

Title: 13/14 PSI - Quantum Gravity Review - Lecture 12

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URL: <http://pirsa.org/14030019>

Abstract:

Solving the Gauss constraints

Solving the Gauss constraints

$$h_e \rightarrow g(v_f) h_e g^{-1}(v_e)$$

$\underbrace{\hspace{1.5cm}}_{v_e} \xrightarrow{\hspace{1.5cm}} v_f$

Solving the Gauss constraints

$$h_e \rightarrow g(\sigma_f) h_e g^{-1}(\sigma_s) \quad \sigma_s \xrightarrow{\quad} \sigma_f$$

$$\langle e \rangle = \chi(h_e) \longrightarrow \chi(g(\sigma_f) h_e g^{-1}(\sigma_s)) = \langle g | L_{g^{-1}(\sigma_f)} R$$

Gauss constraints

$$g(u_f) h_e g^{-1}(u_s)$$


$$\psi(h_e) \longrightarrow \psi(g(u_f) h_e g^{-1}(u_s)) = \langle g | L_{g^{-1}(u_f)} R_{g^{-1}(u_s)} | \psi_e \rangle$$

Solving the Gauss constraints

$$h_e \rightarrow g(\sigma_f) h_e g^{-1}(\sigma_s)$$


$$\langle h_e | \chi_e \rangle = \chi(h_e) \longrightarrow \chi(g(\sigma_f) h_e g^{-1}(\sigma_s)) = \chi(g(\sigma_f) h_e)$$

Group average $|\chi_e\rangle$:

$$\delta(\widehat{g_{\sigma_s}}) \delta(\widehat{g_{\sigma_f}}) |\chi_e\rangle = \int_{G \times G} L$$

Gauss constraints

$$g(\sigma_f) h_e g^{-1}(\sigma_s)$$


$$\psi(h_e) \longrightarrow \psi(g(\sigma_f) h_e g^{-1}(\sigma_s)) = \langle g | L_{g^{-1}(\sigma_f)} R_{g^{-1}(\sigma_s)} | \psi \rangle$$

$$\delta(\widehat{g_{\sigma_s}}) \delta(\widehat{g_{\sigma_f}}) |\psi_e\rangle = \int_{G \times G} L_{g_1} R_{g_2} |\psi_e\rangle dg_1 dg_2$$

$\downarrow$
 $\psi_e(h) \rightarrow \psi_e(g^{-1})$

→ σ_4

$$^{-1}(\sigma_5)) = \langle g | L_{g^{-1}(\sigma_4)} R_{g^{-1}(\sigma_5)} | \psi_e \rangle$$

$$= \int_{G \times G} L_{g_1} R_{g_2} | \psi_e \rangle dg_1 dg_2 = c_{\pm} | 1 \rangle$$

$$\psi_e(h) \rightarrow \psi_e(g_1^{-1} h g_2)$$

Solving the Gauss constraints

$$h_e \rightarrow g(u_f) h_e g^{-1}(u_e) \quad \xrightarrow{u_f} \xrightarrow{u_e}$$

$$\langle h_e | \psi_e \rangle = \chi(h_e) \longrightarrow \chi(g(u_f) h_e g^{-1}(u_e)) = \langle g | L_{g^{-1}(u_f)} R_{g^{-1}(u_e)} | \psi_e \rangle$$

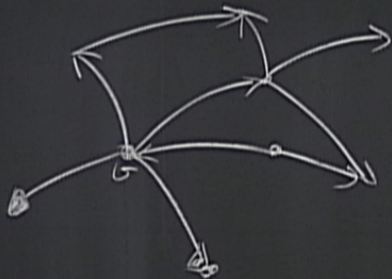
Group average $|\psi_e\rangle$:

$$\widehat{\delta(g_{u_f})} \widehat{\delta(g_{u_e})} |\psi_e\rangle = \int_{h_1, h_2} L_{g_1} R_{g_2} | \psi_e \rangle dg_1 dg_2 = \mathbb{I} | 1 \rangle$$

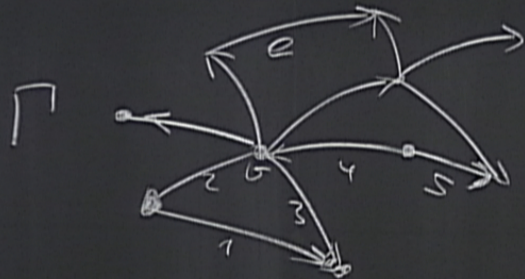
$\psi_e(h) \rightarrow \psi_e(g_1^{-1} h g_2)$

More edges: Graph

More edges: Graph



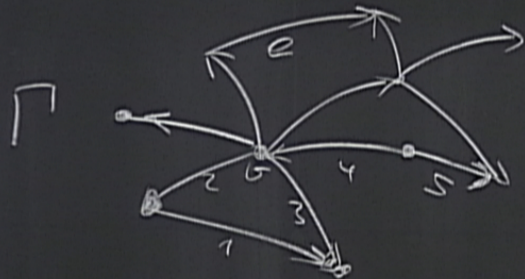
More edges: Graph



Hilbert space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$ $L^2(\Gamma)$

$\psi(g_1, g_2, g_3, \dots, g_n)$, $\int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n$

More edges: Graph

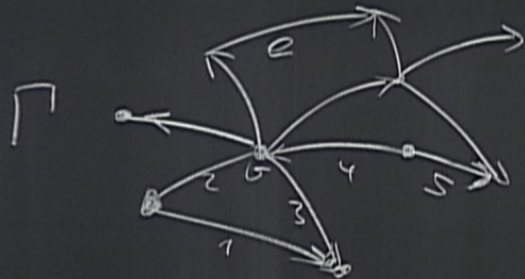


Hilbert space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$ $L^2(\Gamma)$

$\psi(g_1, g_2, g_3, \dots, g_n)$, $\int \overline{\psi}_1 \psi_2 dg_1$

- SNW P

More edges: Graph



Hilbert space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$ $L^2(\Gamma)$

$$\psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n$$

- SNW for one edge $|j\rangle$

- SNW for an edge $|j_a m_e, n_e\rangle$
- Tensor product:

$1, \dots, g_n$

$$|e, n\rangle = |e, g_1, g_2\rangle$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$g_1, \dots, g_n$$

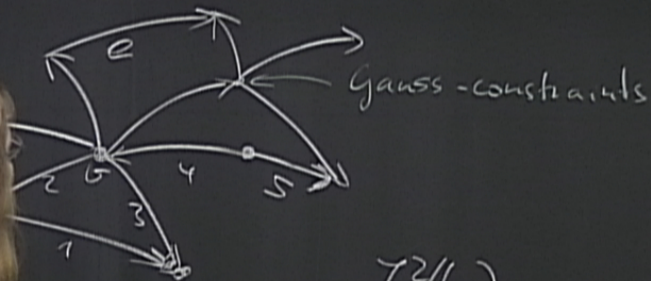
$$|e\rangle \quad |e\rangle$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $(\otimes_e |j_e, m_e, n_e\rangle)$

$$\delta(G_v)$$

$$g_1, \dots, g_n$$

More edges: Graph



$L^2(\mathbb{G})$

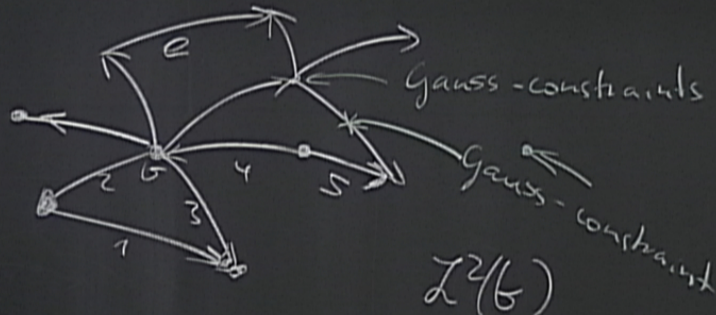
space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$

