Title: The Unreasonable Effectiveness Of Quantum Physics in Modern Mathematics

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Abstract: Mathematics has proven to be "unreasonably effective" in understanding nature. The fundamental laws of physics can be captured in beautiful formulae. In this lecture I want to argue for the reverse effect: Nature is an important source of inspiration for mathematics, even of the purest kind. In recent years ideas from quantum field theory, elementary particles physics and string theory have completely transformed mathematics, leading to solutions of deep problems, suggesting new invariants in geometry and topology, and, perhaps most importantly, putting modern mathematical ideas in a `naturalâ€™ context.

The Unreasonable Effectiveness of Quantum Physics in Modern Mathematics

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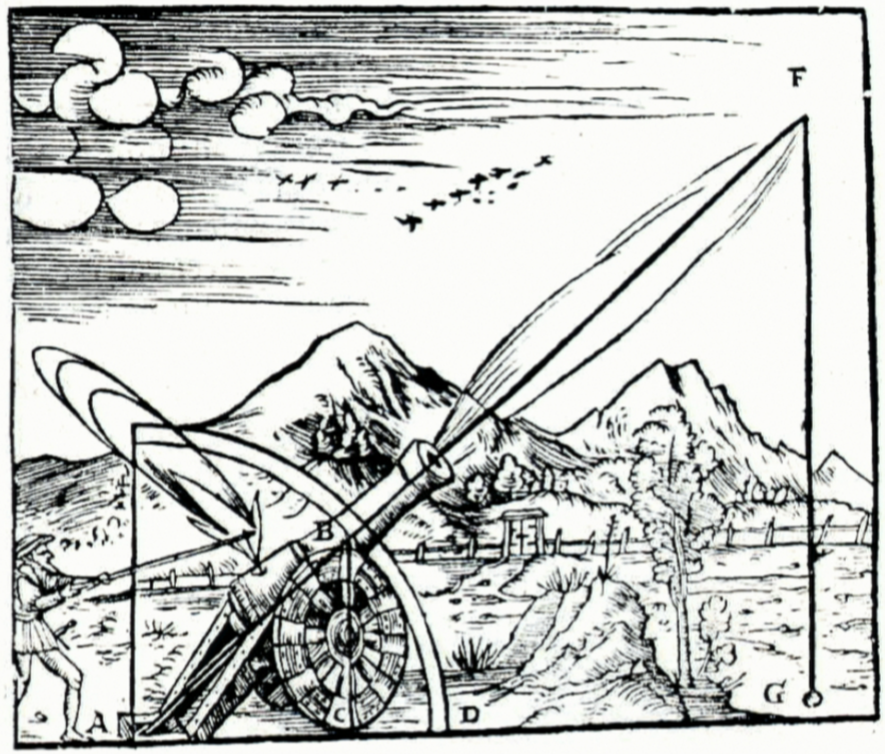
Institute for Advanced Study

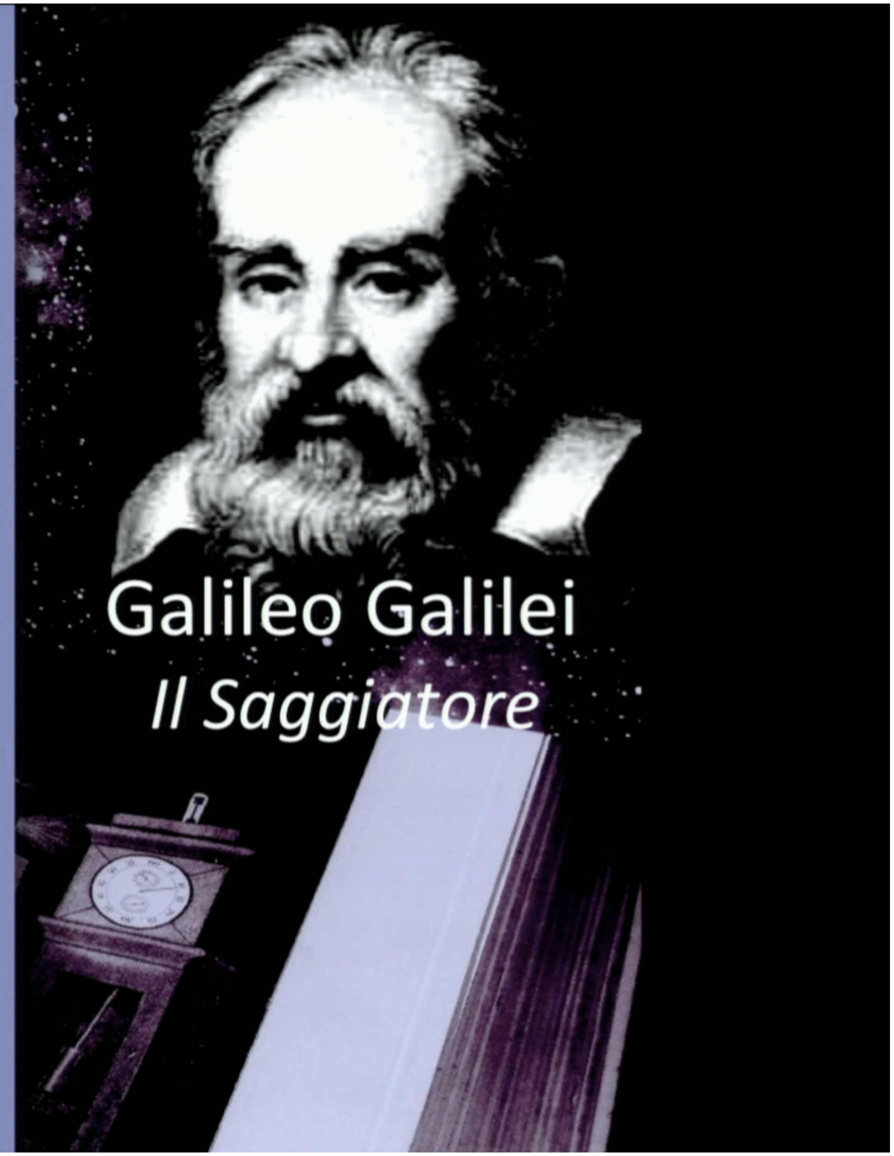
Colloquium, Perimeter Institute, March 5, 2014

Mathematical structures in fundamental physics

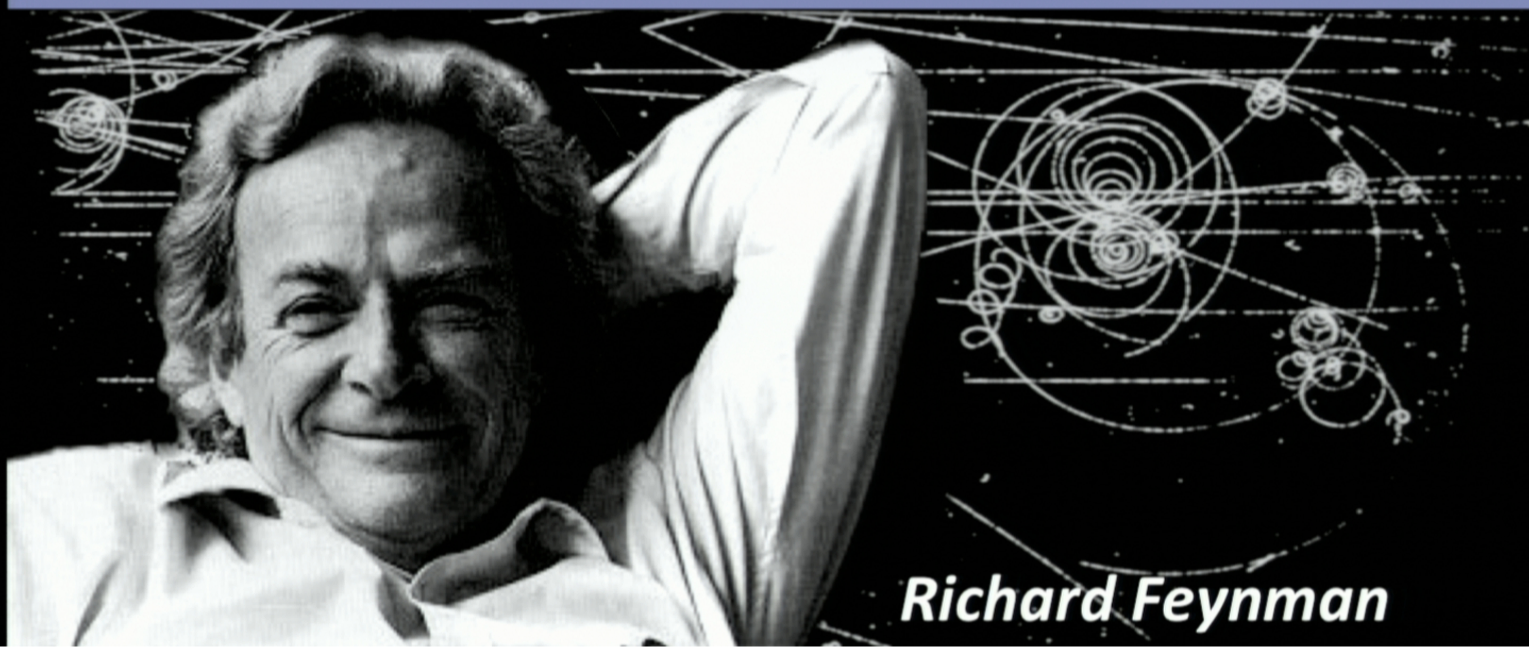



left derived
FUNCTORS
NOW
GROUPS
LIE
ALGEBRAS
HOMOTOPY FOR YOU!
GAUSS
TENSOR
CATEGORIES
HERE!
MORPHISM
MATH
P-ADIC
PROBLEMS
PHYS
mathematics
physics

“Philosophy is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. **It is written in the language of mathematics**, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.”



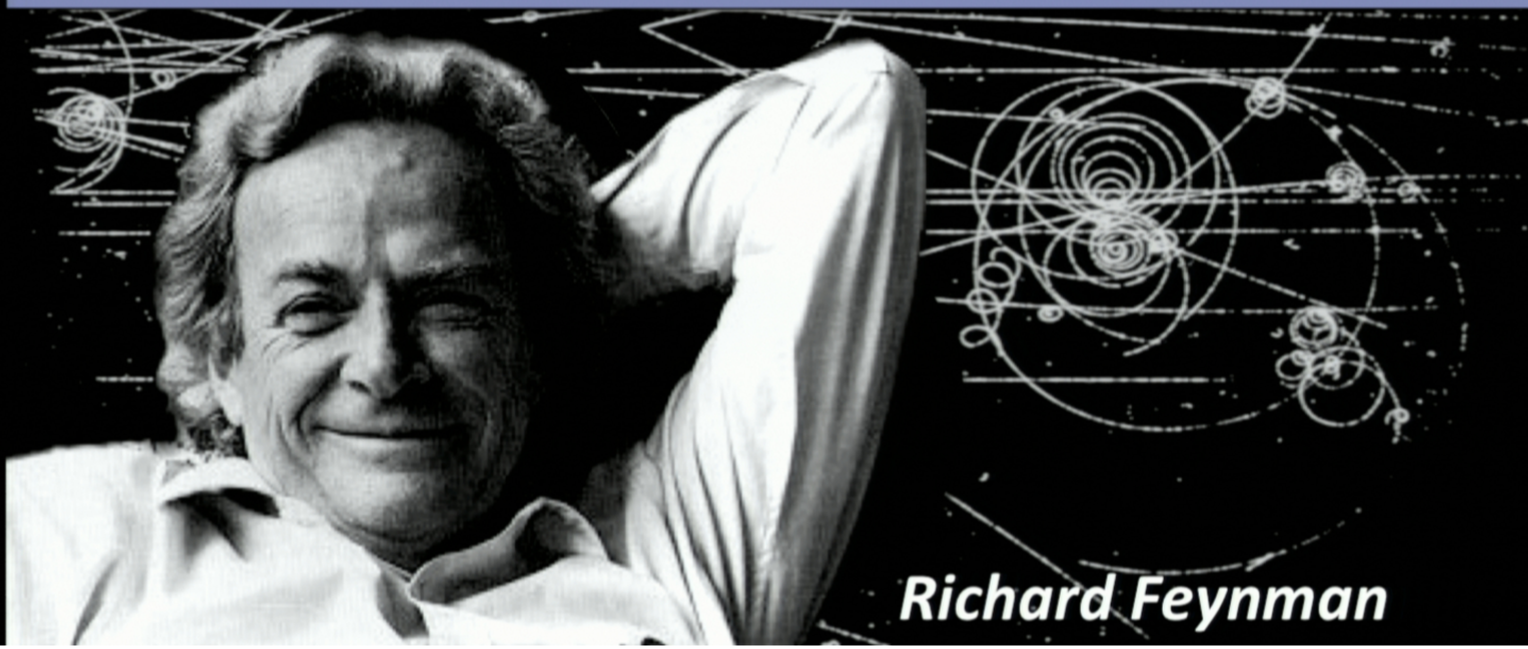
"To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in."



"If all mathematics disappeared today,

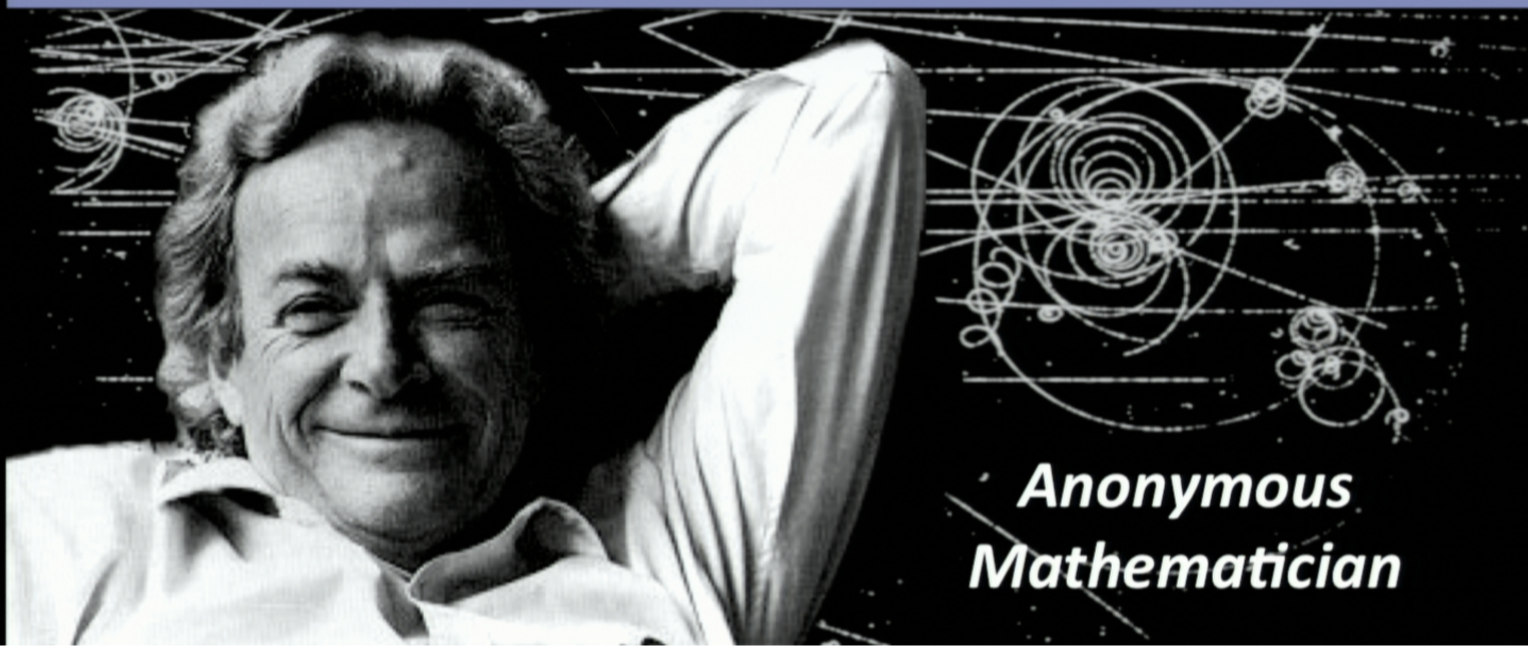
physics would be set back

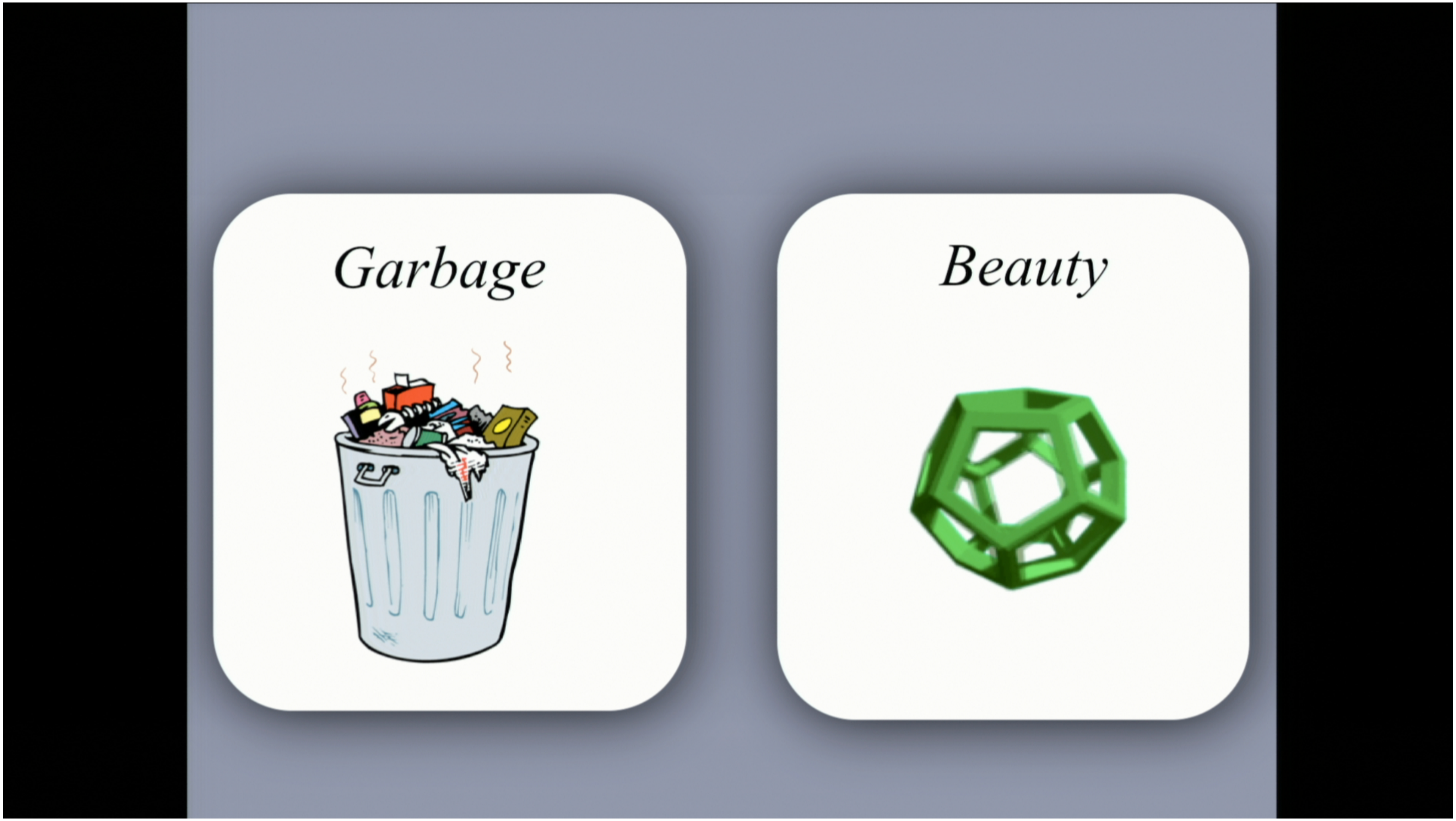
exactly one week."

