Title: 13/14 PSI - Quantum Gravity Review - Lecture 10

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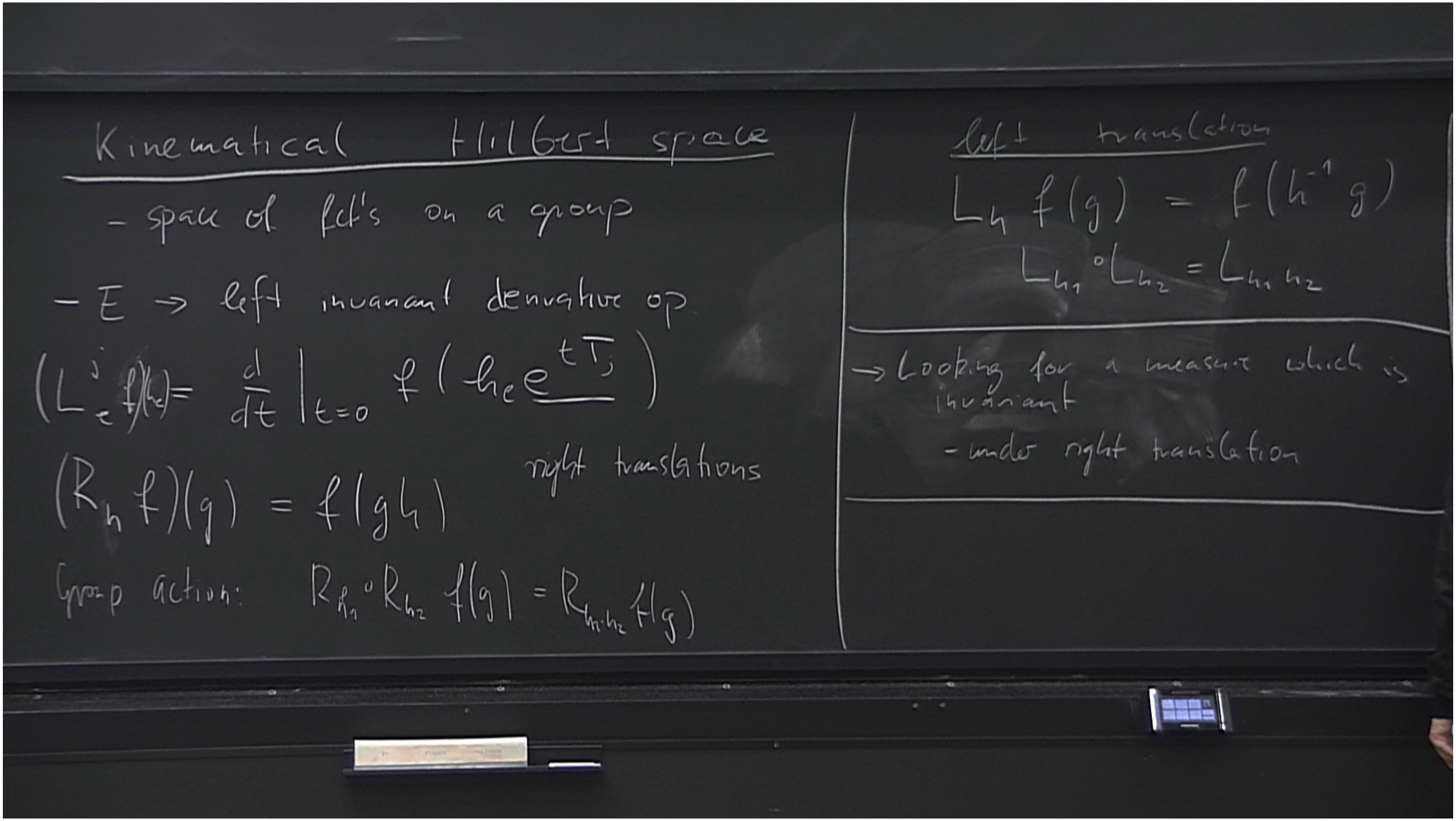
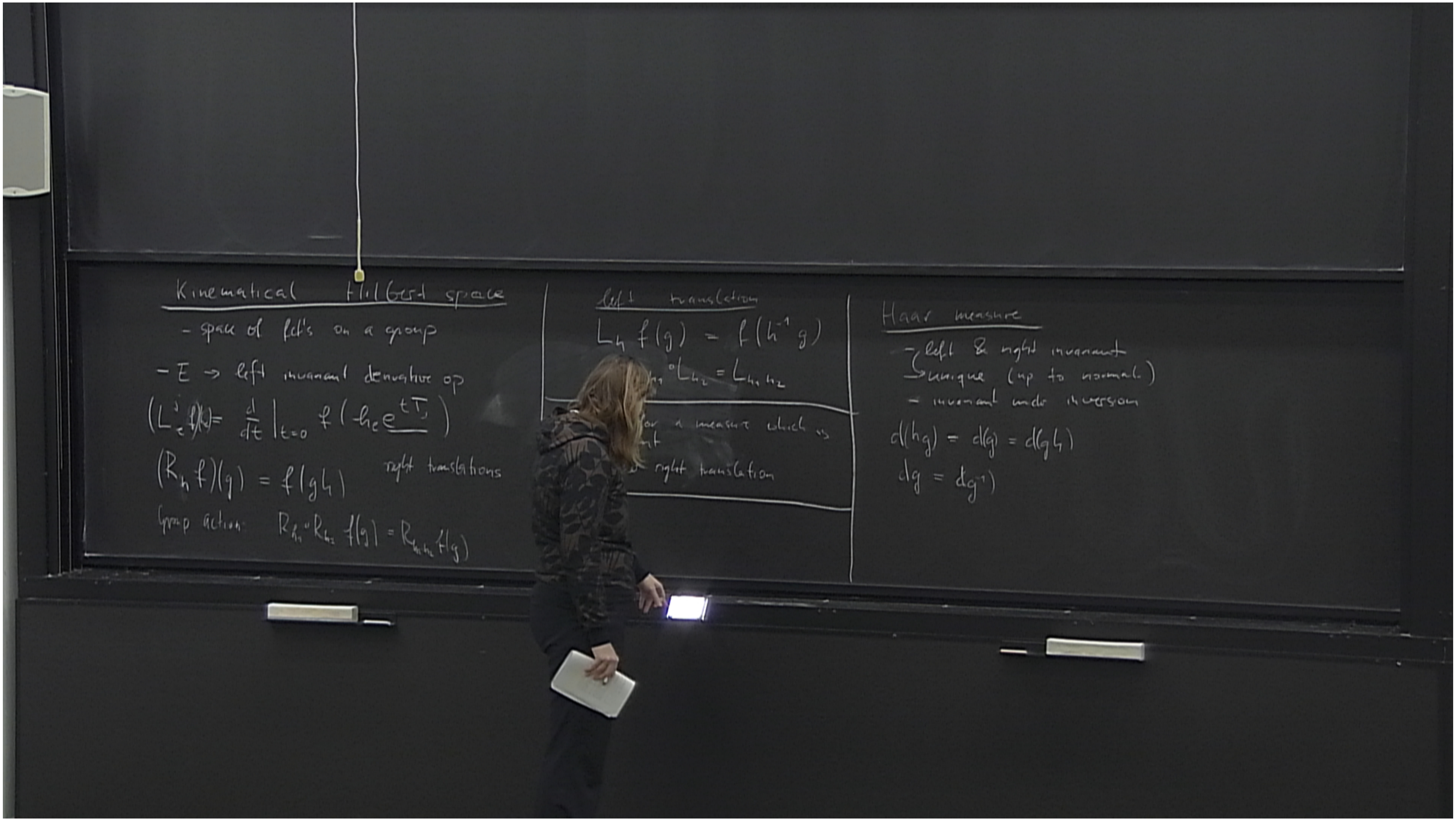
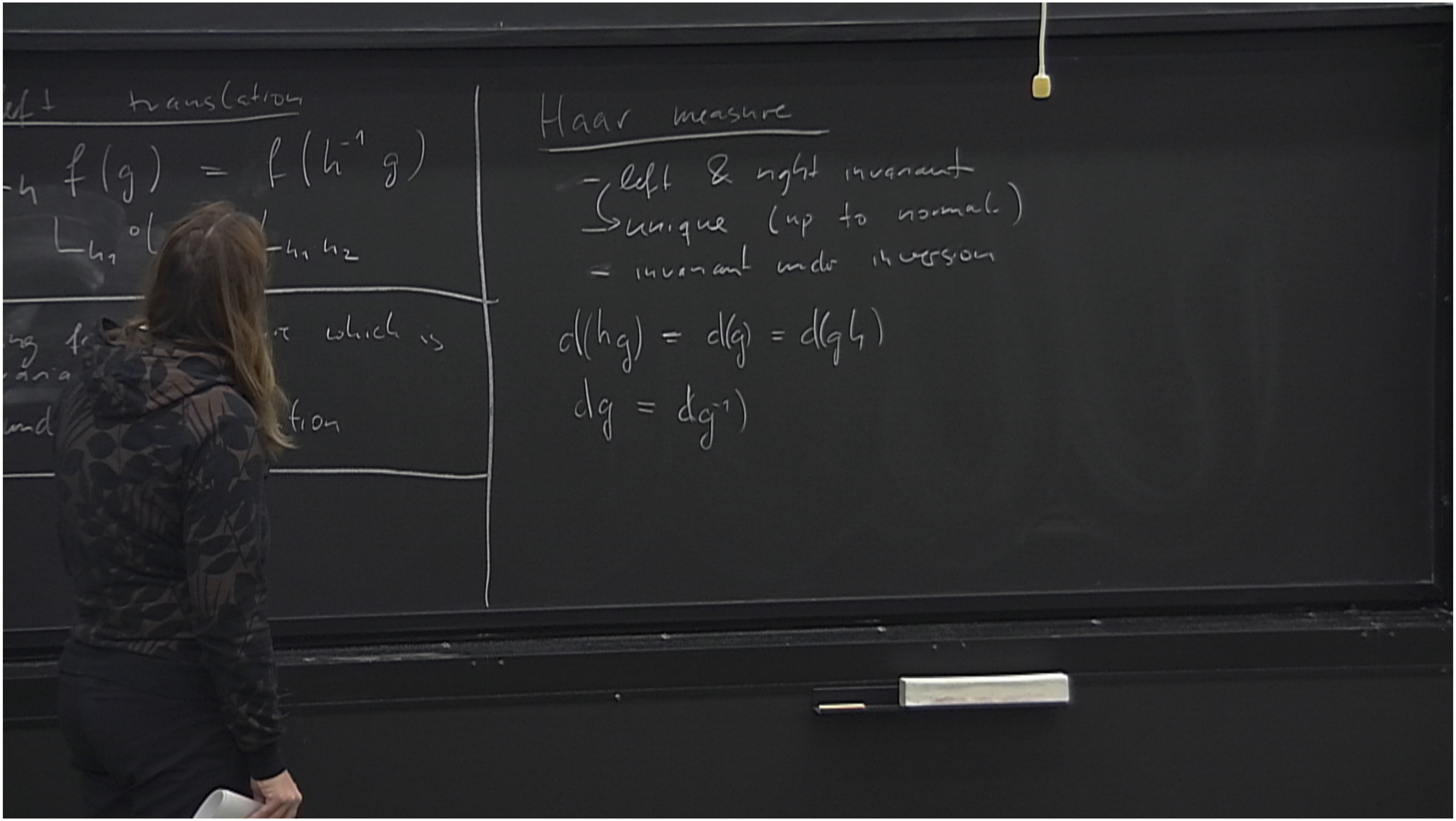
URL: http://pirsa.org/14030017