$(g_1, g_2, \dots, g_n) \mapsto \int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n$

- SNW for an edge $|j_e|$
- Tensor product: $(\bigotimes_e |j_e|)$

$\delta(\mathcal{G}_0)$

More edges: Graph



Hilbert space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$

$$\psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n$$

- SNW for one edge $|j_e\rangle$
- Tensor product: $(\bigotimes_e |j_e\rangle)$

$$\delta(G_v)$$

$$|e\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(G_U)} = \int_G \left(\bigotimes_{e: v=\bar{p}(e)} L_h^e \right)$$

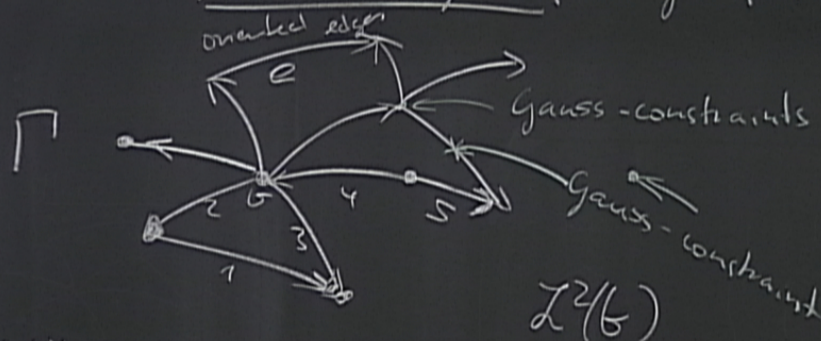
- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $(\otimes_e |j_e, m_e, n_e\rangle)$

$$\widehat{\delta(G_u)} = \int_G \left(\otimes_{\substack{e: v=e_p(e)}} L_h^e \right) \left(\otimes_h R_h^{e'} \right)$$


- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $(\bigotimes_e |j_e, m_e, n_e\rangle)$

$$\widehat{\delta(G_v)} = \int_G \left(\bigotimes_{\substack{e: v=\sigma_f(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v=\sigma_g(e')}} R_h^{e'} \right) dh$$


More edges: Graph



Hilbert space: $\mathcal{H}_\Gamma = \bigotimes_{e \in \Gamma} \mathcal{H}_e$

$\mathbb{Z}(b)$

$$\psi(g) \rightarrow \psi'(g) = \psi(g^{-1})$$

$\psi(g_1, g_2, g_3, \dots, g_n)$

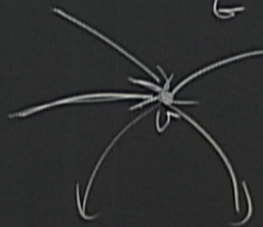
$$\int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n$$

- SNW p. are edge
- Tensor product: $(\otimes_e | \psi_e \rangle)$

$$\delta((\otimes_e | \psi_e \rangle)) = \delta(\otimes_e | \psi_e \rangle)$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(g_v)} = \int_G \left(\bigotimes_{\substack{e: v = \bar{v}_p(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \bar{v}_s(e')}} R_h^{e'} \right) dh$$

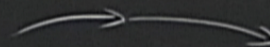
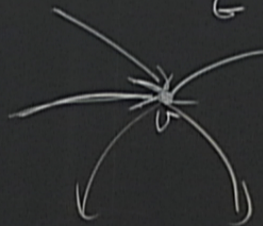


$$g \mapsto \tau'(g) = \tau(g^{-1})$$

$$dg_1 \dots dg_n$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(g_v)} = \int_G \left(\bigotimes_{\substack{e: v = \bar{p}(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \bar{s}(e')}} R_h^{e'} \right) dh$$

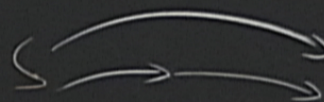
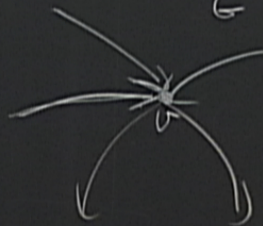


$$g \mapsto \bar{g} = g^{-1}$$

$$dg_1 \dots dg_n$$

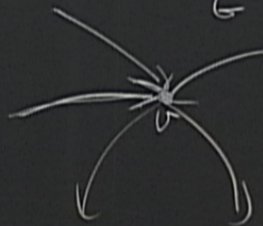
- SNW for an edge $|j_a, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(G_v)} = \int_G \left(\bigotimes_{\substack{e: v = \bar{v}_p(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \bar{v}_s(e')}} R_h^{e'} \right) dh$$



- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(g_v)} = \int_G \left(\bigotimes_{\substack{e: v = \underline{v}_p(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \underline{v}_s(e')}} R_h^{e'} \right) dh$$



$$\begin{array}{ccc} & g_e & \\ \curvearrowright & \searrow & \searrow \\ g_{e_1} & & g_{e_2} \end{array} \quad \begin{array}{c} \psi(g_e) \\ \downarrow \\ \psi(g_{e_1}, g_{e_2}) = \psi(g_{e_2} \circ g_{e_1}) \end{array}$$

$$\psi(g^{-1}) = \psi(g^{-1})$$

$$dg_1 \dots dg_n$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(g_v)} = \left(\bigotimes_{\substack{e: v = \bar{v}_p(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \bar{v}_s(e')}} R_h^{e'} \right) dh$$

$$\begin{array}{ccc} & g_e & \\ \curvearrowright & \searrow & \nearrow \\ g_{e_1} & \xrightarrow{g_{e_2}} & \gamma(g_e) \\ & \searrow & \nearrow \\ & g_{e_2} & \end{array}$$

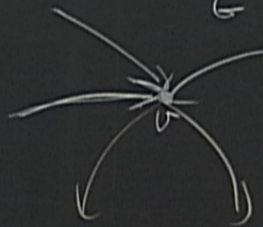
$$\gamma(g_{e_1}, g_{e_2}) = \gamma(g_{e_2} \cdot g_{e_1})$$

$$\gamma \mapsto \gamma'(g) = \gamma(g^{-1})$$

$$\int dg_1 \dots dg_n$$

- SNW for an edge $|j_e, m_e, n_e\rangle$
- Tensor product: $\left(\bigotimes_e |j_e, m_e, n_e\rangle \right)$

$$\widehat{\delta(g_v)} = \int_G \left(\bigotimes_{\substack{e: v = \bar{v}_p(e)}} L_h^e \right) \left(\bigotimes_{\substack{e': v = \bar{v}_s(e')}} R_h^{e'} \right) dh$$