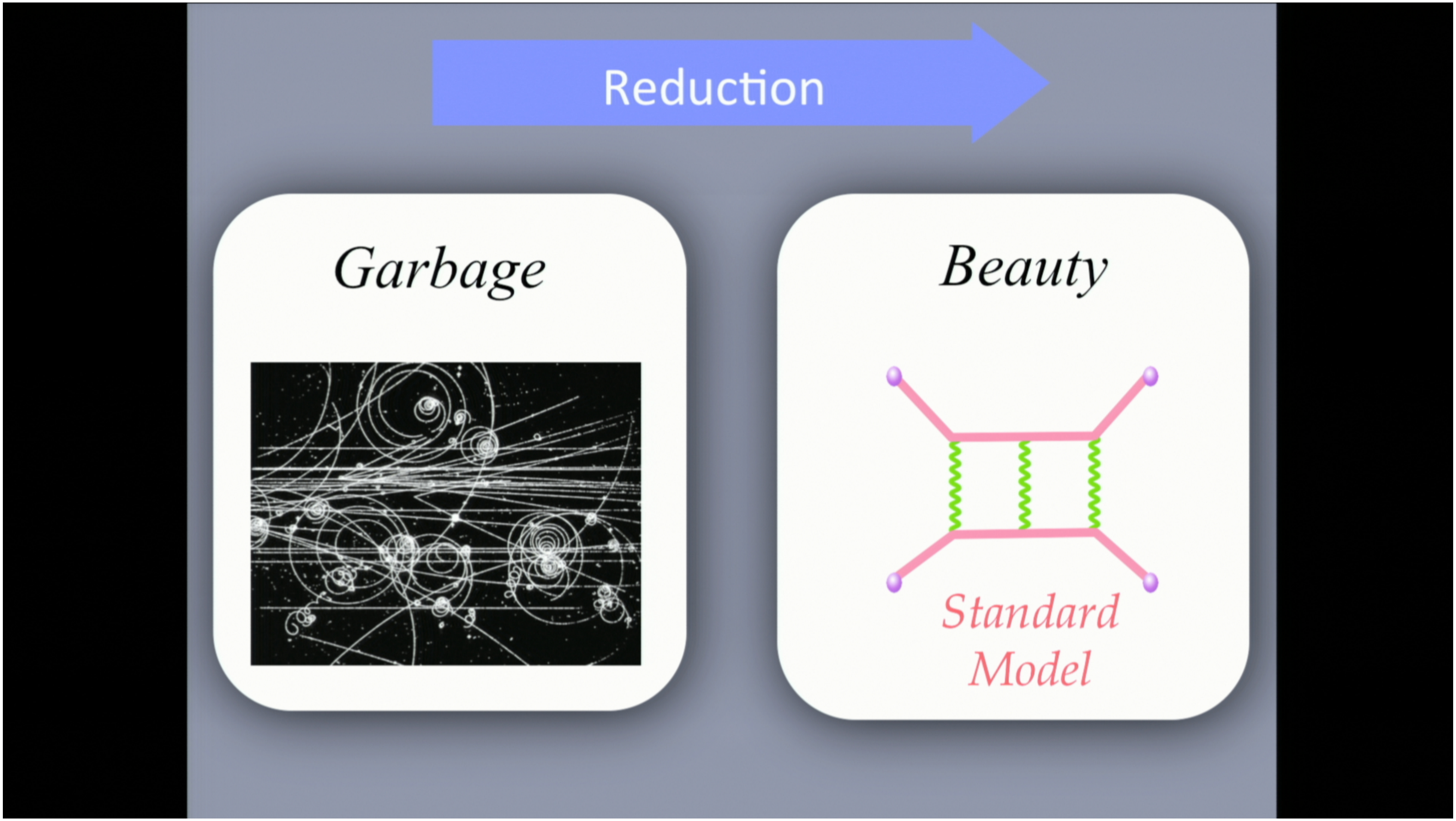


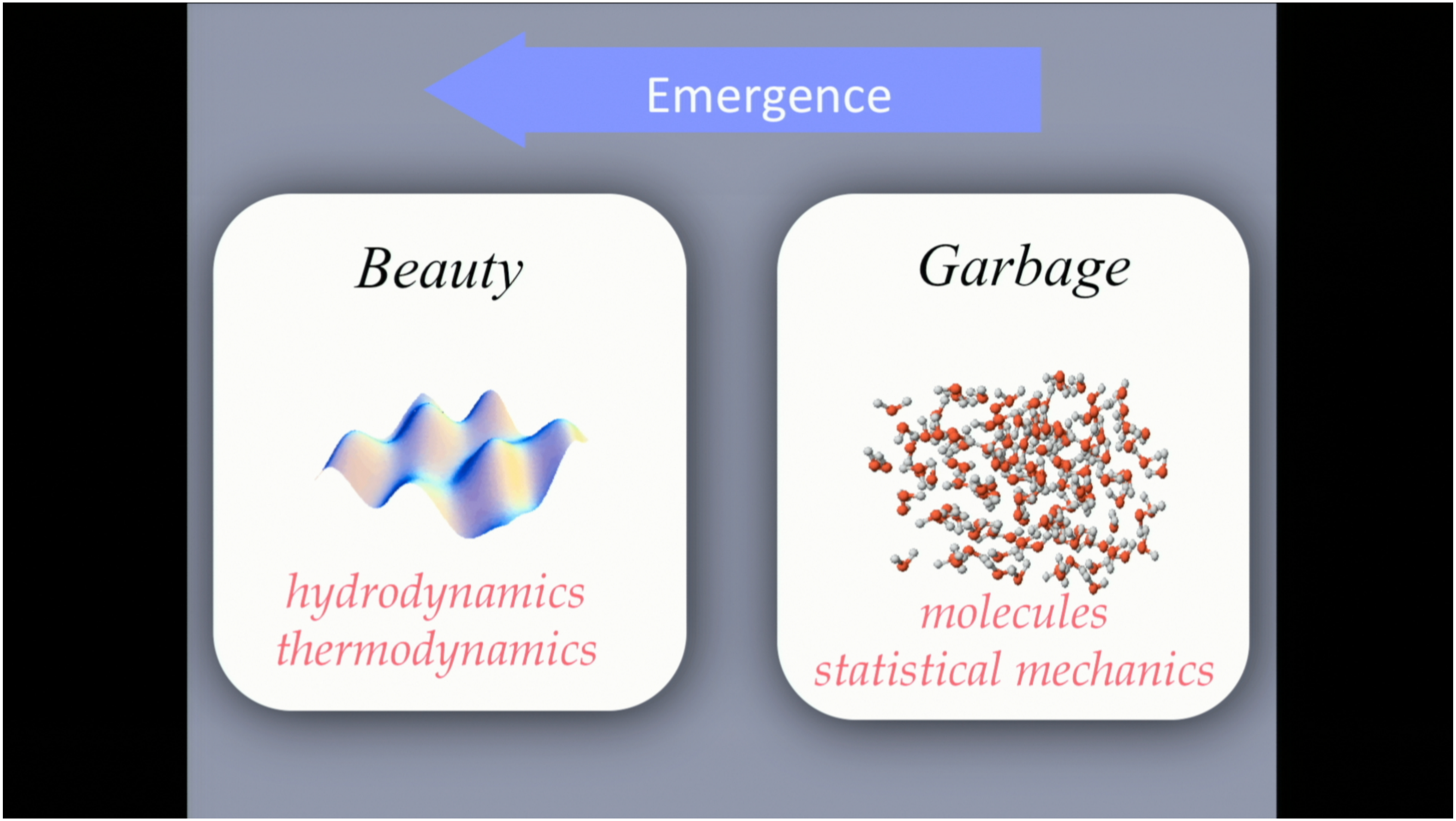
"That was the week

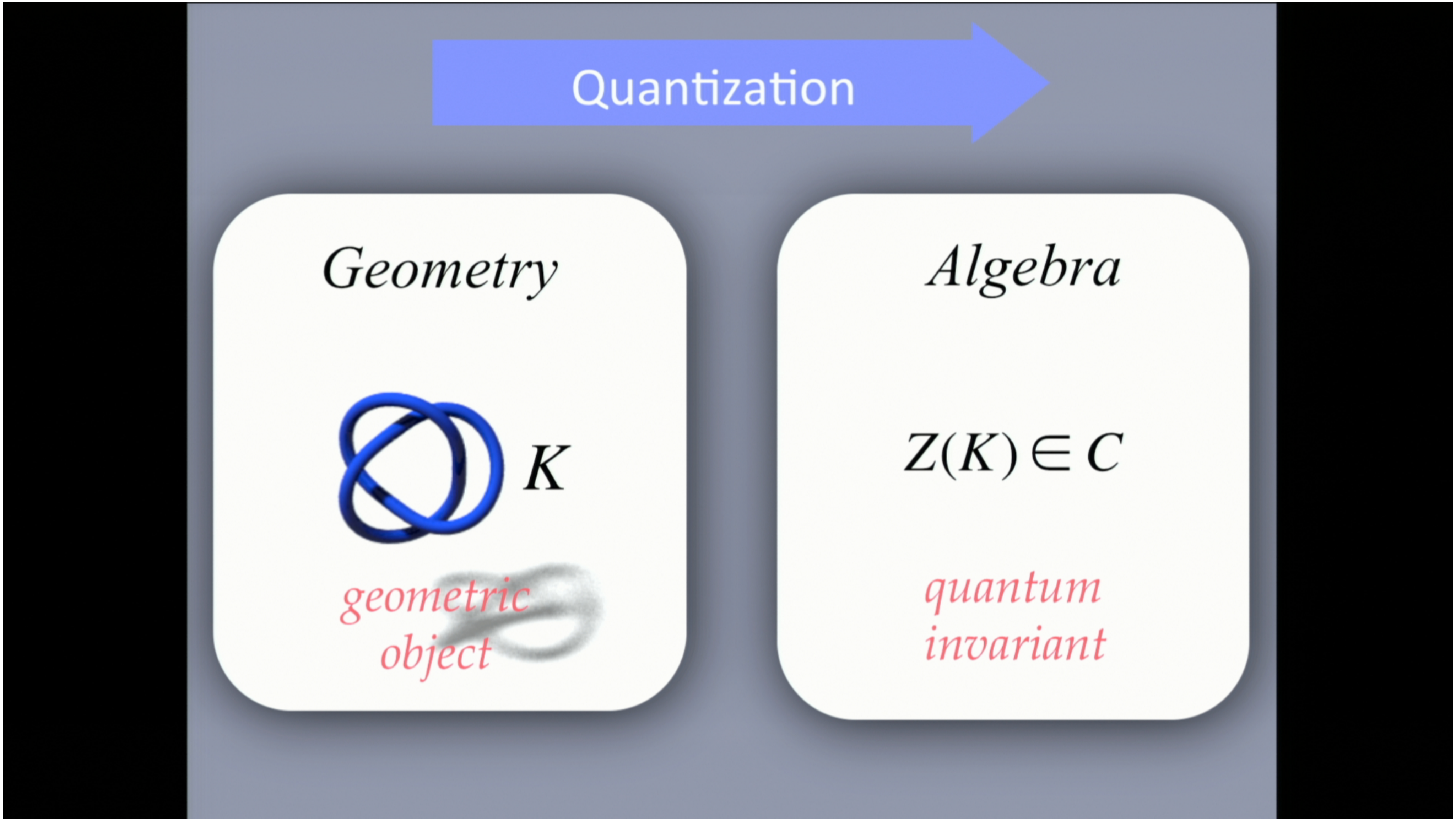
that God created the world."

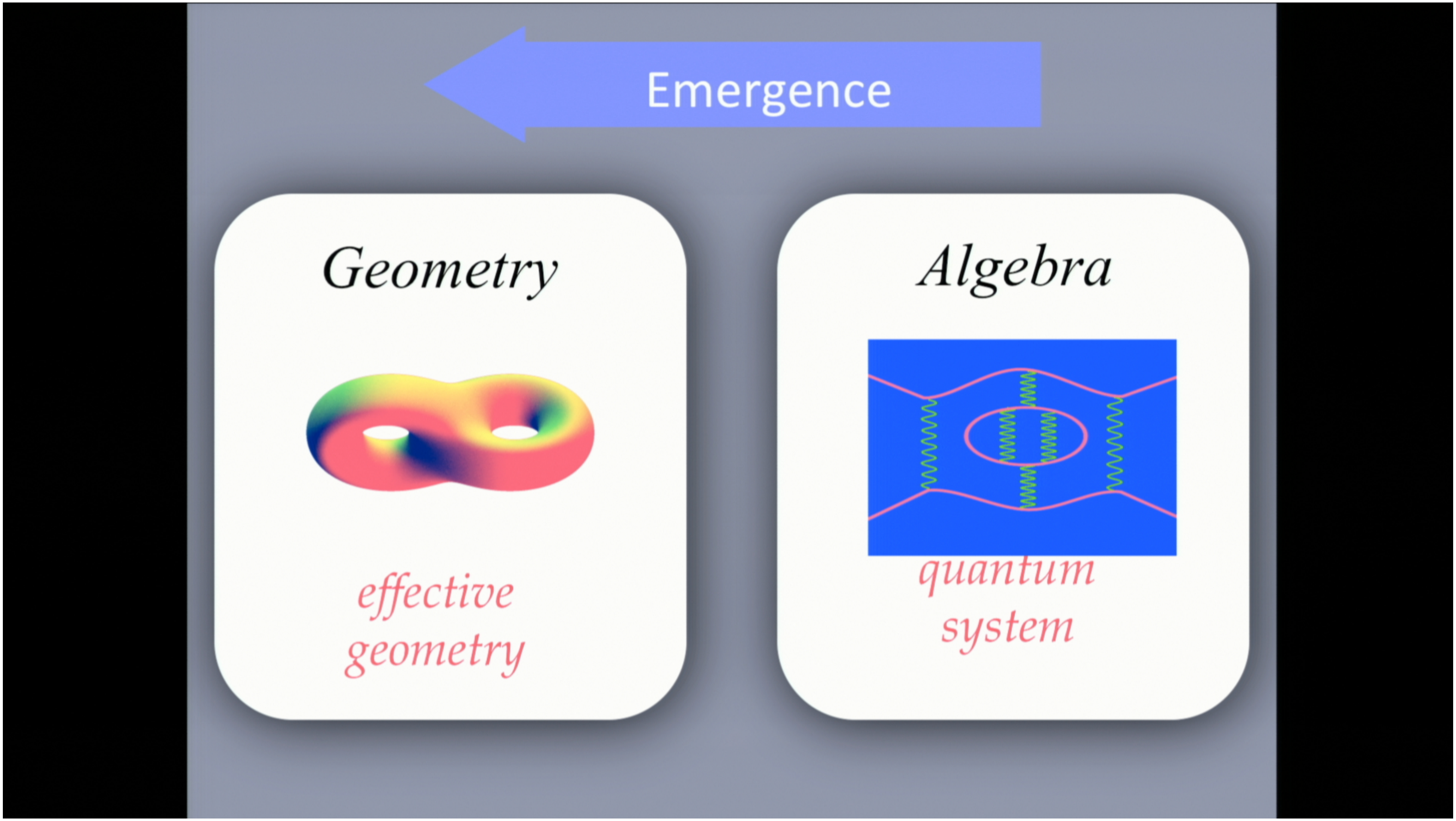


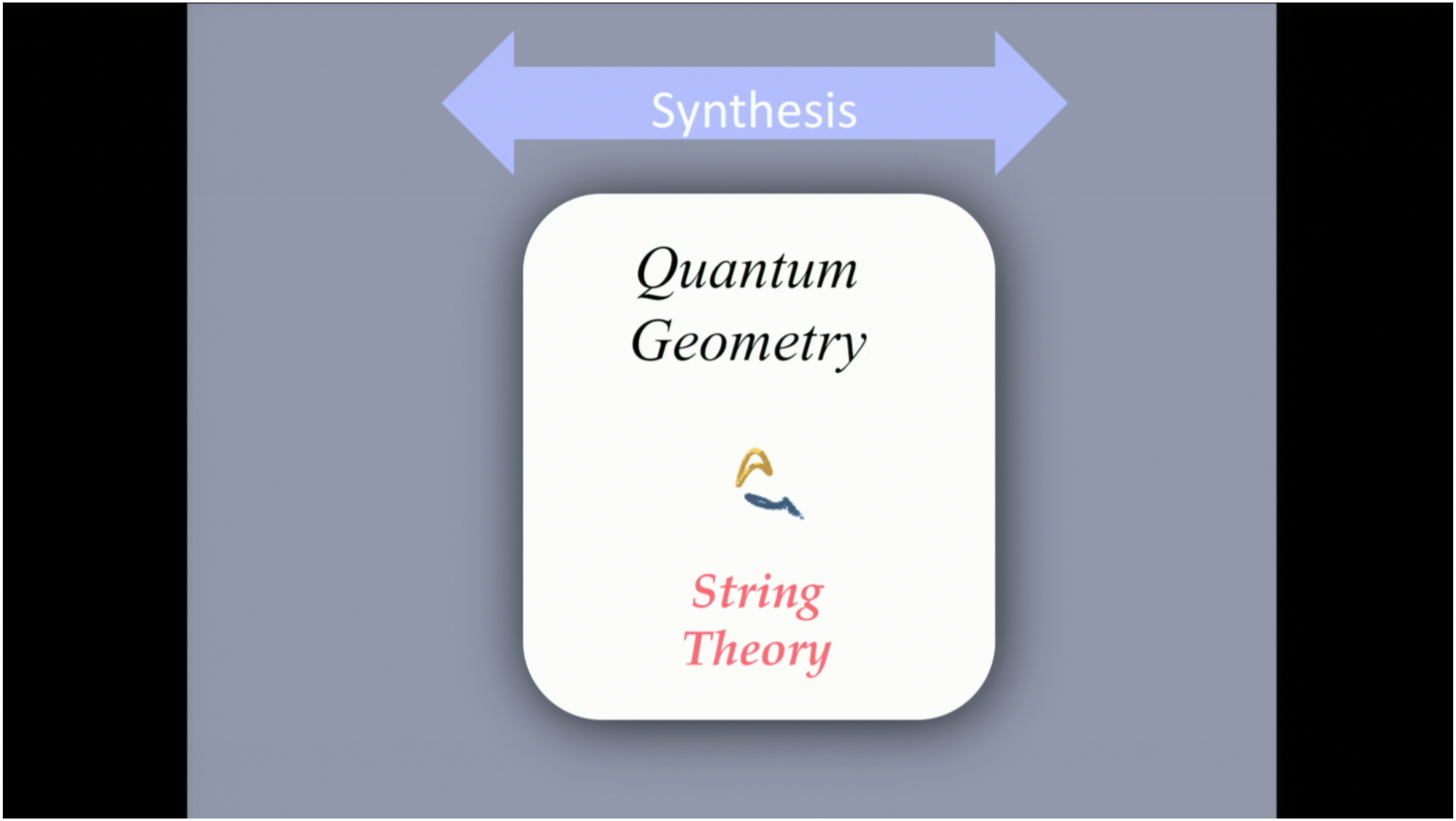

Garbage
Beauty

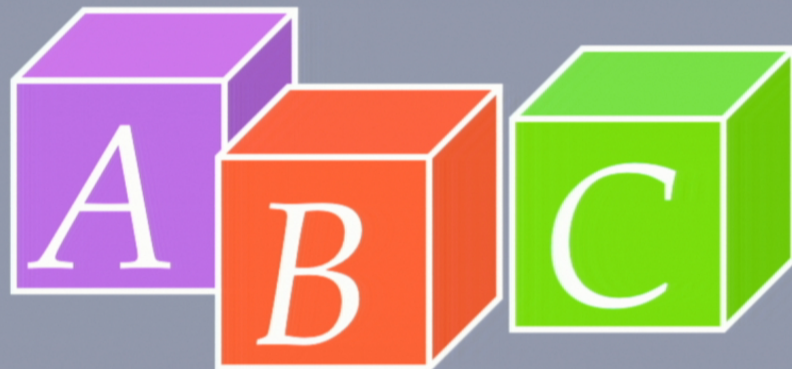

Reduction
Garbage
Beauty
Standard
Model


Emergence
Beauty
hydrodynamics
thermodynamics
Garbage
molecules
statistical mechanics


Quantization
Geometry
K
geometric object
Algebra
Z(K) ∈ C
quantum invariant

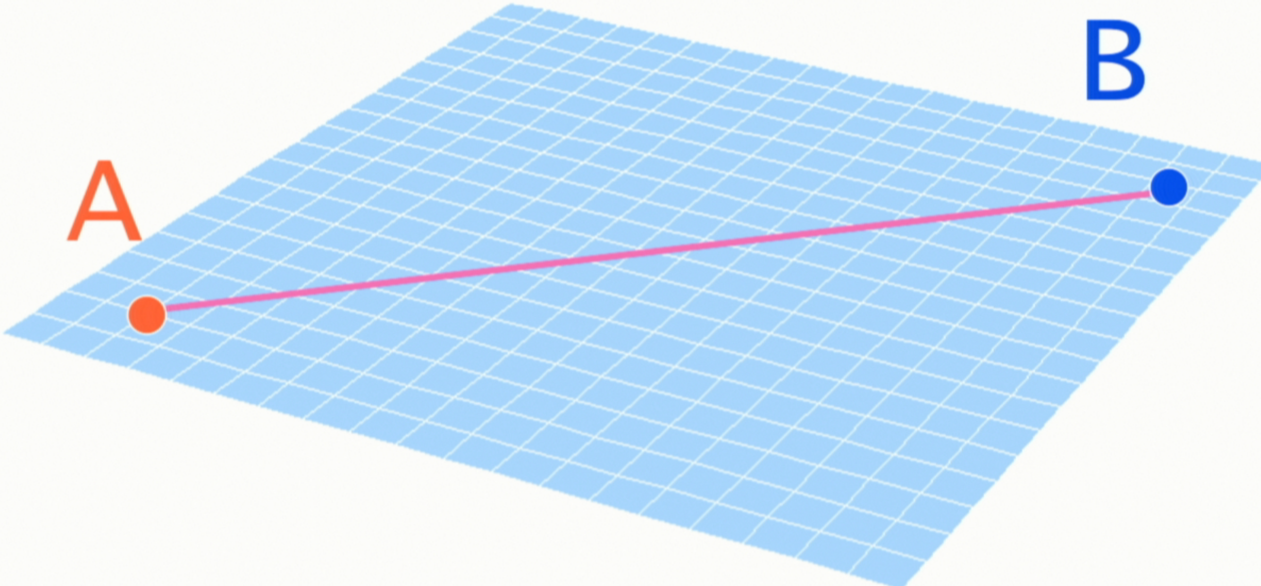

Emergence
Geometry
effective geometry
Algebra
quantum system


Synthesis
Quantum
Geometry
String
Theory

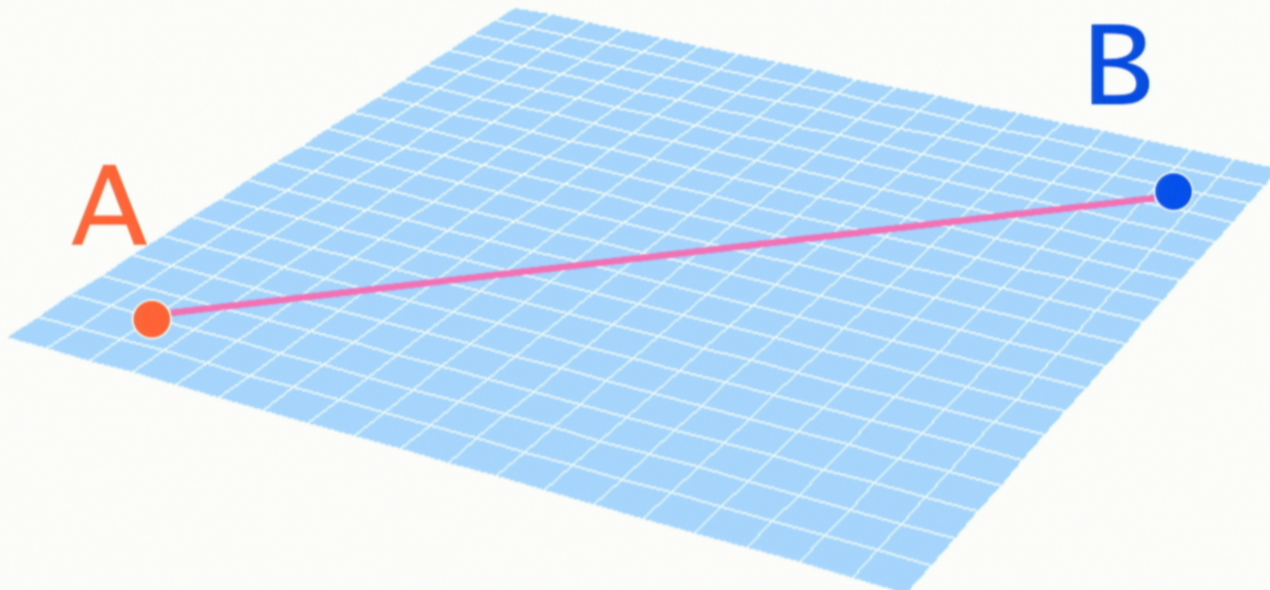


of Physics for Mathematicians

Classical Mechanics

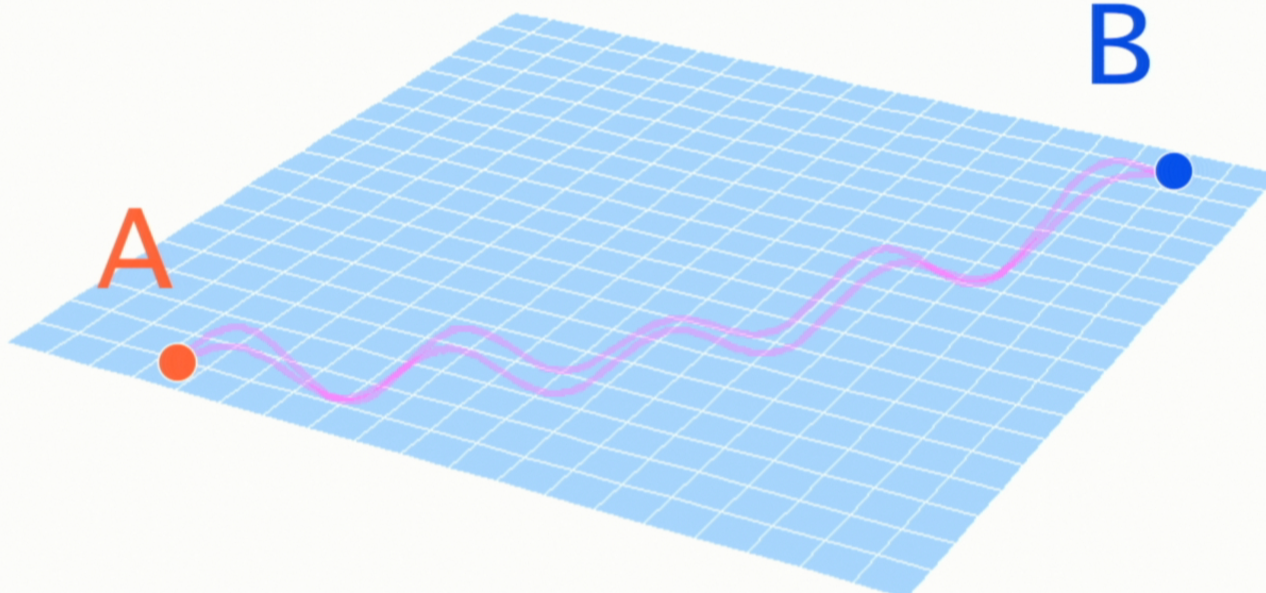


Classical Mechanics



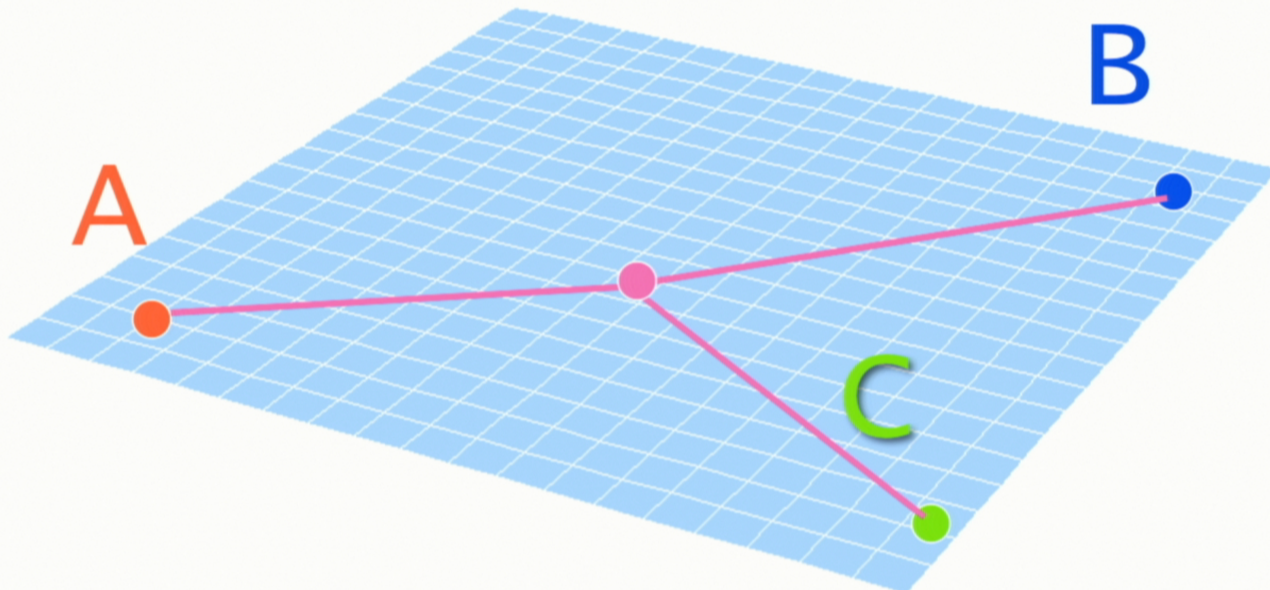
calculus, geometry,
dynamical systems, chaos,...

