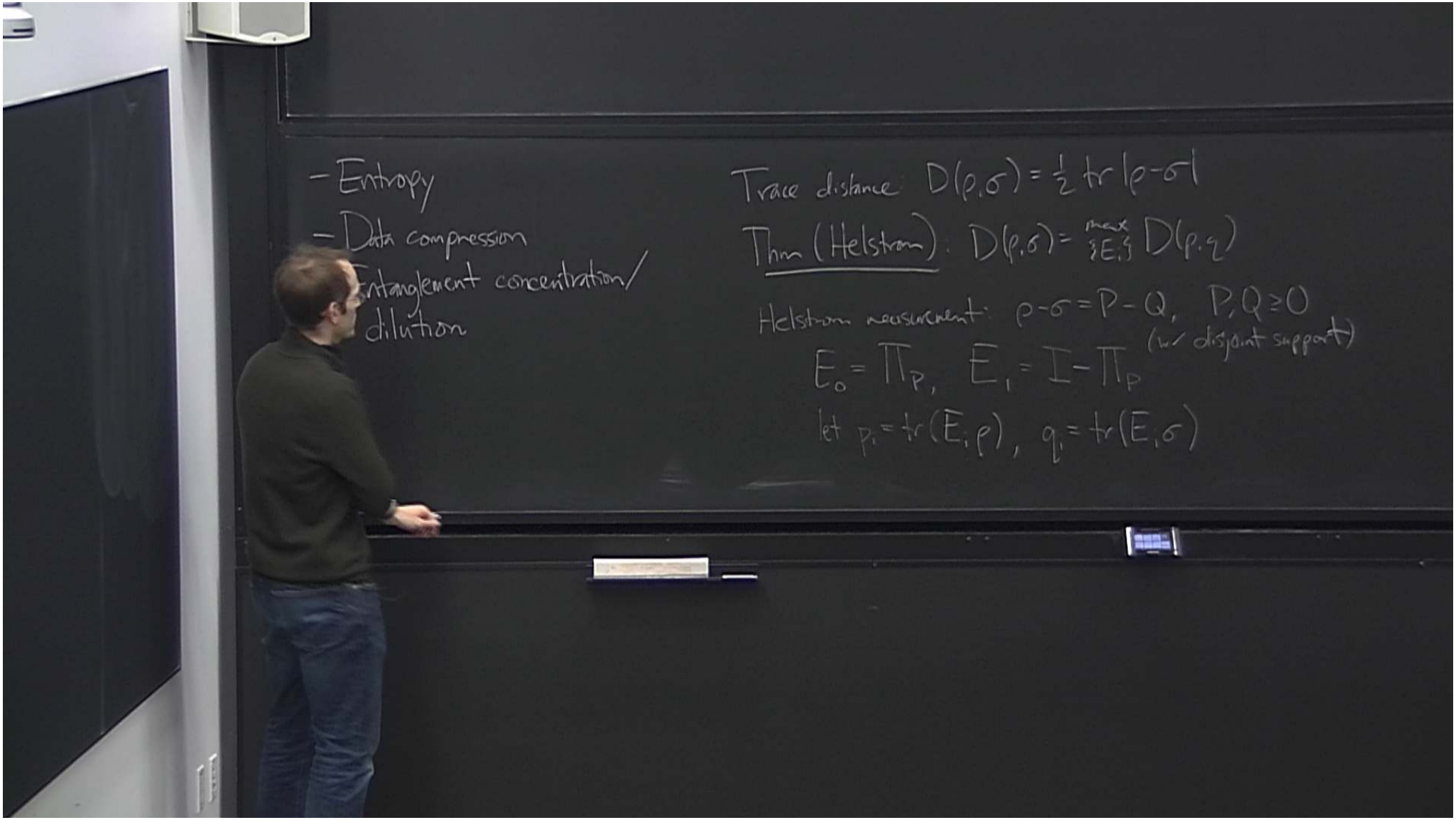
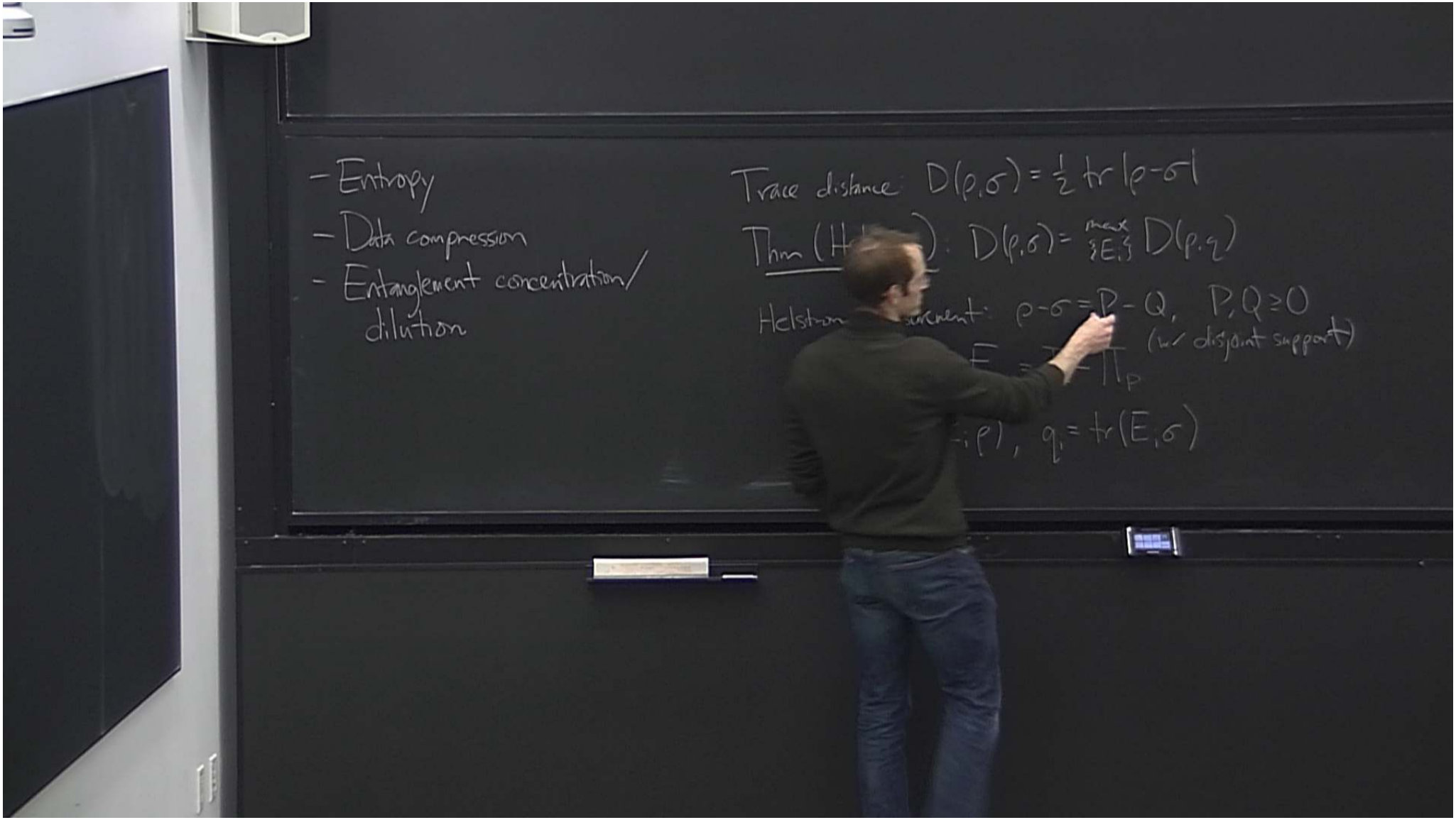
Title: 13/14 PSI - Quantum Information Review - Lecture 11

Date: Mar 03, 2014 11:30 AM

URL: http://pirsa.org/14030024

Abstract:


- Entropy
- Data compression
- Entanglement concentration/
dilution
Trace distance: $D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho-\sigma|$
Thm (Helstrom): $D(\rho,\sigma) = \max_{\{E_i\}} D(p,q)$
Helstrom measurement: $\rho-\sigma = P-Q$, $P,Q \geq 0$ (w/ disjoint support)
$E_0 = \Pi_P$, $E_1 = I - \Pi_P$
let $p_i = \mathrm{tr}(E_i\rho)$, $q_i = \mathrm{tr}(E_i\sigma)$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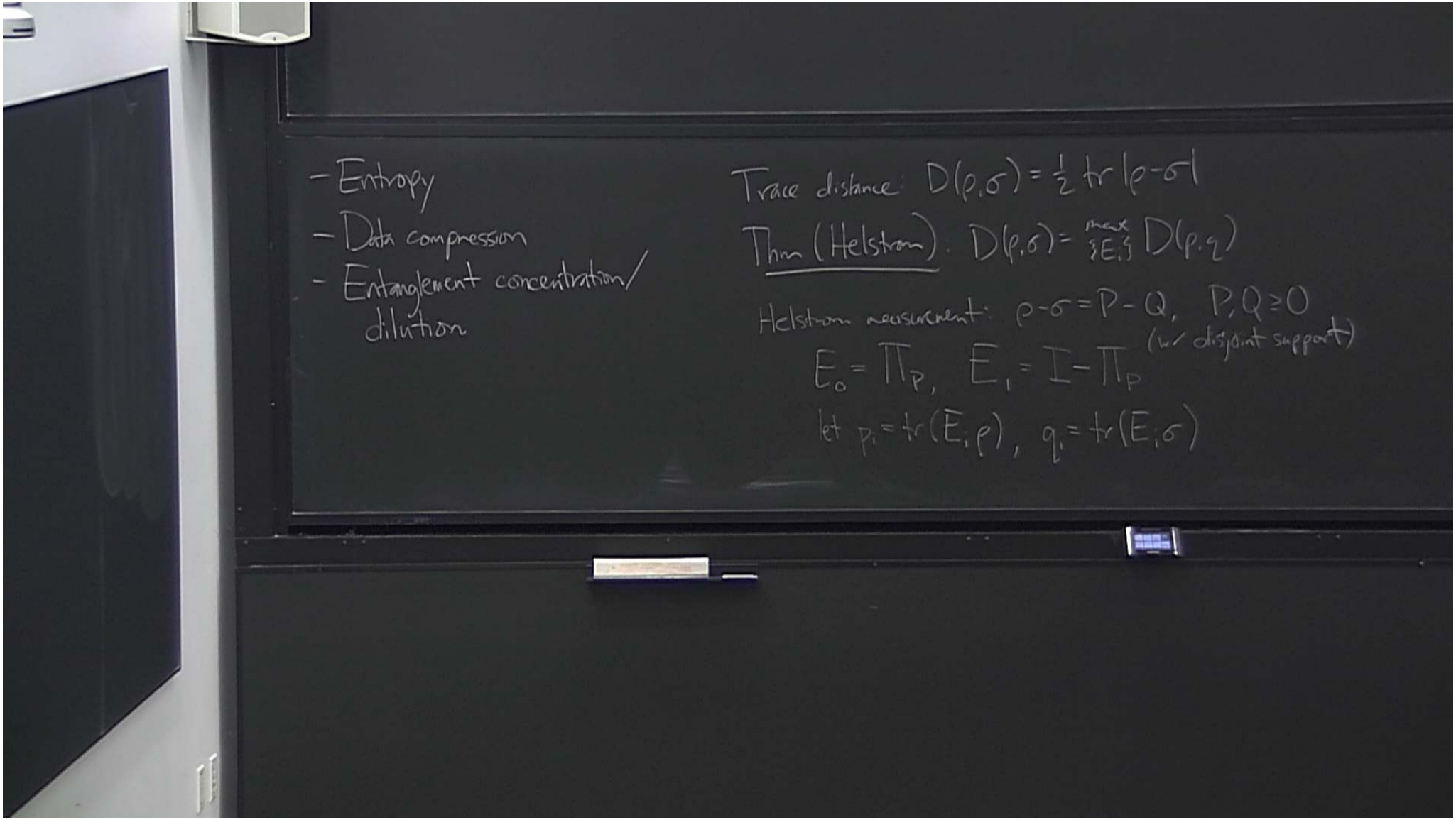
$D(\rho,\sigma) = \max_{\{E_j\}} D(p,q)$
$\rho-\sigma = P - Q,\ P,Q \geq 0$
(w/ disjoint support)
Π_P
$q_i = \mathrm{tr}(E_i\sigma)$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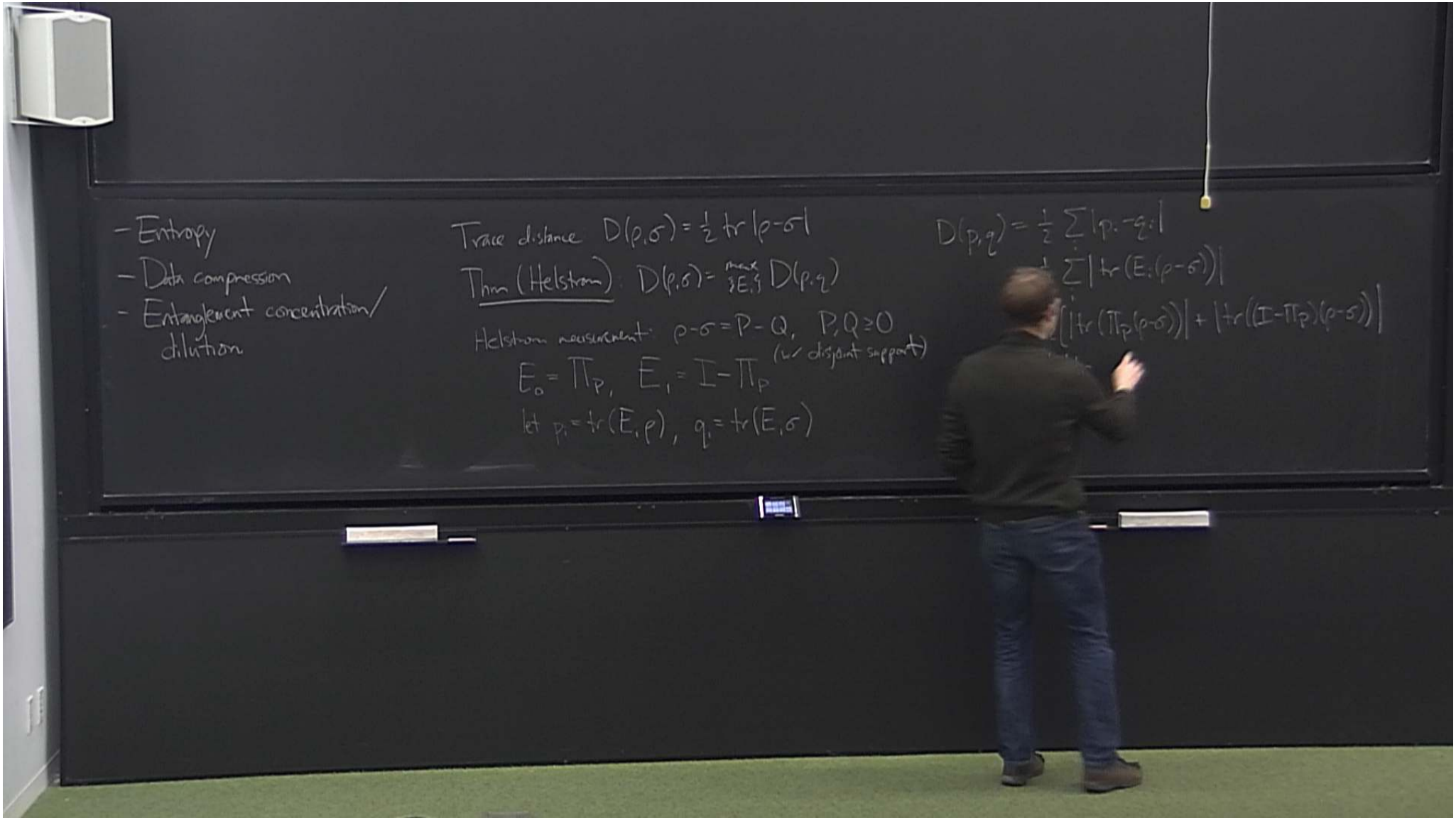
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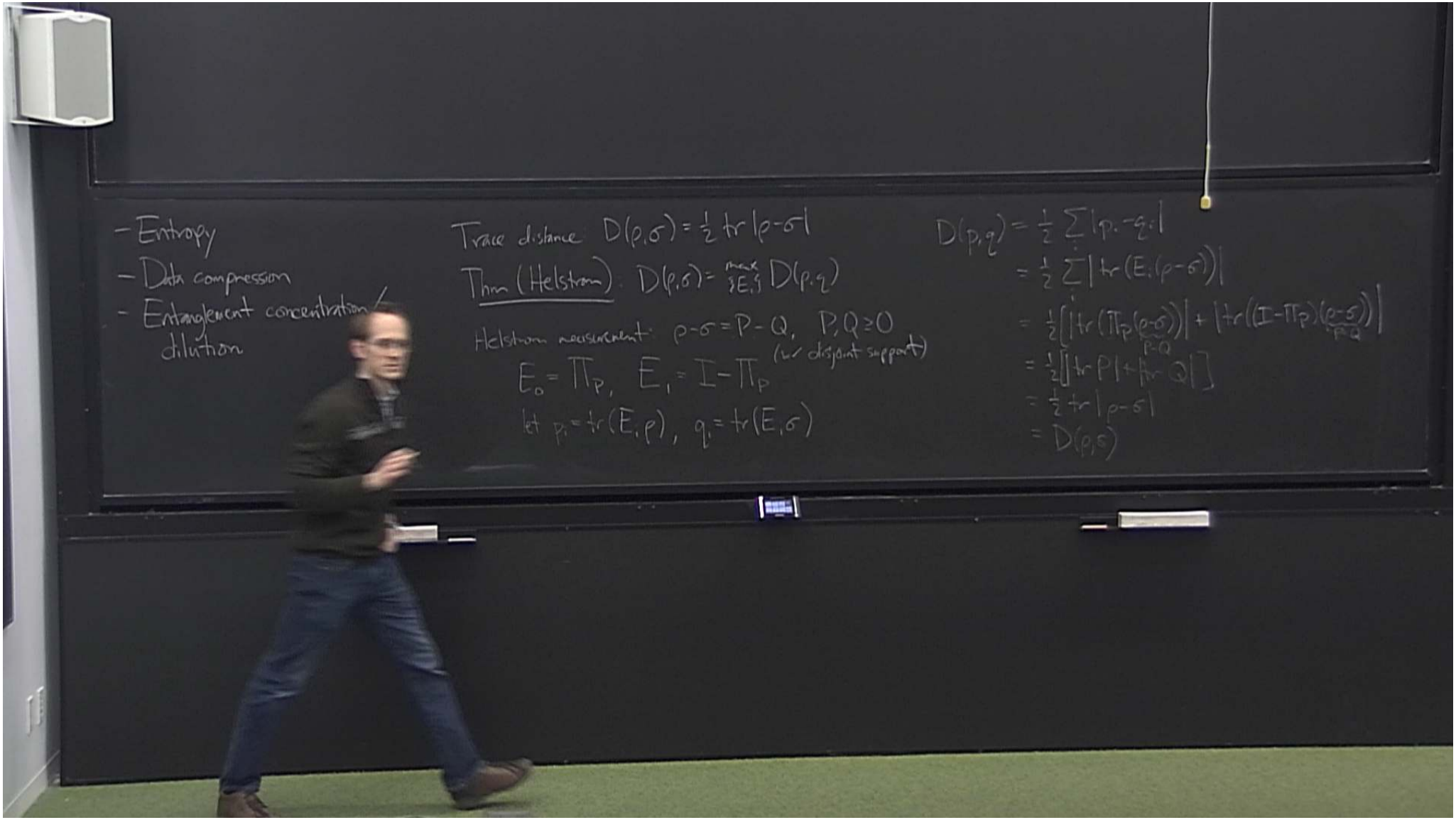
$E_0 = \Pi_P$, $E_1 = I-\Pi_P$
let $p_i = \mathrm{tr}(E_i\rho)$, $q_i = \mathrm{tr}(E_i\sigma)$
$D(p,q) = \frac{1}{2}\sum|p_i - q_i|$
$\frac{1}{2}\sum|\mathrm{tr}(E_i(\rho-\sigma))|$
$\frac{1}{2}\left[|\mathrm{tr}(\Pi_P(\rho-\sigma))| + |\mathrm{tr}((I-\Pi_P)(\rho-\sigma))|\right.$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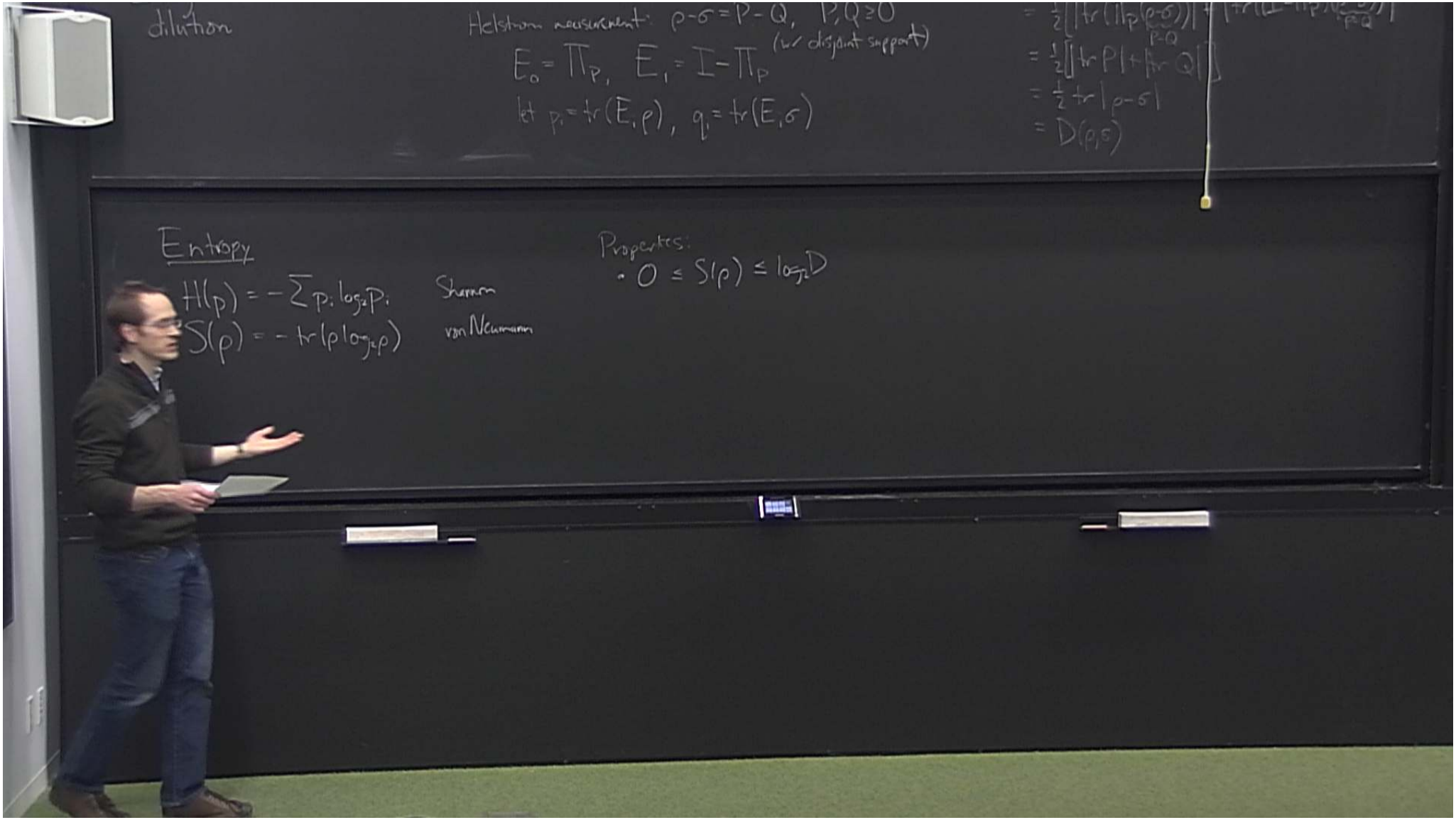
let $p_i = \mathrm{tr}(E_i\rho)$, $q_i = \mathrm{tr}(E_i\sigma)$
$D(p,q) = \frac{1}{2}\sum_i |p_i - q_i|$
$= \frac{1}{2}\sum_i |\mathrm{tr}(E_i(\rho-\sigma))|$
$= \frac{1}{2}\left[|\mathrm{tr}(\Pi_P(\rho-\sigma))| + |\mathrm{tr}((I-\Pi_P)(\rho-\sigma))|\right]$