Abstract:


Kinematical Hilbert space
- space of fct's on a group
- E → left invariant derivative op
$(L^j_e f)(h_e) = \frac{d}{dt}\Big|_{t=0} f(h_e e^{tT_j})$
$(R_h f)(g) = f(gh)$
right translations
Group action: $R_{h_1} \circ R_{h_2} f(g) = R_{h_1 h_2} f(g)$
left translation
$L_h f(g) = f(h^{-1} g)$
$L_{h_1} \circ L_{h_2} = L_{h_1 h_2}$
→ Looking for a measure which is invariant
- under right translation


Kinematical Hilbert space
- space of fct's on a group
- E → left invariant derivative op
$(L^i_e f)(h) = \frac{d}{dt}\Big|_{t=0} f(h_e e^{tT_i})$
$(R_h f)(g) = f(gh)$ right translations
Group action: $R_{h_1} \circ R_{h_2} f(g) = R_{h_1 h_2} f(g)$
left translation
$L_h f(g) = f(h^{-1} g)$
$L_{h_1} \circ L_{h_2} = L_{h_1 h_2}$
or a measure which is
nt
right translation
Haar measure
- left & right invariant
- unique (up to normal.)
- invariant under inversion
$d(hg) = d(g) = d(gh)$
$dg = d(g^{-1})$


translation
$L_h f(g) = f(h^{-1} g)$
which is
Haar measure
- left & right invariant
unique (up to normal.)
- invariant under inversion
$d(hg) = d(g) = d(gh)$
$dg = d(g^{-1})$

$$\langle \psi_1, \psi_2 \rangle = \int_G \overline{\psi_2(g)}\, \psi_1(g)\, d\overset{\swarrow \text{Haar measure}}{g}$$

$$\langle R_h \psi_1, R_h \psi_2 \rangle = \int_G \overline{\psi_2(gh)}\, \psi_1(\underbrace{gh}_{g'})\, \underbrace{dg}_{\shortparallel}$$

$$\Rightarrow g = g'h^{-1} \qquad dg'h^{-1} \overset{\text{Haar}}{\downarrow} = dg'$$

$$= \int_G \overline{\psi_2(g')}\, \psi_1(g')\, dg'$$

$$= \langle \psi_1, \psi_2 \rangle$$

Haar measure on SU(2)

$SU(2) \longrightarrow$ 3-sphere

$$g = \exp(\alpha\, n^i T_i)$$

$\alpha \to [0, 2\pi)$

$n^i \to$ 3D unit vector $\to$ 2 angles

... eters on $SU(2)$

$(\theta, \phi) \longrightarrow \vec{n}$

$\phi \in [0, 2\pi)$

$\theta \in [0, \pi]$

$$\langle \psi_1 , \psi_2 \rangle = \int_G \overline{\psi_2(g)}\, \psi_1(g)\, dg$$ Haar measure

$$\langle R_h \psi_1 , R_h \psi_2 \rangle = \int_G \overline{\psi_2(gh)}\, \psi_1(\underbrace{gh}_{g'})\, \underbrace{dg}_{\shortparallel}$$ Haar

