K_{ii}^{(s)} = C(\alpha_4, \alpha_3, \beta_i^{(s)})\, C(Q_0 - \beta_i^{(s)}, -b_0/2, \alpha_1)\,, \quad \beta_i^{(s)} = \alpha_1 \pm \frac{b_0}{2}\,, \;\; i = 1, 2$$

$\to$ we will need to prescribe the gluing $\left|\left|(\cdots)\right|\right|_*^2$

impose crossing symmetry



$$K_{11}^{(s)}\left|\left|I_1^{(s)}\right|\right|_*^2 + K_{22}^{(s)}\left|\left|I_2^{(s)}\right|\right|_*^2 = K_{11}^{(u)}\left|\left|I_1^{(u)}\right|\right|_*^2 + K_{22}^{(u)}\left|\left|I_2^{(u)}\right|\right|_*^2$$

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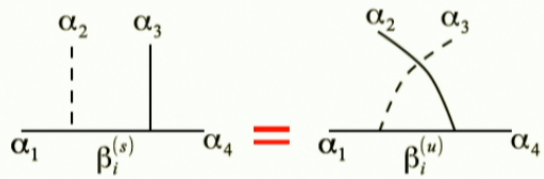

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analytic continuation $I_i^{(s)} = \sum_{j=1}^2 M_{ij} I_j^{(u)}$, $\tilde{I}_i^{(s)} = \sum_{j=1}^2 \tilde{M}_{ij} \tilde{I}_j^{(u)}$ yields:

$$\sum_{k,l=1}^2 K_{kl}^{(s)} \tilde{M}_{ki} M_{lj} = K_{ij}^{(u)}$$

Now we need an ansatz for the gluing rule:

- S^5 gluing rule $\rightarrow$ 3-point function $C_S(\alpha_1, \alpha_2, \alpha_3)$
- $S^4 \times S^1$ gluing rule $\rightarrow$ 3-point function $C_{id}(\alpha_1, \alpha_2, \alpha_3)$

Checks

- 5d SQCD, $SU(2)$, $N_f = 4$ theory $\Leftrightarrow$ 4-point correlator

$$Z^{SQCD}_{S^4\times S^1} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4}\rangle_{id} = \int d\alpha C_{id}(\alpha_1, \alpha_2, \alpha) C_{id}(Q_0 - \alpha, \alpha_3, \alpha_4) \prod_{k=1}^{2} (\mathcal{F}^{\alpha_i}_{\alpha})_k$$

$$Z^{SQCD}_{S^5} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4}\rangle_{S} = \int d\alpha C_{S}(\alpha_1, \alpha_2, \alpha) C_{S}(E - \alpha, \alpha_3, \alpha_4) \prod_{k=1}^{3} (\mathcal{F}^{\alpha_i}_{\alpha})_k$$

- 5d $\mathcal{N} = 1^*$ $SU(2)$ theory $\Leftrightarrow$ 1-point torus correlator

$$Z^{SQCD}_{\mathcal{N}=1^*} = \langle V_{\alpha_1}\rangle_{id} = \int d\alpha C_{id}(Q_0 - \alpha, \alpha_1, \alpha) \prod_{k=1}^{2} (\mathcal{F}^{\alpha_1}_{\alpha})_k$$

$$Z^{\mathcal{N}=1^*}_{S^5} = \langle V_{\alpha_1}\rangle_{S} = \int d\alpha C_{S}(E - \alpha, \alpha_1, \alpha) \prod_{k=1}^{3} (\mathcal{F}^{\alpha_1}_{\alpha})_k$$

Brief summary and open questions

The factorisation of 5d partition functions in terms of 5d holomorphic blocks $\mathcal{B}^{5d}$ and their identification with chiral $\mathcal{V}ir_{qt}$ blocks, suggest to map 5d partition functions to q-deformed Liouville correlators.

We defined q-deformed Liouville correlators in terms of $\mathcal{V}ir_{qt}$ blocks and 3-point functions and showed that indeed they can be mapped to 5d partition functions.

To prove the consistency of our construction we still need to show that correlators are invariant under the Moore-Seiberg groupoid dualities (change of pant-decomposition). This is our big challenge ahead.

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