$= \frac{1}{2}[\mathrm{tr}P + \mathrm{tr}Q]$
$= \frac{1}{2}\mathrm{tr}|\rho-\sigma|$
$= D(\rho,\sigma)$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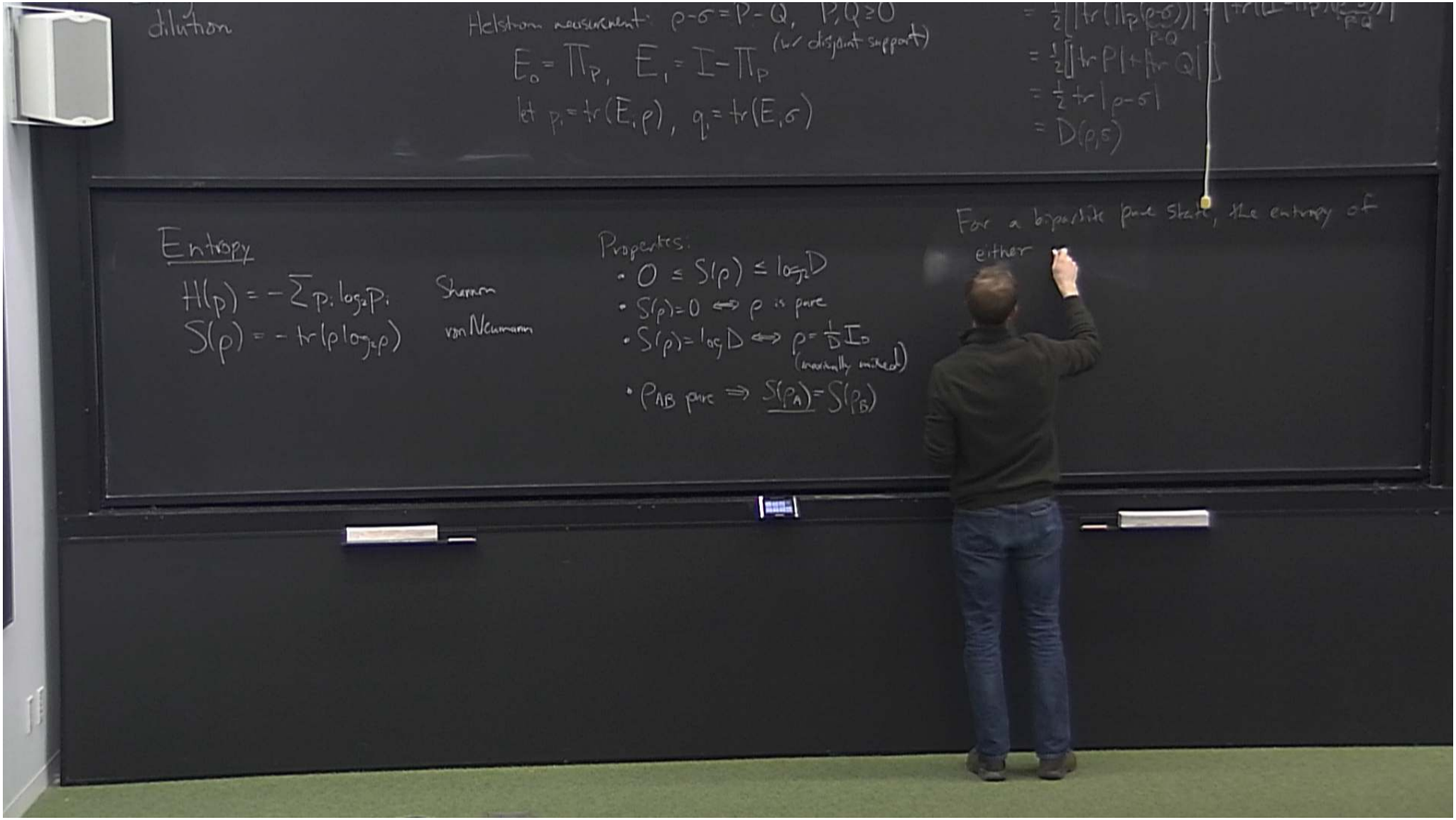
$E_0 = \Pi_P$, $E_1 = I - \Pi_P$
let $p_i = \mathrm{tr}(E_i \rho)$, $q_i = \mathrm{tr}(E_i \sigma)$
$= \frac{1}{2}\left[\left|\mathrm{tr}(\Pi_P(\rho-\sigma))\right| + \left|\mathrm{tr}((I-\Pi_P)(\rho-\sigma))\right|\right]$
$= \frac{1}{2}\left[|\mathrm{tr}\,P| + |\mathrm{tr}\,Q|\right]$
$= \frac{1}{2}\mathrm{tr}|\rho-\sigma|$
$= D(\rho,\sigma)$
Entropy
$H(p) = -\sum p_i \log_2 p_i$ Shannon
$S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$ von Neumann
Properties:
• $0 \leq S(\rho) \leq \log_2 D$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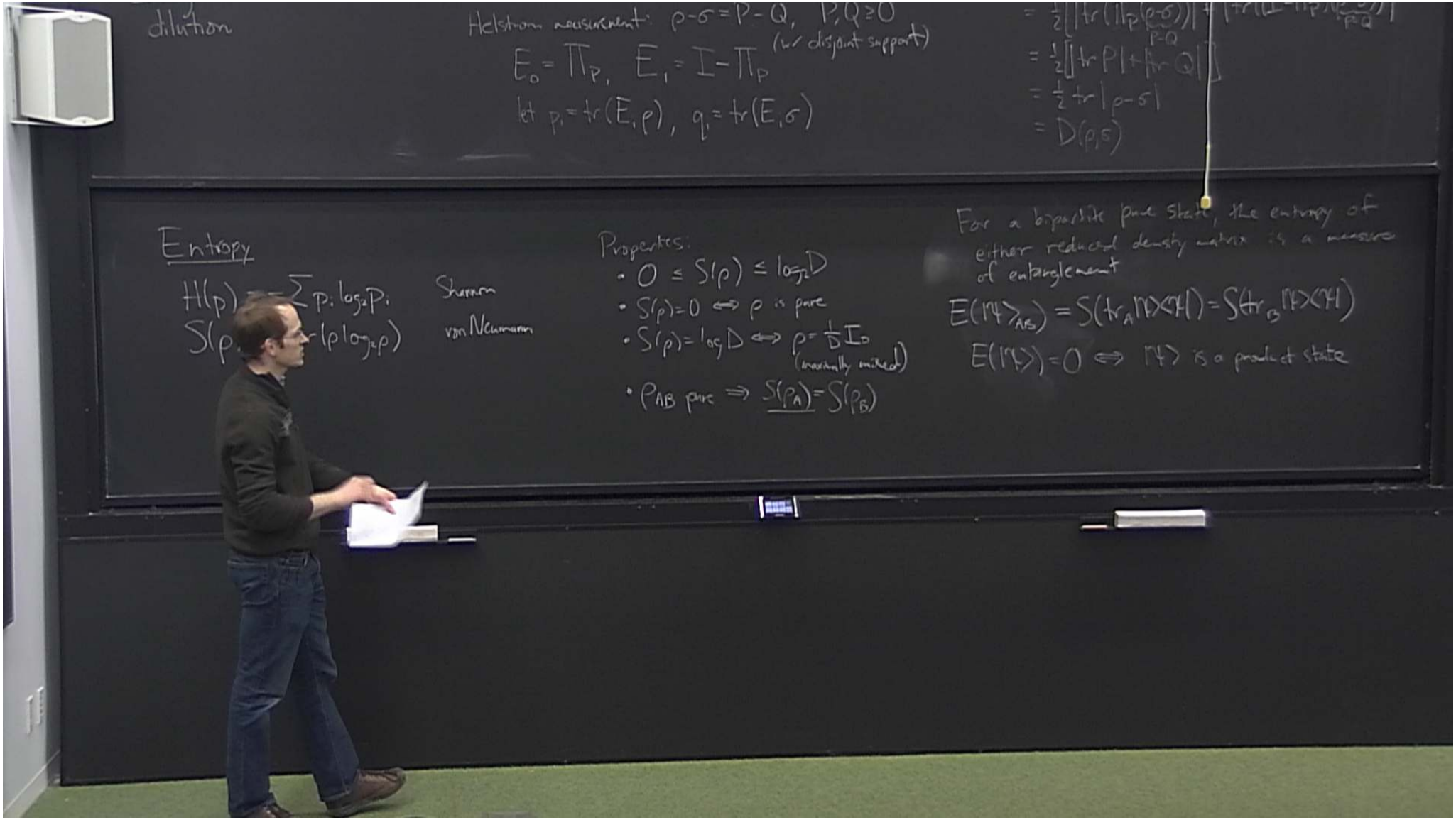
Properties:
• $0 \leq S(\rho) \leq \log_2 D$
• $S(\rho) = 0 \iff \rho$ is pure
• $S(\rho) = \log D \iff \rho = \frac{1}{D} I_D$ (maximally mixed)
• ρ_{AB} pure $\Rightarrow S(\rho_A) = S(\rho_B)$
For a bipartite pure state, the entropy of either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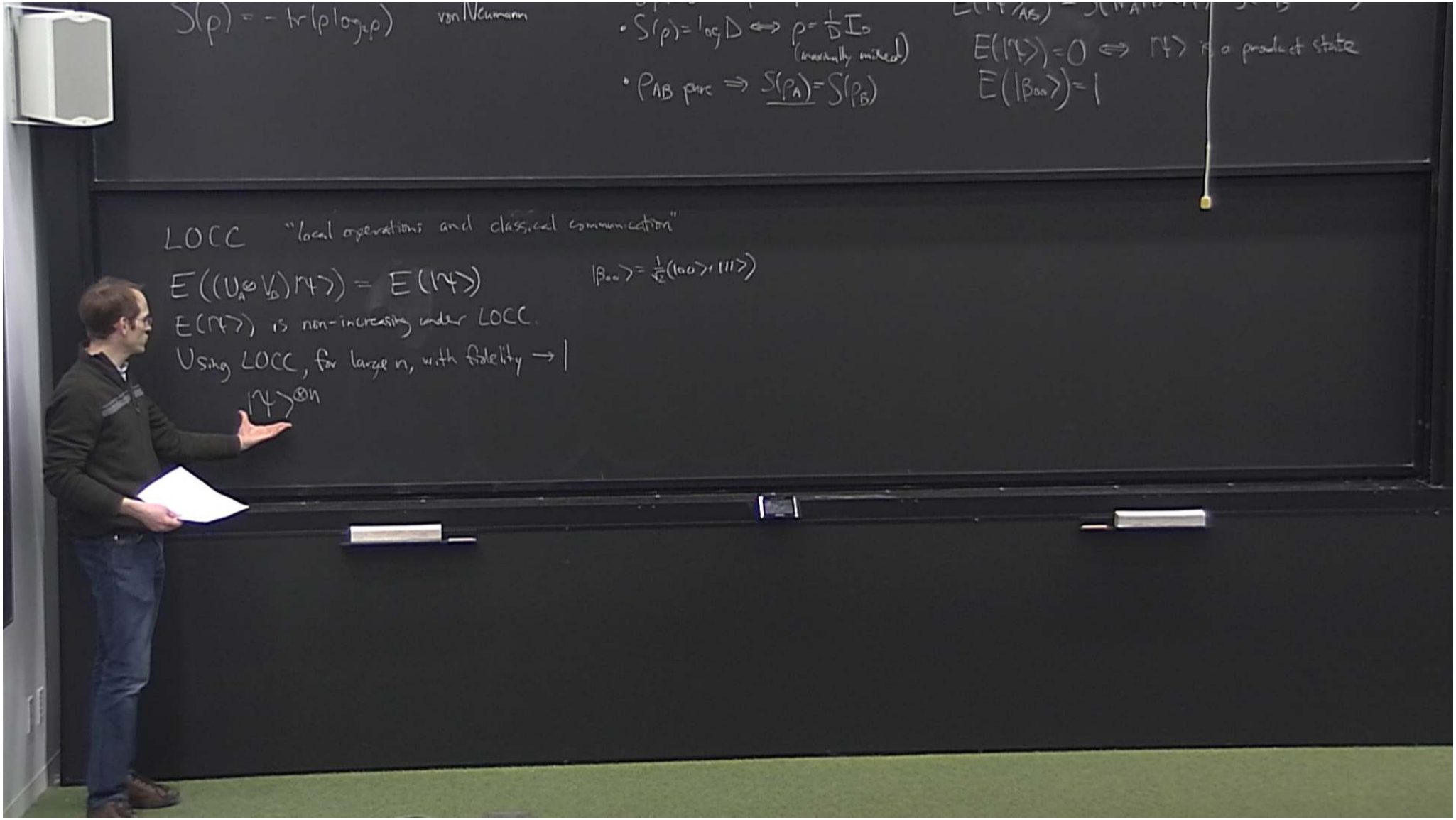
• ρ_{AB} pure $\Rightarrow S(\rho_A) = S(\rho_B)$
For a bipartite pure state, the entropy of either reduced density matrix is a measure of entanglement
$E(|\psi\rangle_{AB}) = S(\mathrm{tr}_A |\psi\rangle\langle\psi|) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$