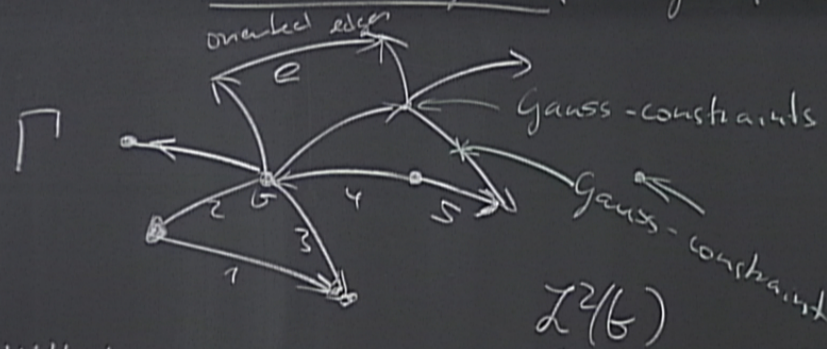


$$\begin{array}{ccc} & g_e & \\ g_{e_1} \searrow & \xrightarrow{\quad} & \nearrow g_{e_2} \\ & \chi & \end{array} \quad \begin{array}{c} \psi(g_e) \\ \downarrow \\ \psi(g_{e_1}, g_{e_2}) = \psi(g_{e_2} \cdot g_{e_1}) \end{array} \quad \begin{array}{c} h \quad h^{-1} \\ \downarrow \quad \downarrow \end{array}$$

$$\psi(g) \rightarrow \psi'(g) = \psi(g^{-1})$$

$$\int dg_1 \dots dg_n$$

More edges: Graph



Hilbert space: $\mathcal{H}_P = \bigotimes_{e \in P} \mathcal{H}_e$ $L^2(\mathcal{G})$

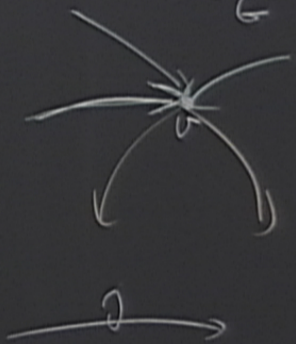
$\psi(g_1, g_2, g_3, \dots, g_n)$

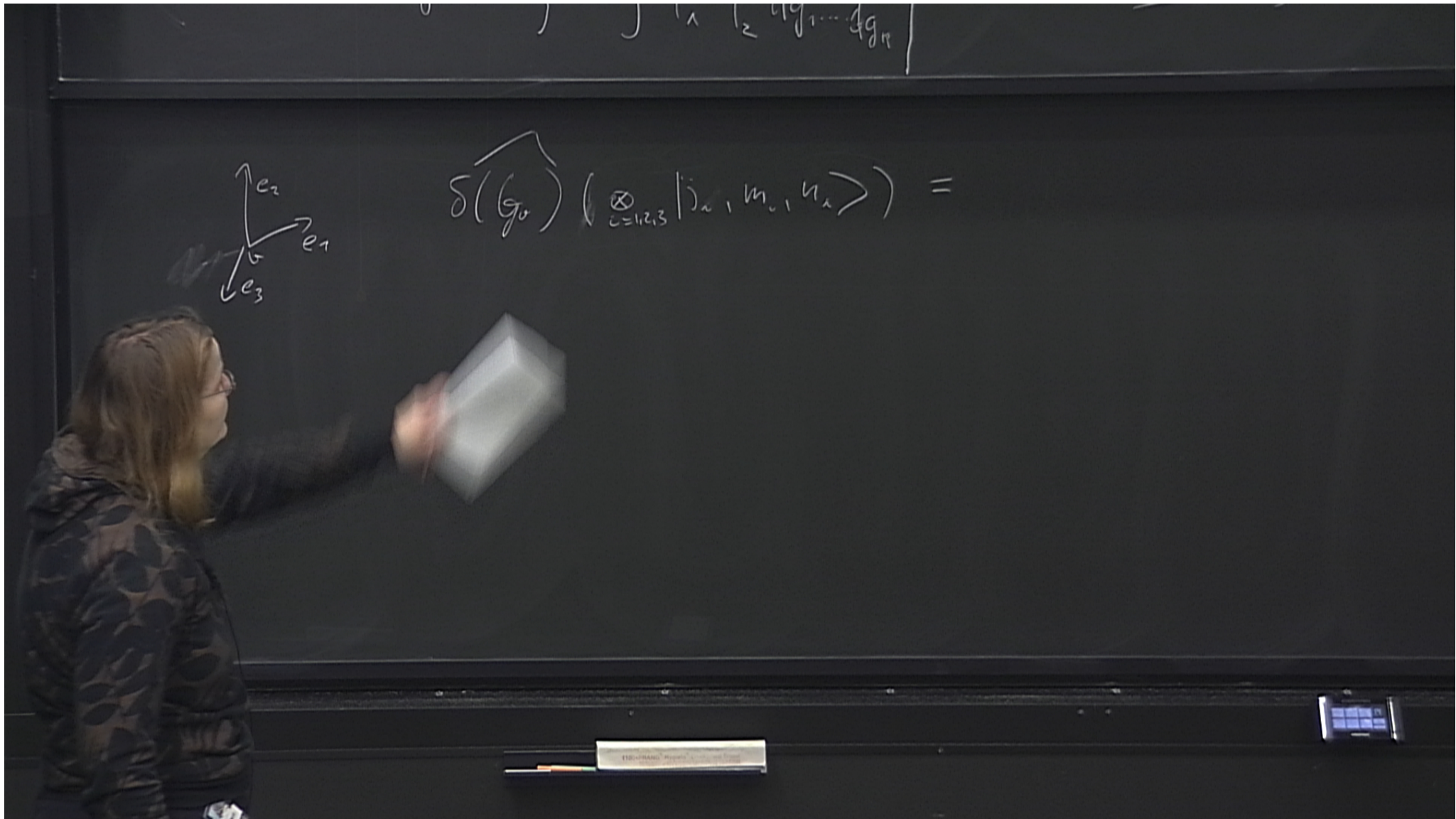
$\int \overline{\psi_1} \psi_2 dg_1 \dots dg_n$