Quantum Mechanics

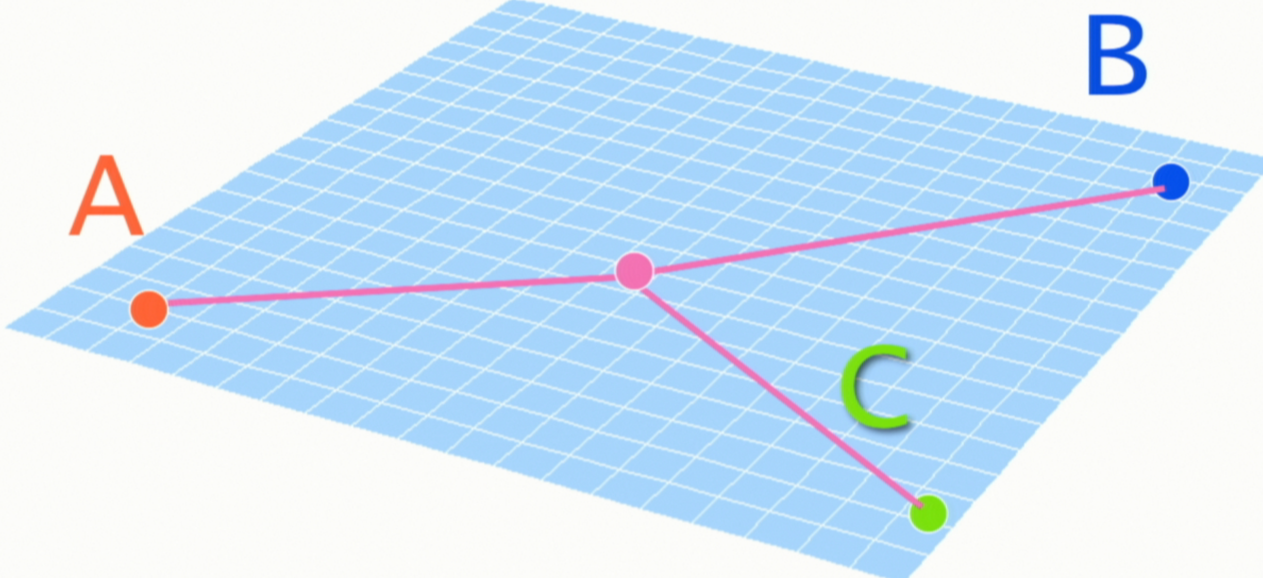


Sum over histories $\sum e^{-Action/h}$

Quantum Field Theory

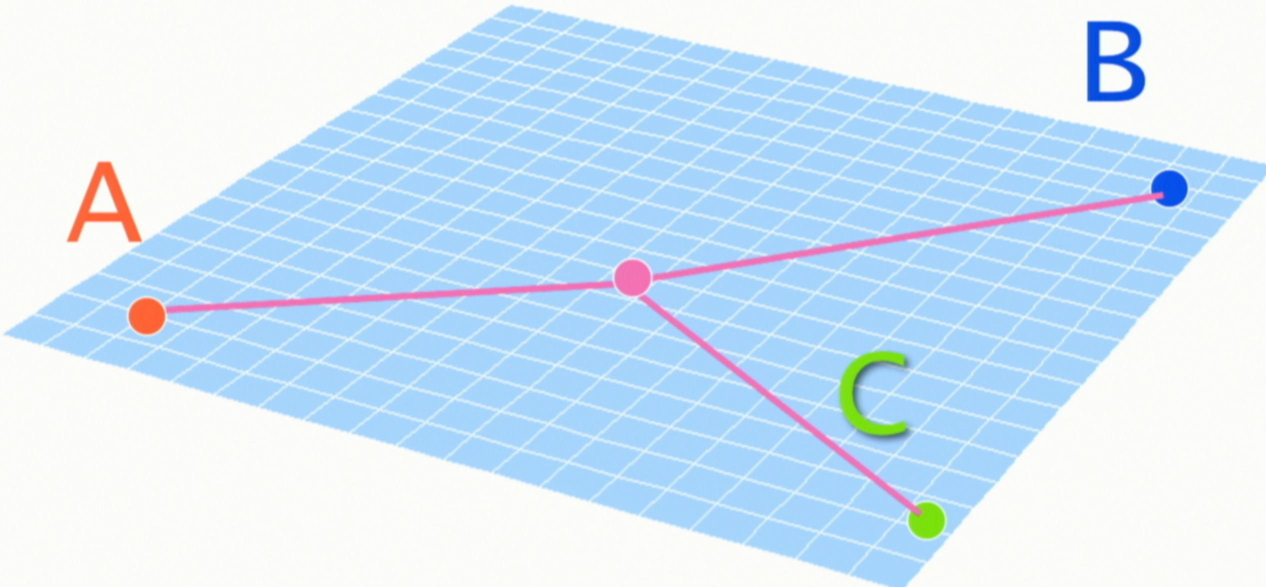


Quantum Field Theory



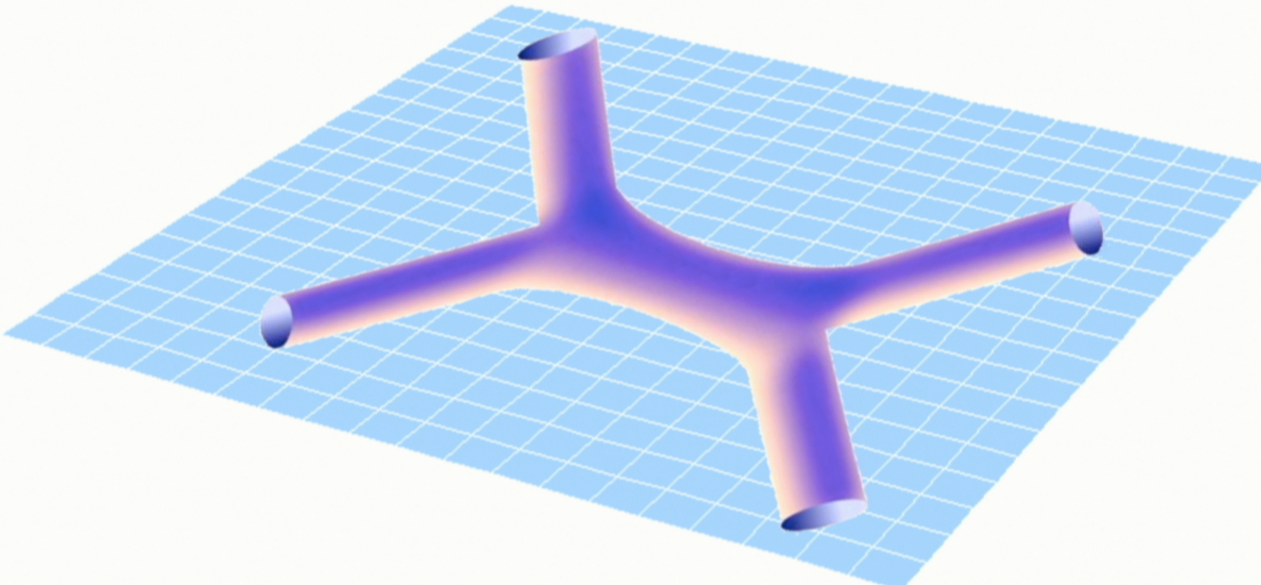
creation/annihilation of particles

Quantum Field Theory

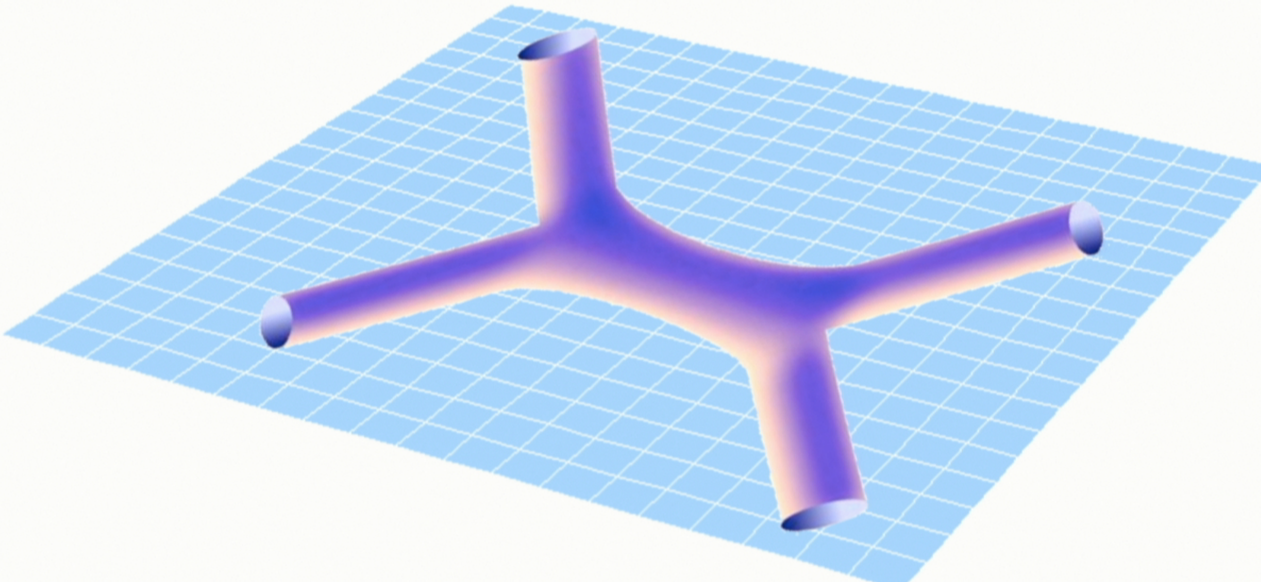


quantum topology:
knots, 3-manifolds, 4-manifolds

Perturbative String Theory

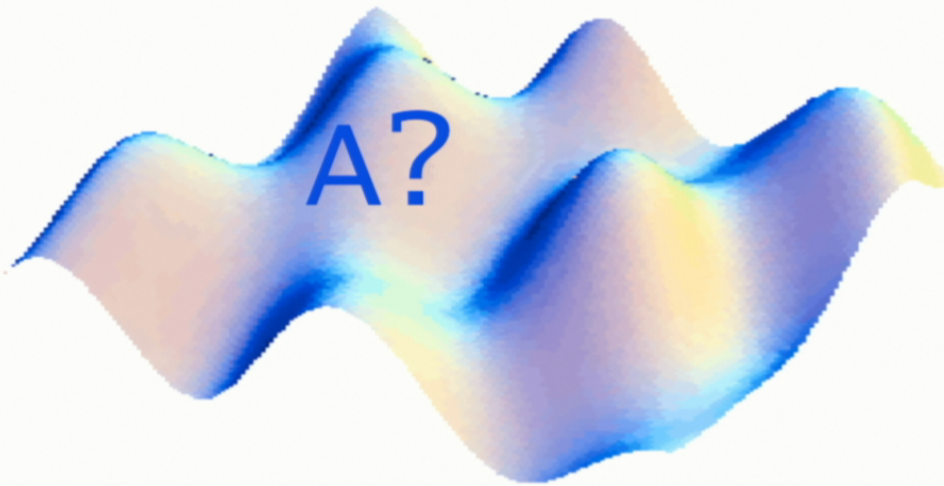


Perturbative String Theory

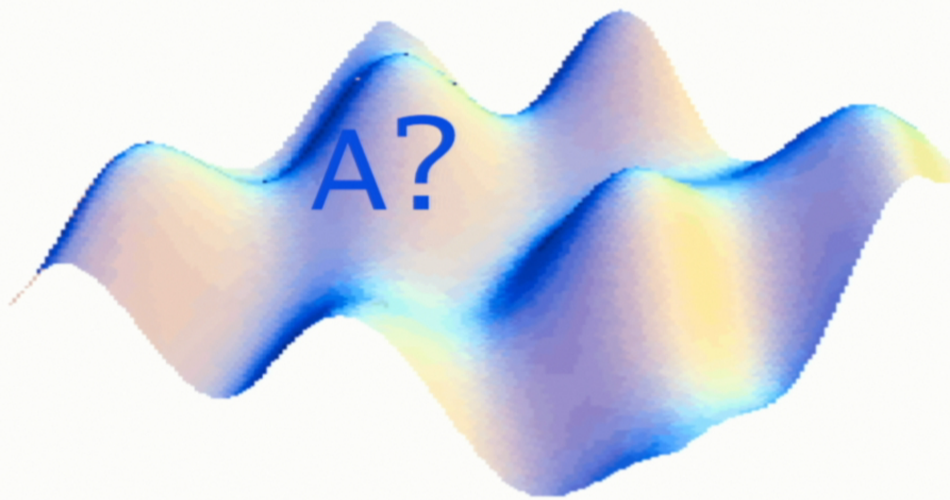


conformal field theory, algebraic curves, moduli spaces, mirror symmetry, …

Quantum Gravity



Quantum Gravity

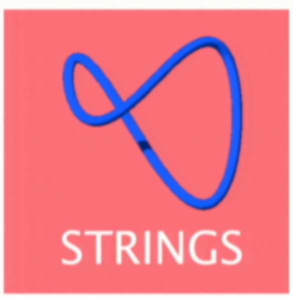


non-commutative geometry, Donaldson-Thomas invariants, automorphic forms, categorification,…

Three Principles

Gravity, General Relativity
(pseudo) Riemannian geometry and topology of space & time

Quantum Mechanics, Gauge Theory
functional analysis, operator algebra
gauge principle $\psi \rightarrow e^{i\varphi} \psi$
non-commutativity, action U(N) on Hilbert space



Strings, Branes, Extended Objects
quantization of loop spaces (Hamiltonian formalism)
Riemann surfaces (Lagrangian formalism)
non-perturbative completion (M-theory): branes

What is Quantum Geometry?

- Non-commutativity: $[x^i,x^j]\neq 0$, in particular $[space,time] \neq 0$
- Deformation of classical geometry, to be recovered in limit $\ell_{\text{Planck}} \rightarrow 0$.

What is Quantum Geometry?

- Non-commutativity: $[x^i, x^j] \neq 0$, in particular $[space, time] \neq 0$
- Deformation of classical geometry, to be recovered in limit $\ell_{\text{Planck}} \to 0$.
- Quantum foam: quantized topological features of size $n\,\ell_{\text{Planck}}$ (à la Bohr-Sommerfeld orbits).
- Holography: geometry is emergent in $N \to \infty$ limit, cf statistical mechanics vs thermodynamics (universality).
- Geometry depends on the probe. Need to split system in (big) space-time and (small) observer/test object.

Role of Mathematics

Role of Mathematics

- What are the appropriate mathematical concepts, if they have yet been found at all?
- What new fields of mathematics should be involved (number theory & arithmetic geometry, logic, ...)?
- Are we looking for a single universal structure (global definition), or several complementary points of view (local definition, charts + maps, "duality frames")?
- *Quantum Geometry* looks like a mathematical ``program'' such as the Langlands program (many non-trivial examples, strange relations, dualities, automorphic forms, tying together diverse fields, vast generalizations, open ended, ...).

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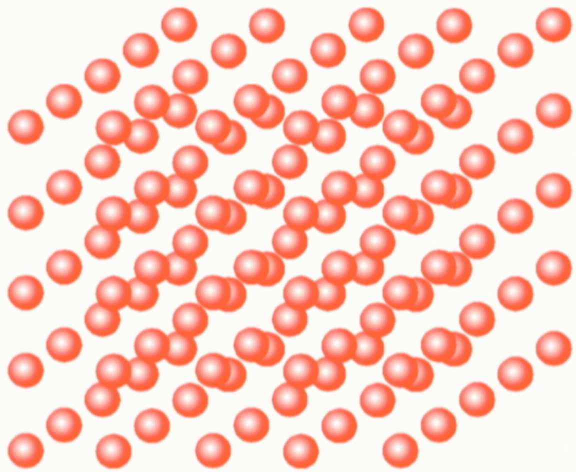
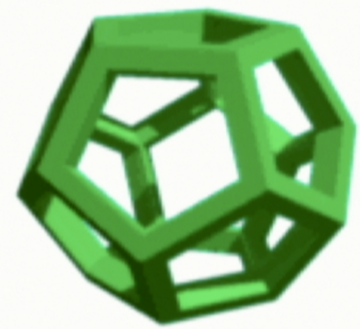
- *Quantum Geometry* looks like a mathematical ``program'' such as the Langlands program (many non-trivial examples, strange relations, dualities, automorphic forms, tying together diverse fields, vast generalizations, open ended, ...).
- Is mathematical elegance a guiding light or a Siren, whose song draws the Ship of Physics onto the cliffs?

(What Kind of) Mathematical Beauty



universal
calculus,
Hilbert spaces

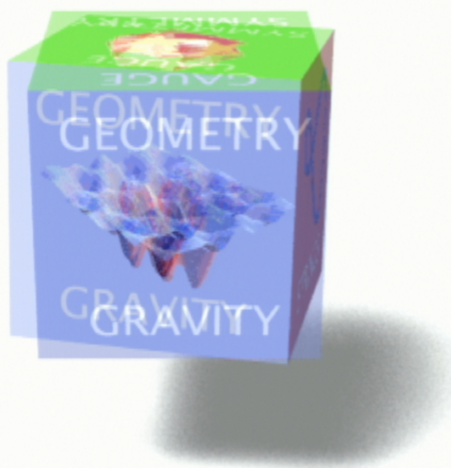


exceptional
E_8, Monster,
Calabi-Yau

Mathematics of String Theory

Connects gravity and gauge theory in a natural way

- Closed strings = gravity
- Open strings = gauge field



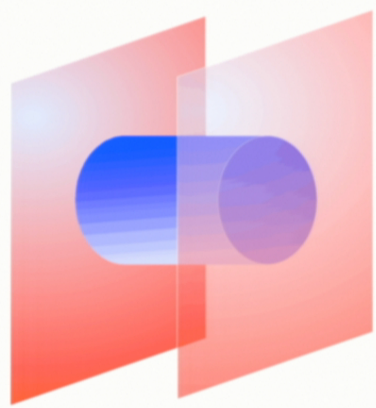
Atiyah-Singer Index Theorem

analysis ⇔ geometry

Atiyah-Singer Index Theorem

analysis ⇔ geometry

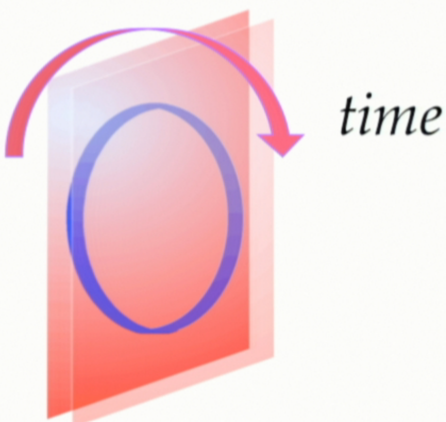
$$\text{Index}\, \not{D}_{E_1 \otimes E_2} = \int ch(E_1 \otimes E_2^*)\hat{A}(X)$$

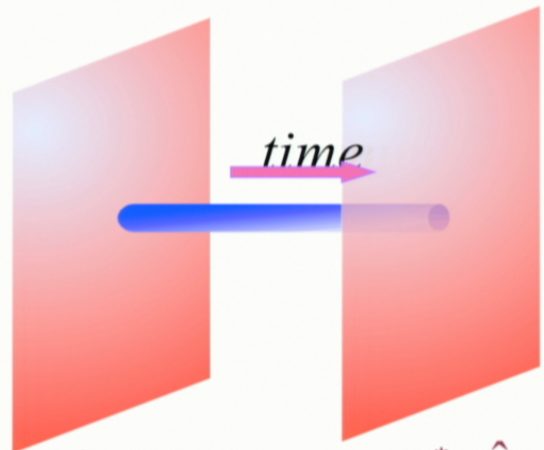


Atiyah-Singer Index Theorem

analysis ⇔ geometry

$$\text{Index}\, \not{D}_{E_1 \otimes E_2} = \int ch(E_1 \otimes E_2^*)\hat{A}(X)$$



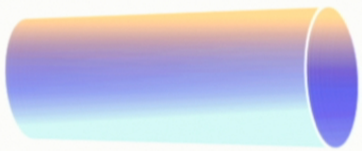


$$\text{Tr}(-1)^F = \text{Index}\, \not{D}_{E_1 \otimes E_2}$$

$$\langle E_1, E_2 \rangle = \int ch(E_1 \otimes E_2^*)\hat{A}(X)$$

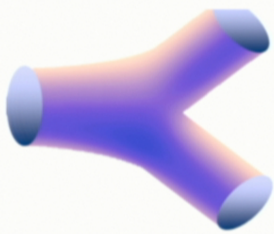
Two fundamental parameters in string theory