$E(|\psi\rangle) = 0 \iff |\psi\rangle$ is a product state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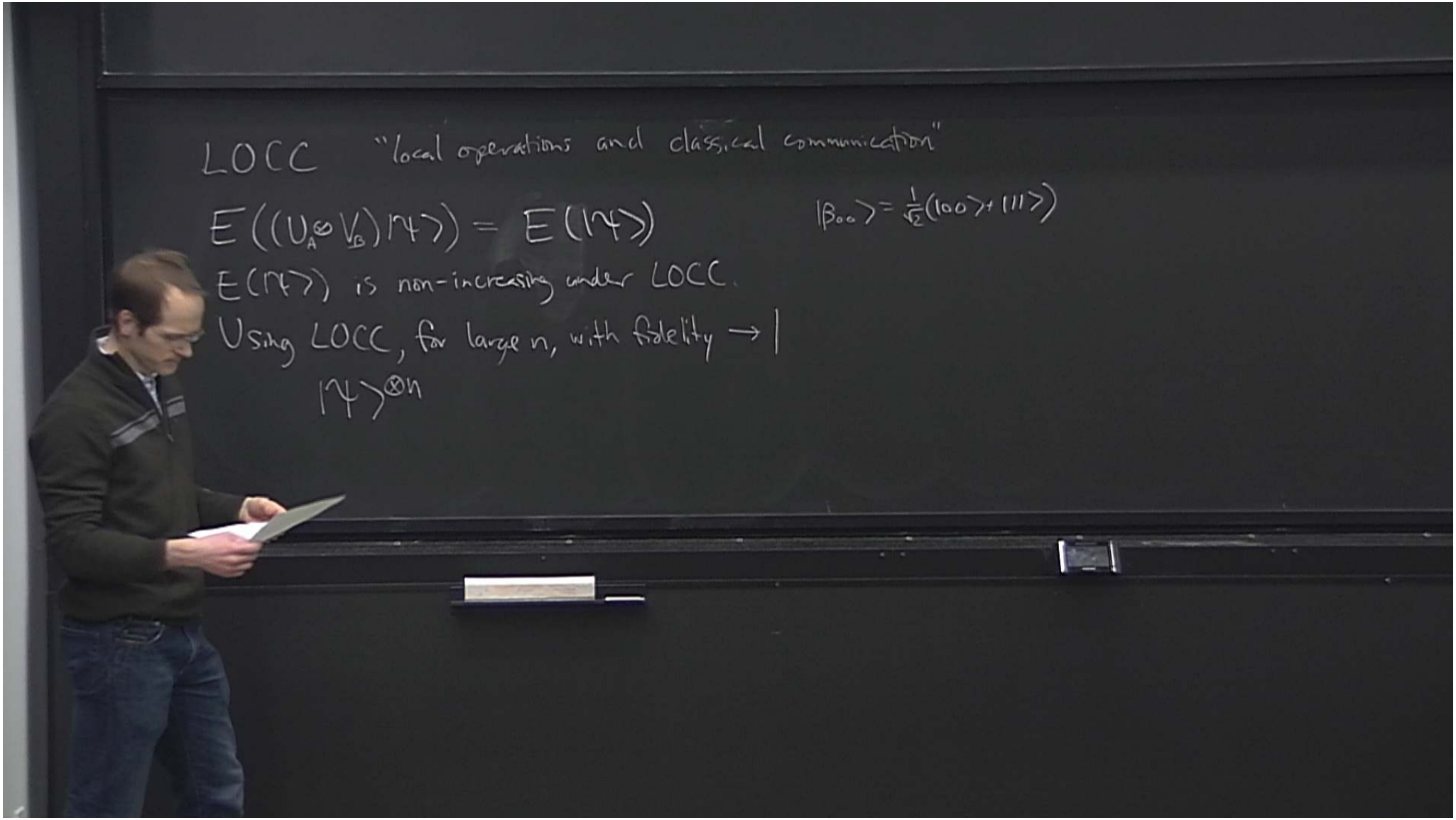
$E(|\psi\rangle) = 0 \Leftrightarrow |\psi\rangle$ a product state
$E(|\beta_{00}\rangle) = 1$
LOCC "local operations and classical communication"
$E((U_A \otimes V_B)|\psi\rangle) = E(|\psi\rangle)$
$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
$E(|\psi\rangle)$ is non-increasing under LOCC.
Using LOCC, for large n, with fidelity $\to 1$
$|\psi\rangle^{\otimes n}$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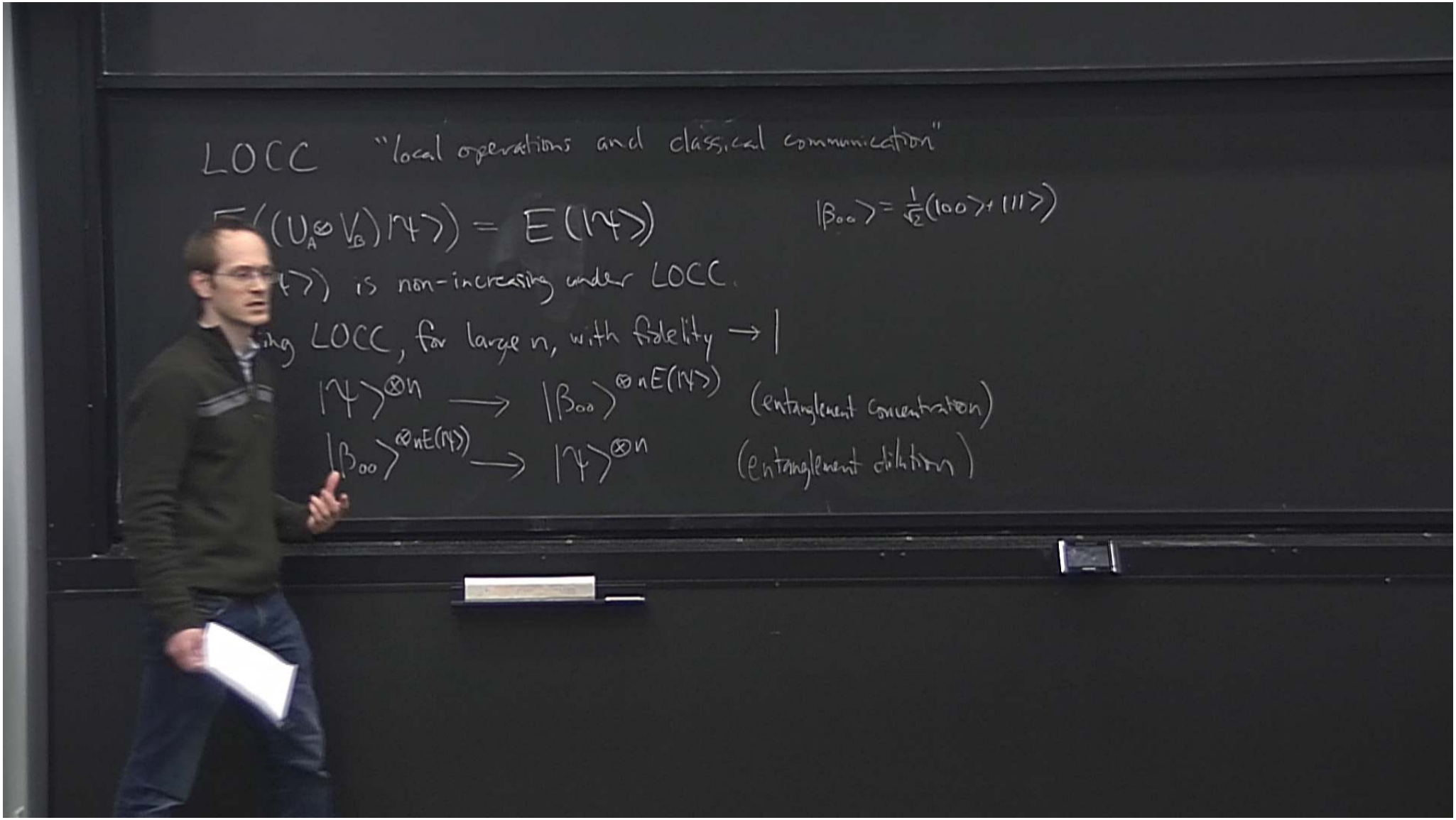
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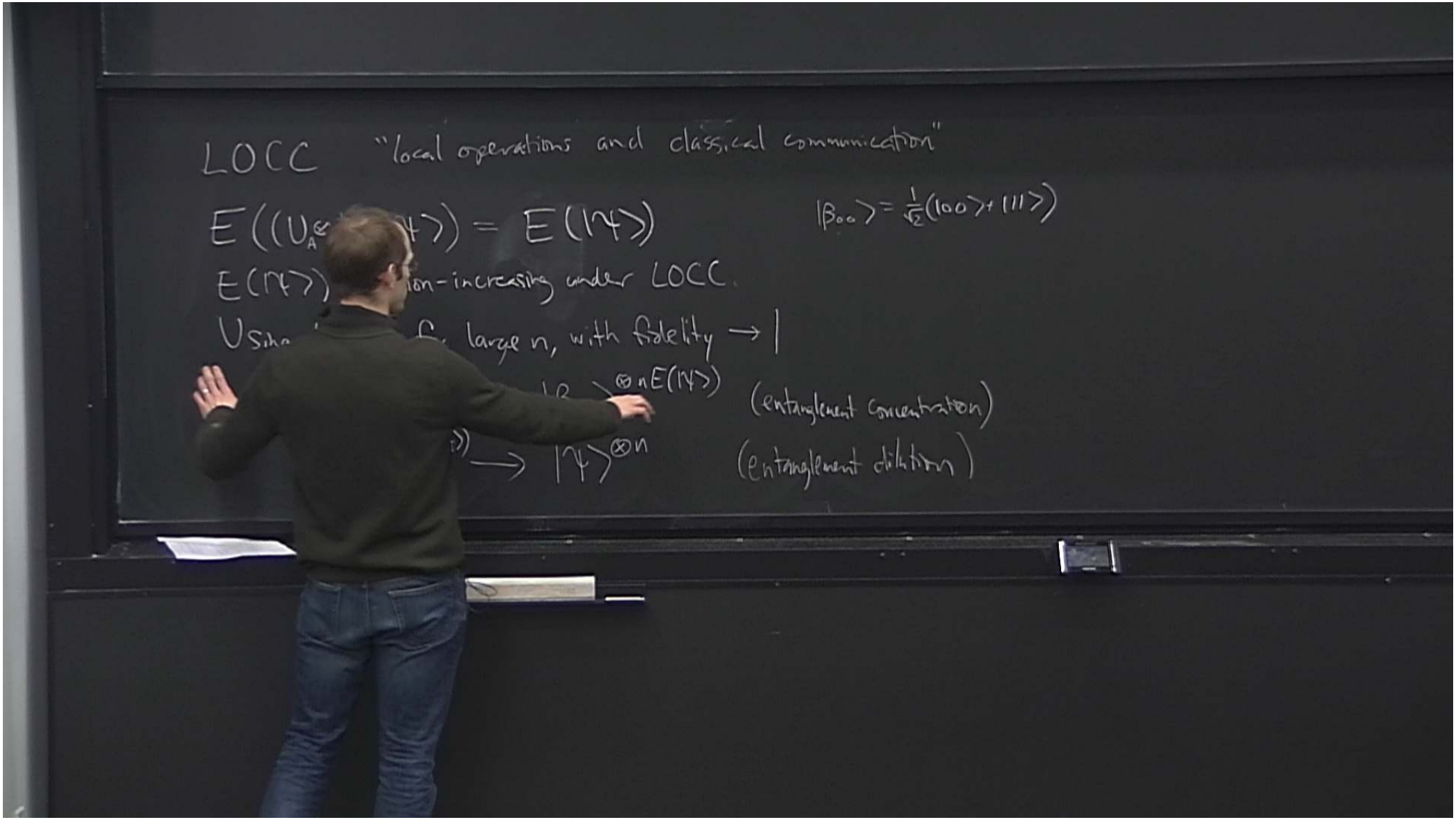
$|\psi\rangle^{\otimes n} \longrightarrow |\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$ (entanglement concentration)
$|\beta_{00}\rangle^{\otimes nE(|\psi\rangle)} \longrightarrow |\psi\rangle^{\otimes n}$ (entanglement dilution)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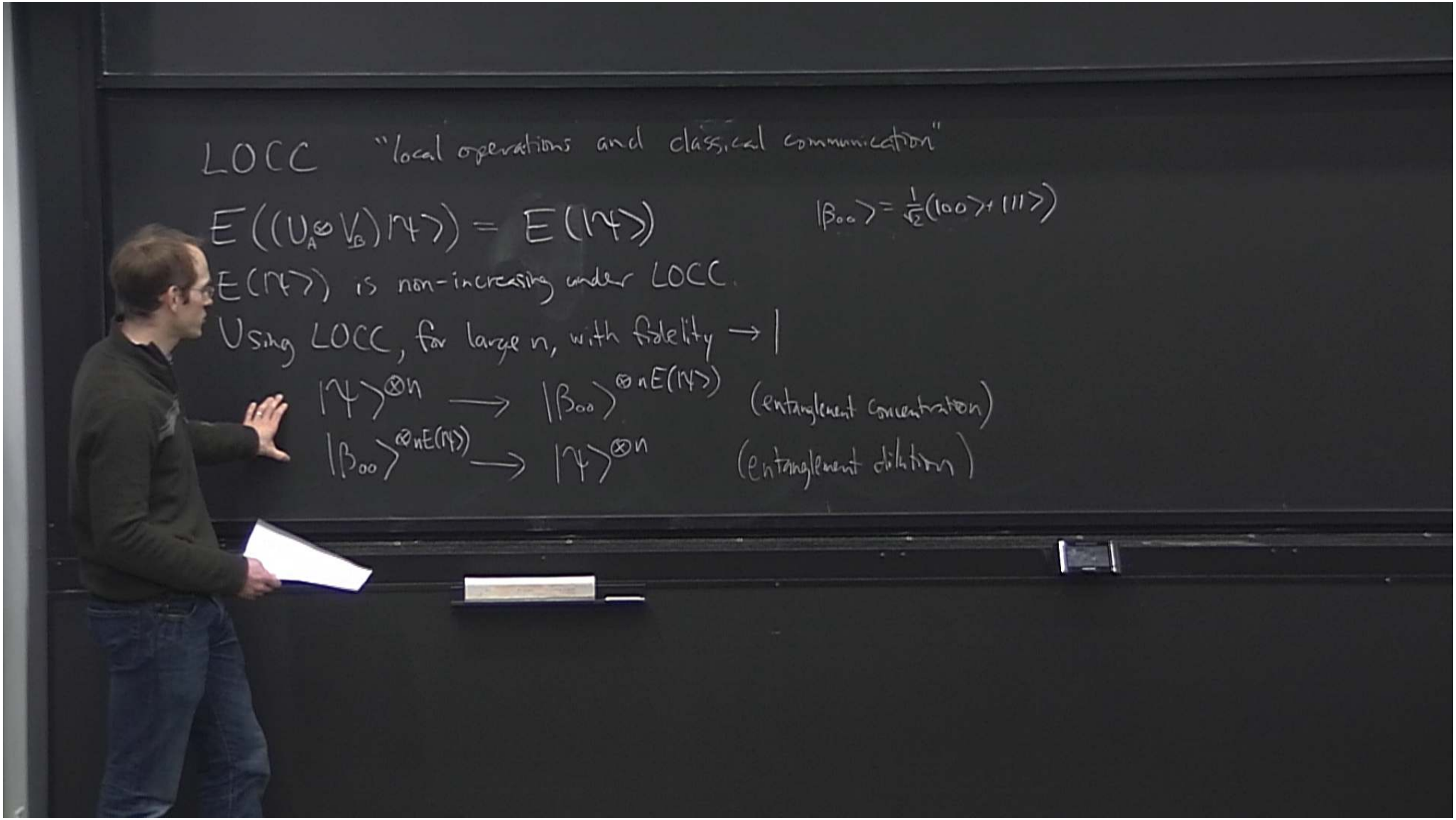
(entanglement concentration)
→ |ψ>^⊗n
(entanglement dilution)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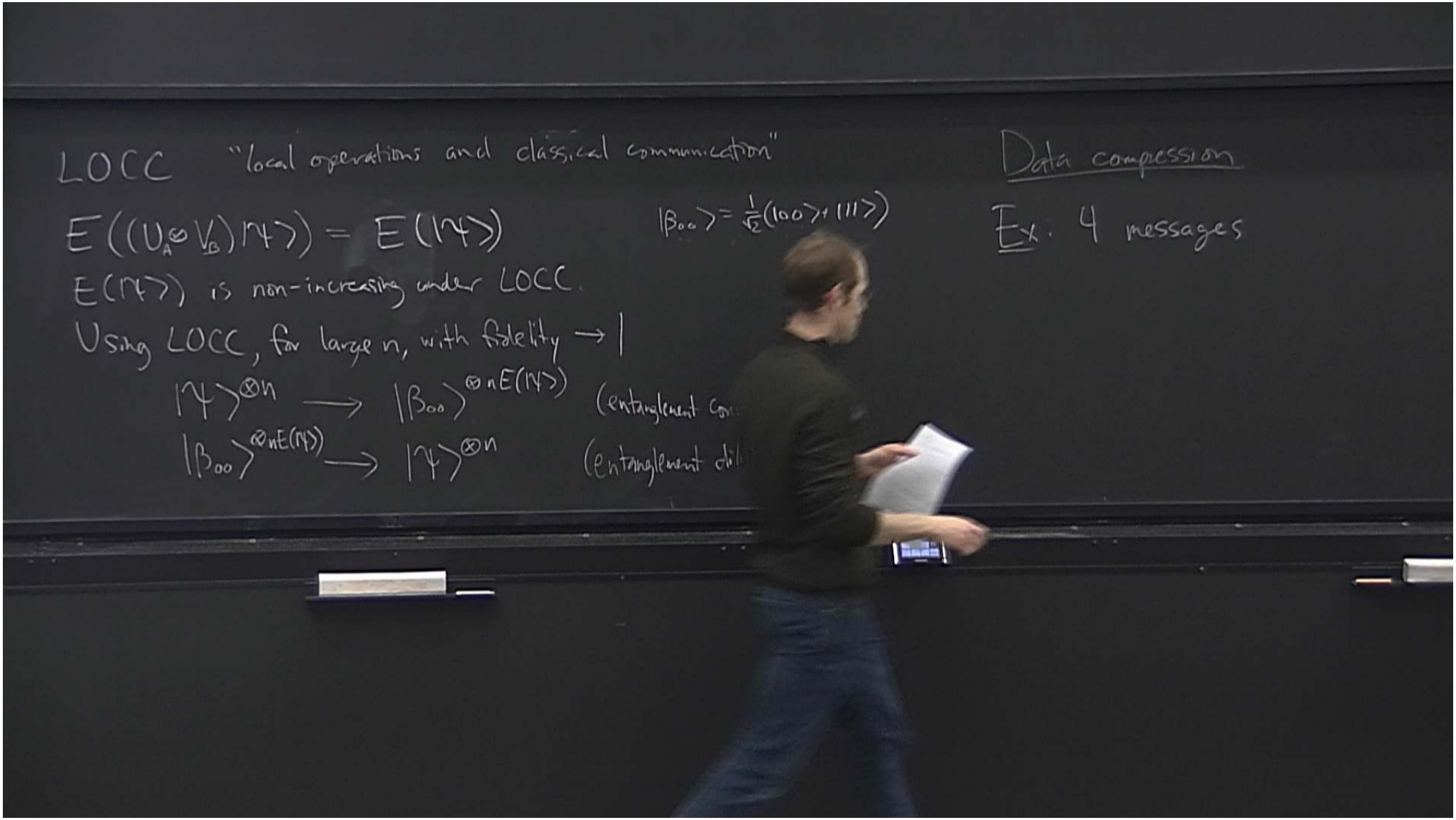
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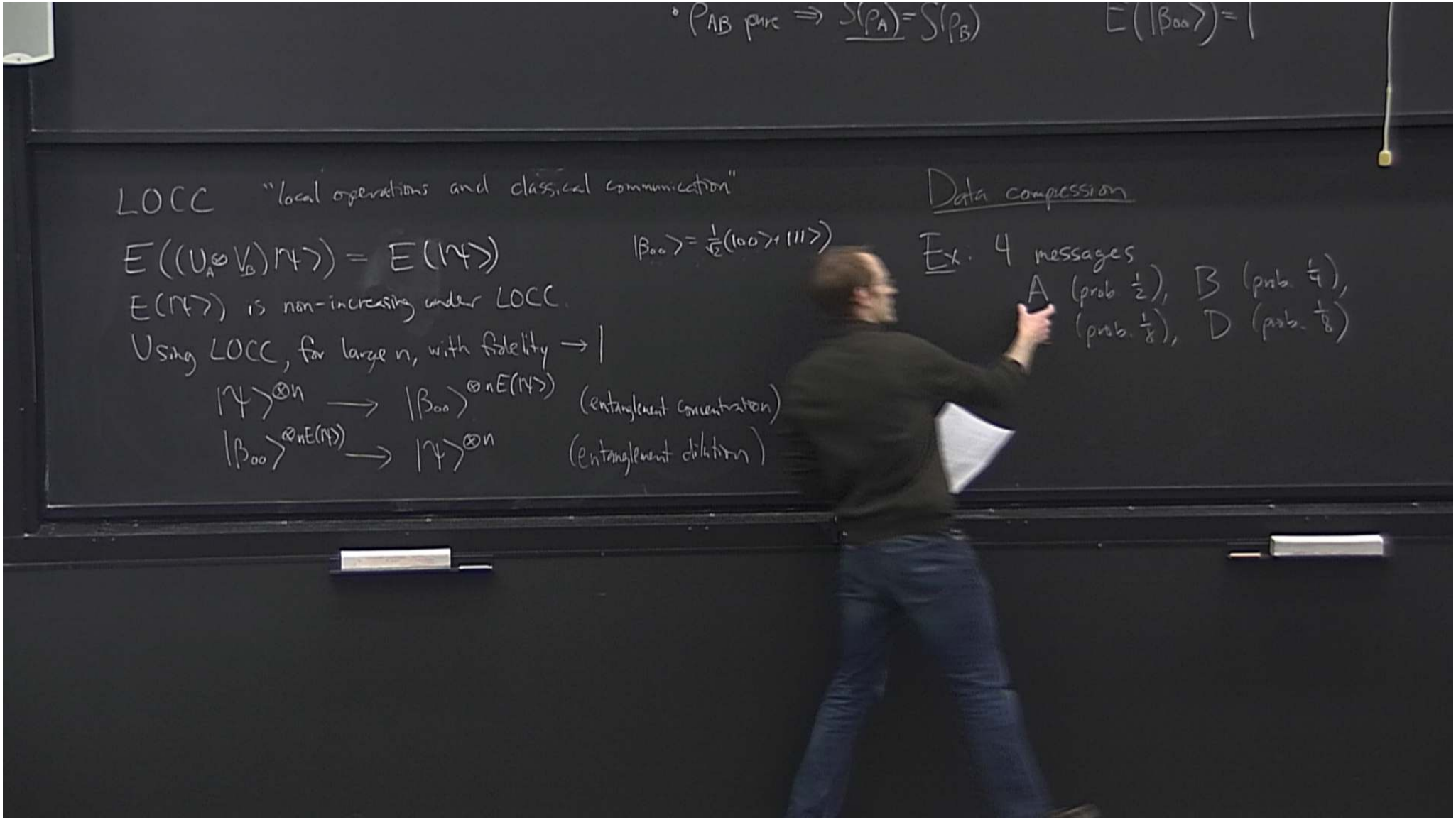
Data compression
Ex. 4 messages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Data compression
Ex. 4 messages
A (prob $\frac{1}{2}$), B (prob. $\frac{1}{4}$),
(prob. $\frac{1}{8}$), D (prob. $\frac{1}{8}$)