$\psi(g) \rightarrow \psi'(g) = \psi(g^{-1})$

- SNW for one edge
- Tensor product: $(\otimes_e | \psi_e$

$\delta(\mathcal{G}_v) = \int_{\mathcal{G}} (\otimes_e | \psi_e$
 $e.v=v$

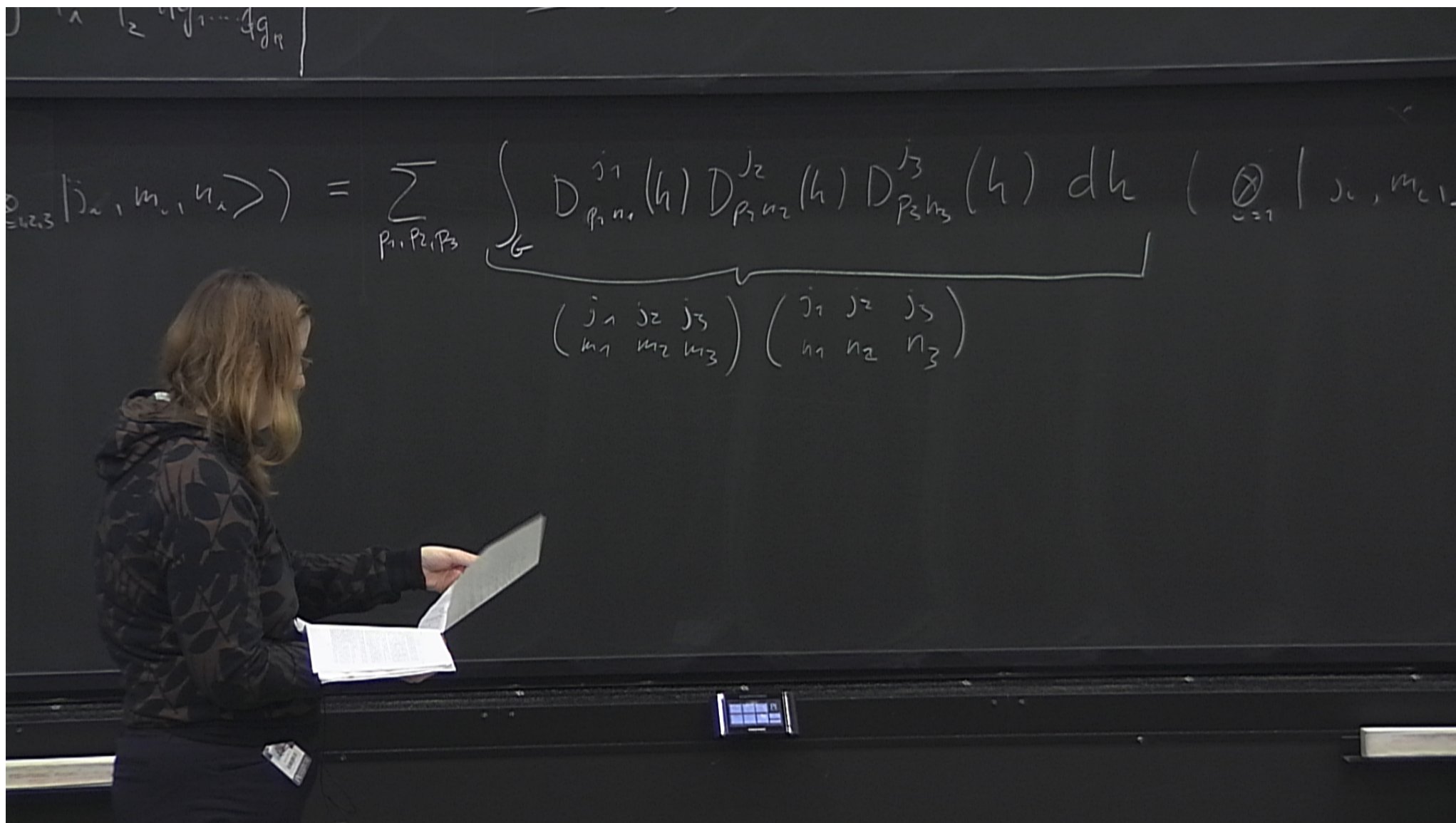




$$\widehat{\delta(g_0)} \left(\bigotimes_{i=1,2,3} |j_i, m_i, n_i\rangle \right) = \sum_{p_1, p_2, p_3} \int_G D_{p_1 n_1}^{j_1}(h) D_{p_2 n_2}^{j_2}(h) D_{p_3 n_3}^{j_3}(h) dh$$

$$D_{p_1 n_1}^{j_1}(h) D_{p_2 n_2}^{j_2}(h) D_{p_3 n_3}^{j_3}(h) dh \quad (\otimes_{i=1}^3 |j_i, m_i, \underline{p_i}\rangle)$$

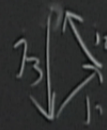
$$|j_1, m_1, n_1\rangle = \sum_{p_1, p_2, p_3} \int_G D_{p_1 n_1}^{j_1}(h) D_{p_2 n_2}^{j_2}(h) D_{p_3 n_3}^{j_3}(h) dh \left(\bigotimes_{i=1}^3 |j_i, m_i\rangle \right)$$



$\psi\rangle = \sum_{p_1, p_2, p_3} \int_G D_{p_1 m_1}^{j_1}(h) D_{p_2 m_2}^{j_2}(h) D_{p_3 m_3}^{j_3}(h) dh \left(\bigotimes_{i=1}^3 |j_i, m_i, \underline{p_i}\rangle \right)$

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ Wigner 3j symbols

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle$



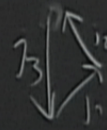
4g_n

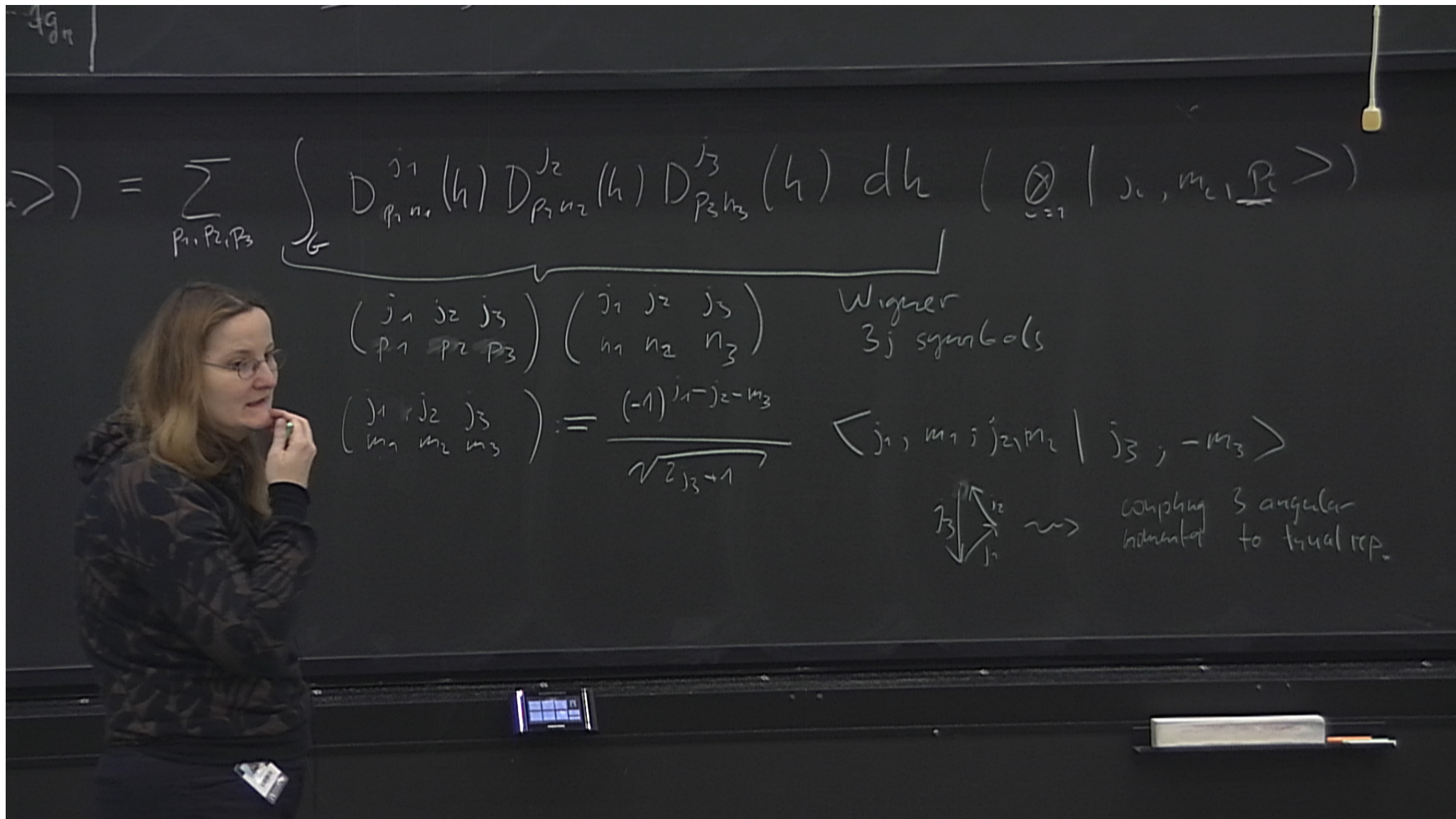
$$| \psi \rangle = \sum_{p_1, p_2, p_3} \int_G D_{p_1 m_1}^{j_1}(h) D_{p_2 m_2}^{j_2}(h) D_{p_3 m_3}^{j_3}(h) dh \left(\bigotimes_{i=1}^3 | j_i, m_i, \underline{p_i} \rangle \right)$$