$\Rightarrow g = g'h^{-1}$ $\quad dg'h^{-1} \overset{\downarrow}{=} dg'$

$$= \int_G \overline{\psi_2(g')}\, \psi_1(g')\, dg'$$

$$= \langle \psi_1 , \psi_2 \rangle$$

Normalization $\int 1 dg = 1$

Haar m

$SU(2)$

$g =$

$\Rightarrow$ param

$\alpha,$

asure on $SU(2)$

$\longrightarrow$ 3-sphere

$\exp(\alpha\, n^i T_i)$

$\alpha \to [0, 2\pi)$

$n^i \to$ 3D unit vector $\to$ 2 angles

tors on $SU(2)$

$(\psi, \phi) \longrightarrow \vec{n}$

$\phi \in [0, 2\pi)$

$\psi \in [0, \pi]$

$$n^1 = \sin\psi \cos\phi$$
$$n^2 = \cos\psi$$
$$n^3 = \sin\psi \sin\phi$$

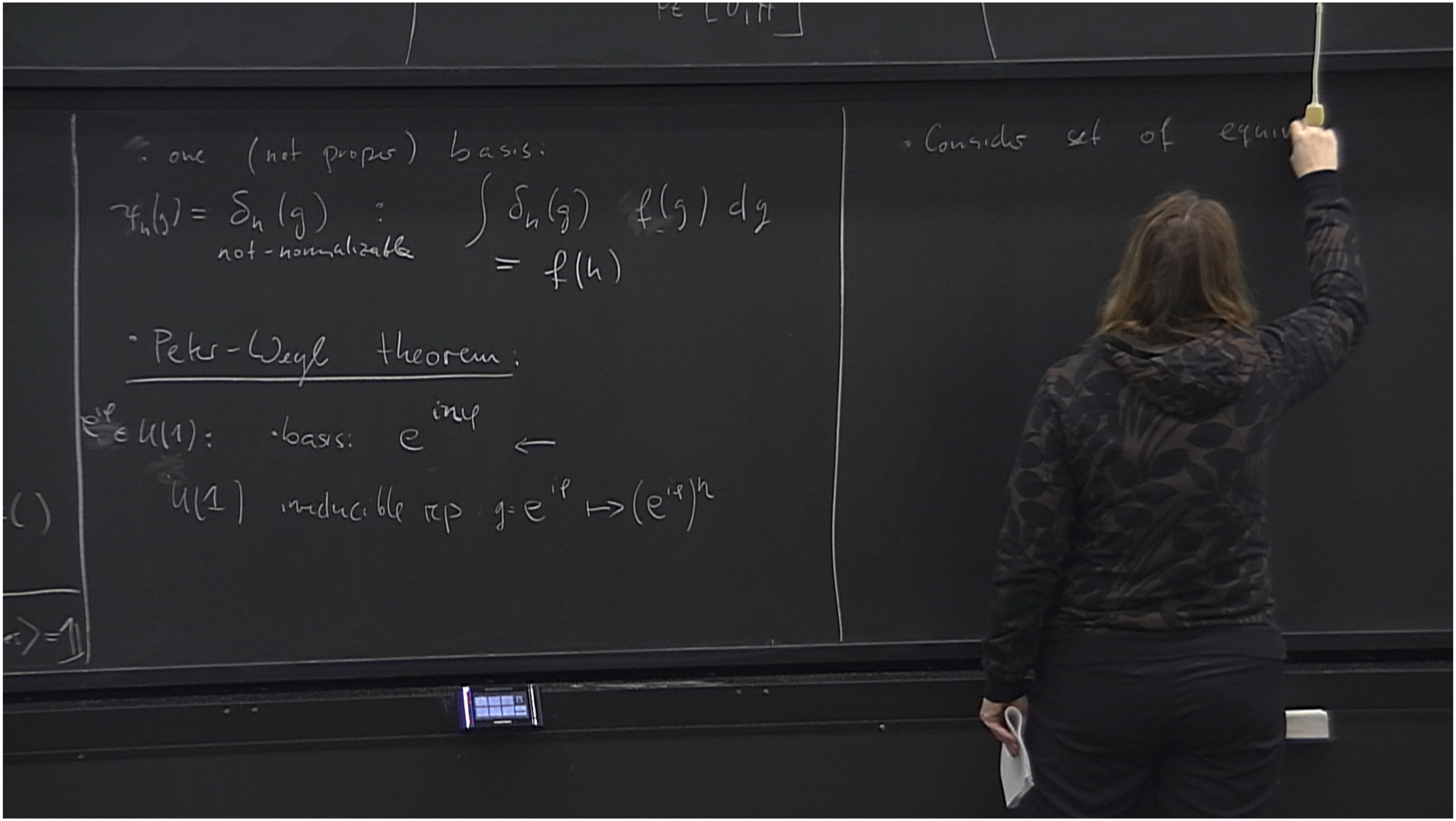
$$\boxed{dg = \frac{1}{4\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin(\psi)\, d\alpha\, d\psi\, d\phi}$$