String length *(Planck's constant on world sheet)*



$\ell_{string}, \ \alpha' = \ell^2_{string}$

String coupling *(Planck's constant in space time)*

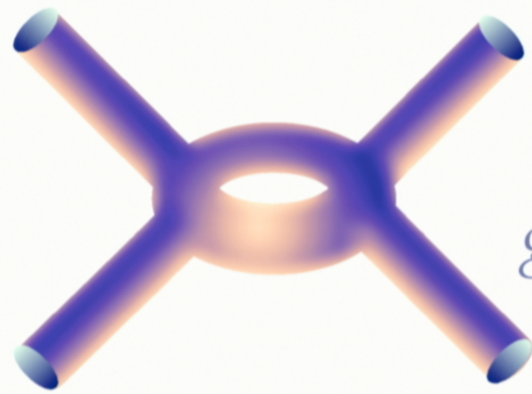


g_{string}

String Partition Function

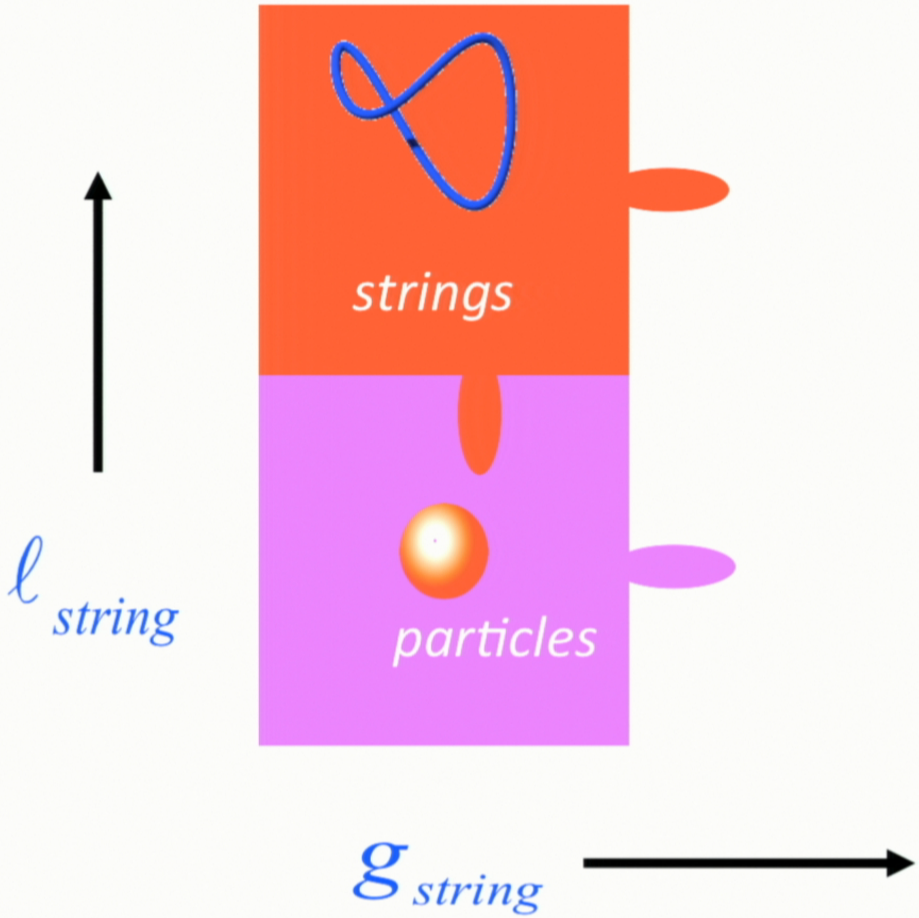
$$Z_{string} = \exp \sum_{g \geq 0} g_{string}^{2g-2} F_g(t)$$

$g_{string} \approx 0$

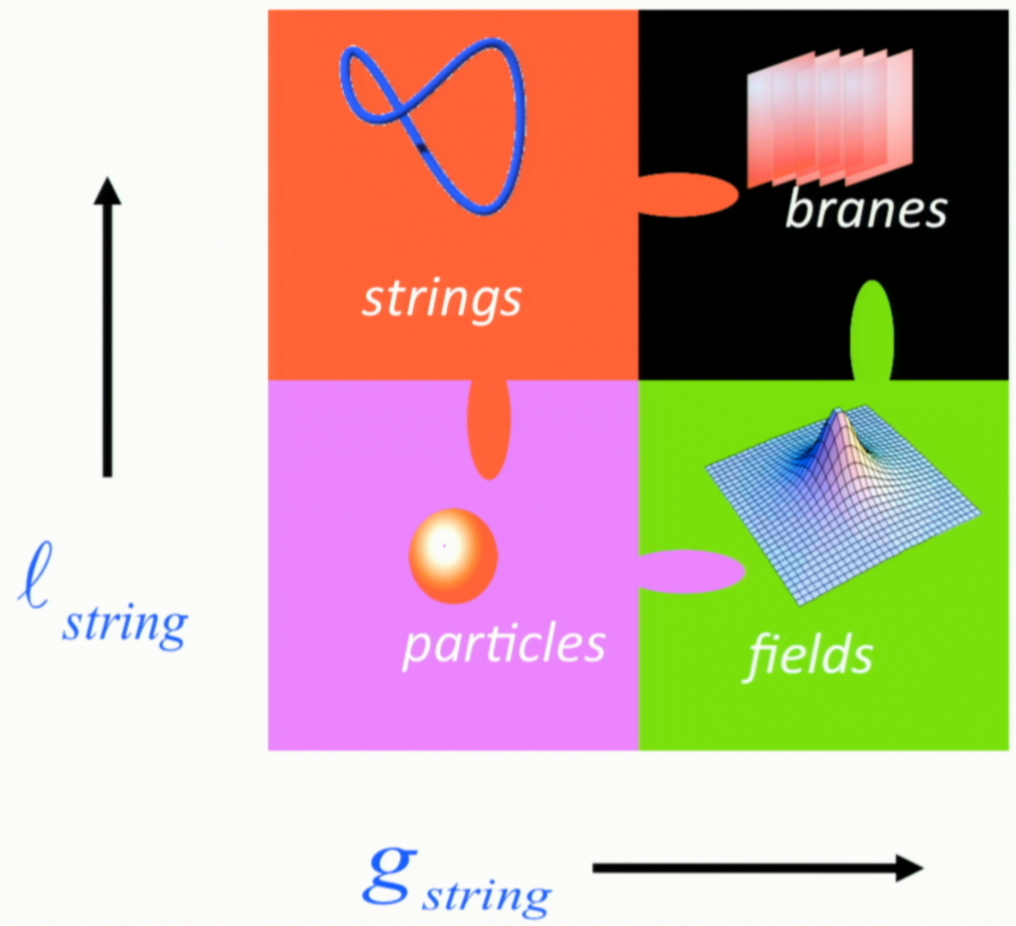


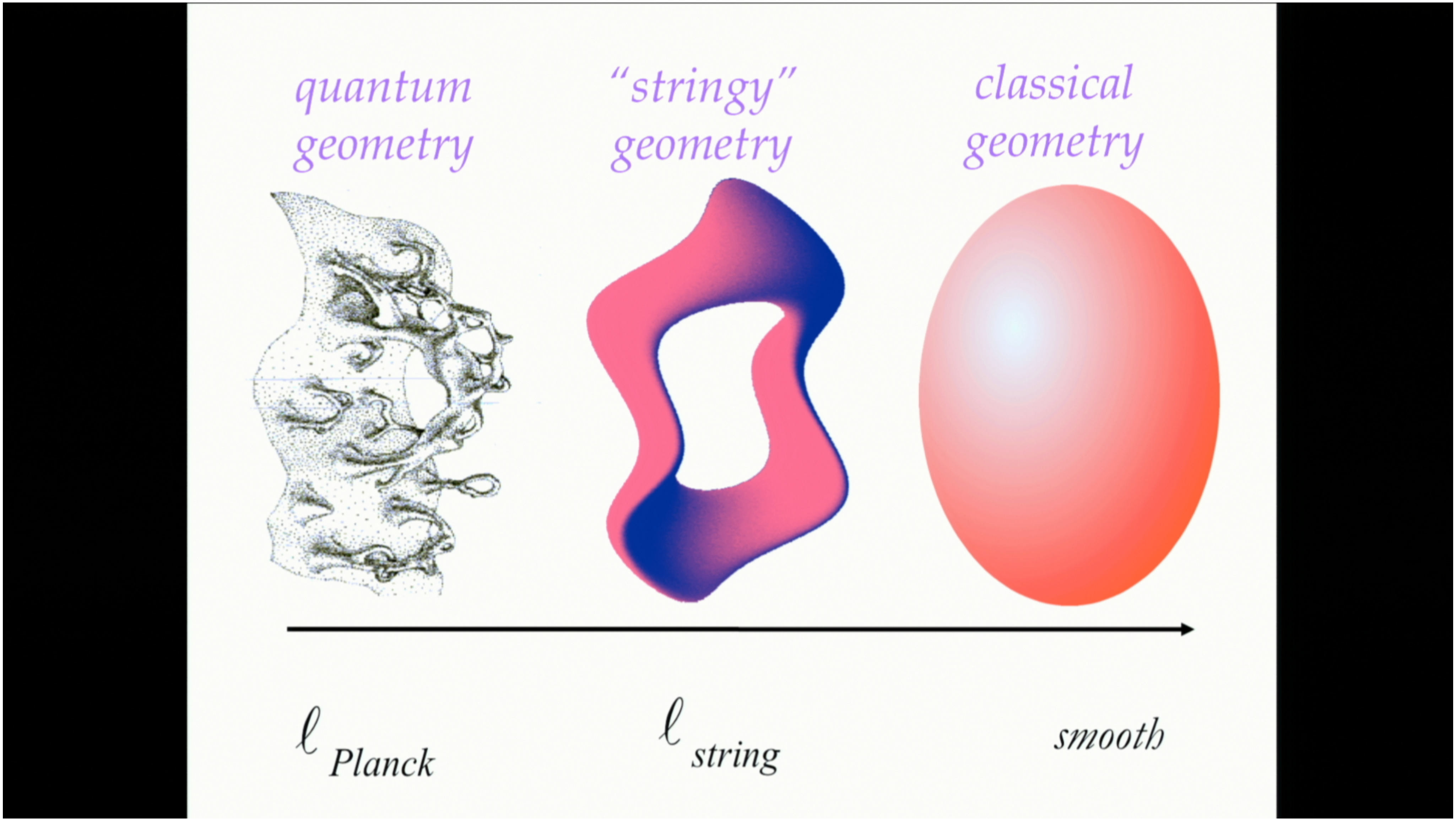
Moduli spaces of algebraic curves

Phase Diagram



Phase Diagram




quantum
geometry
"stringy"
geometry
classical
geometry
ℓ_{Planck}
ℓ_{string}
smooth

(Supersymmetric) QM

differential geometry & topology

$$\text{Hilbert space} = \Omega^*(X)$$
$$H = -\Delta = -(dd^* + d^*d)$$

ground states = harmonic forms

$$Harm^*(X) \cong H^*(X)$$

(Supersymmetric) QM

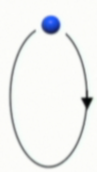
differential geometry & topology

$$Hilbert\ space = \Omega^*(X)$$
$$H = -\Delta = -(dd^* + d^*d)$$

ground states = harmonic forms

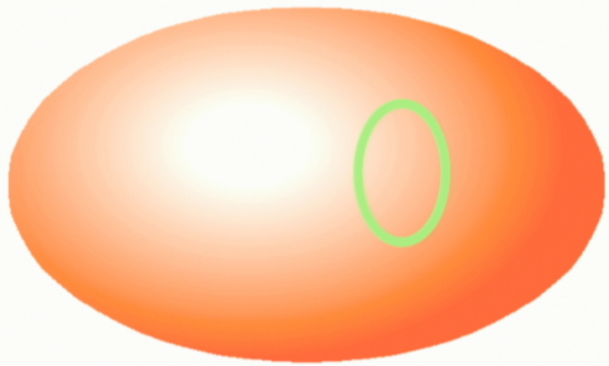
$$Harm^*(X) \cong H^*(X)$$

partition function = Witten index



$$Tr((-1)^{\deg} e^{-tH}) = Euler(X)$$

Conformal Field Theory



Loop Space $\mathcal{L}X$

Quantize loop space

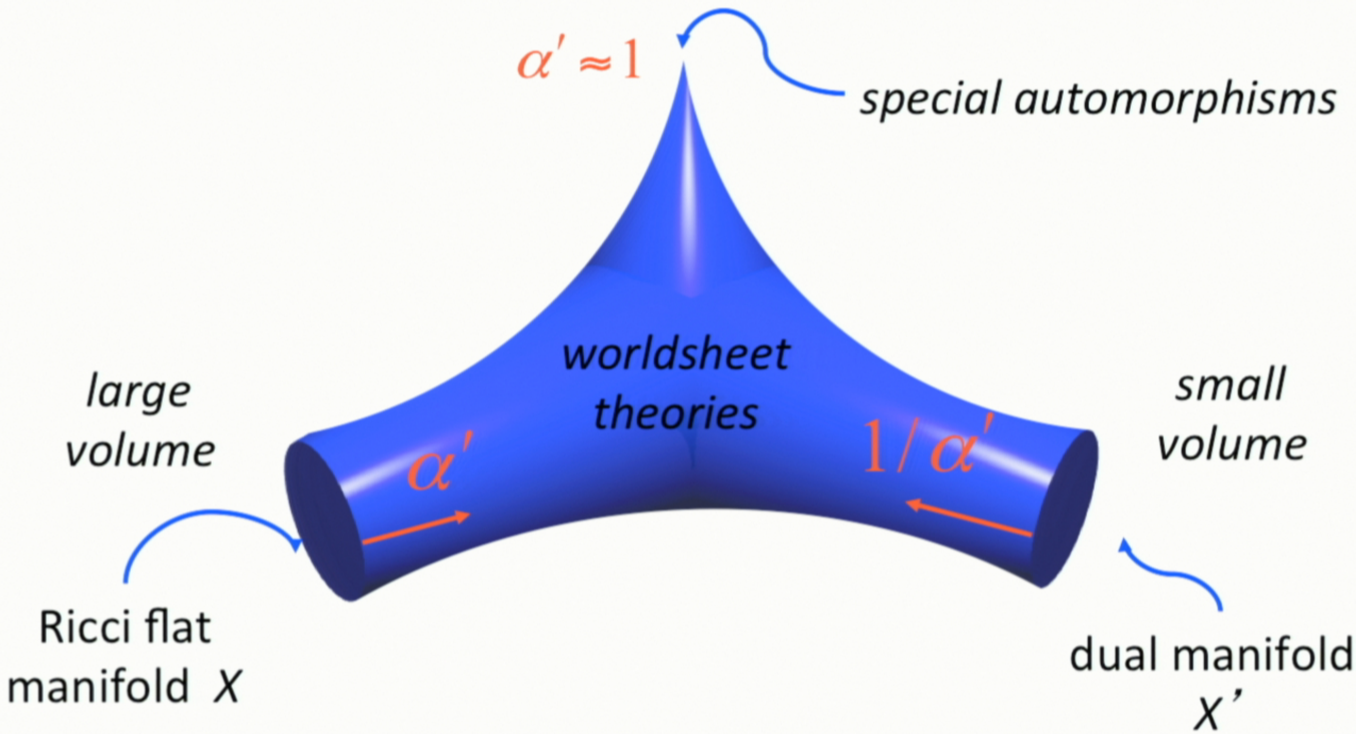
$$\mathcal{H} = L^2(X, \mathcal{F} \otimes \mathcal{F})$$

Fock space created by string oscillators
∞-dimensional analysis

thicken $\mathrm{X} \subset \mathrm{LX}$

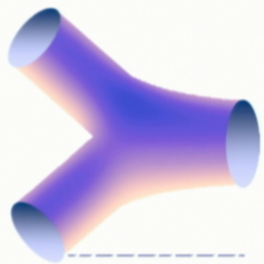
Moduli Space of D=2 CFTs

Moduli Space of D=2 CFTs



Classification of CFT

Algebra



$$A \otimes A \to A$$

∞-dim chiral vertex algebras
classification reps Virasoro

Geometry

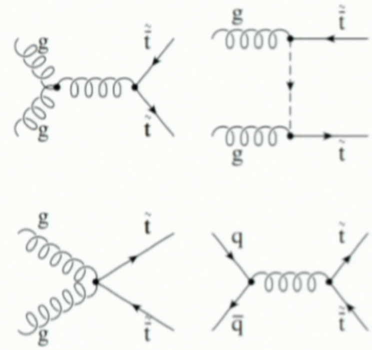
higher genus
modular invariance
*SL(2,**Z**)*
completeness of d.o.f.