$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$

Wigner 3j symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle$$





$\psi\rangle = \sum_{p_1, p_2, p_3} \int_G D_{p_1 n_1}^{j_1}(h) D_{p_2 n_2}^{j_2}(h) D_{p_3 n_3}^{j_3}(h) dh \left(\bigotimes_{i=1}^3 |j_i, m_i, \underline{p_i}\rangle \right)$

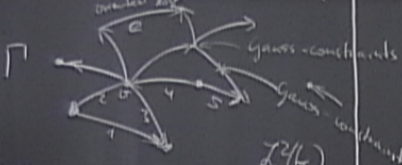
$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ Wigner 3j symbols

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle$

$j_3 \begin{matrix} \nearrow j_2 \\ \searrow j_1 \end{matrix} \rightsquigarrow$ coupling 3 angular momenta to total rep.

$V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \longrightarrow [V_0] \oplus \dots$

More edges: graph



Hilbert space: $\chi_P = \bigotimes_{e \in \Gamma} \chi_e$

$$\psi(g_1, g_2, g_3, g_4)$$

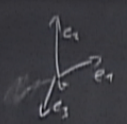
$$\psi(g) \rightarrow \psi'(g) = \psi(g')$$

- Tensor product: $(\bigotimes_e |j_e, m_e, n_e\rangle)$

$$\delta(g_v) = \int_G \left(\bigotimes_{e \in \text{sup}(v)} L_h^e \right) \left(\bigotimes_{e \in \text{inf}(v)} R_h^e \right) dh$$



$$\psi(g_e) \xrightarrow{g_e} \psi(g_e) \quad \psi(g_{e_1}, g_{e_2}) = \psi(g_{e_2} g_{e_1})$$



$$\delta(g_v) \left(\bigotimes_{e \in \Gamma} |j_e, m_e, n_e\rangle \right) = \sum_{p_1, p_2, p_3} \int_G D_{p_1, n_1}^{j_1}(h) D_{p_2, n_2}^{j_2}(h) D_{p_3, n_3}^{j_3}(h) dh \left(\bigotimes_{e \in \Gamma} |j_e, m_e, p_e\rangle \right)$$

$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ Wigner 3j symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - m_1 - m_2}}{\sqrt{2j_3 + 1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle$$

$j_3 \uparrow \downarrow \rightarrow$ coupling 3 angular momenta to total rep.
 $V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \rightarrow [V_0] \oplus \dots$

$$\sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$

$$|j_i, m_i\rangle = \sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$

$$|j_i, m_i\rangle = \sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$



$$|j_i, m_i\rangle = \sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$

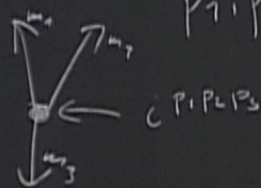

is normalized

$$|j_i, m_i\rangle = \sum_{p_1, p_2, p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$


is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

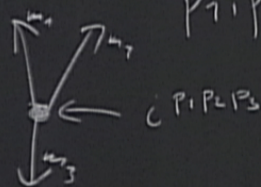
$$|j_1, j_2, j_3, m_1, m_2, m_3\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{C \\ p_1, p_2, p_3}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$



is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

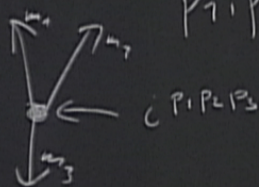
$$|j_1, j_2, j_3, m_1, m_2, m_3\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{C_{p_1 p_2 p_3} \\ \text{interwiner}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$



is normalized

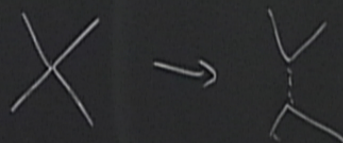
$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

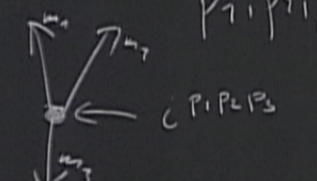
$$|j_1, j_2, j_3, m_1, m_2, m_3\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{C_{p_1 p_2 p_3} \\ \text{interwiner}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$



is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

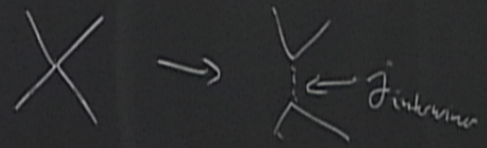


$$|j_i, m_i, p_i\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{p_1, p_2, p_3 \\ \text{interwiner}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$


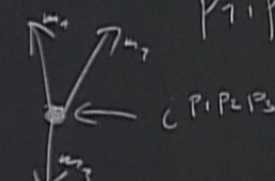
• is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

• $X \rightarrow \text{interwiner}$



• coupling conditions.

$$|j_i, m_i, p_i\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{p_1, p_2, p_3 \\ \text{interwiner}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$


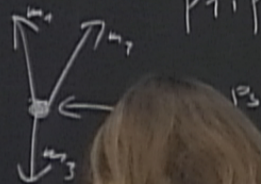
• is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$



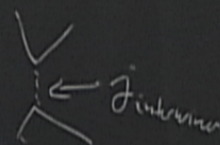
• coupling conditions. $j_1 + j_2 \geq j_3 \quad \forall \text{ permutation}$

$$\langle k_{ij}, m_i \rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{p_1, p_2, p_3 \\ \text{interwiners}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$