Normalization $\int 1 dg = 1$

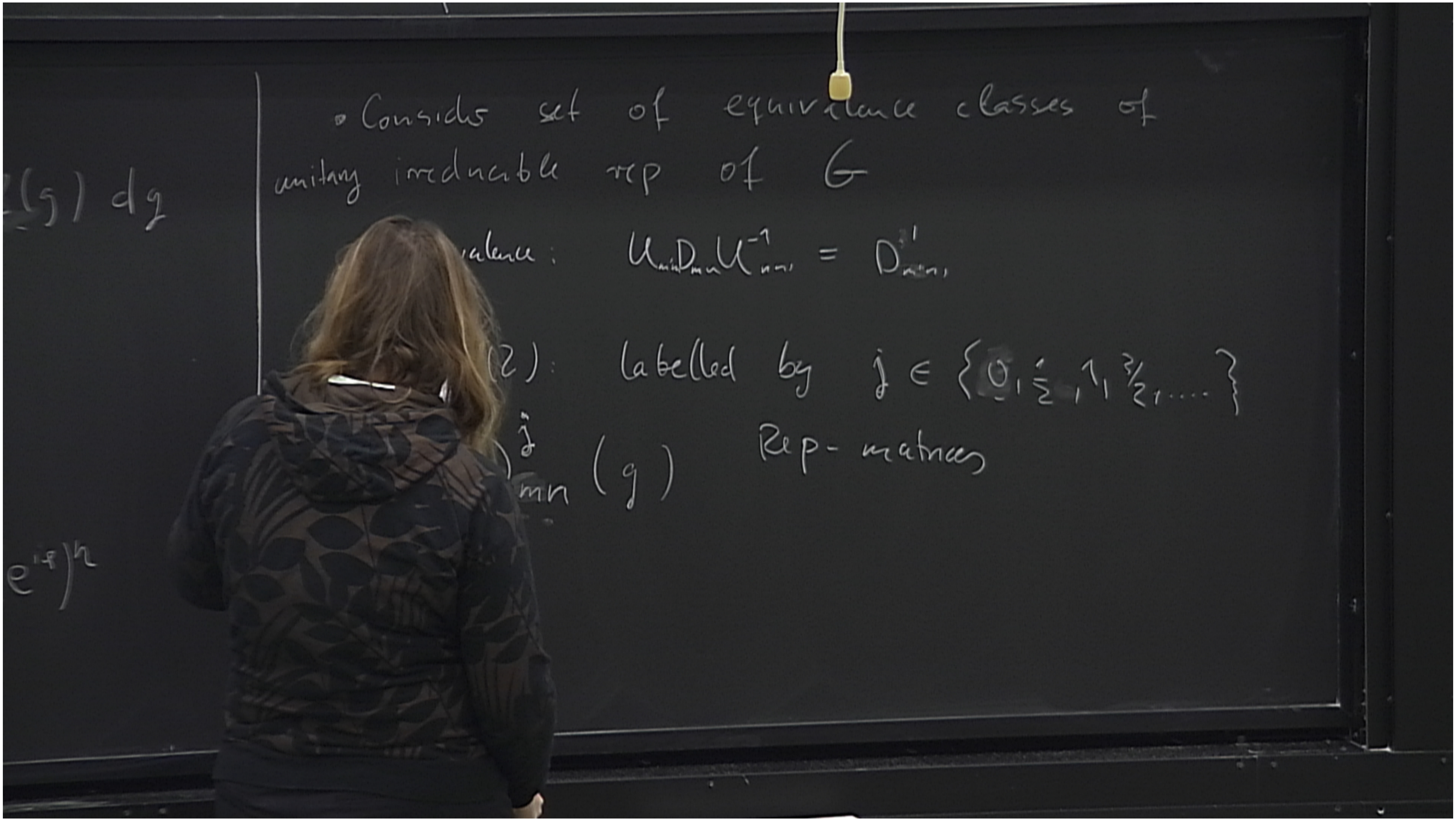
$SU(2) \rightarrow$ compact

- $f \equiv 1 \rightarrow$ normalizable
- $|1\rangle \in \mathcal{L}^2(G, dg)$
- $|1\rangle \longrightarrow \langle E \cdot E \rangle_{|1\rangle} = 0$
- (Ashtekar-Lewandowski) - Vacuum

- alternative (recent) vacuum $\langle$ closed holonomies $\rangle = \mathbb{1}$



· one (not proper) basis:
ψ_h(g) = δ_h(g) : ∫ δ_h(g) f(g) dg
not-normalizable
= f(h)
· Peter-Weyl theorem:
e^{iφ} ∈ U(1): ·basis: e^{inφ} ←
U(1) irreducible rep g = e^{iφ} ↦ (e^{iφ})^n
· Consider set of


(g) dg
• Consider set of equivalence classes of
unitary irreducible rep of G
valence: $U_{mm'} D_{mn} U^{-1}_{nn'} = D'_{m'n'}$
(2): labelled by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$
$D^{j}_{mn}(g)$ Rep- matrices
$e^{i\varphi})^{n}$

$(g)\, dg$

• Consider set of equivalence classes of unitary irreducible rep of G

equivalence: $U_{mm'} D_{mn} U^{-1}_{nn'} = D'_{m'n'}$

• for $SU(2)$: labelled by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

$\hookrightarrow$ $D^{j}_{mn}(g)$ Rep-matrices

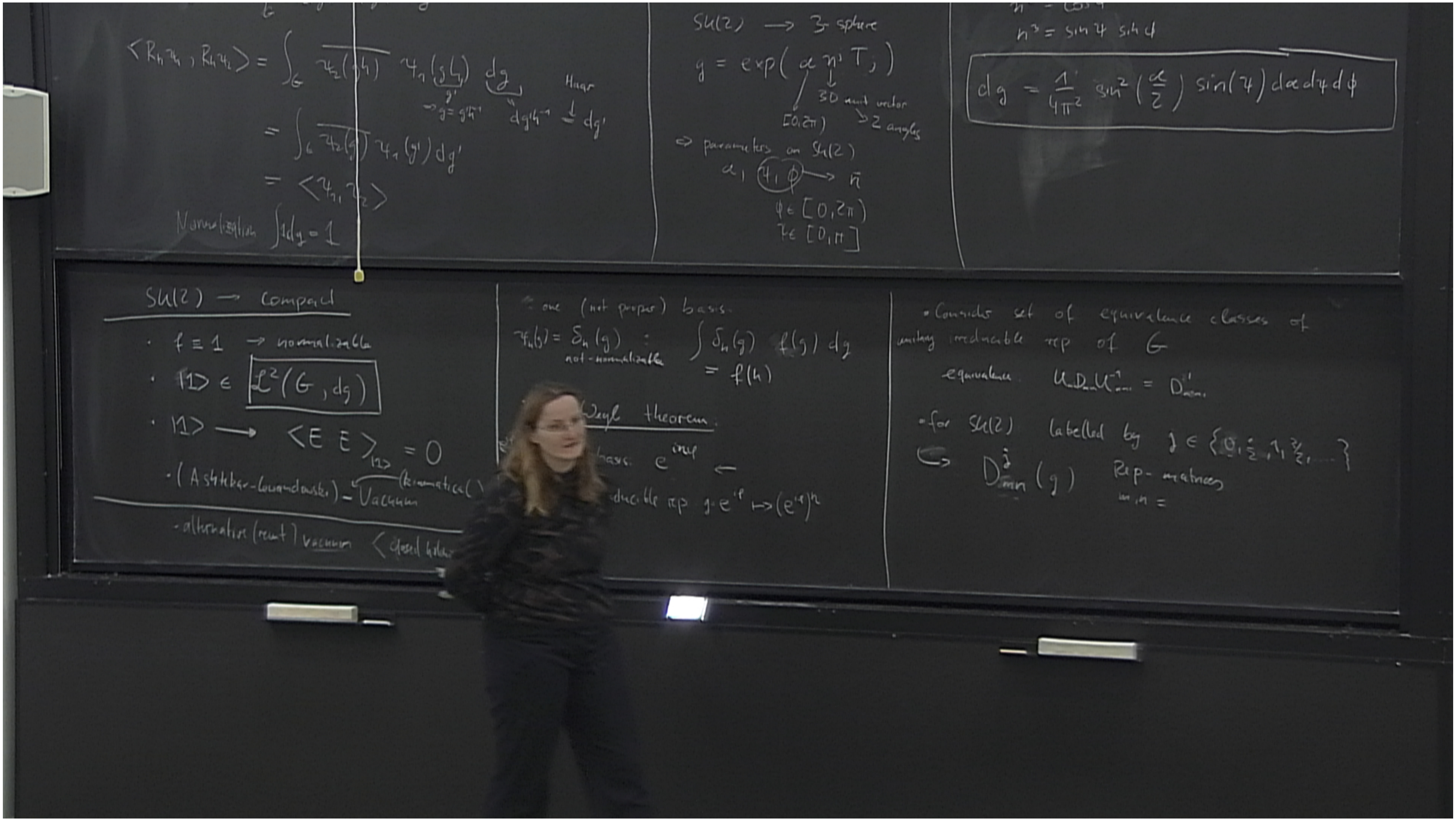
- Consider set of equivalence classes of unitary irreducible rep of G