ρ_{AB} pure $\Rightarrow S(\rho_A) = S(\rho_B)$ $\quad E(|\beta_{00}\rangle) = 1$

...rations and classical communication"

$E(|\psi\rangle)$ $\quad |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

...easing under LOCC

...ge n, with fidelity $\to 1$

$\to |\beta_{00}\rangle^{\otimes nE(|\psi\rangle)}$ (entanglement concentration)

$\to |\psi\rangle^{\otimes n}$ (entanglement dilution)

Data compression

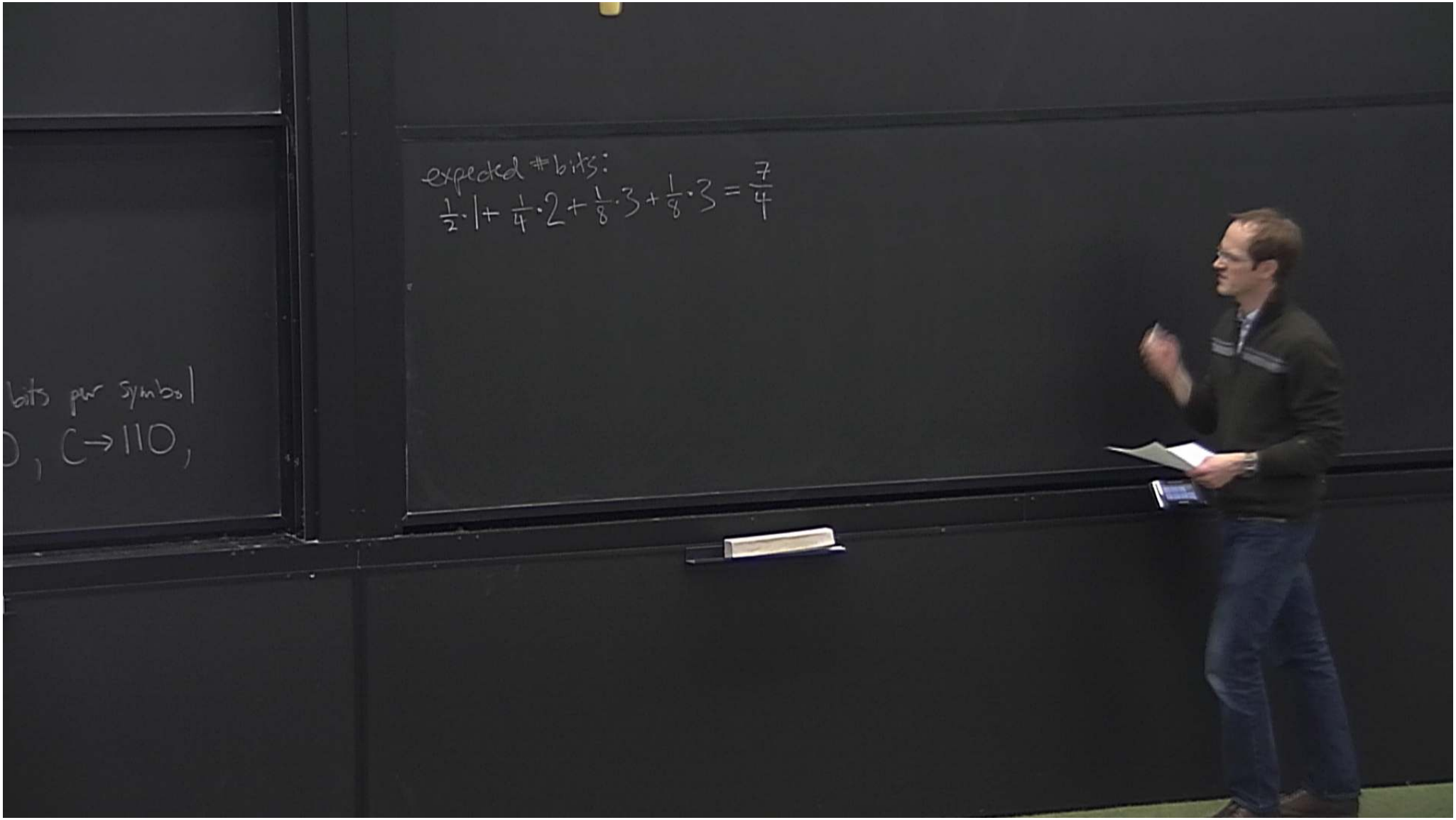
Ex: 4 messages

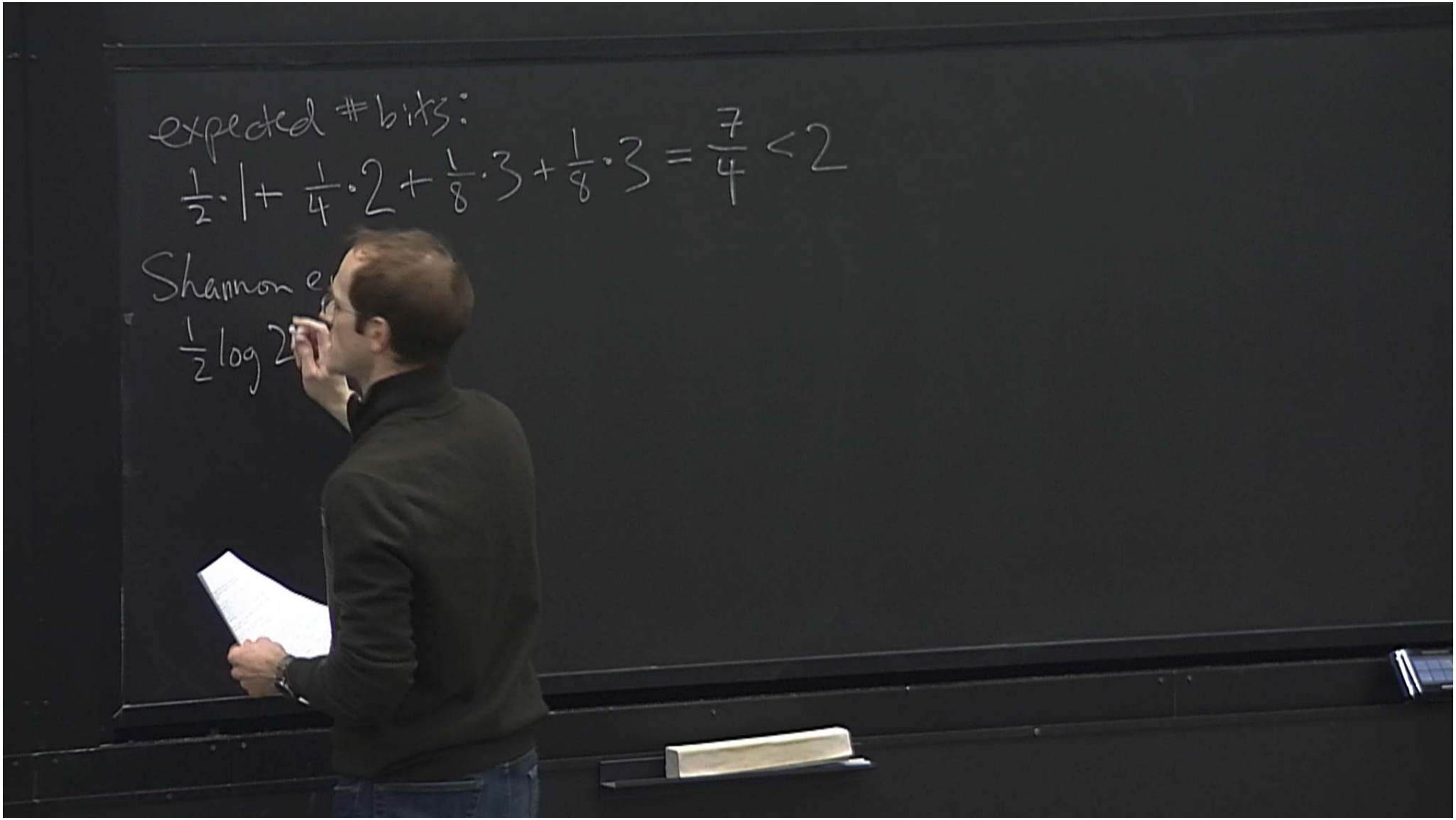
A (prob $\frac{1}{2}$), B (prob. $\frac{1}{4}$),

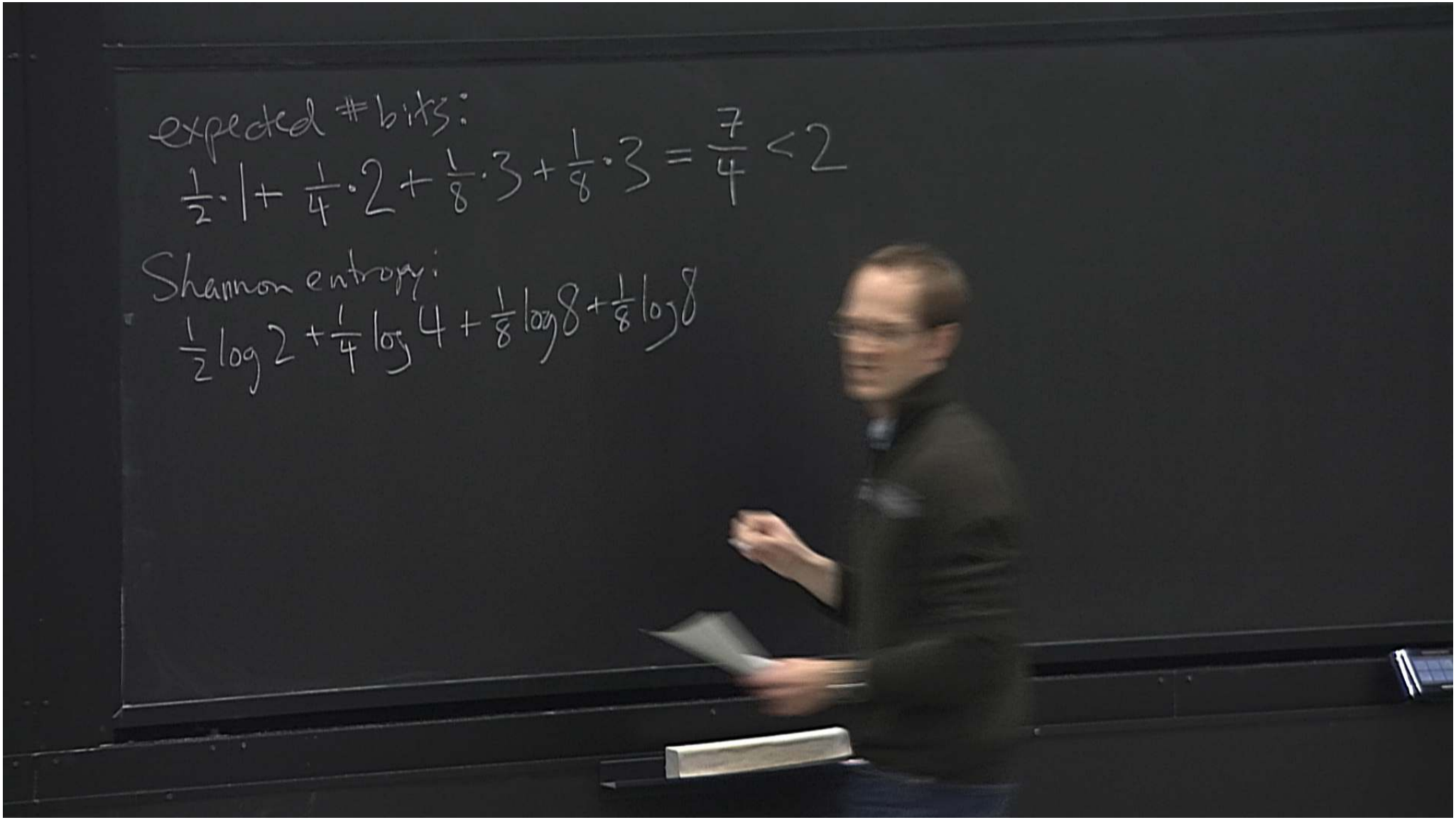
C (prob. $\frac{1}{8}$), D (prob. $\frac{1}{8}$)