Quantum Field Theory

Algebra

$$\langle O_1(x_1) \ldots . O_n(x_n) \rangle$$

local operators
scattering amplitudes

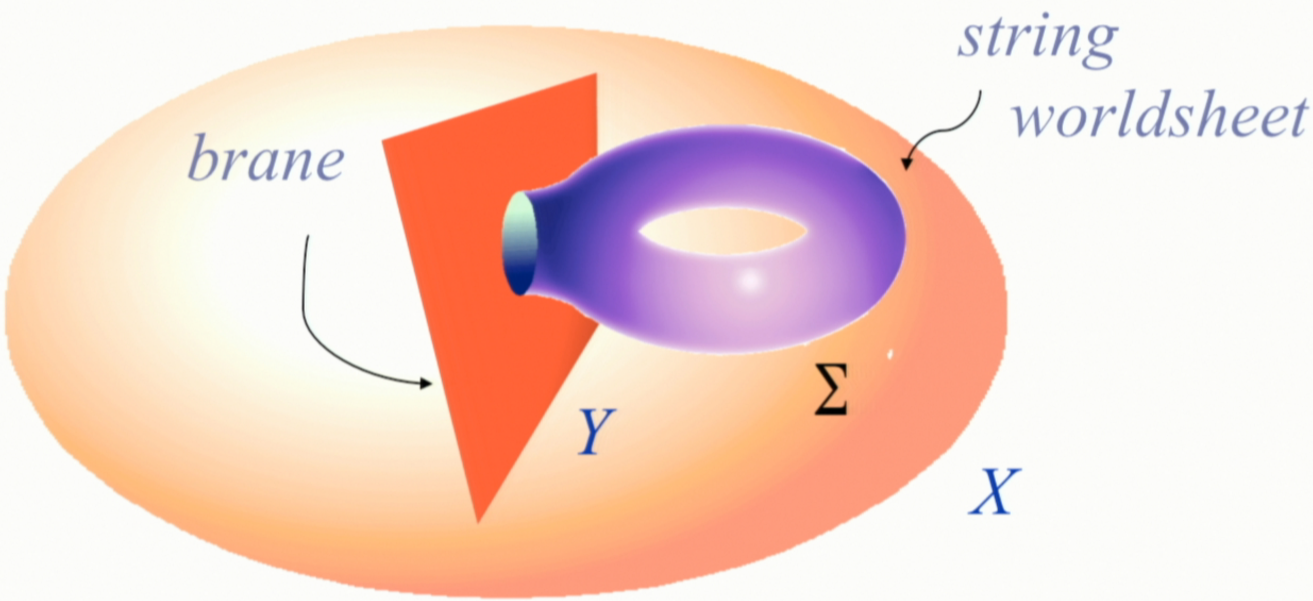


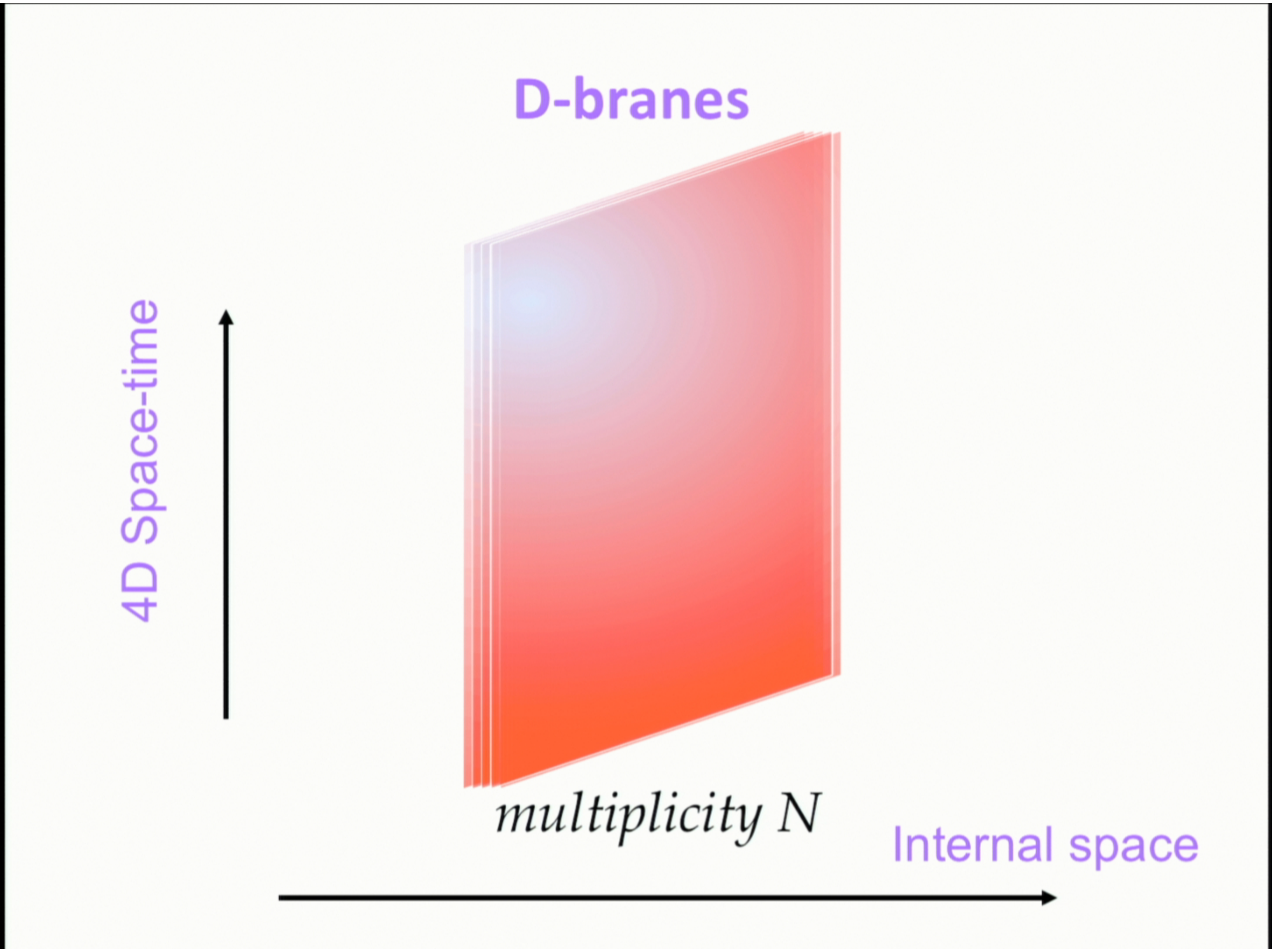
Geometry

topological indices
defect operators

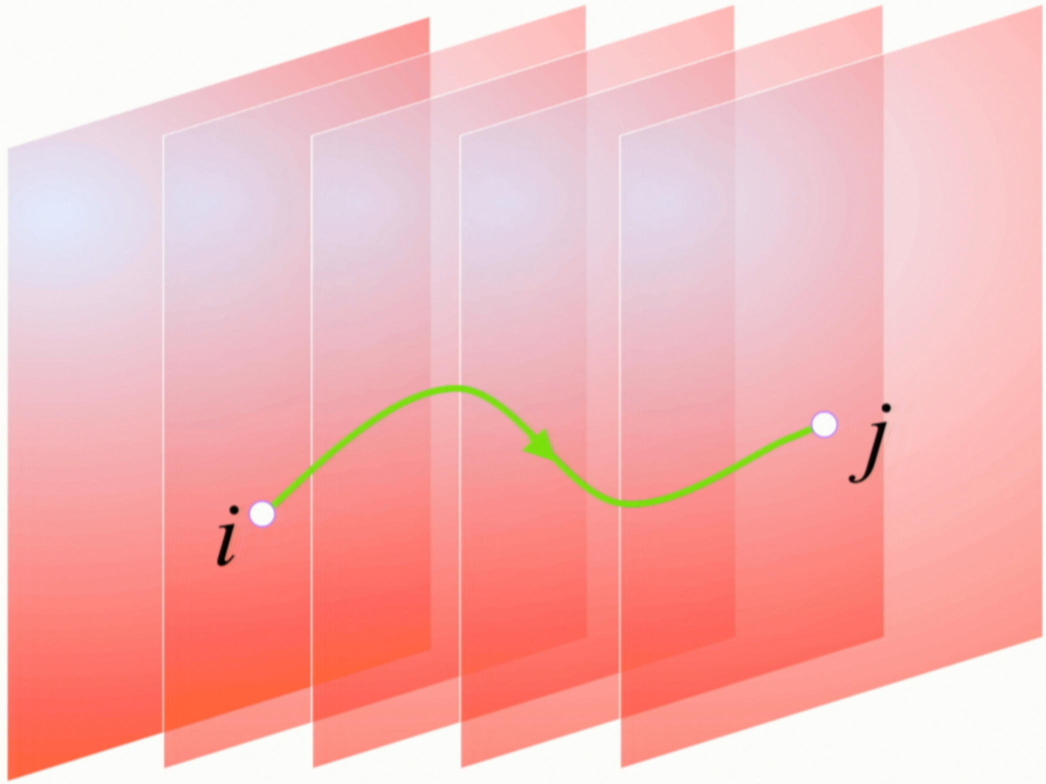
D-branes: Relative CFT

Space-time picture

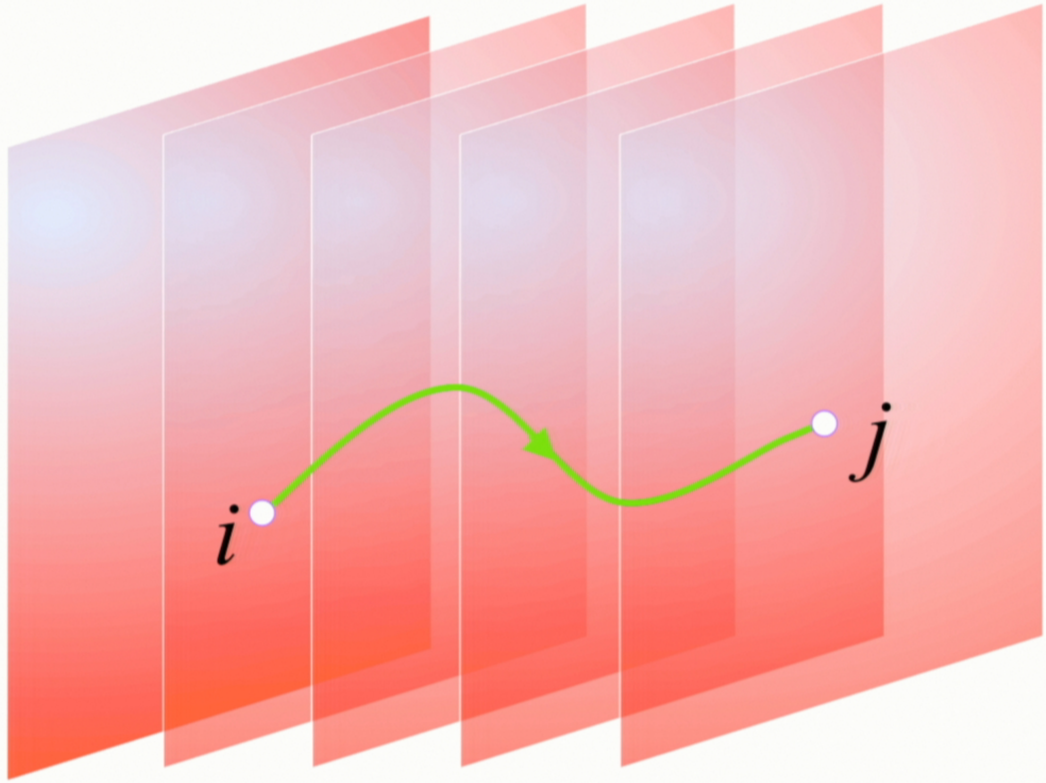



D-branes
4D Space-time
multiplicity N
Internal space

U(N) Yang-Mills Theory

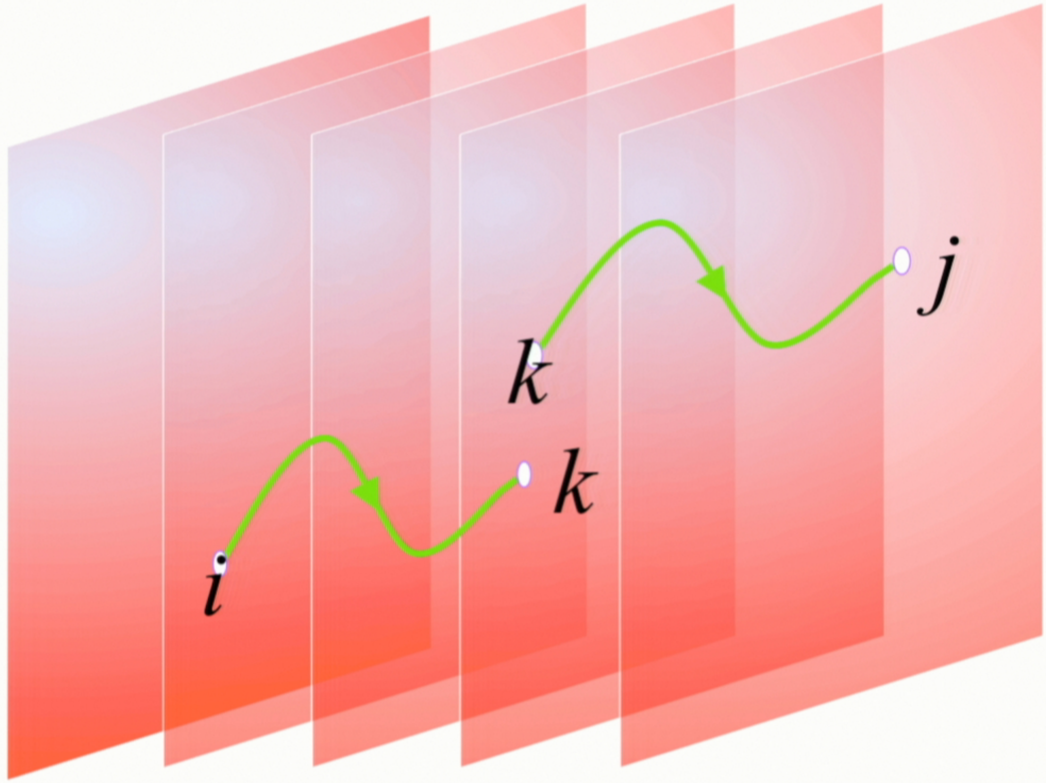


U(N) Yang-Mills Theory

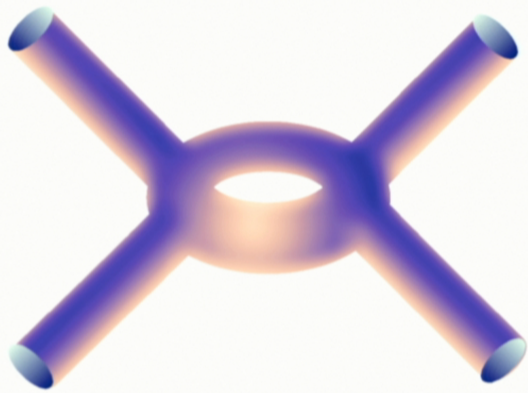


$N \times N$ *matrix of strings* A_{ij}

U(N) Yang-Mills Theory



S-Dualities



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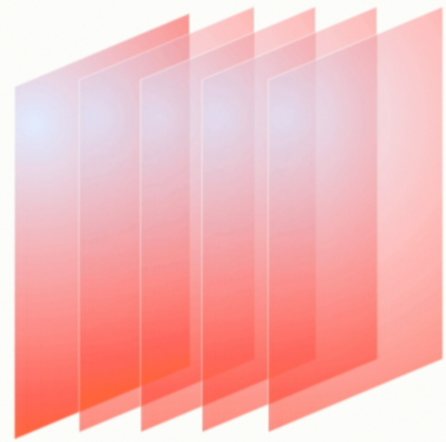
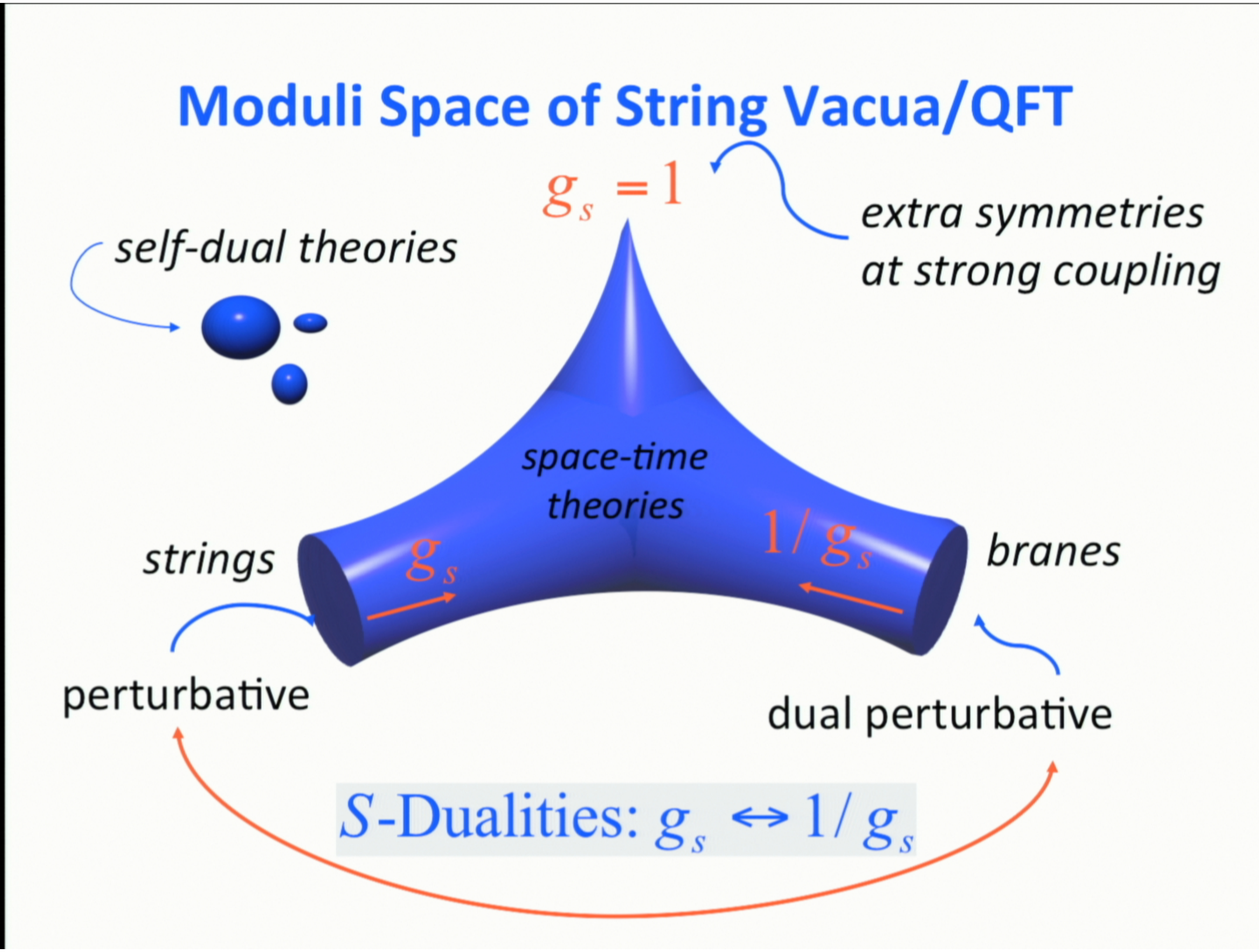


inf-dim Lie algebras
loop spaces
Virasoro algebra
genus expansion

D-branes

$g_s \approx \infty$

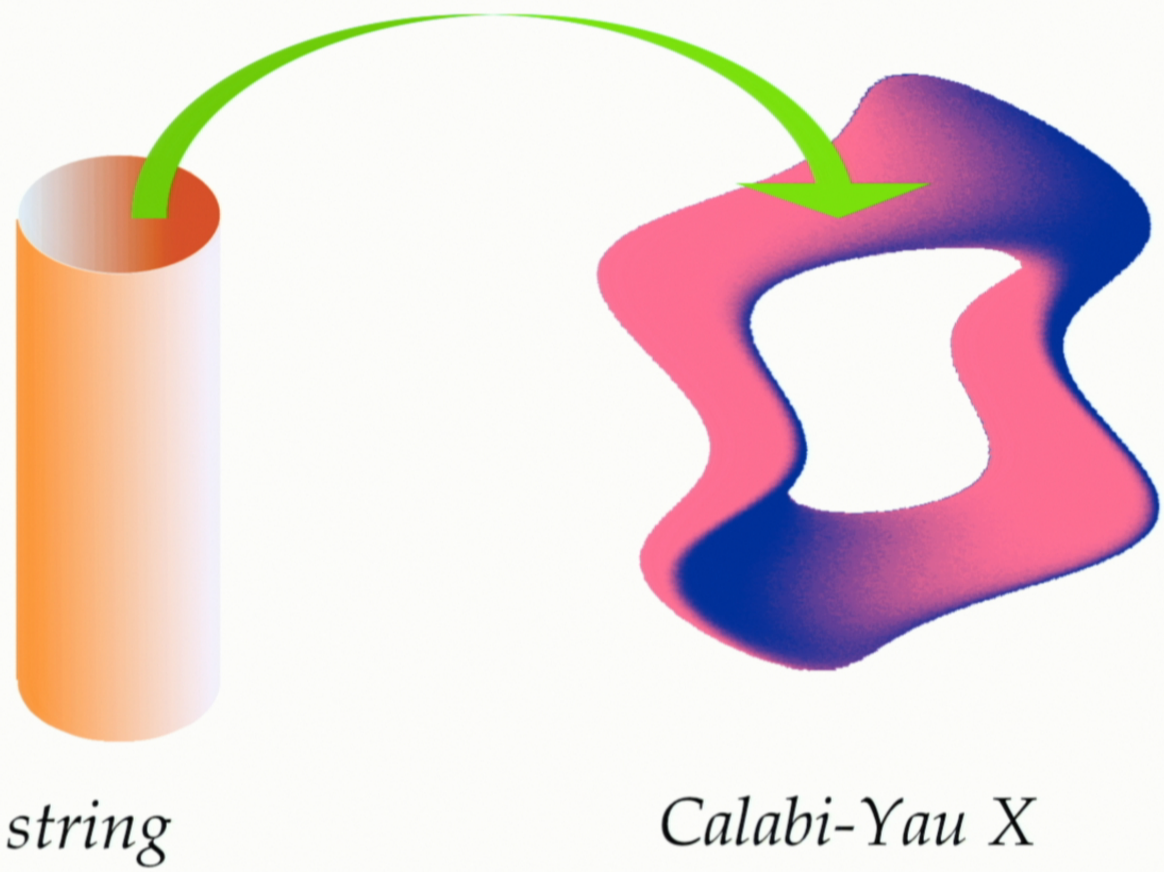


Moduli Space of String Vacua/QFT
$g_s = 1$
extra symmetries at strong coupling
self-dual theories
space-time theories
strings
g_s
$1/g_s$
branes
perturbative
dual perturbative
S-Dualities: $g_s \leftrightarrow 1/g_s$

Topological Strings

- Toy model that allows exact computations (cf topology versus geometry)
- Exact BPS/supersymmetric sector of superstrings (holomorphic)
- Critical dim = 6, Calabi-Yau manifolds

Quantum Amplitude F



Instanton sum = exact

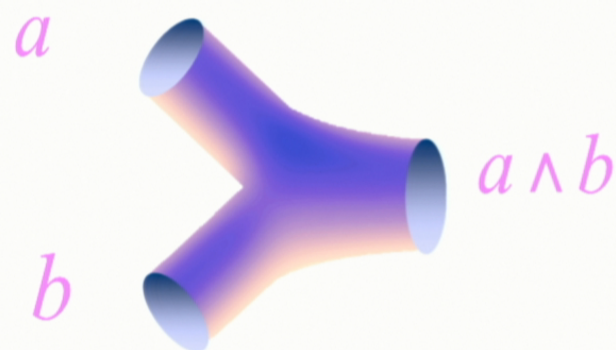
$$F(t) = \sum_{d \geq 0} N_d e^{-dt}$$

$d = \mathrm{deg}$

$t = area / \ell^2_{string}$

"Stringy" Geometry

physical fields = cohom classes $a,b \in H^*(X)$



$$H^* \otimes H^* \to H^*$$

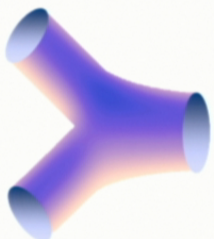
Classical Intersection Product

$\ell_s = 0$

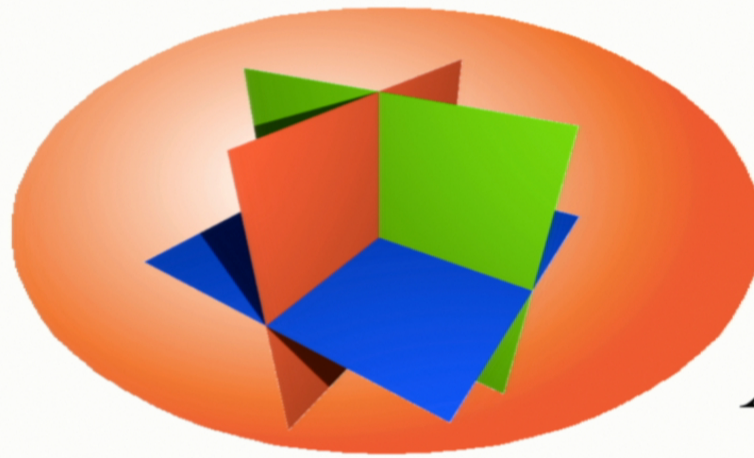
dual cycles

a

b



c =

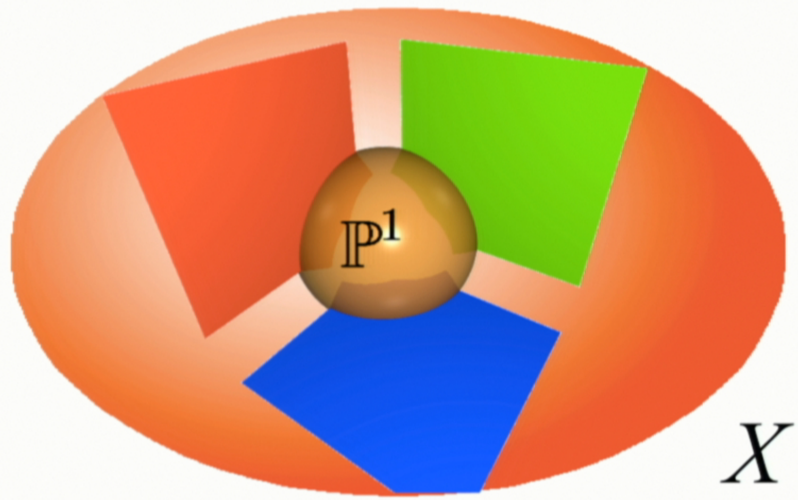


X

$$= \int_X a \wedge b \wedge c$$

$\ell_s > 0$ **Quantum Cohomology**

$$= \sum_{\substack{\text{rat curves} \\ \text{degree } d}} e^{-dt/\ell_s^2}$$



$\ell_s > 0$ **Quantum Cohomology**

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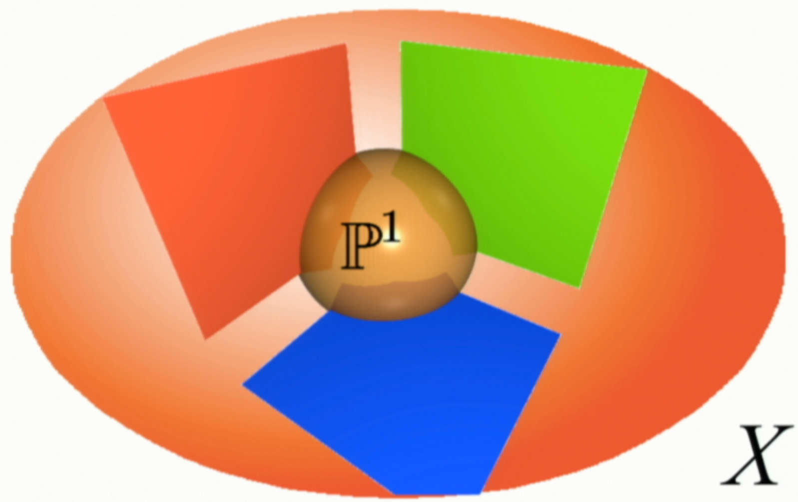
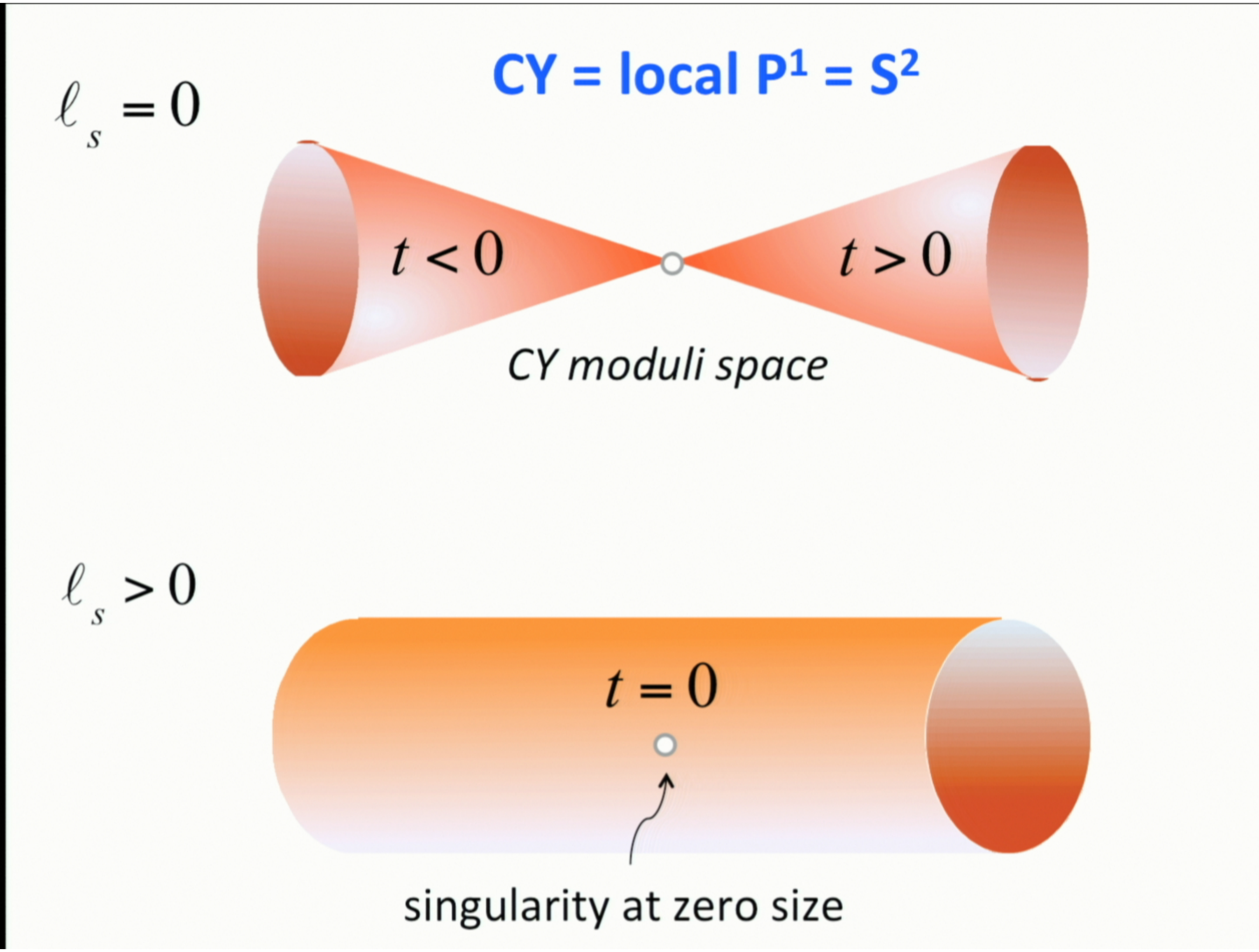
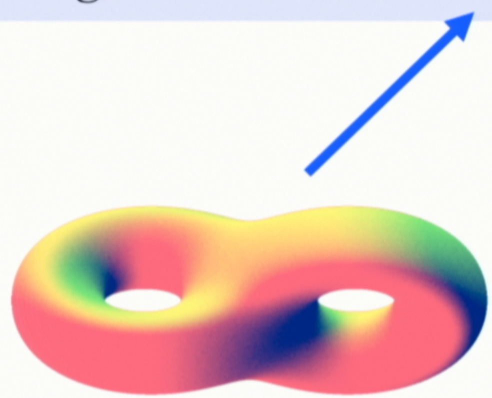


for $\boldsymbol{P}^n$ $\quad x^{n+1} = 0 \quad \Rightarrow \quad x^{n+1} = e^{-t}$