equivalence: $U D U^{-1} = D'$

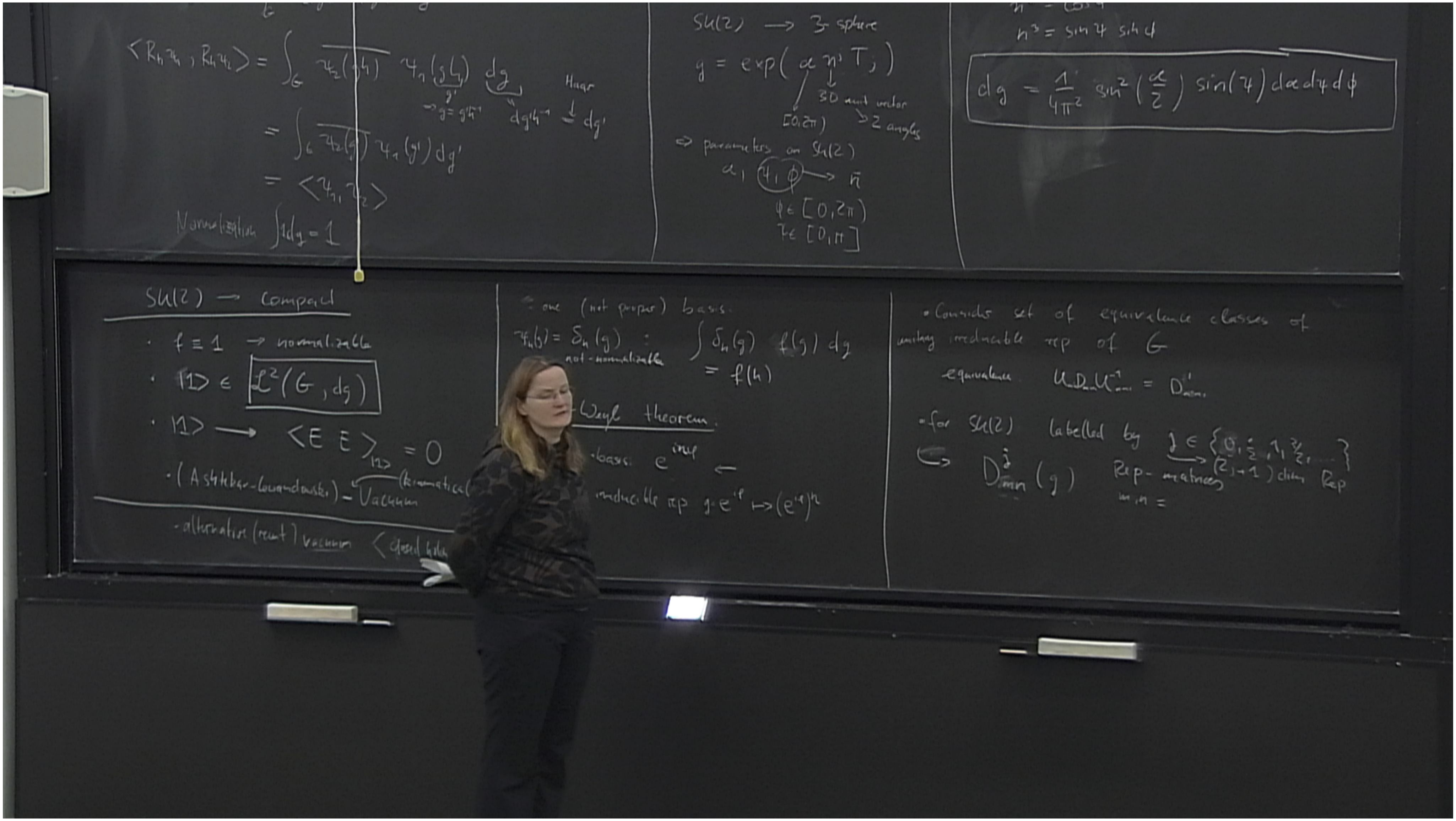
- for $SU(2)$: labelled by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

$\hookrightarrow$ $D^{j}_{mn}(g)$ Rep-matrices

$m, n =$



⟨R_h ψ_1, R_h ψ_2⟩ = ∫_G ψ_2(gh) ψ_1(gh) dg
Haar
= ∫_G ψ_2(g′) ψ_1(g′) dg′
= ⟨ψ_1, ψ_2⟩
Normalization ∫ 1 dg = 1
SU(2) → 3-sphere
g = exp(α n^i T_i)
3D unit vector
[0, 2π)
2 angles
⇒ parameters on SU(2)
α, (ψ, φ) → n
φ ∈ [0, 2π)
ψ ∈ [0, π]
n³ = sin ψ sin φ
dg = 1/(4π²) sin²(α/2) sin(ψ) dα dψ dφ
SU(2) → compact
· f ≡ 1 → normalizable
· |1⟩ ∈ L²(G, dg)
· |1⟩ ⟶ ⟨E E⟩_{|1⟩} = 0
· (Ashtekar-Lewandowski) – Vacuum (kinematical)
- alternative (recent) vacuum
one (not proper) basis
ψ_h(g) = δ_h(g) : ∫ δ_h(g) f(g) dg = f(h)
not-normalizable
Weyl theorem
basis e^{inφ}
irreducible rep g·e^{iφ} ↦ (e^{iφ})^n
· Consider set of equivalence classes of unitary irreducible rep of G
equivalence U D U^{-1} = D′
· for SU(2) labelled by j ∈ {0, 1/2, 1, 3/2, ...}
D^j_{mn}(g) Rep-matrices



SU(2) → 3-sphere
g = exp(α n^i T_i)
3D unit vector
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⇔ parameters on SU(2)
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Haar
Normalization ∫ dg = 1
SU(2) → compact
f ≡ 1 → normalizable
|1> ∈ L^2(G, dg)
|1> → <E E>_{|1>} = 0
(Ashtekar-Lewandowski) - Vacuum (kinematical)
alternative (recent) vacuum
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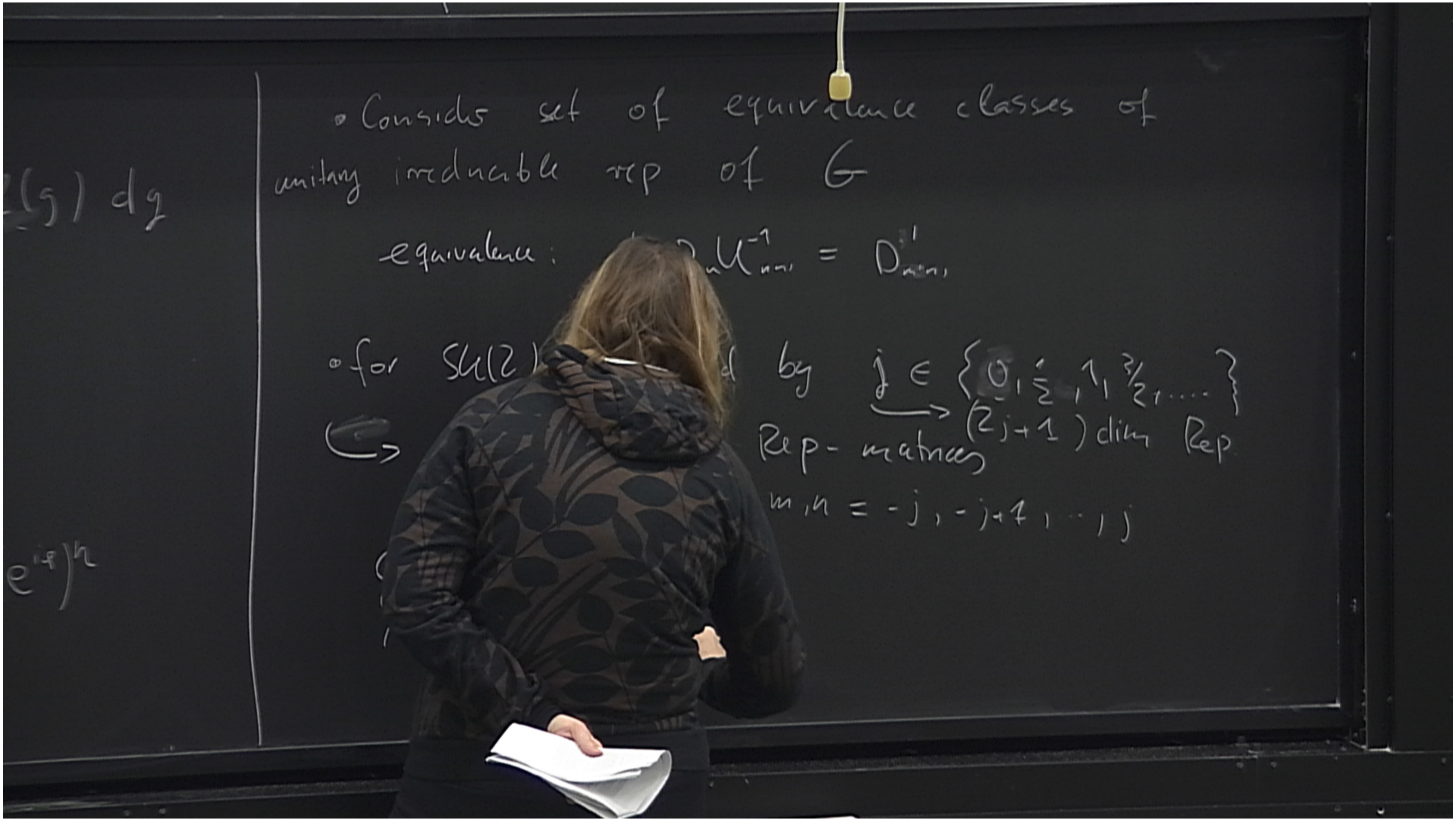
Weyl theorem
basis
irreducible rep
Consider set of equivalence classes of unitary irreducible rep of G
equivalence
for SU(2) labelled by j ∈ {0, 1/2, 1, 3/2, ...}
D^j_{mn}(g) Rep-matrices (2j+1) dim Rep