Sending unencoded, we need $\log_2 4 = 2$ bits per symbol

instead, encode


expected # bits:
$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 = \frac{7}{4}$
bits per symbol
C→110,


expected # bits:
$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 = \frac{7}{4} < 2$
Shannon e
$\frac{1}{2}\log 2$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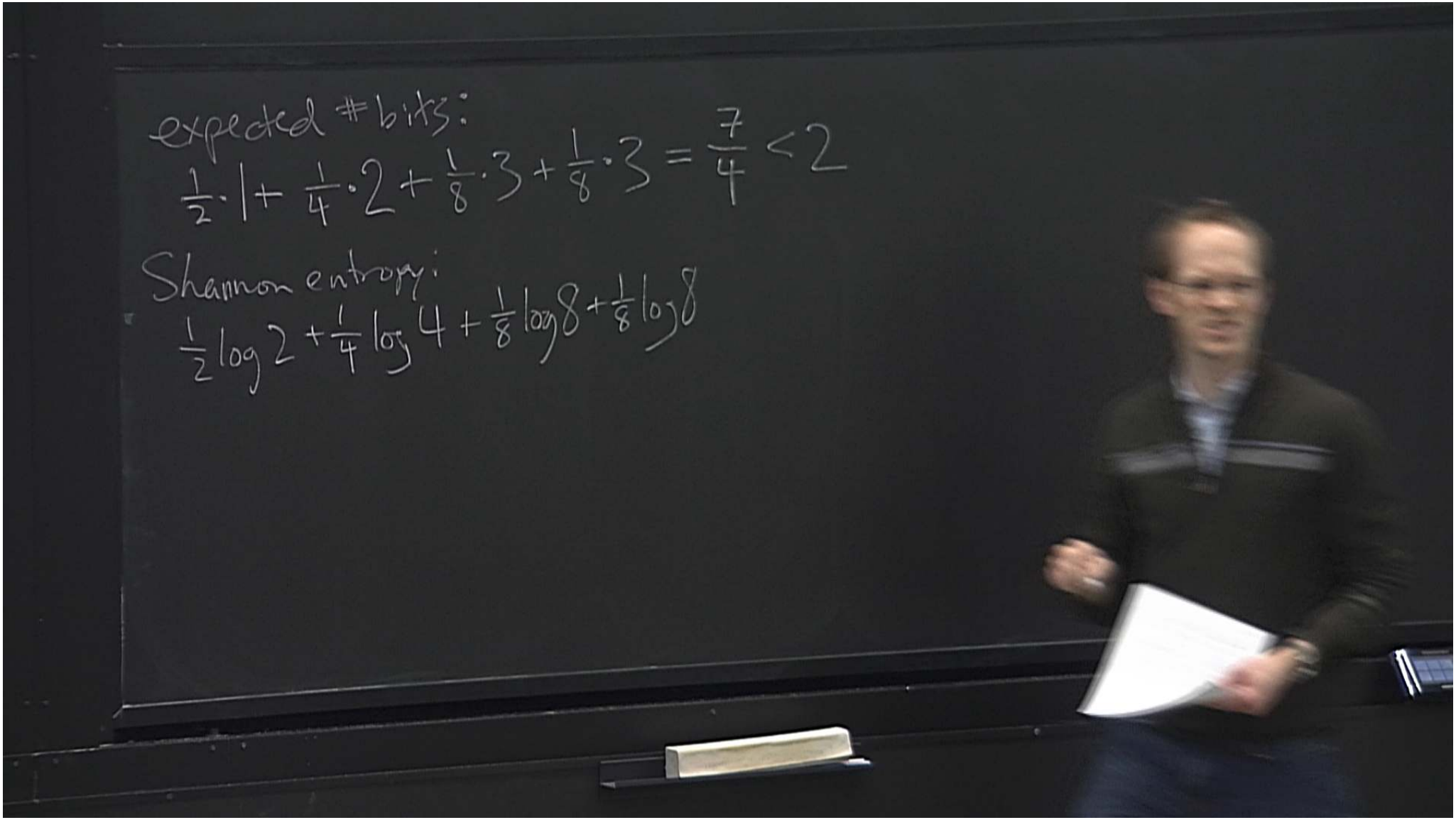
Shannon entropy:
$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8$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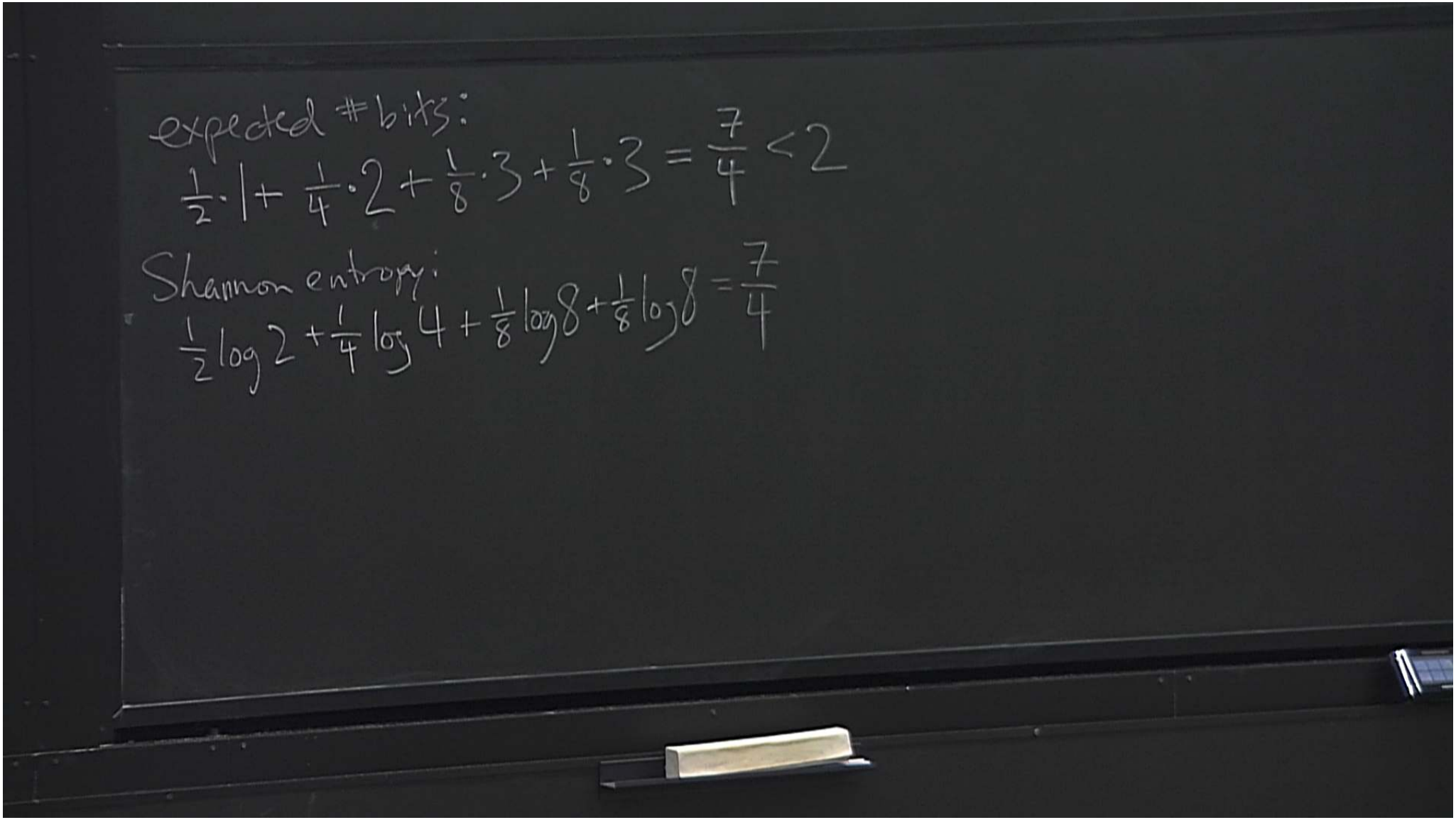
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$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 = \frac{7}{4}$

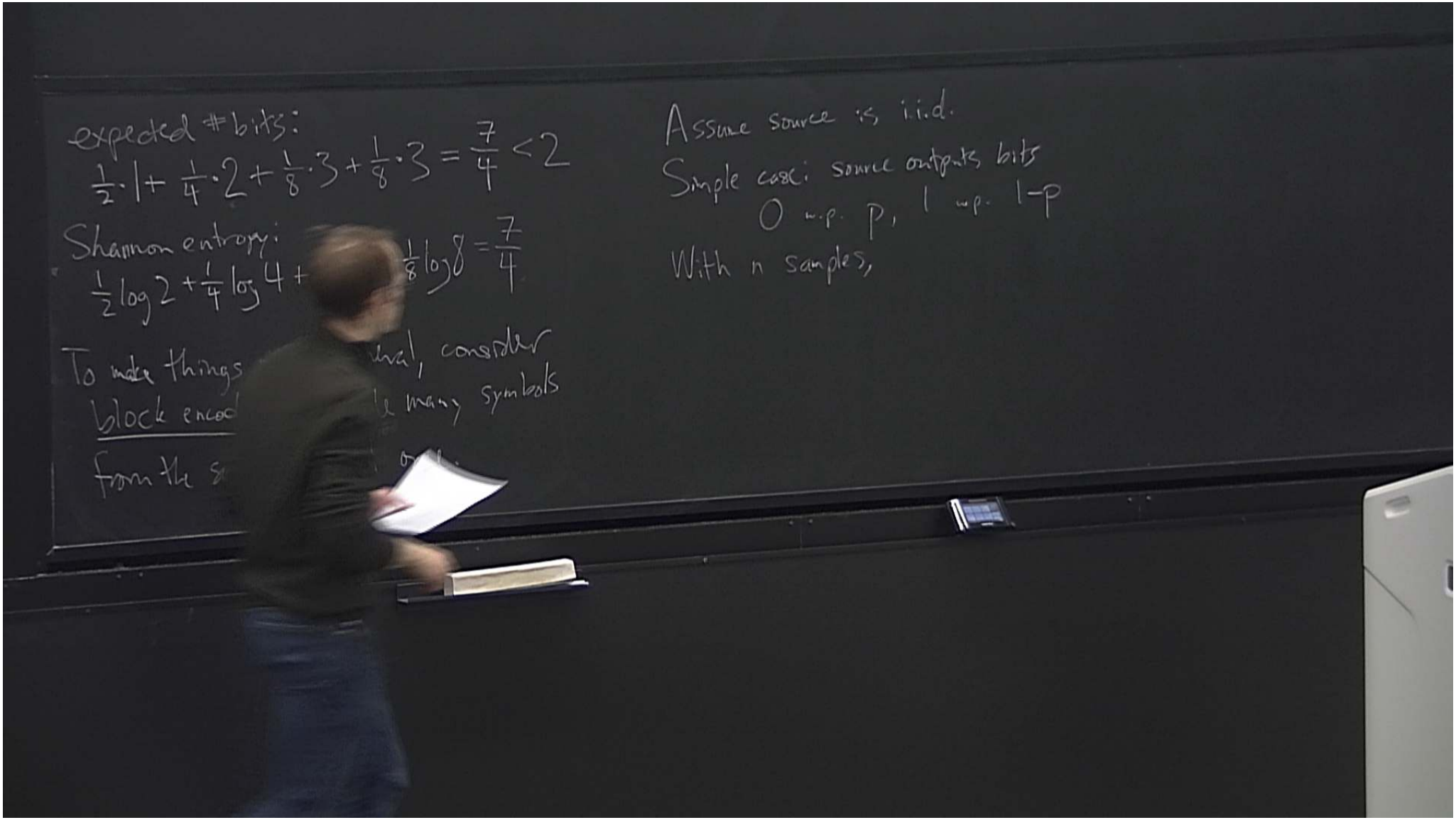
expected # bits:

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 = \frac{7}{4} < 2$$

Shannon entropy:

$$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 = \frac{7}{4}$$

To make things nice in general, consider block encodings: encode many symbols from the same source at once.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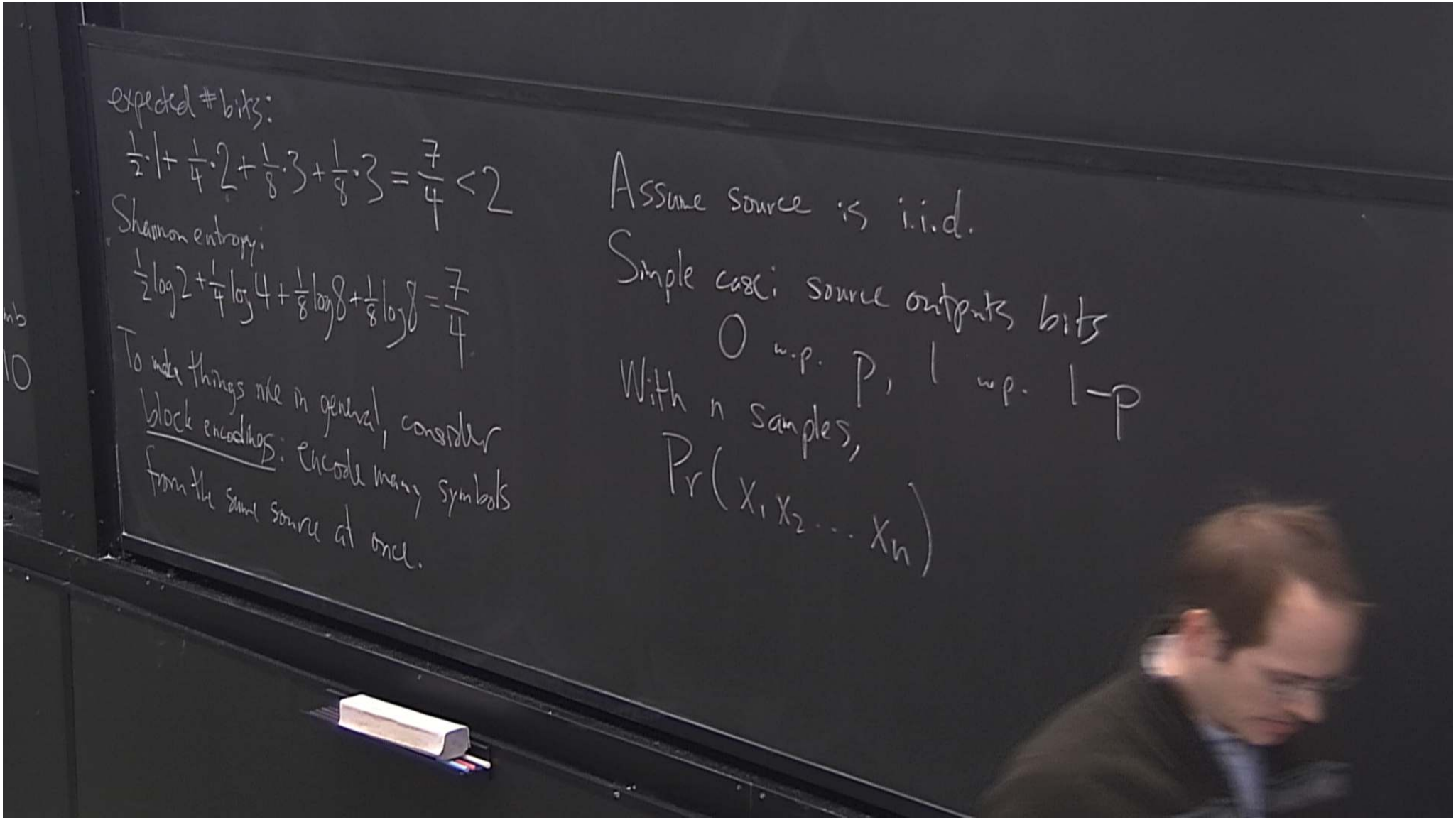
Shannon entropy:
$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 +$ $\frac{1}{8}\log 8 = \frac{7}{4}$
To make things
block encod
from the s
Assume source is i.i.d.
Simple case: source outputs bits
0 w.p. p, 1 w.p. 1-p
With n samples,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Shannon entropy:
$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 = \frac{7}{4}$
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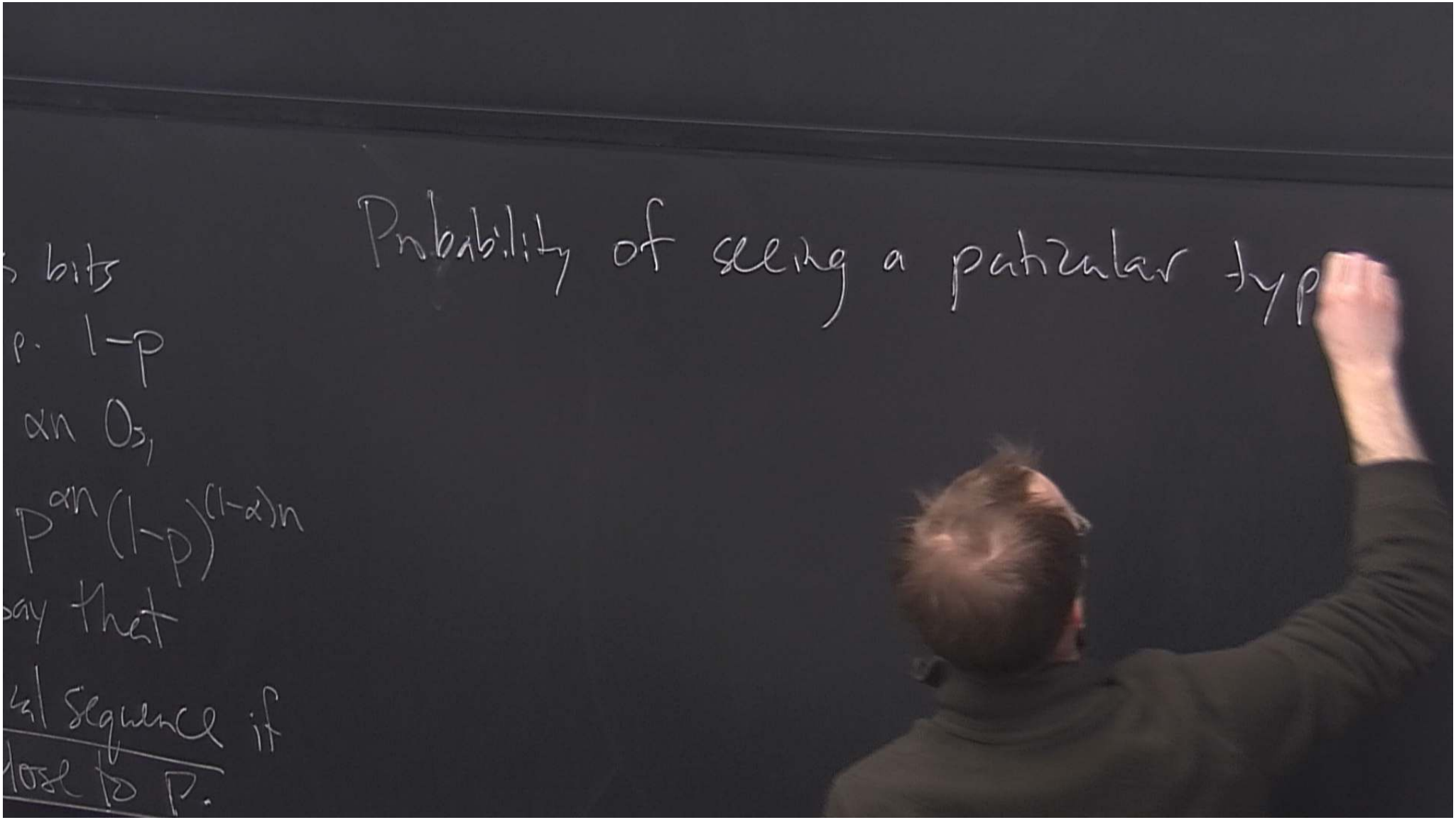
With n samples,
$Pr(X_1 X_2 \ldots X_n)$


Probability of seeing a particular typ
bits
p. 1-p
an 0s,
$p^{an}(1-p)^{(1-\alpha)n}$
say that
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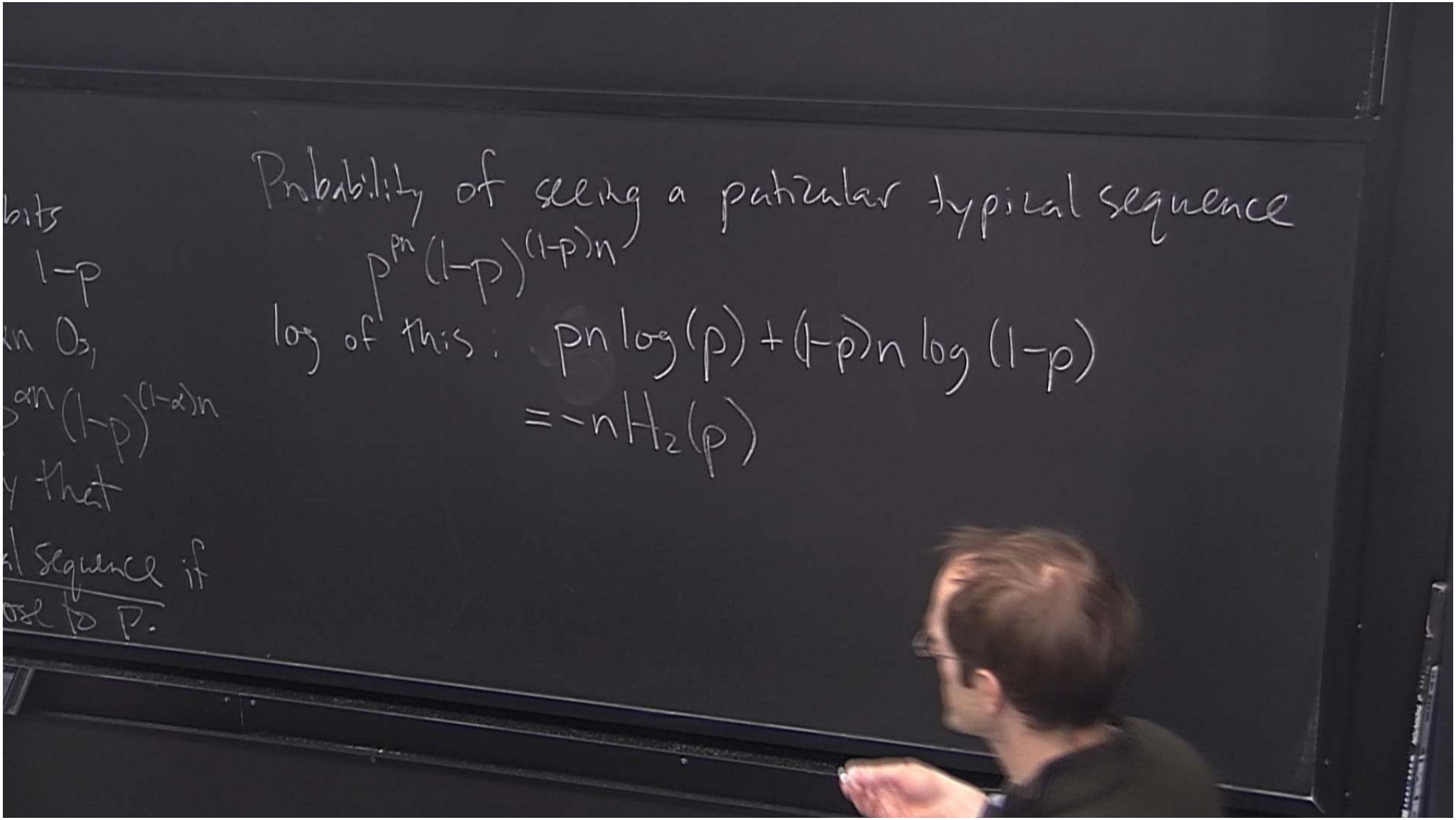
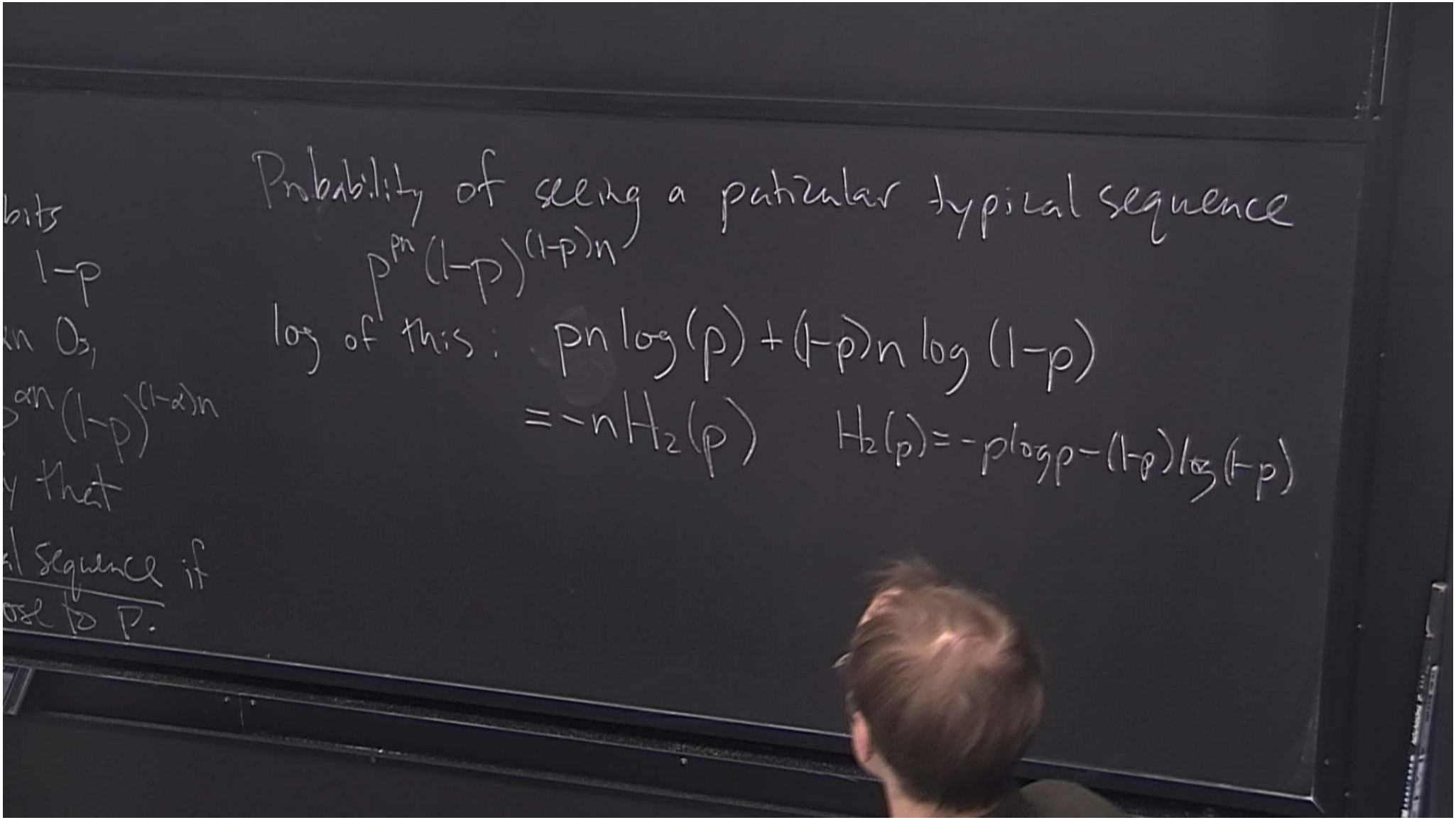
lose to p.