CY = local $P^1 = S^2$
$\ell_s = 0$
$t < 0$
$t > 0$
CY moduli space
$\ell_s > 0$
$t = 0$
singularity at zero size

Topological string partition function

$$Z_{top} \underset{g_s \approx 0}{=} \exp \sum_{g \geq 0} g_s^{2g-2} F_g(t)$$

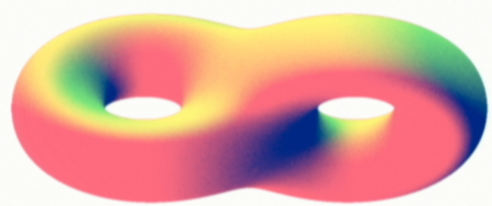


genus g

Topological string partition function

$$Z_{top} \;=\; \exp \sum_{g \geq 0} g_s^{2g-2} F_g(t)$$

$g_s \approx 0$

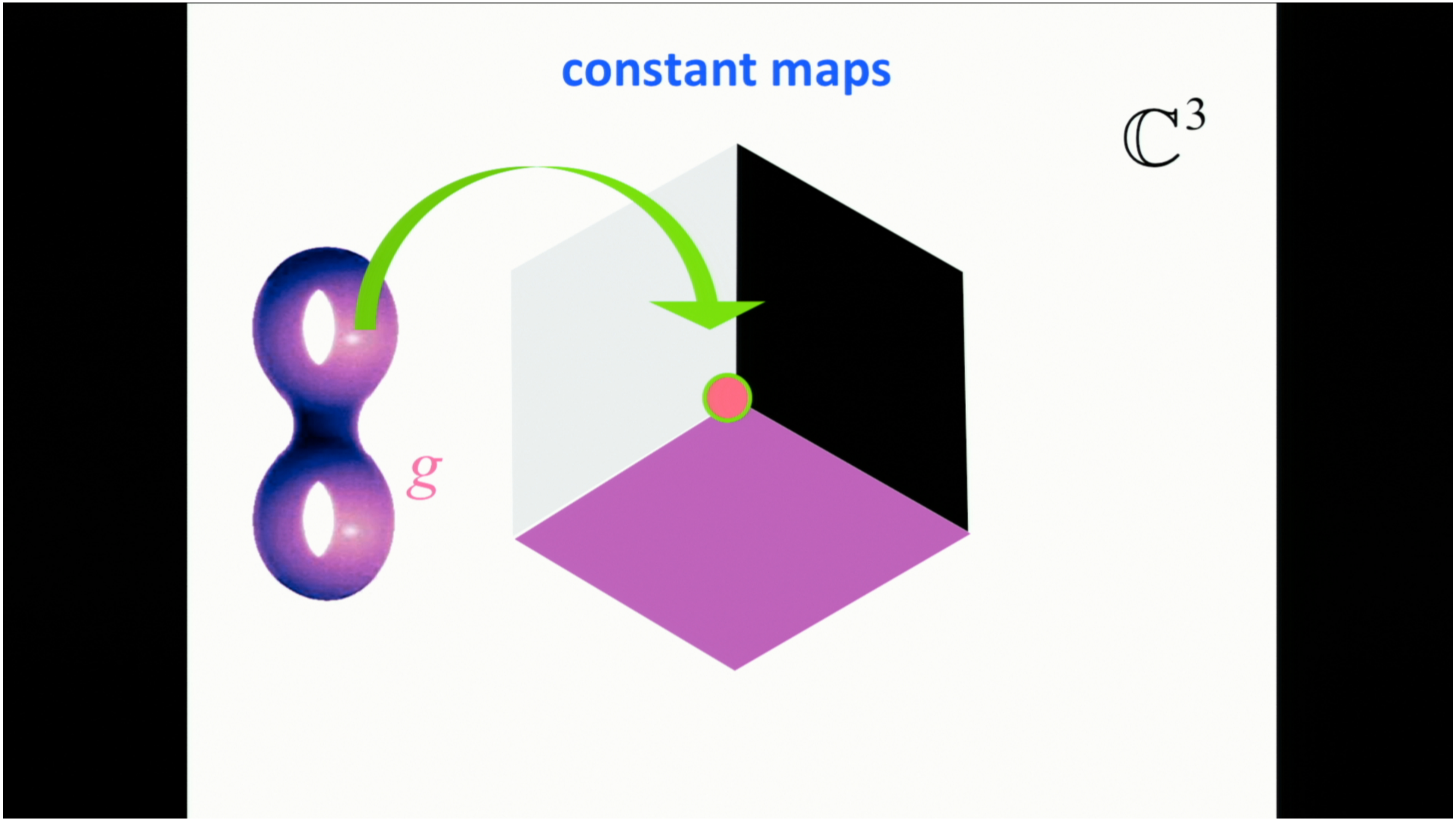


genus g

Simplest Calabi-Yau

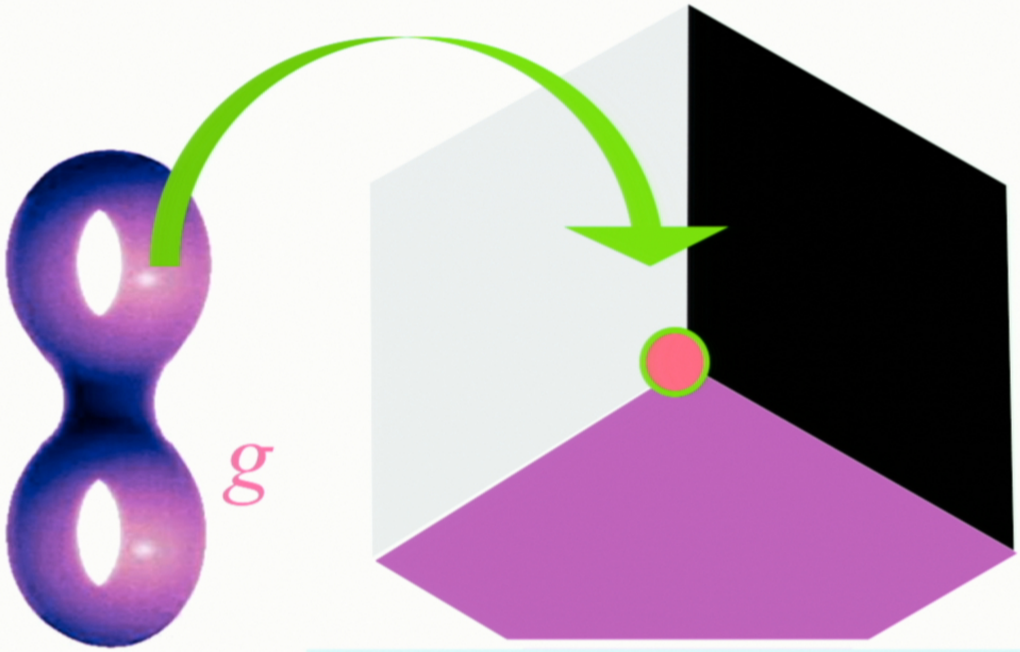
$\mathbb{C}^3$

$$\left(|z_1|,|z_2|,|z_3|\right) \in \mathbb{R}^3$$


constant maps
$\mathbb{C}^3$
g

constant maps

$\mathbb{C}^3$



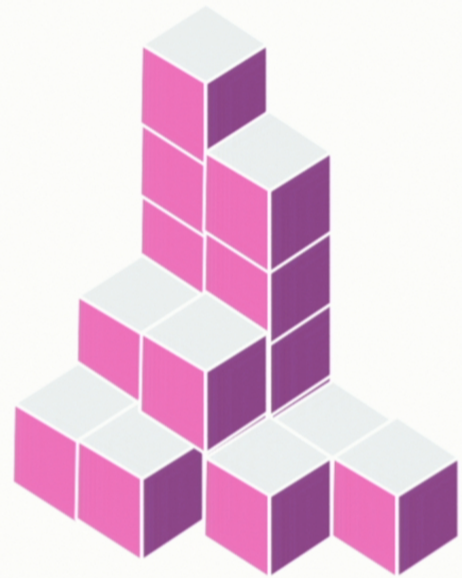
$$N_{g,0} = \int_{\bar{M}_g} \lambda_{g-1}^3 = \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}$$

3d Partitions

$$Z = \exp \sum_{g \geq 0} N_g g_s^{2g-g}$$

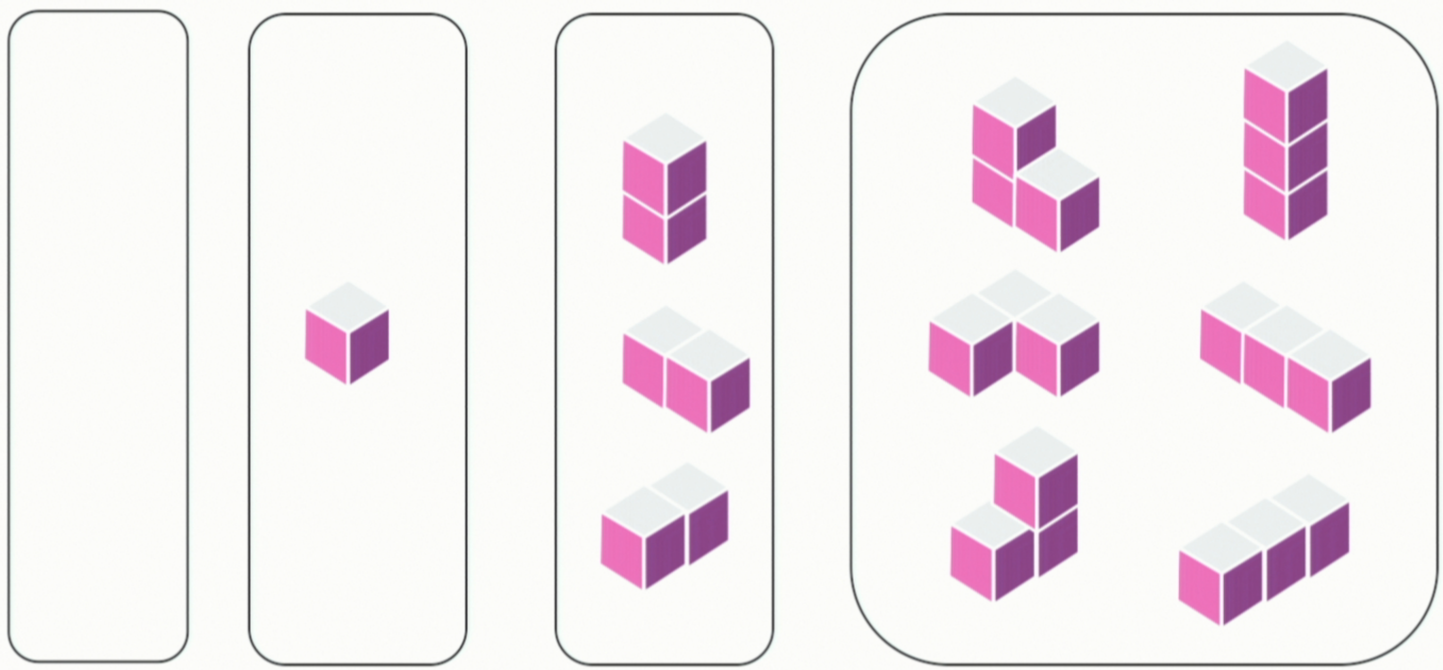
$$= \prod_{n>0} \left(1 - q^n\right)^{-n}$$

$$= \sum_{\text{3d partitions of } N} q^N$$

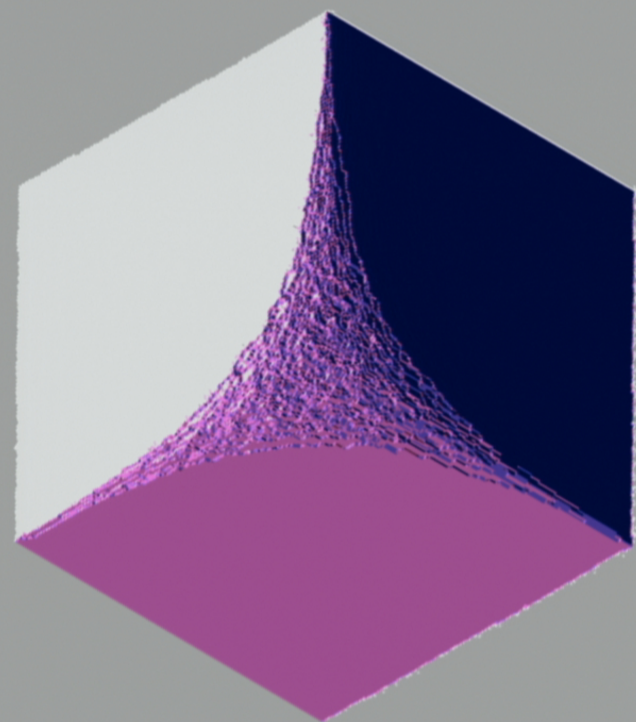


$$q = e^{-g_s}$$

$$Z_{top} = \prod_{n>0} \left(1 - q^n\right)^{-n} = 1 + q + 3q^2 + 6q^3 + \ldots$$

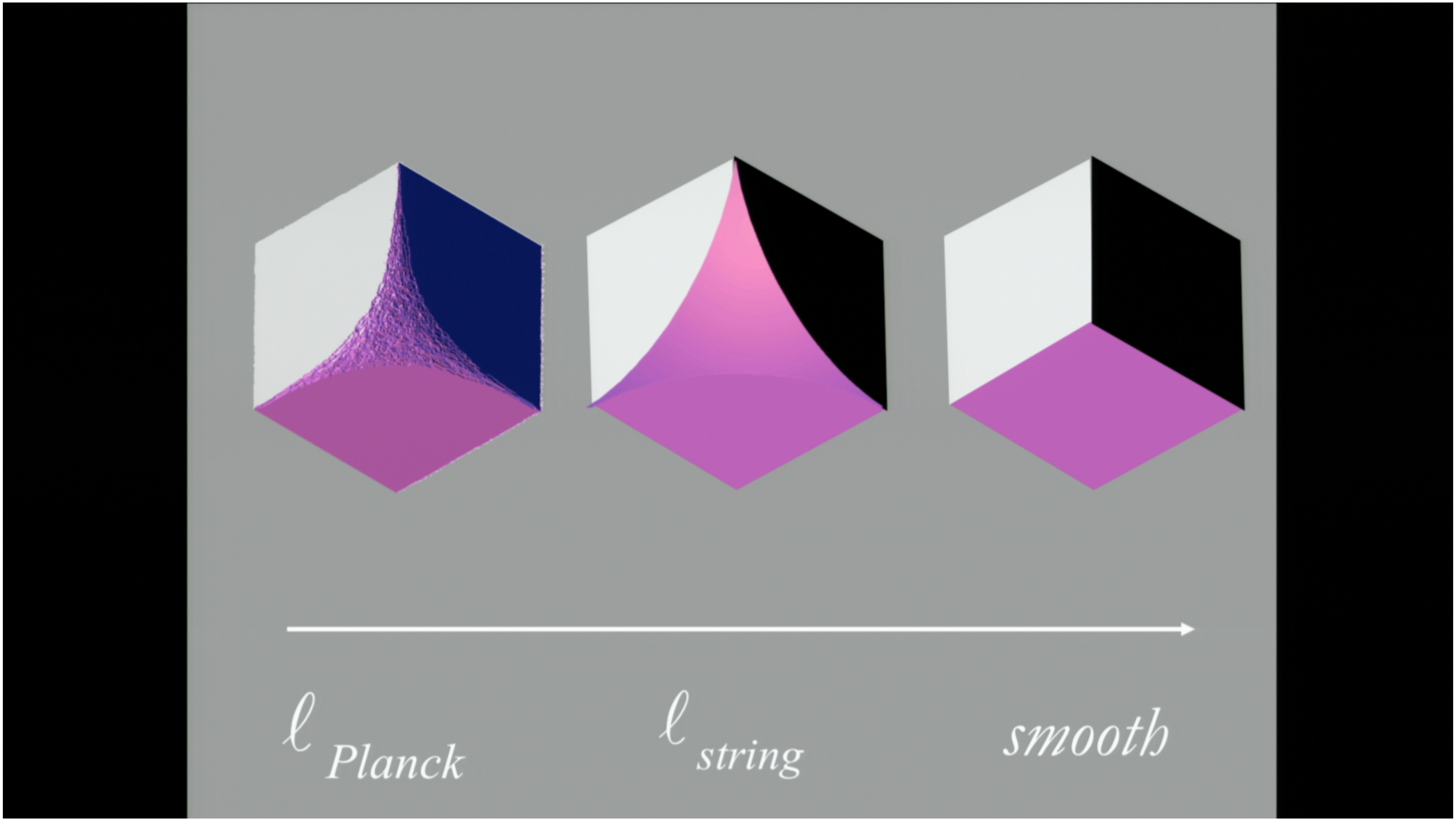


Melting Crystals



Reshetikhin, Okounkov, Vafa, Nekrasov,...

Limit Shape = Mirror Manifold


ℓ_{Planck}
ℓ_{string}
smooth

Wigner's Random Matrix Model

$$\lim_{N \to \infty} Z_N, \quad Z_N = \int_{N \times N} d\Phi \cdot e^{-Tr\Phi^2 / g_s}$$

Wigner's Random Matrix Model

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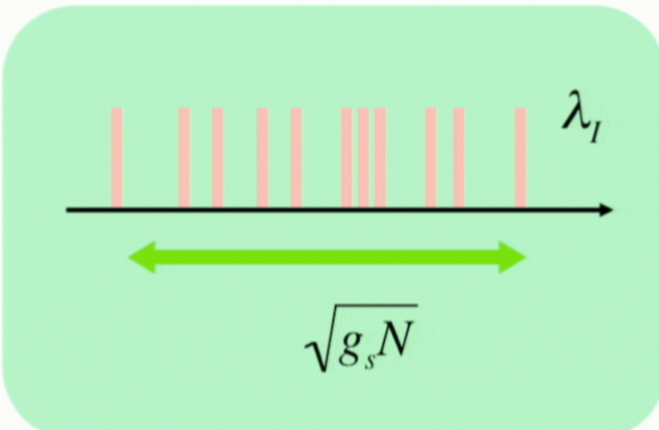
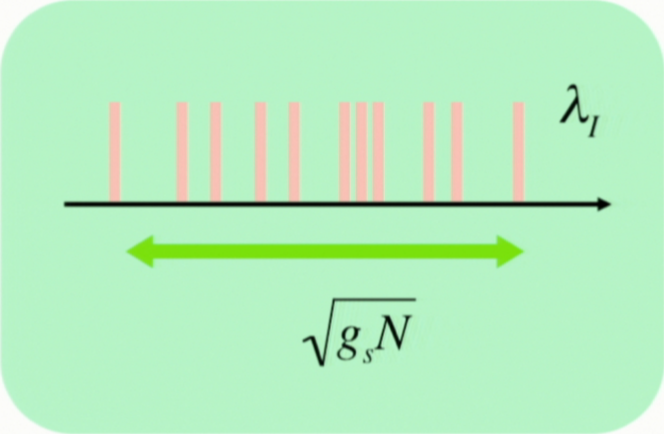


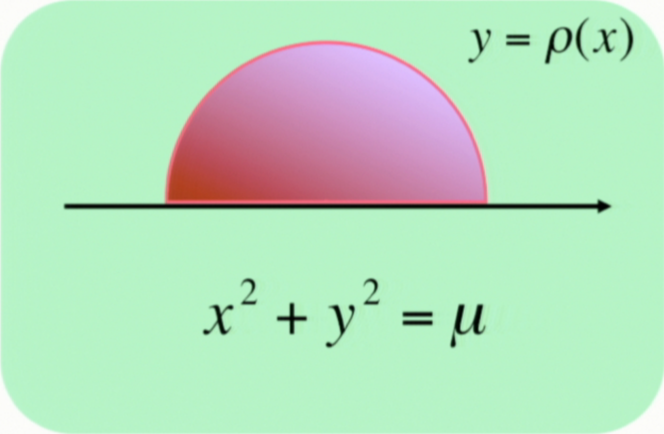
Wigner's Random Matrix Model

$$\lim_{N \to \infty} Z_N, \quad Z_N = \int_{N \times N} d\Phi \cdot e^{-Tr\Phi^2 / g_s}$$

Eigenvalue distribution in 't Hooft limit

$$N \to \infty, \quad g_s \to 0, \quad Ng_s = \mu = fixed$$



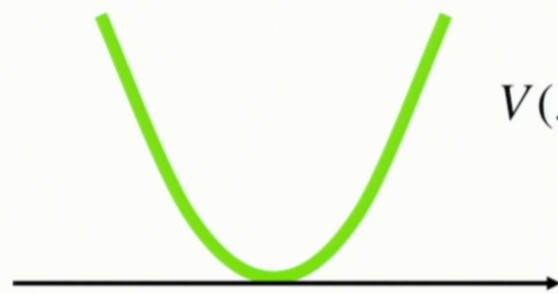


Eigenvalue Dynamics

$$Z_{matrix} = \int d^N\lambda \cdot \prod (\lambda_I - \lambda_J)^2 \cdot e^{-\sum_I \lambda_I^2 / g_s}$$

Effective action (repulsive Coulomb force)