Considr set of equivalence classes of
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by j ∈ {0, 1/2, 1, 3/2, ...}
(2j+1) dim Rep
Rep- matrices
m,n = -j, -j+1, ..., j

$(g)\, dg$

$e^{i\varphi})^n$

• Consider set of equivalence classes of unitary irreducible rep of G

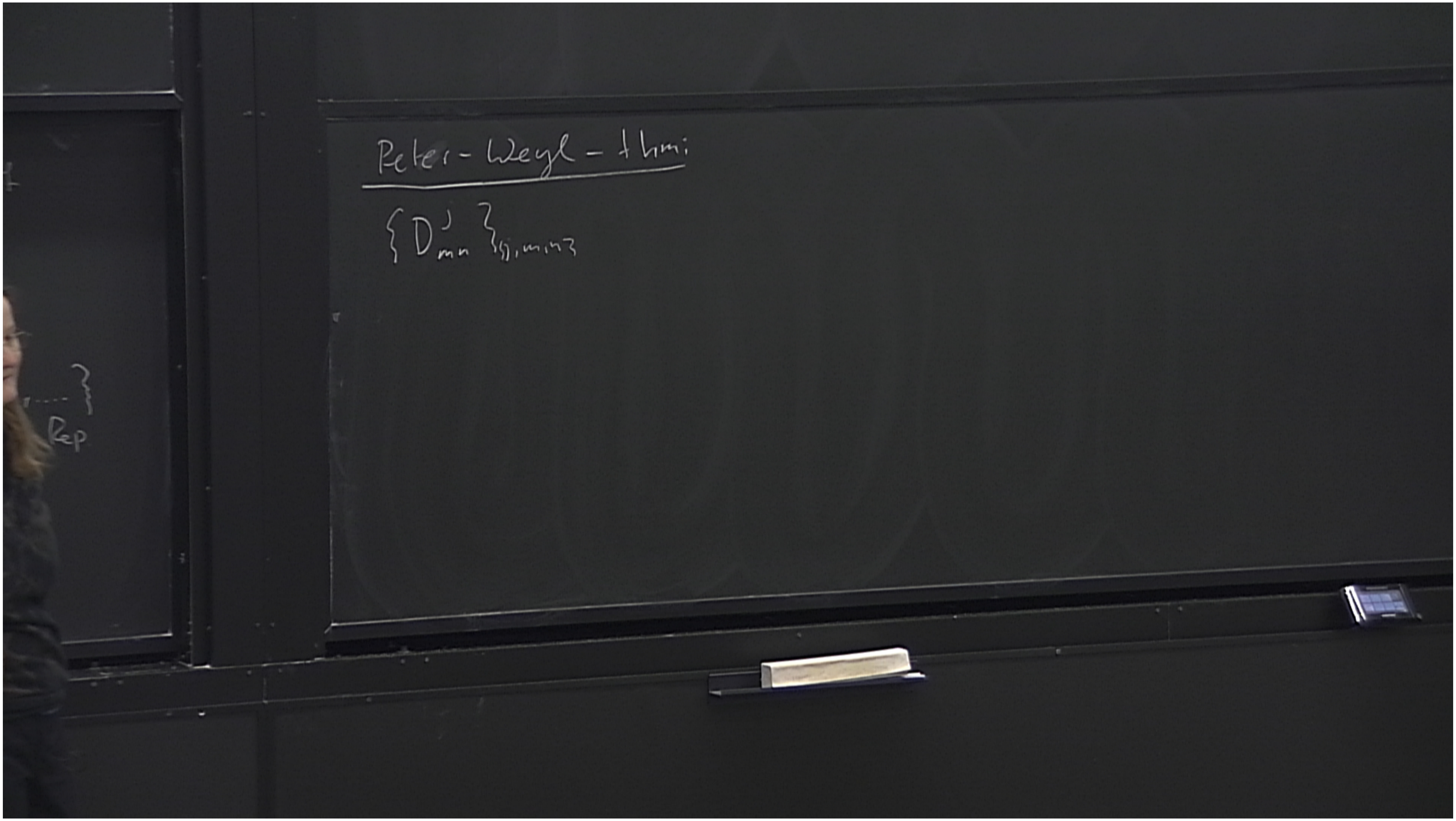
equivalence: $U_{mm'} D_{mn} U^{-1}_{nn'} = D'_{m'n'}$