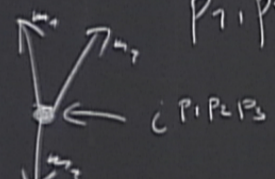
normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$



conditions.

$$j_1 + j_2 \geq j_3 \quad \forall \text{ permutations}; \quad j_1 + j_2 + j_3 \in \mathbb{N}$$

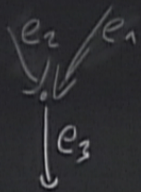
$$|j_i, m_i\rangle = \sum_{p_1, p_2, p_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}}_{\substack{p_1, p_2, p_3 \\ \text{interwiner}}} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$$


• is normalized

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

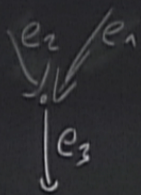


• Coupling conditions. $j_1 + j_2 \geq j_3 \quad \forall \text{ permutation}; \quad j_1 + j_2 + j_3 \in \mathbb{N}$



$\Rightarrow$

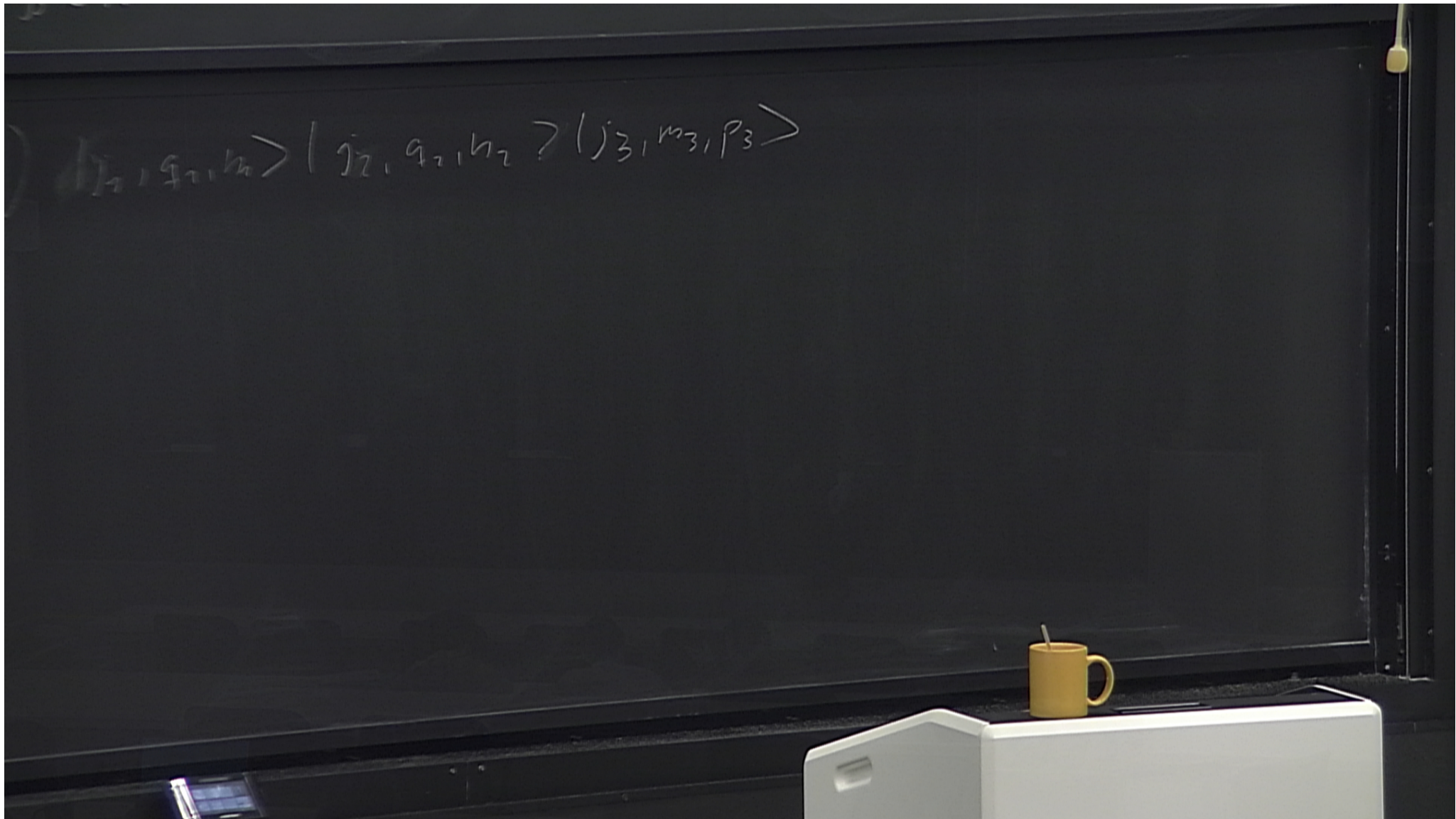
$$\sum_{q_1, q_2, p_3} (-1)$$



$\Rightarrow$

$$\sum_{q_1, q_2, p_3}$$

$$(-1)^{j_1 - q_1} (-1)^{j_2 - q_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ -q_1 & -q_2 & p_3 \end{pmatrix} |j_1, q_1, j_2\rangle$$

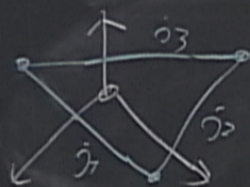


$\psi(q_1, q_2, q_3, \dots, q_n)$

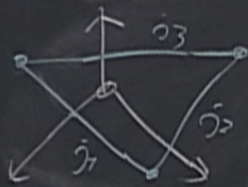
$$= \int \overline{\psi}_1 \psi_2 dq_1 \dots dq_n$$

 $\rightarrow$


$$\psi(g_1, g_2, g_3, \dots, g_n) = \int \overline{\psi}_1 \psi_2 dg_1 \dots dg_n \quad \xrightarrow{2}$$

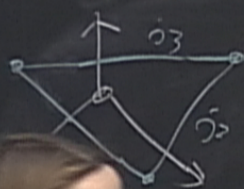


$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \overline{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{I}$$

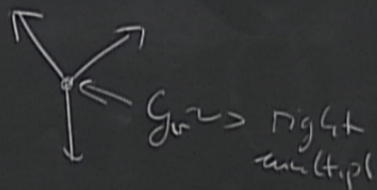


- Gauss-constant: derivative of gauge action
 $\partial E + \epsilon EA$

$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{2}$$

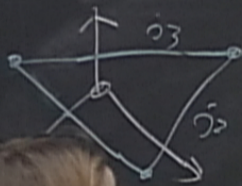


- Gauss-constant: derivative of gauge action
 $\partial E + \epsilon E A$

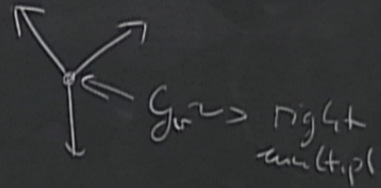


g_u

$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

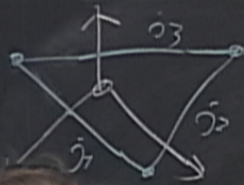


- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon E A$

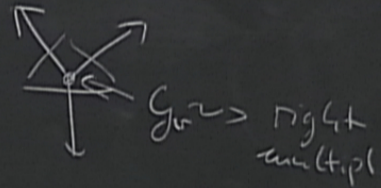


$$G_u^k = \hat{E}_1^k + \hat{E}_2^k + \hat{E}_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