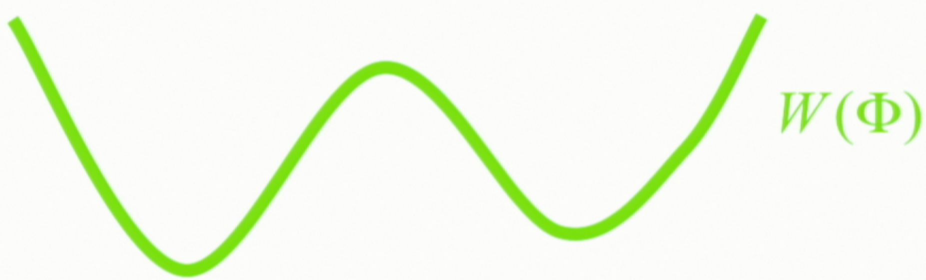
$$S_{eff} = \sum_I \lambda_I^2 - 2g_s \sum_{I<J} \log(\lambda_I - \lambda_J)$$



$$V(x) = x^2$$

General Matrix Model

$$Z_{matrix} = \int d\Phi \cdot e^{TrW(\Phi)/g_s}$$



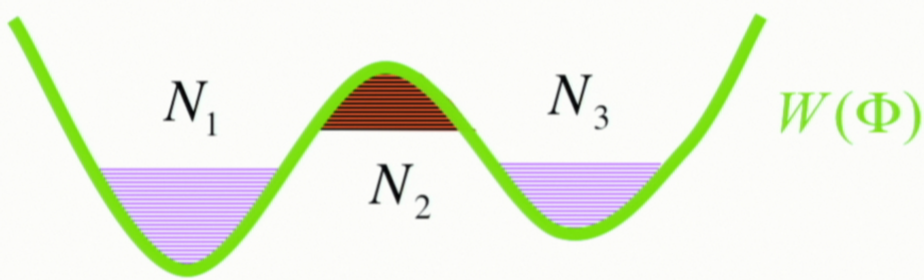
General Matrix Model

$$Z_{matrix} = \int d\Phi \cdot e^{TrW(\Phi)/g_s}$$

’t Hooft limit

$$N_I \to \infty,\ \ g_s \to 0,\ \ N_I g_s = \mu_I = fixed$$

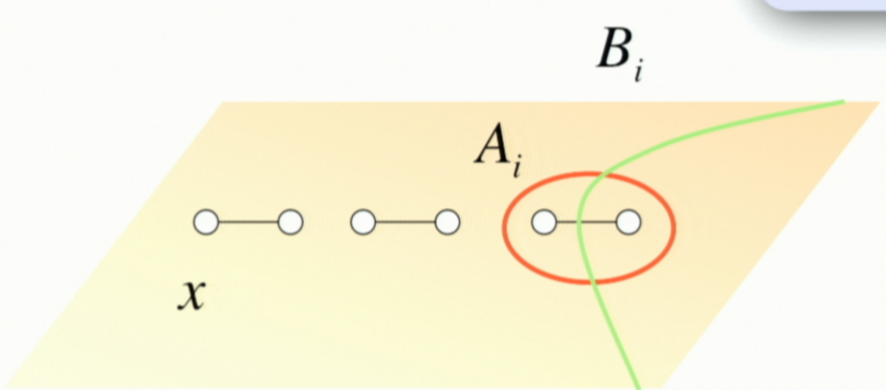
Filling fractions

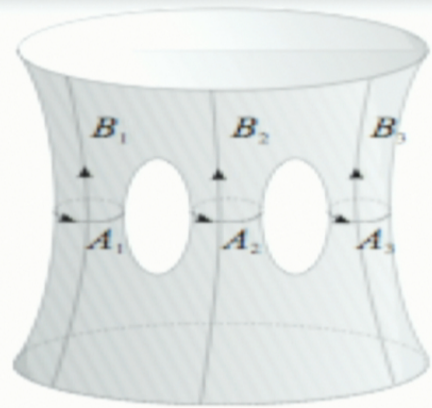


Spectral Curve

Hyperelliptic curve

$$C : y^2 = W'(x)^2 + f(x)$$





Spectral Curve

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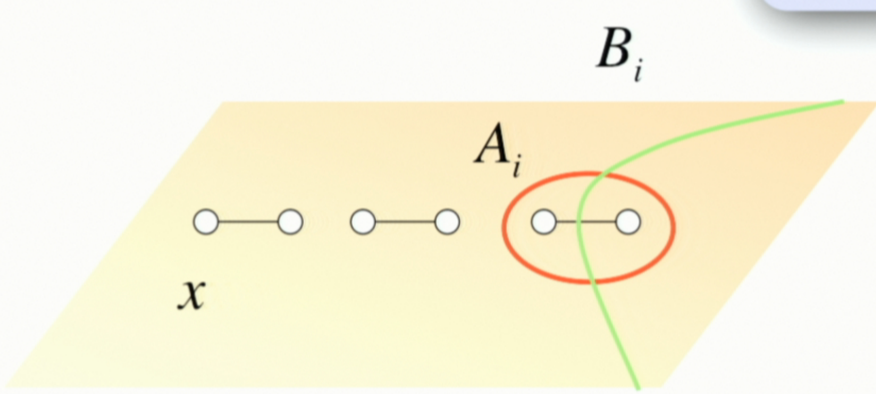
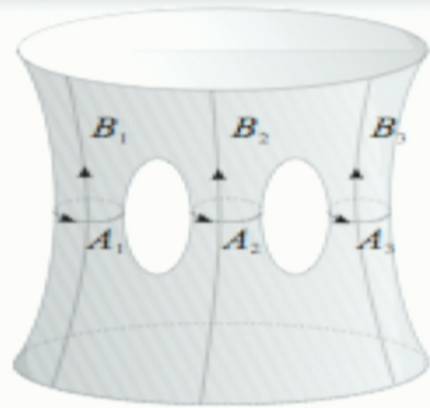




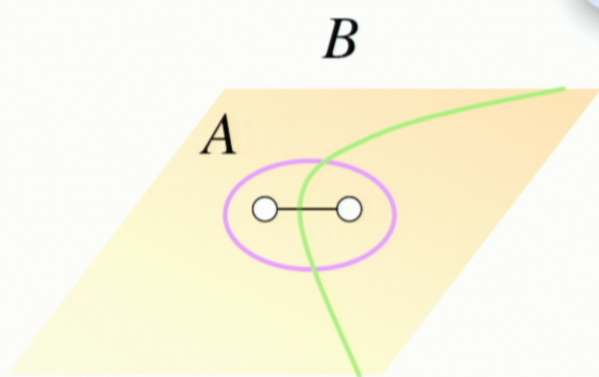
Quantum invariants of $C, \;\; \omega = ydx$

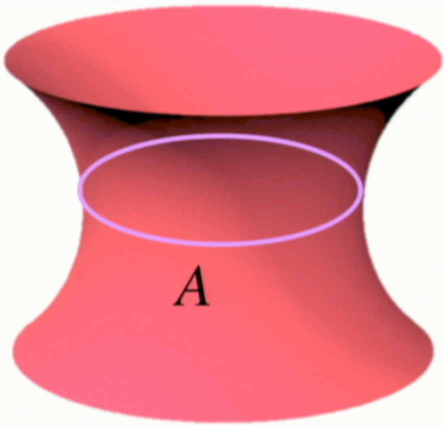
$$\mu_i = \oint_{A_i} ydx \qquad \frac{\partial F_0}{\partial \mu_i} = \oint_{B_i} ydx$$

Wigner's semi-circle

Rational spectral curve

$$C : y^2 = x^2 + \mu$$





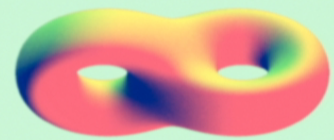
$$\frac{\partial F_0}{\partial \mu} = \mu \log \mu$$

Gauge Theories

M^4

self-dual connections
on 4-manifolds

Conformal Field Theory



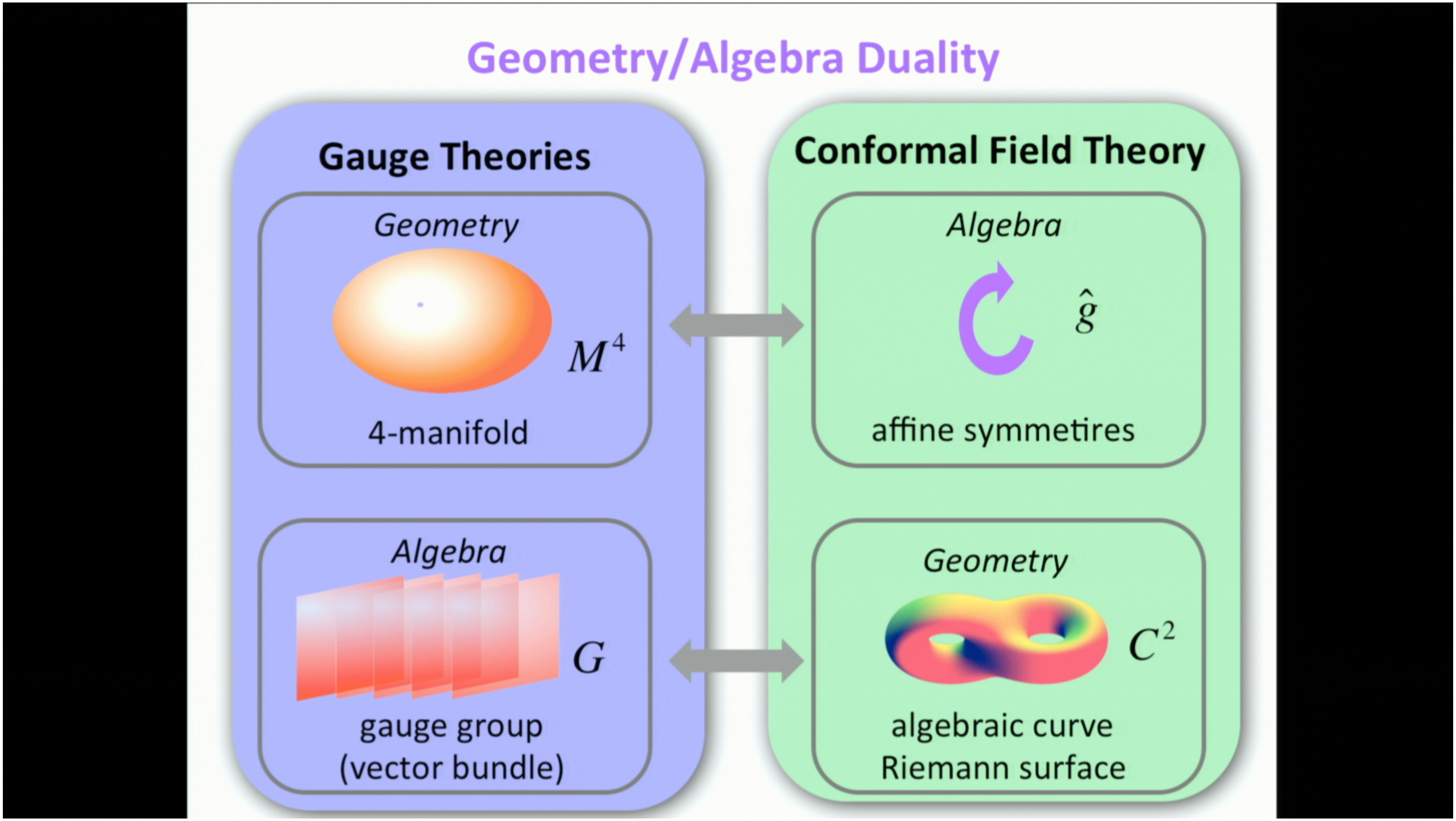
C^2

2d QFT on algebraic curves
reps of infinite dim algebras

Matrix Models

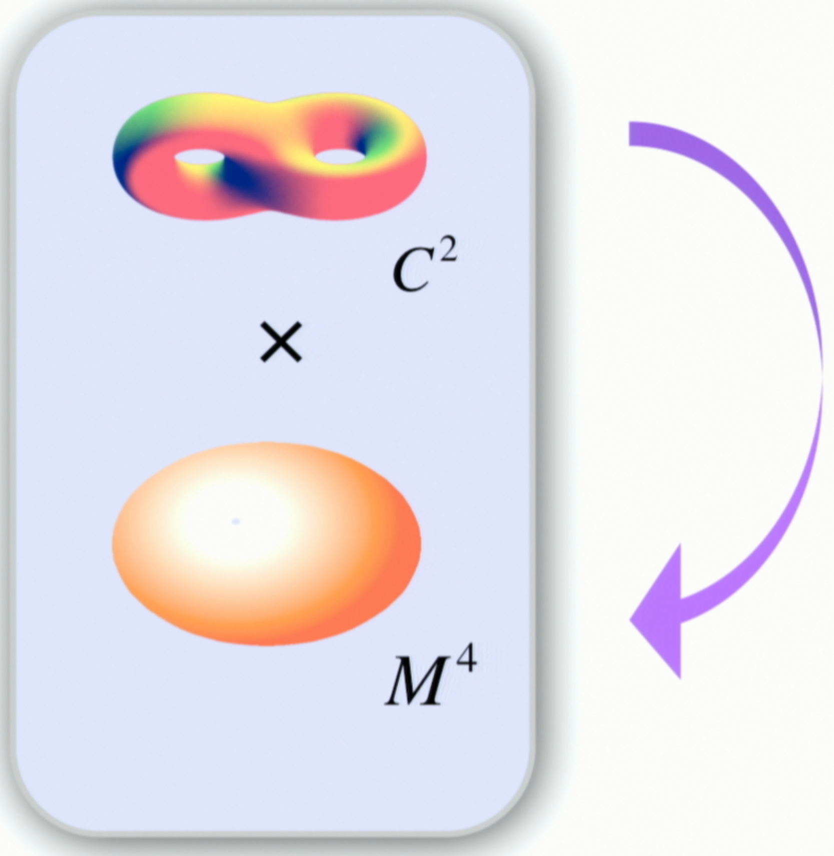
$$\int_{N\times N} d\Phi \cdot e^{TrW(\Phi)}$$

Eigenvalue distribution

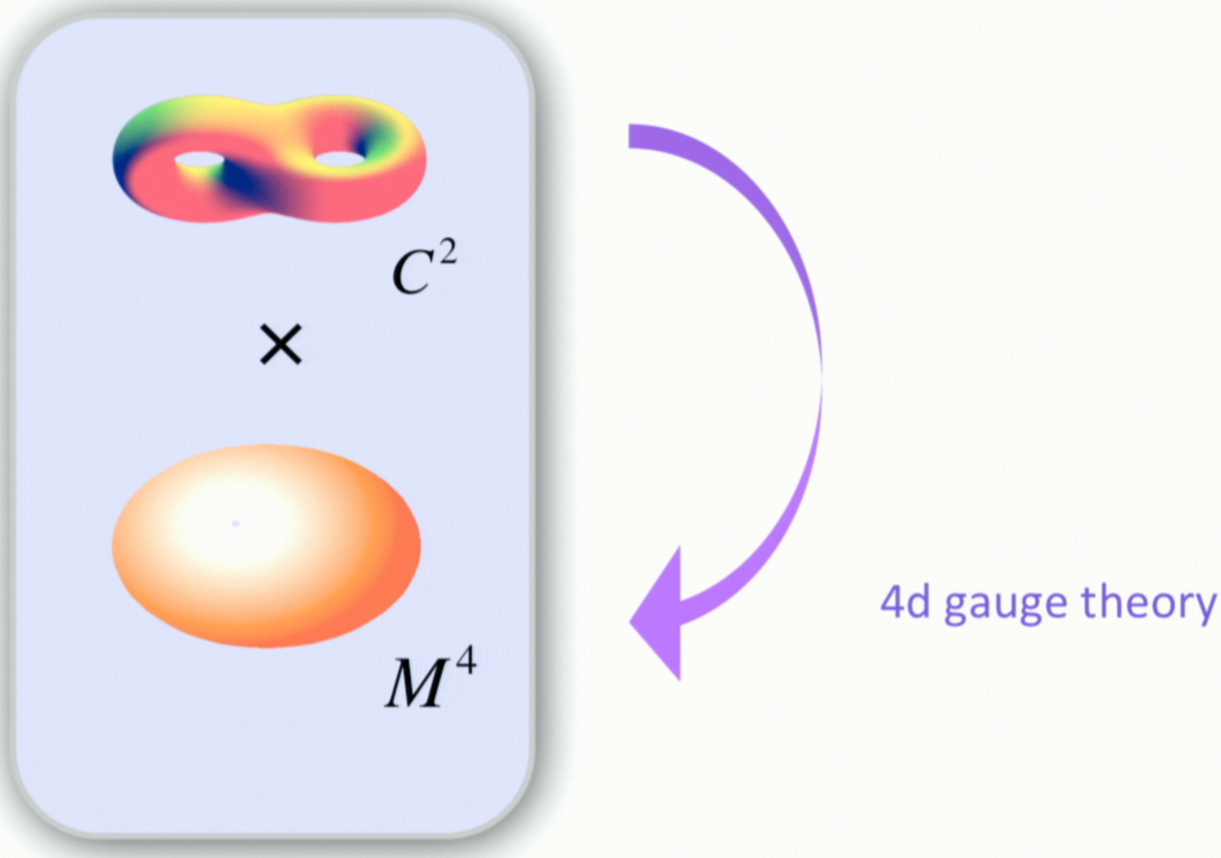


Geometry/Algebra Duality
Gauge Theories
Geometry
M^4
4-manifold
Algebra
G
gauge group
(vector bundle)
Conformal Field Theory
Algebra
$\hat{g}$
affine symmetires
Geometry
C^2
algebraic curve
Riemann surface

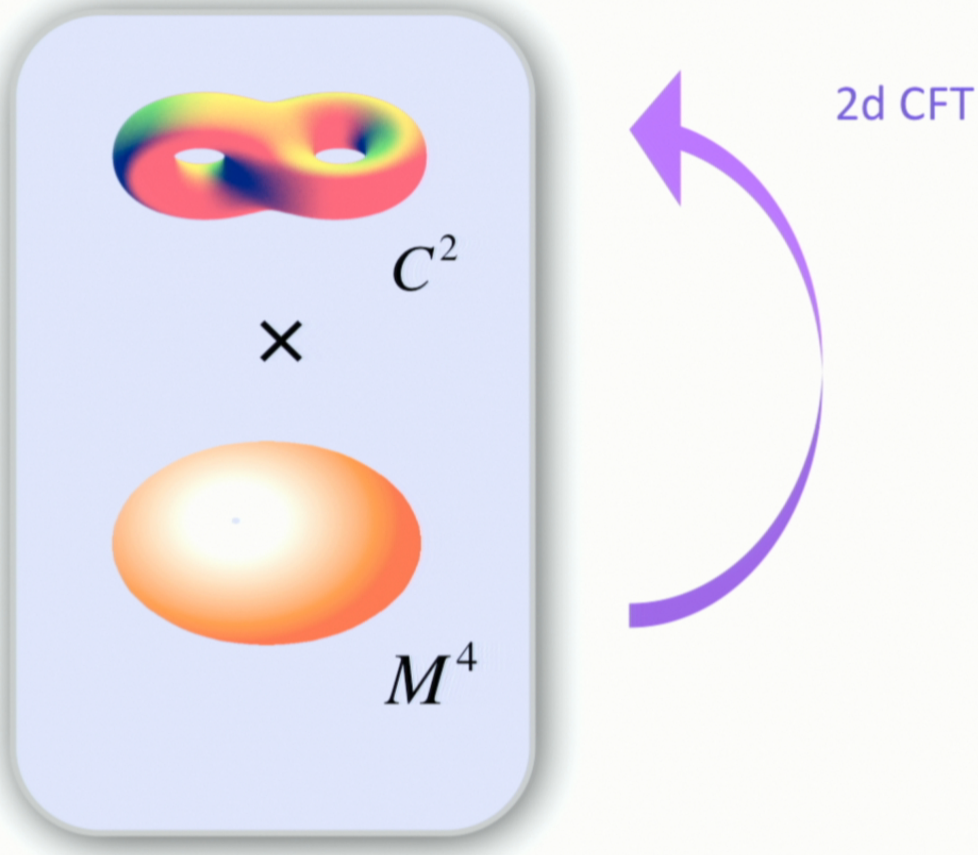
D=6 Tensor (2-form) "Gauge" Theory



D=6 Tensor (2-form) "Gauge" Theory



D=6 Tensor (2-form) “Gauge” Theory

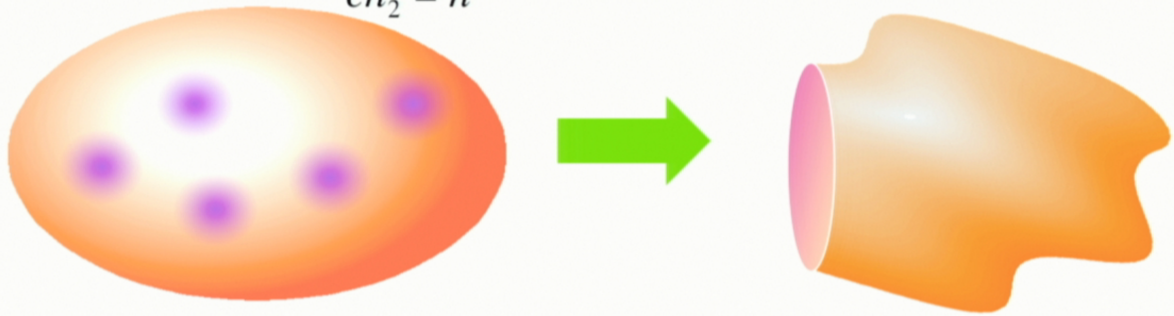


4D Gauge Theories

Instantons: self-dual connection $F = *F$

4-manifold M^4 *moduli space* $Mod_{N,n}(M)$

$ch_2 = n$



Action

$$S = \frac{4\pi}{g^2}\int TrF \wedge *F + \frac{\theta}{8\pi^2}\int TrF \wedge F = -n \cdot 2\pi i \tau$$

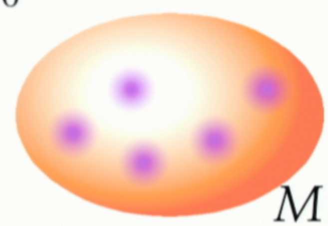
Gauge coupling

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \in H$$

Partition Function

Path integral localizes to instantons

$$Z_{gauge} = \int DA \cdot e^{-S(A)} = \sum_{n \geq 0} d(n) q^n, \quad q = e^{2\pi i \tau}$$

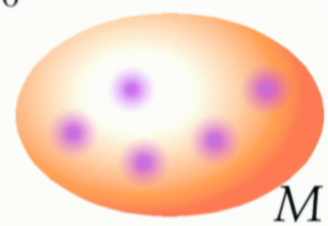


Contribution of moduli space $Mod_{N,n}(M)$

Partition Function

Path integral localizes to instantons

$$Z_{gauge} = \int DA \cdot e^{-S(A)} = \sum_{n \geq 0} d(n) q^n, \quad q = e^{2\pi i \tau}$$



Contribution of moduli space $Mod_{N,n}(M)$

N=4 SUSY Gauge Theory

Vafa-Witten: partition function is a modular form

$$Z_{gauge}(M;q) = \sum_{n\geq 0} d(n) q^n, \quad d(n) = Euler\left(M_{N,n}\right)$$

N=4 SUSY Gauge Theory

Vafa-Witten: partition function is a modular form

$$Z_{gauge}(M;q) = \sum_{n\geq 0} d(n) q^n, \quad d(n) = Euler\left(M_{N,n}\right)$$

- *SL*(2,**Z**) S-duality in N=4 gauge theories ↔ modular invariance of a quantum CFT on 2-torus

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

T^2

- Related to 2d CFT characters

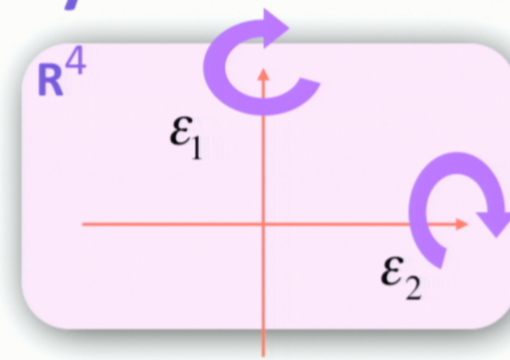
$$Z_{gauge}(q) = Tr_V q^{L_0} = \chi(q)$$