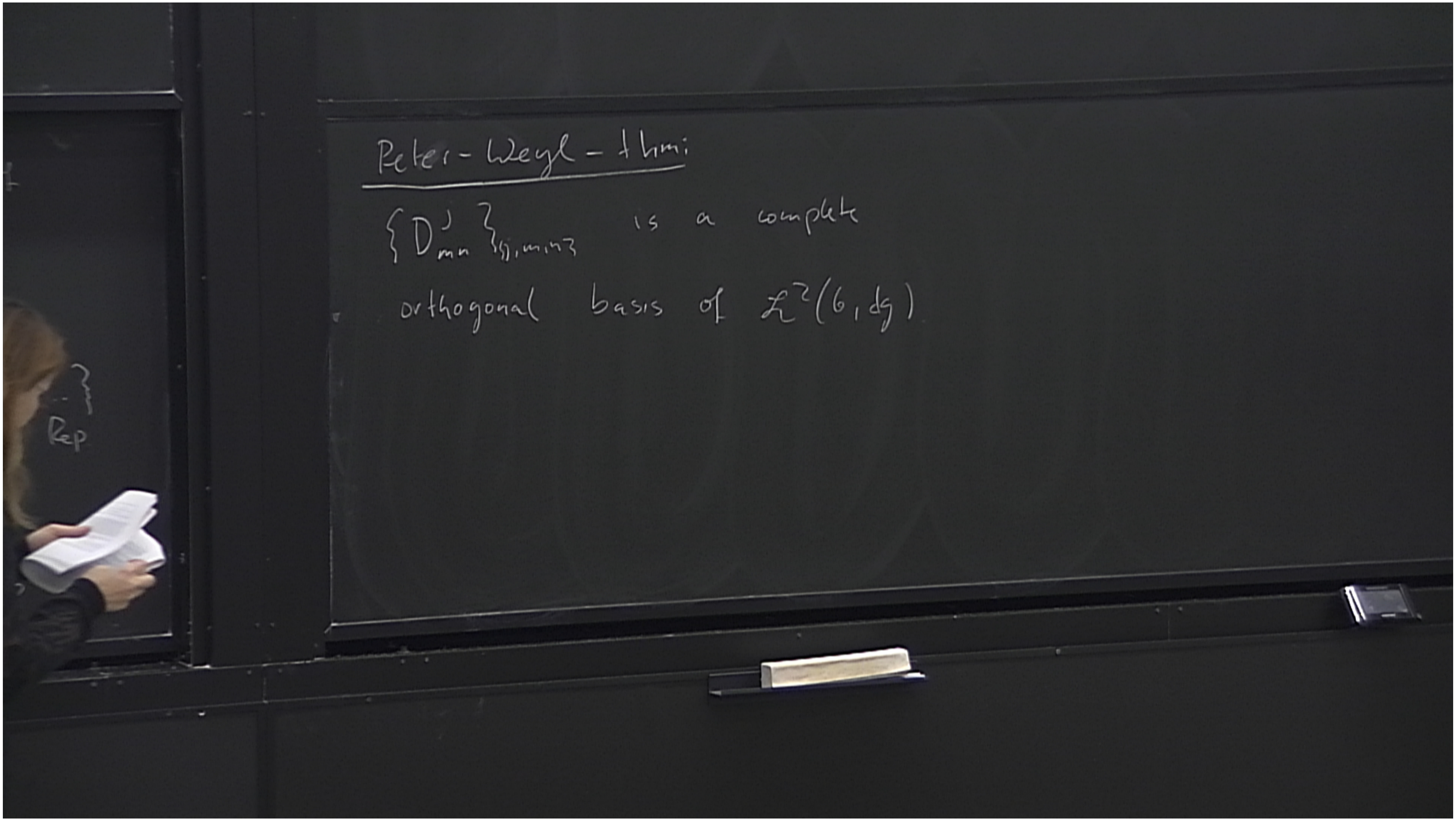
• for $SU(2)$: labelled by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ $\longrightarrow$ $(2j+1)$ dim Rep

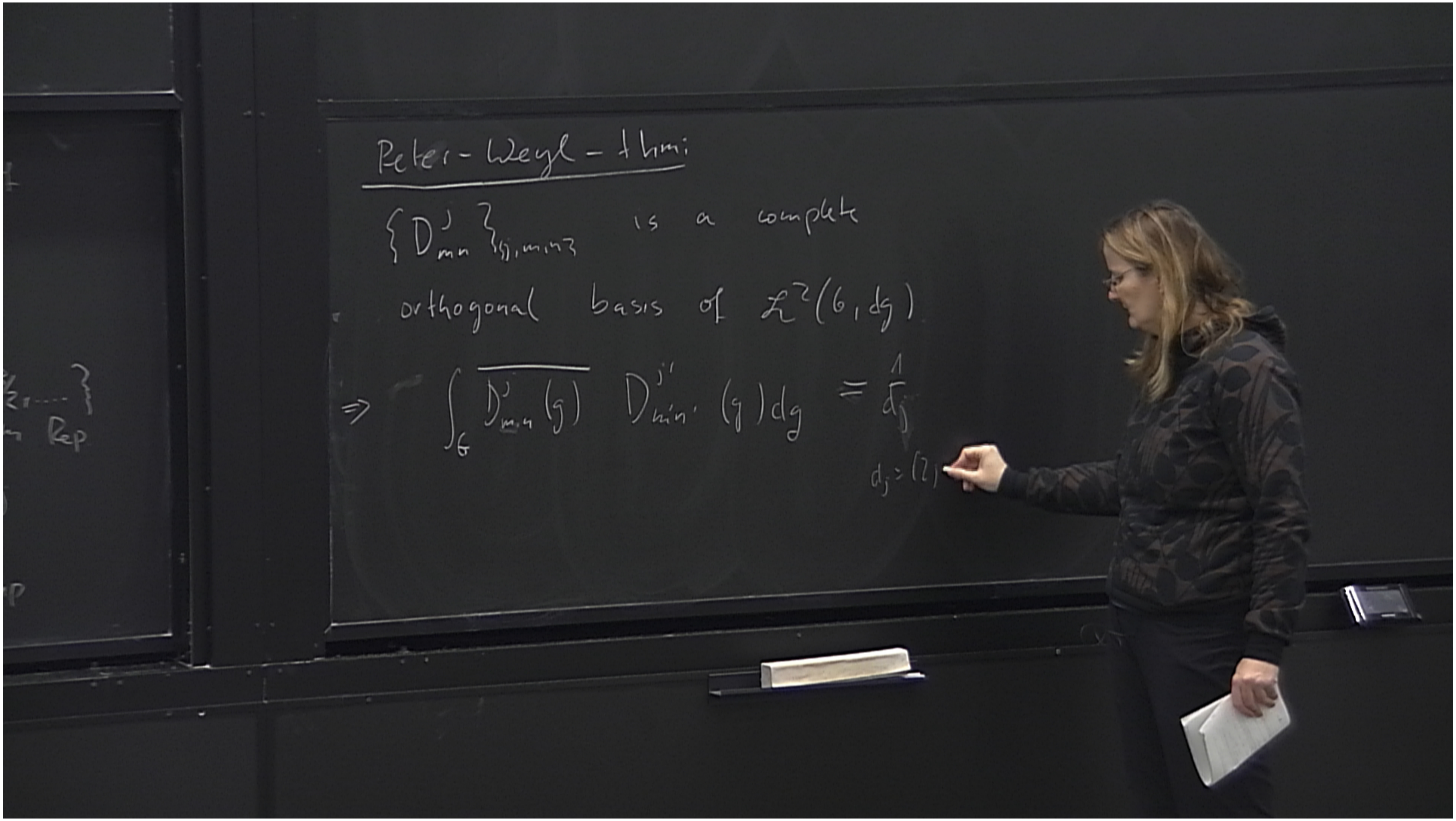
$\hookrightarrow$ $D^{j}_{mn}(g)$ Rep-matrices

$m, n = -j, -j+1, \dots, j$

$\{D^{j}_{mn}(g)\}_{\{j,m,n\}}$ = set of fcts on the group


Peter - Weyl - thm:
$\{D^j_{mn}\}_{j,m,n}$
Rep


Peter-Weyl-thm:
$\{D^j_{mn}\}_{\{j,m,n\}}$ is a complete
orthogonal basis of $\mathcal{L}^2(G, dg)$
Rep