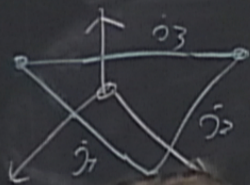


- Gauss-constant: derivative of gauge action
 $\partial E + \epsilon EA$

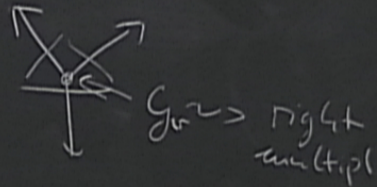


$$G^k = \hat{E}_1^k + \hat{E}_2^k + \hat{E}_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

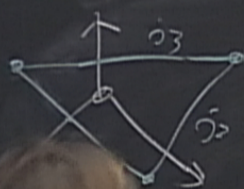


- Gauss-constants: derivative of gauge action
 $\partial E + \epsilon E A$

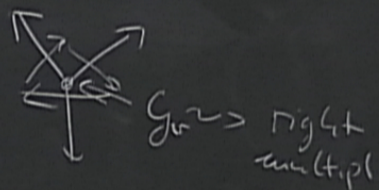
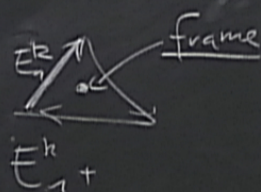


$$\vec{g}^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad ; \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

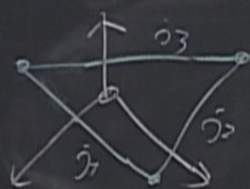


- Gauss - constraint: $\partial E + \epsilon E A$ derivative of gauge action

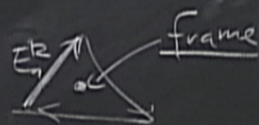


$$G^k = \hat{E}_1^k + \hat{E}_2^k + \hat{E}_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad ; \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

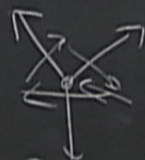


- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon EA$



$$E_1 + E_2 + E_3 = 0;$$

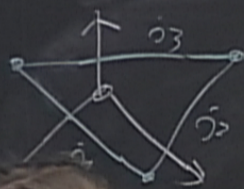
closure of triangle



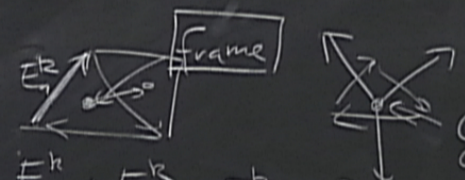
$G_{\mu\nu} \rightarrow$ right
multipl

$$G_{\mu\nu}^k = E_1^k + E_2^k + E_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$



- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon E A$



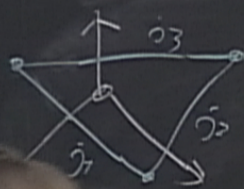
$$E_1^k + E_2^k + E_3^k = 0;$$

closure of triangle

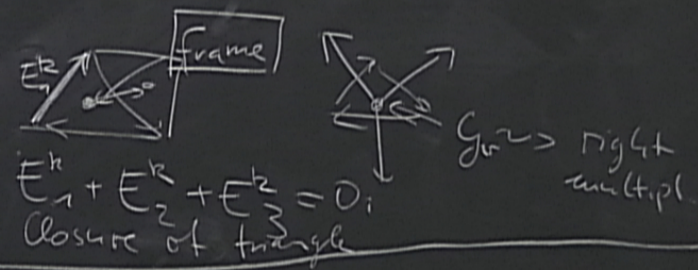
$g_{\mu\nu} \rightarrow$ right multipl.

$$\vec{G}^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$

$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$



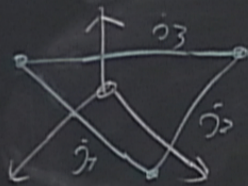
- Gauss - constraints: derivative of gauge action
 $\partial E + \epsilon E A$



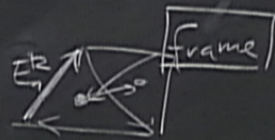
$$\vec{g}_u^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$



$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

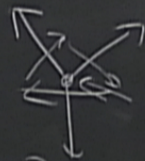


- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon EA$



$$E_1^k + E_2^k + E_3^k = 0;$$

closure of triangle

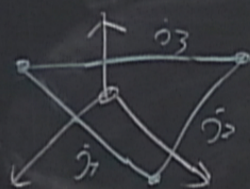


$g_{\mu\nu} \rightarrow$ right
multipl

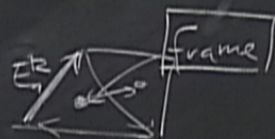
$$\vec{g}_\mu^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$



$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

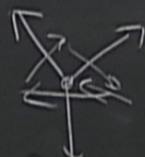


- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon EA$



$$E_1^k + E_2^k + E_3^k = 0$$

closure of triangle

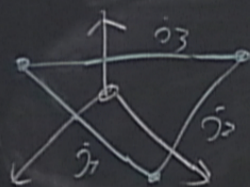


$g_{12} \rightarrow$ right multipl

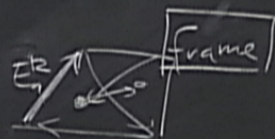
$$G^k = E_1^k + E_2^k + E_3^k$$



$$\Psi(g_1, g_2, g_3, \dots, g_n) = \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$

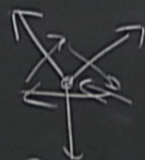


- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon EA$



$$E_1^k + E_2^k + E_3^k = 0;$$

closure of triangle

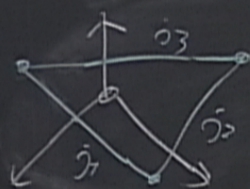


$G_{\mu\nu} \rightarrow$ right
multipl

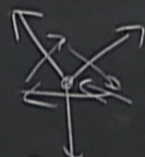
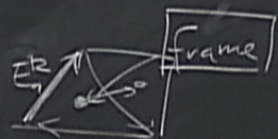
$$G_{\mu\nu}^k = E_1^k + E_2^k + E_3^k$$



$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad , \quad \int \overline{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$



- Gauss - constraints: derivative of gauge action
 $\partial E + \epsilon EA$



$G_{\mu\nu} \rightarrow$ right
multipl

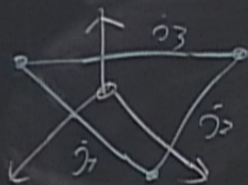
$$E_1^k + E_2^k + E_3^k = 0;$$

closure of triangle

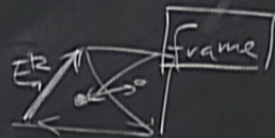
$$\vec{G}_{\mu\nu}^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$



$$\Psi(g_1, g_2, g_3, \dots, g_n) \quad ; \quad \int \bar{\Psi}_1 \Psi_2 dg_1 \dots dg_n \quad \xrightarrow{J}$$