Probability of seeing a particular typical sequence
$p^{pn}(1-p)^{(1-p)n}$
log of this: $pn\log(p) + (1-p)n\log(1-p)$
$= -nH_2(p)$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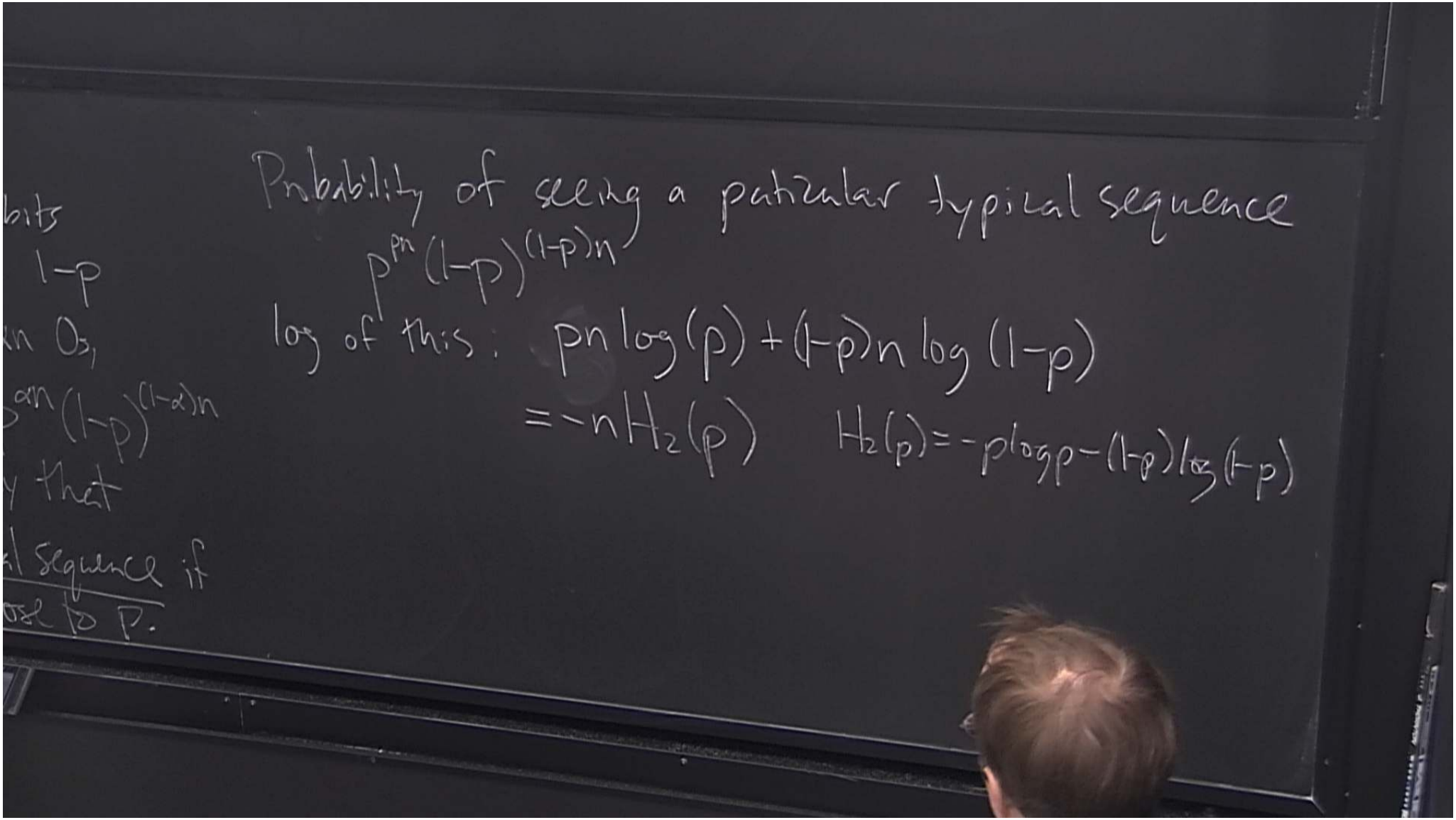
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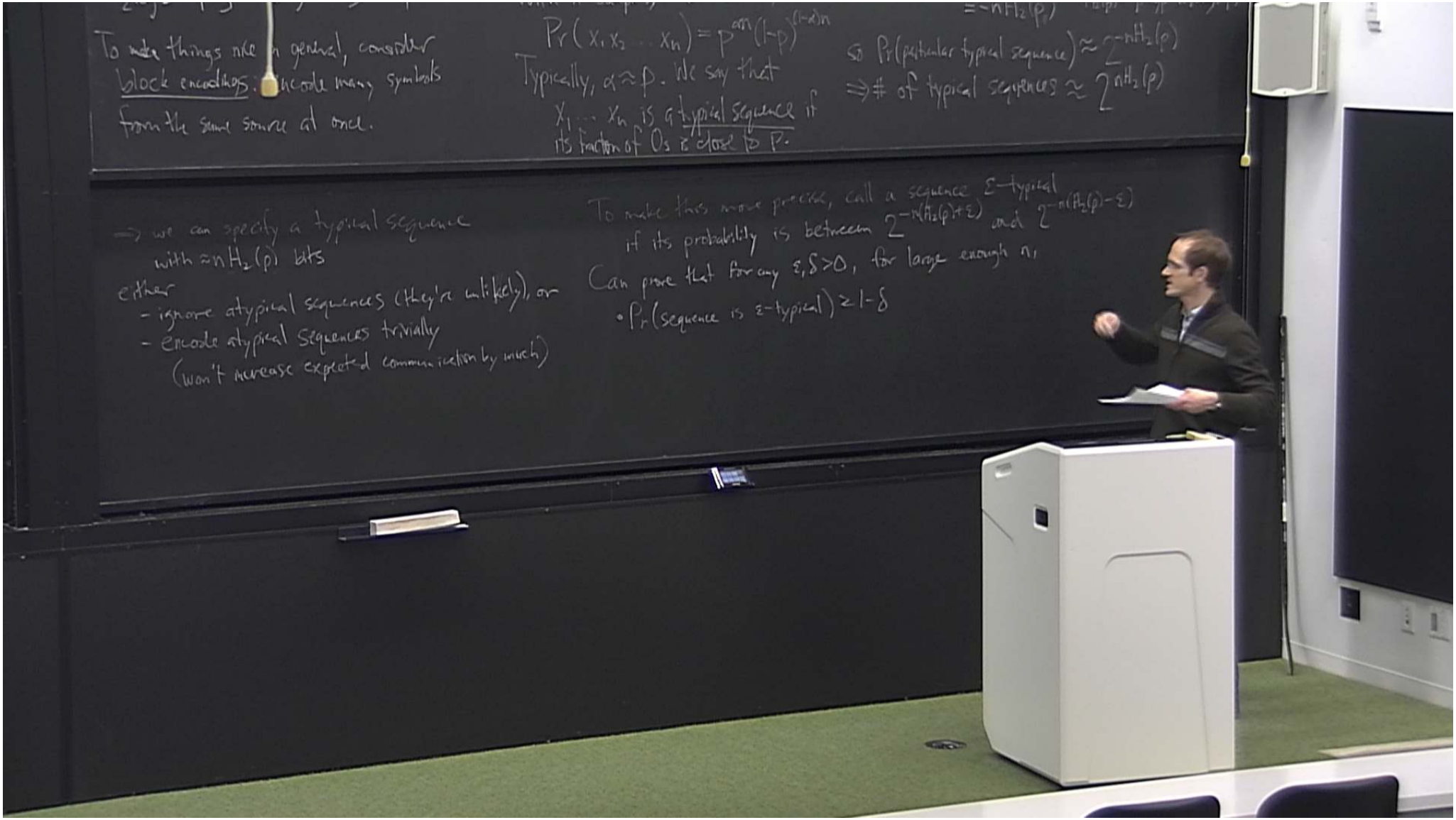
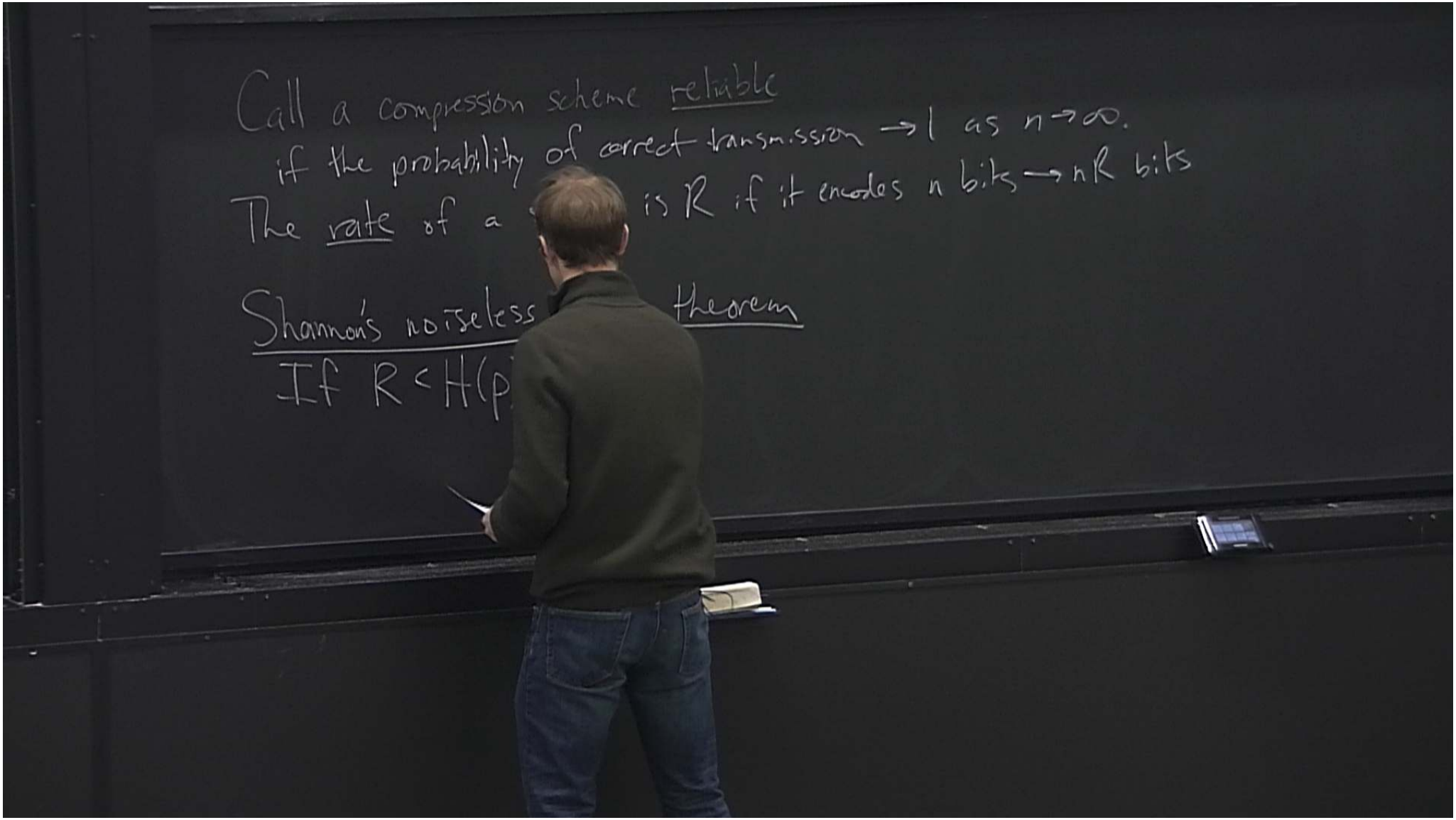

To make things nice in general, consider block encodings: encode many symbols from the same source at once.
$Pr(x_1 x_2 \dots x_n) = p^{an}(1-p)^{(1-a)n}$
Typically, $a \approx p$. We say that $x_1 \dots x_n$ is a typical sequence if its fraction of 0s is close to p.
So $Pr(\text{particular typical sequence}) \approx 2^{-nH_2(p)}$
$\Rightarrow$ # of typical sequences $\approx 2^{nH_2(p)}$
$\Rightarrow$ we can specify a typical sequence with $\approx nH_2(p)$ bits
either
- ignore atypical sequences (they're unlikely), or
- encode atypical sequences trivially (won't increase expected communication by much)
To make this more precise, call a sequence ε-typical if its probability is between $2^{-n(H_2(p)+\varepsilon)}$ and $2^{-n(H_2(p)-\varepsilon)}$
Can prove that for any $\varepsilon, \delta > 0$, for large enough n,
- $Pr(\text{sequence is } \varepsilon\text{-typical}) \geq 1-\delta$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if the probability of correct transmission → 1 as n → ∞.
The rate of a
is R if it encodes n bits → nR bits
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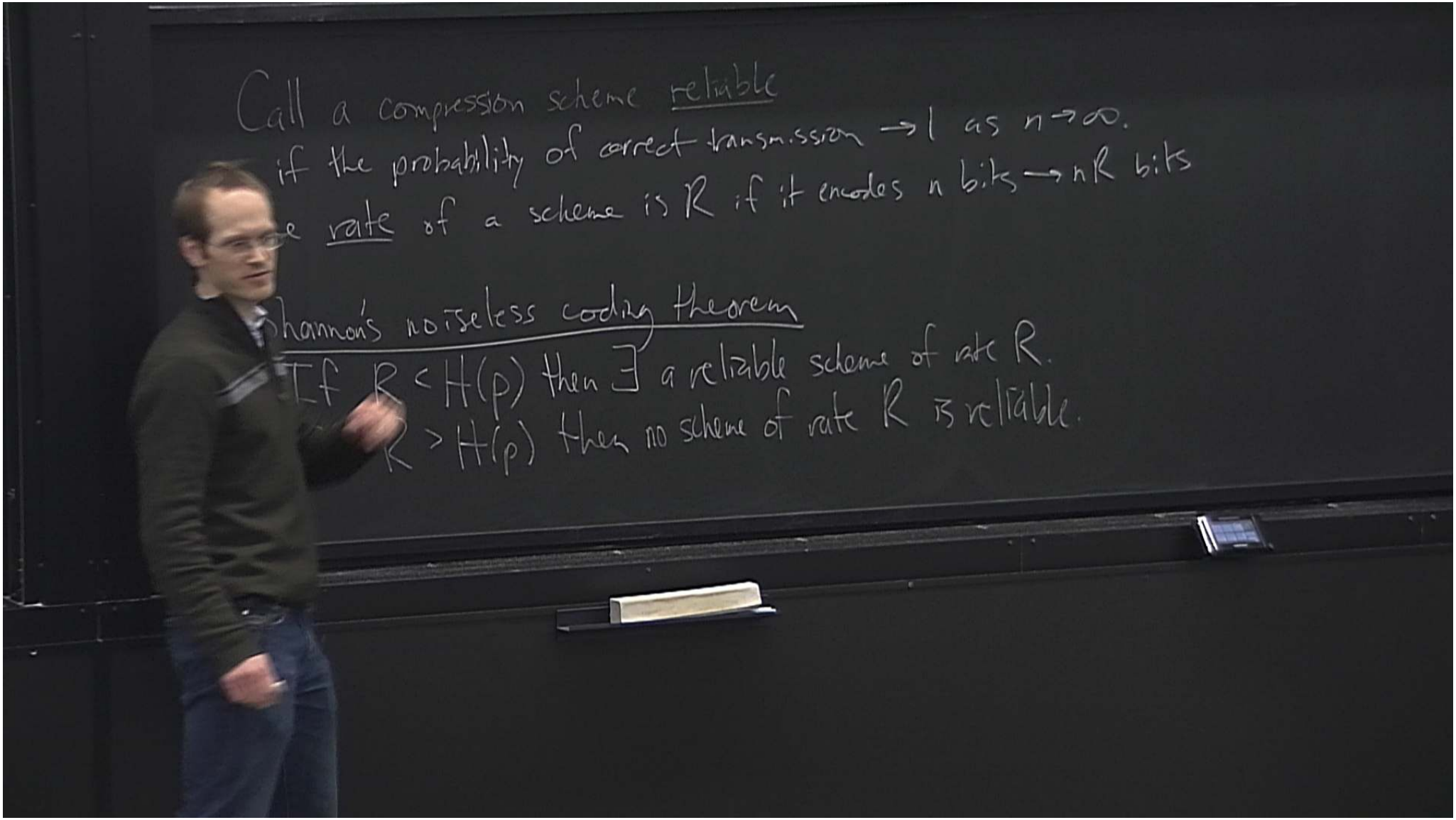
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