N=2 SUSY Gauge Theory

Equivariant action of SO(4) x U(N) on

$$Mod_{N,n}(R^4)$$

$$(\varepsilon_1,\varepsilon_2;a) \in T^2 \times T^N$$



$$U(1) \times U(1) \subset SO(4)$$

Nekrasov partition function: equivariant fundamental class

$$Z_{gauge}(q;\varepsilon_1,\varepsilon_2,a) = \sum_{n\geq 0} q^n \int_{M_{N,n}} 1(\varepsilon_1,\varepsilon_2,a)$$

Traditional (supersymmetric) case $\varepsilon_1 = -\varepsilon_2 = \varepsilon = g_s$

Alday-Gaiotto-Tachikawa (AGT)

Gauge theory = 2d CFT = correlator of vertex operators in Liouville theory

$$Z_{gauge} = Z_{CFT}$$

Alday-Gaiotto-Tachikawa (AGT)

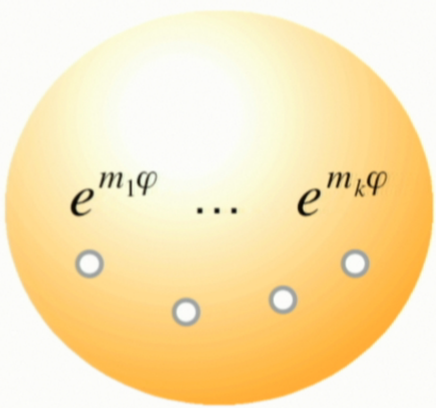
Gauge theory = 2d CFT = correlator of vertex operators in Liouville theory

$$Z_{gauge} = Z_{CFT}$$

$$Z_{gauge} = \left\langle e^{m_1\varphi} \cdots e^{m_k\varphi} \right\rangle = \int D\varphi \cdot e^{-S[\varphi]} \cdot e^{m_1\varphi} \cdots e^{m_k\varphi}$$

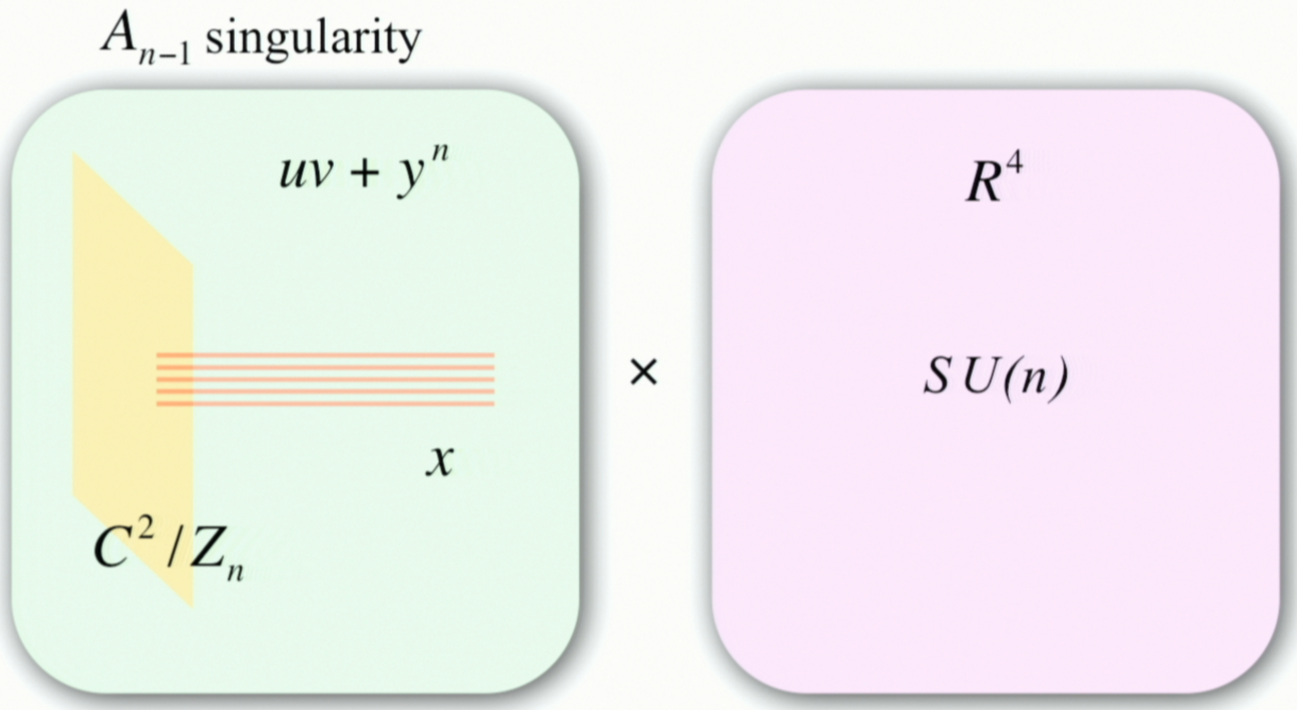
$$S = \frac{1}{g_s^2} \int (\partial\varphi)^2 + QR\varphi + e^{b\varphi}$$

$$Q = b + \frac{1}{b}, \quad b^2 = \frac{\varepsilon_1}{\varepsilon_2}, \quad g_s = \sqrt{\varepsilon_1 \varepsilon_2}$$



vertex operators insertions

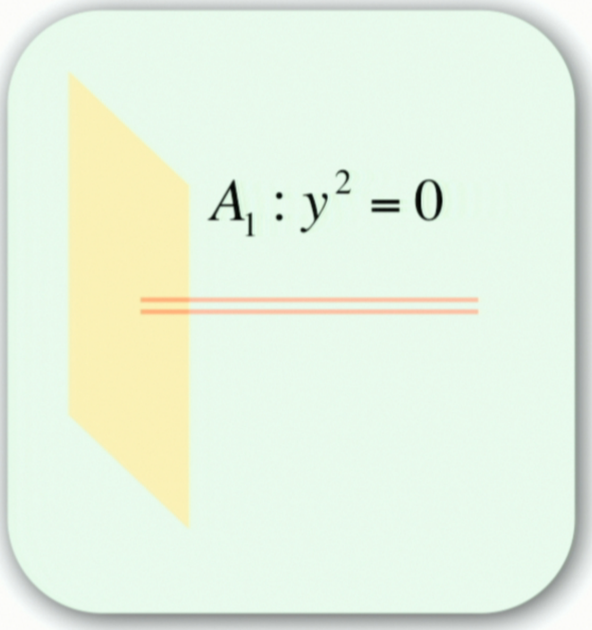
Geometry of SU(n) Gauge Theory



Calabi-Yau
(internal space)

Space-Time

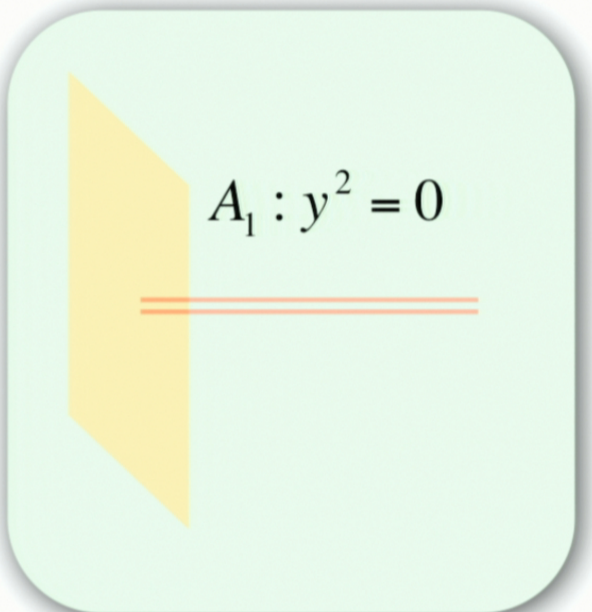
Singularities & Matrix Models



$$\int d\Phi \cdot e^{aTr\Phi^2}$$

$$\lim_{a \to 0} \left\{ y^2 = a^2(x^2 + \mu) \right\}$$

Singularities & Matrix Models



$$\int d\Phi \cdot e^{aTr\Phi^2}$$

$$\lim_{a\to 0} \left\{ y^2 = a^2(x^2 + \mu) \right\}$$

Matrix Models & Liouville Theory

$$Z_{matrix} = \int d^N x \cdot \prod \left(x_I - x_J\right)^2 \cdot e^{\sum W(x_I)/g_s}$$

CFT correlator

$$Z_{matrix} = \left\langle \int d^N x \cdot \prod e^{i\varphi}(x_i) \cdot e^{\oint \partial\varphi W / g_s} \right\rangle_N$$

Screening charge

$$Z_{matrix} = \left\langle e^{\int e^{i\varphi}} \cdot e^{\oint \partial\varphi W / g_s} \right\rangle_N$$

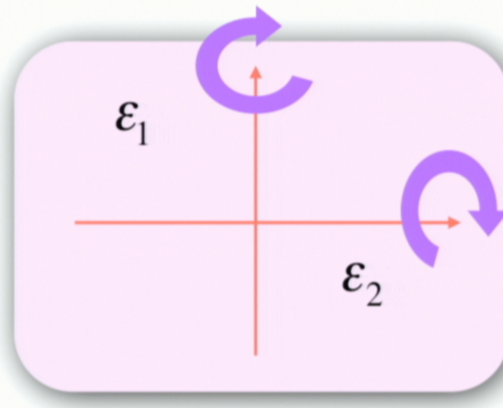
Large N : Coulomb gas picture.

Nekrasov's Omega-Deformation

Equivariant instantons U(1) x U(1)

$$Z_{gauge}(\varepsilon_1, \varepsilon_2)$$

Space-Time R^4



Twisting CFT: Background charge

$$S[\varphi] + \int d^2 z (\varepsilon_1 + \varepsilon_2) R^{(2)} \varphi \qquad Q = b + \frac{1}{b} = (\varepsilon_1 + \varepsilon_2)/g_s$$

Beta Ensemble

General Liouville theory $b \neq i,\ \varepsilon_1 \neq -\varepsilon_2$

$$Z = \left\langle \exp \int e^{b\varphi} \right\rangle_N$$

Generalized matrix model measure

$$Z = \int d^N x \cdot \prod \left(x_I - x_J \right)^{2\beta} e^{\sum_I W(x_I)}, \quad \beta = -b^2$$

$$\beta = 1/2 : SO(N), \quad \beta = 2 : Sp(N)$$

Liouville Picture

Vertex operators

$$\left\langle \prod e^{m_i \varphi} \right\rangle_N$$

Liouville Picture

Vertex operators

$$\left\langle \prod e^{m_i \varphi} \right\rangle_N$$

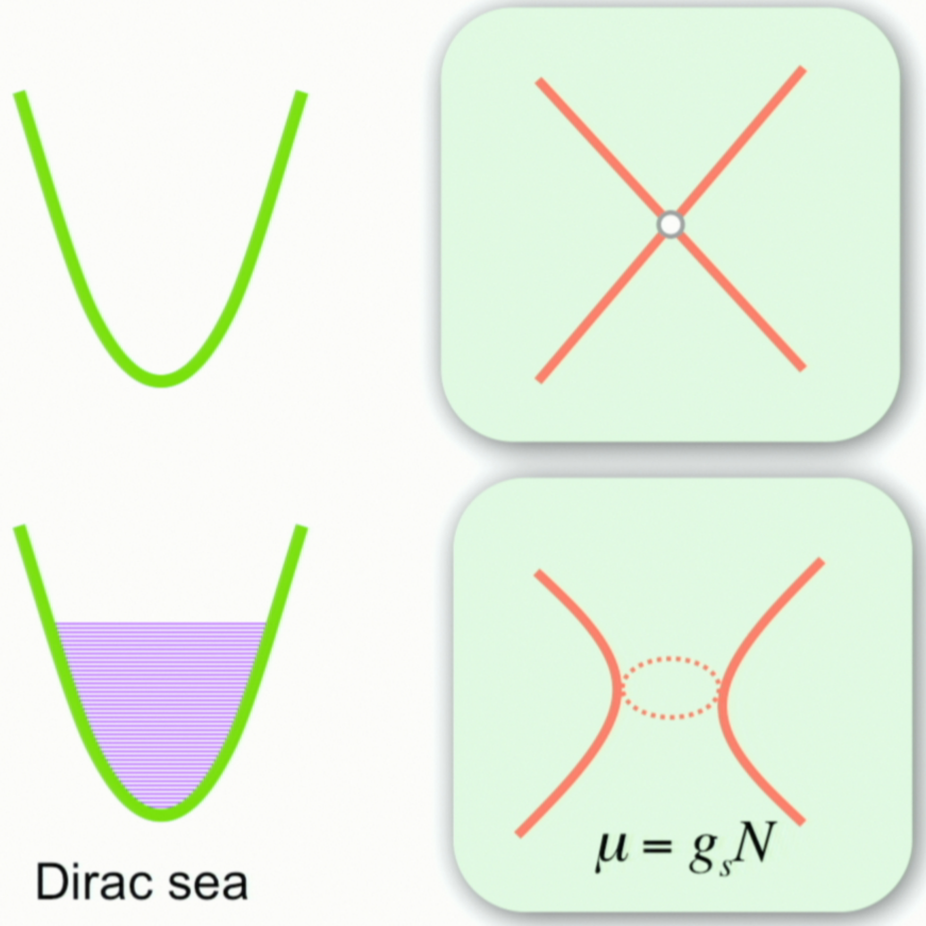
Matrix model representation

$$Z = \int d\Phi \cdot \prod \det\left(\Phi - x_i\right)^{m_i}$$

Effective potential (Penner matrix model)

$$W(\Phi) = \sum_i m_i \log\left(\Phi - x_i\right)$$

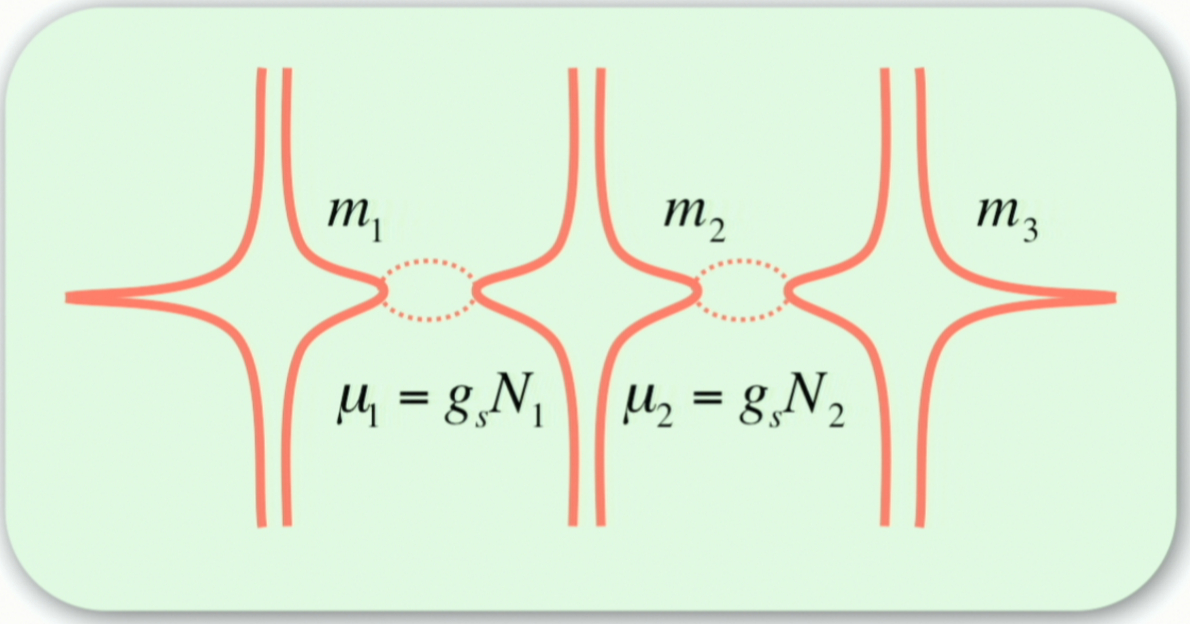
Resolving Double Points



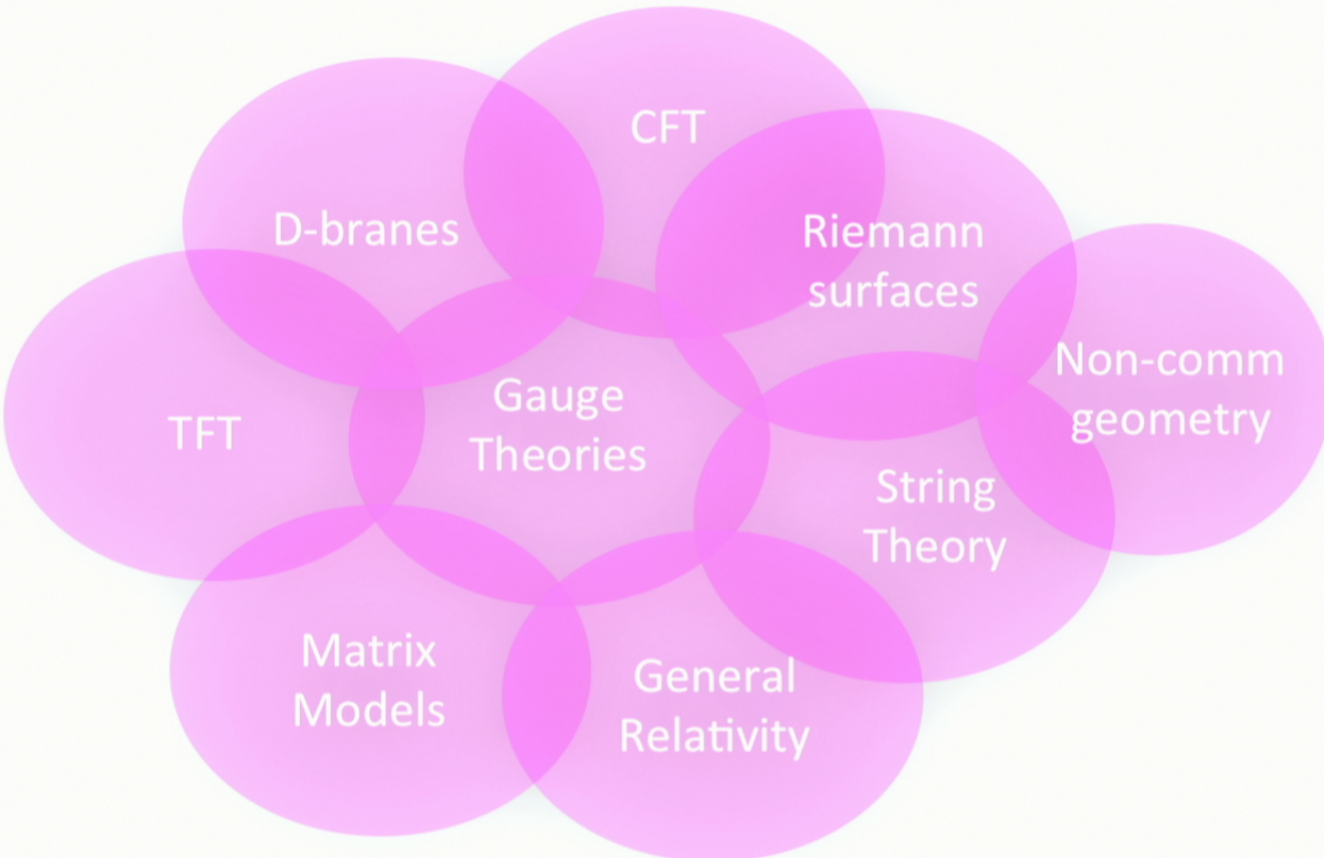
$$y^2 = x^2$$

$$y^2 = x^2 + \mu$$

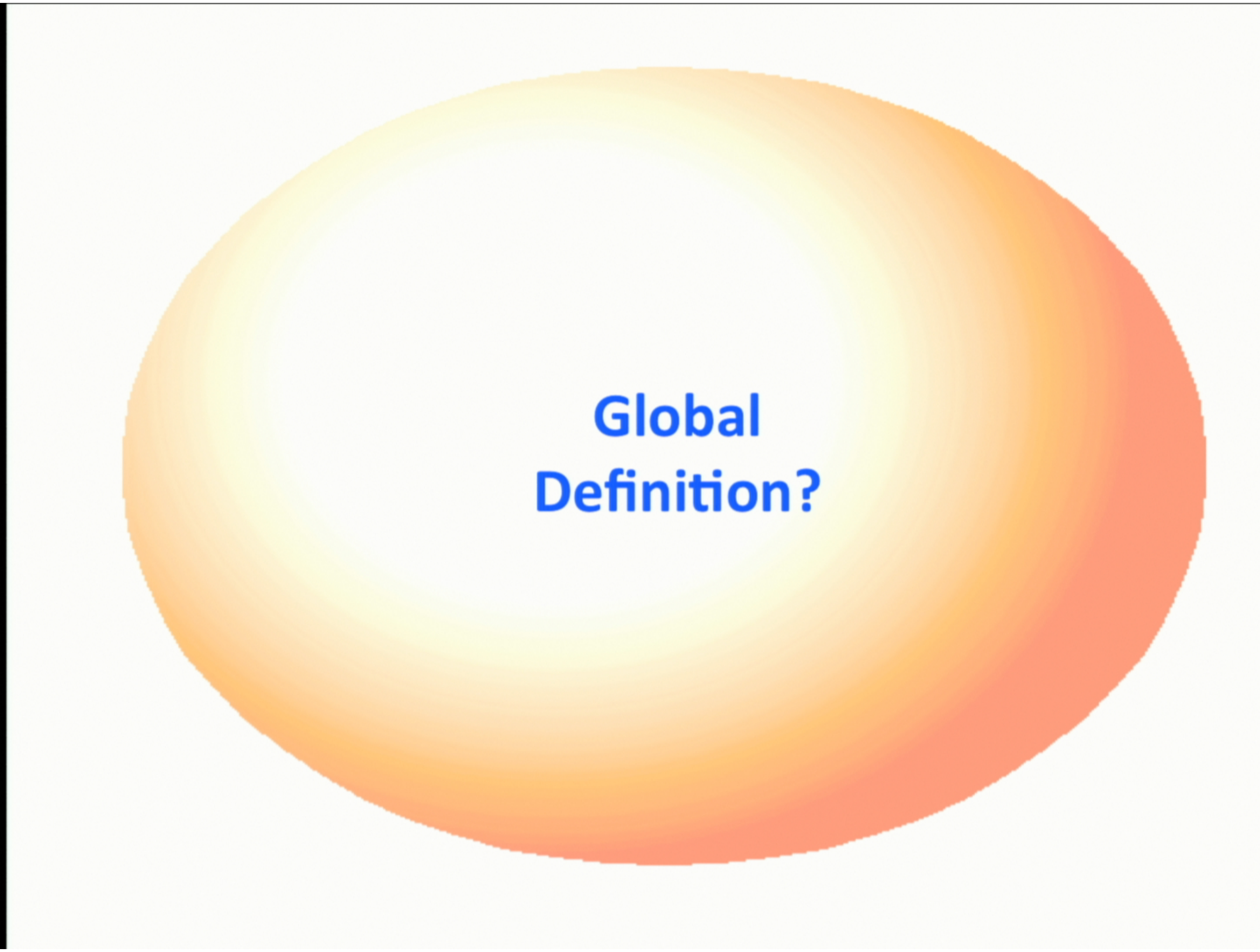
Spectral Curve



$$y^2 = \left(\sum_i \frac{m_i}{x - x_i} \right)^2 + \sum_i \frac{f_i}{x - x_i}$$



Local Definitions


Global
Definition?