- Gauss - constraint: derivative of gauge action
 $\partial E + \epsilon EA$



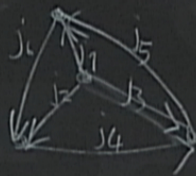
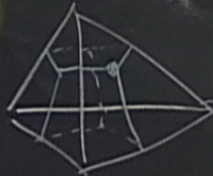
$$E_1^k + E_2^k + E_3^k = 0;$$

closure of triangle



$g_{\mu\nu} \rightarrow$ right
multipl

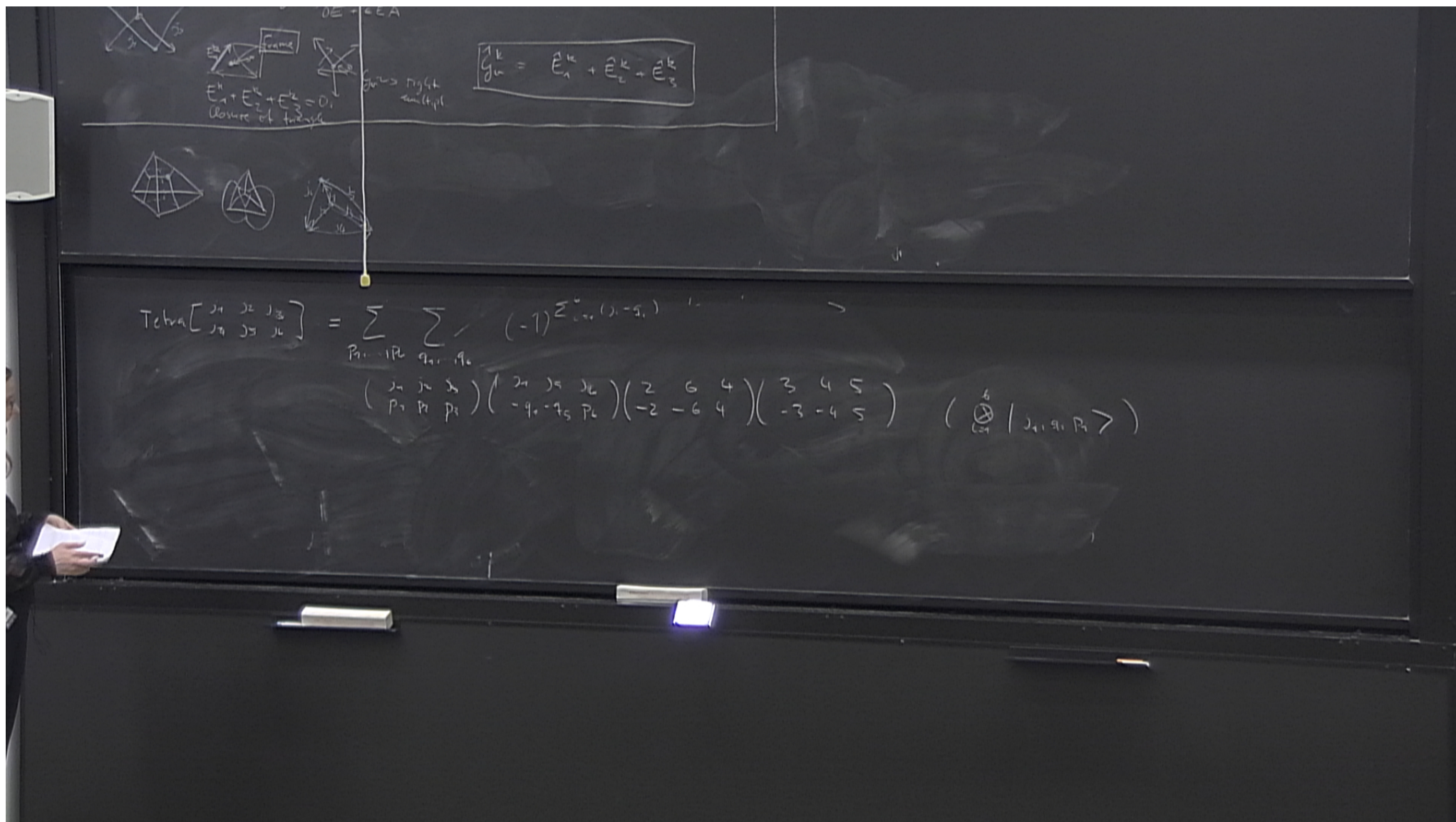
$$\vec{G}^k = \vec{E}_1^k + \vec{E}_2^k + \vec{E}_3^k$$



$$\text{Tetra} \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} = \sum_{p_1, \dots, p_6} \sum_{q_1, \dots, q_6} (-1)^{\sum_{i=1}^6 (j_i - q_i)} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -q_1 & -q_5 & p_6 \end{pmatrix} \begin{pmatrix} 2 & 6 & 4 \\ -2 & -6 & 4 \end{pmatrix} \begin{pmatrix} \end{pmatrix}$$

$j_1 - j_1$

$$\begin{pmatrix} 2 & 6 & 4 \\ -2 & -6 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 5 \end{pmatrix} \quad \left(\bigoplus_{i=1}^6 |j_i, q_i, p_i\rangle \right)$$



$j_1 - j_2$

$$\begin{pmatrix} 2 & 6 & 4 \\ -2 & -6 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 5 \end{pmatrix}$$

$$\left(\bigoplus_{i=1}^6 |j_i, q_i, p_i\rangle \right)$$

$$\downarrow q = 11$$

$$\delta q, p_i$$

$$j_i - g_i)$$

$$\begin{pmatrix} 2 & 6 & 4 \\ -2 & - & \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 5 \end{pmatrix}$$

$$\left(\underbrace{\bigoplus_{i=1}^6 |j_i, g_i, p_i\rangle}_{\downarrow g_i = 11} \right)$$

$$\delta_{4, p_i}$$

$$\text{Tetra} \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} = \sum_{p_1, \dots, p_6} \sum_{q_1, \dots, q_6} (-1)^{\sum_{i=1}^6 (j_i - q_i)}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ -q_1 & -q_5 & p_4 \end{pmatrix} \begin{pmatrix} 2 & 6 & 4 \\ -2 & -6 & 4 \end{pmatrix}$$

$$\rightarrow \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$$

$$\begin{aligned}
 \left[\begin{smallmatrix} 3 \\ j_1 \end{smallmatrix} \right] &= \sum_{p_1, \dots, p_6} \sum_{q_1, \dots, q_6} (-1)^{\sum_{i=1}^6 (j_i - q_i)} \\
 &\quad \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -q_1 & -q_5 & p_4 \end{pmatrix} \begin{pmatrix} 2 & 6 & 4 \\ -2 & -6 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 5 \end{pmatrix} \quad \left(\begin{smallmatrix} 6 \\ \oplus \\ i=1 \end{smallmatrix} \right) \\
 &\rightarrow \left\{ \begin{smallmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{smallmatrix} \right\} \rightarrow \delta(h)
 \end{aligned}$$