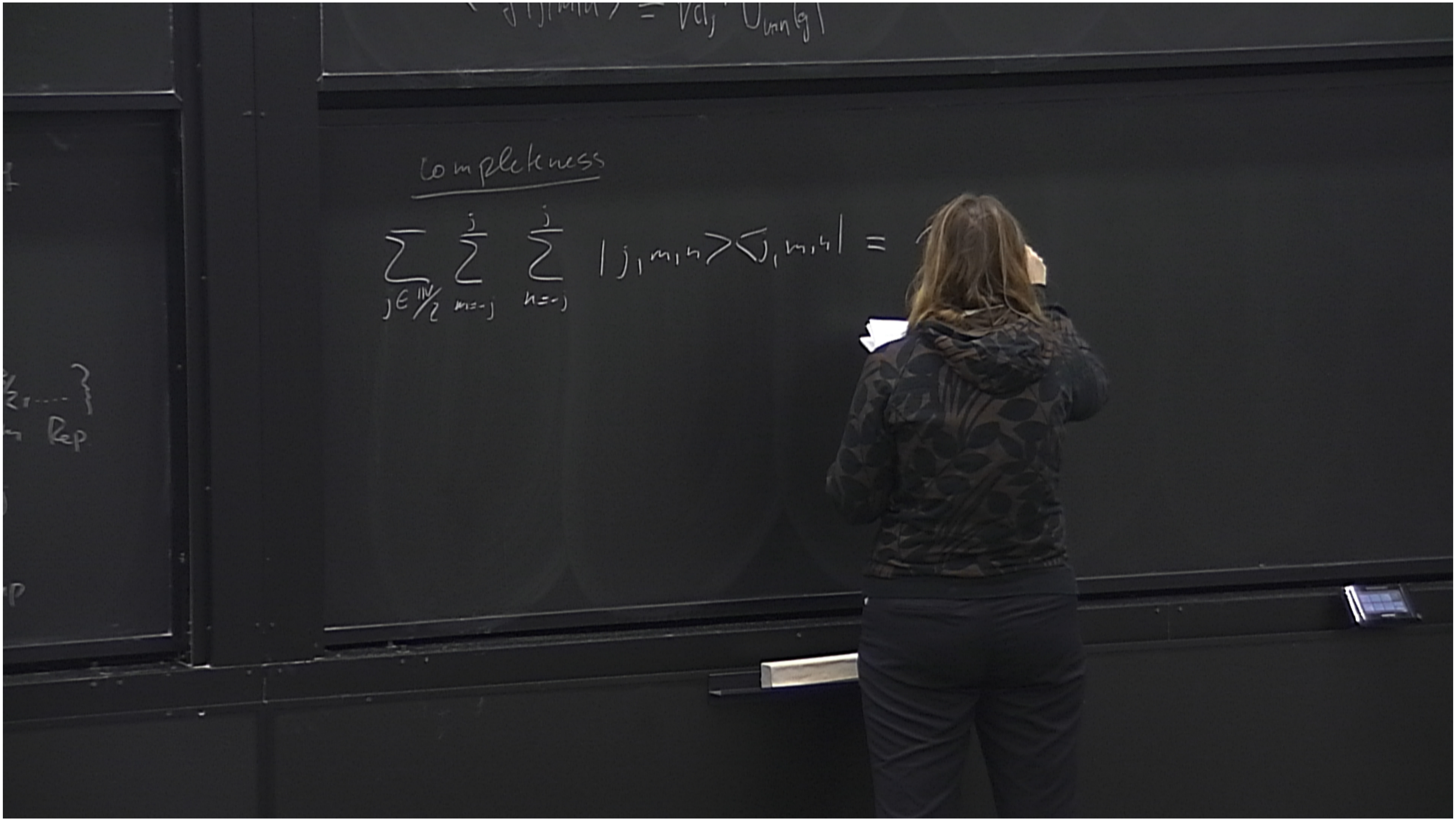

Peter-Weyl-thm:
$\{D^j_{mn}\}_{\{j,m,n\}}$ is a complete
orthogonal basis of $\mathcal{L}^2(G,dg)$
$\Rightarrow \int_G \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j}$
$d_j = (2j$
Rep

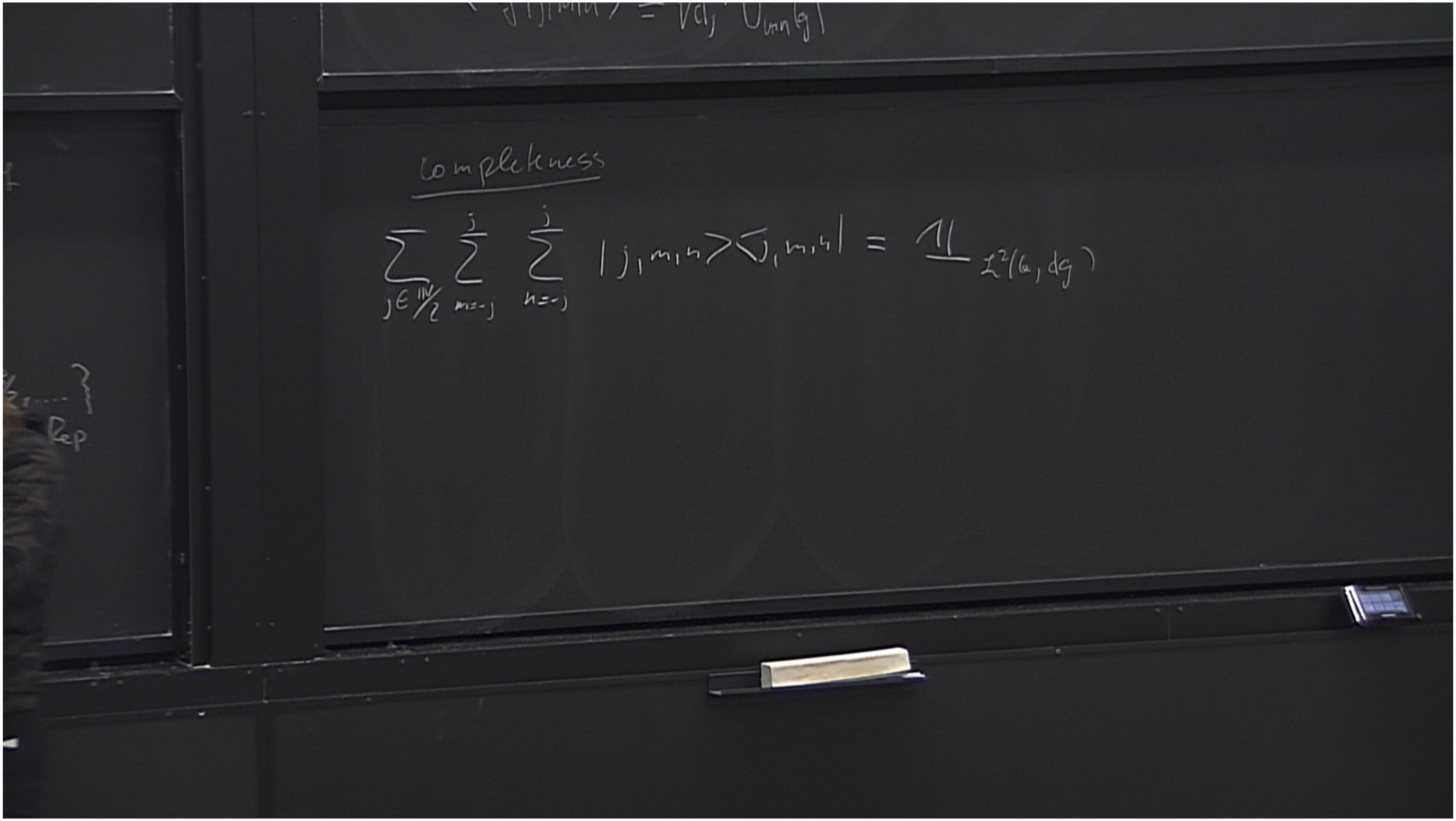
Peter-Weyl-thm:

$\{D^j_{mn}\}_{\{j,m,n\}}$ is a complete orthogonal basis of $\mathcal{L}^2(G, dg)$

$$\Rightarrow \int_G \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\,dg = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'}$$

$d_j = (2j+1)$

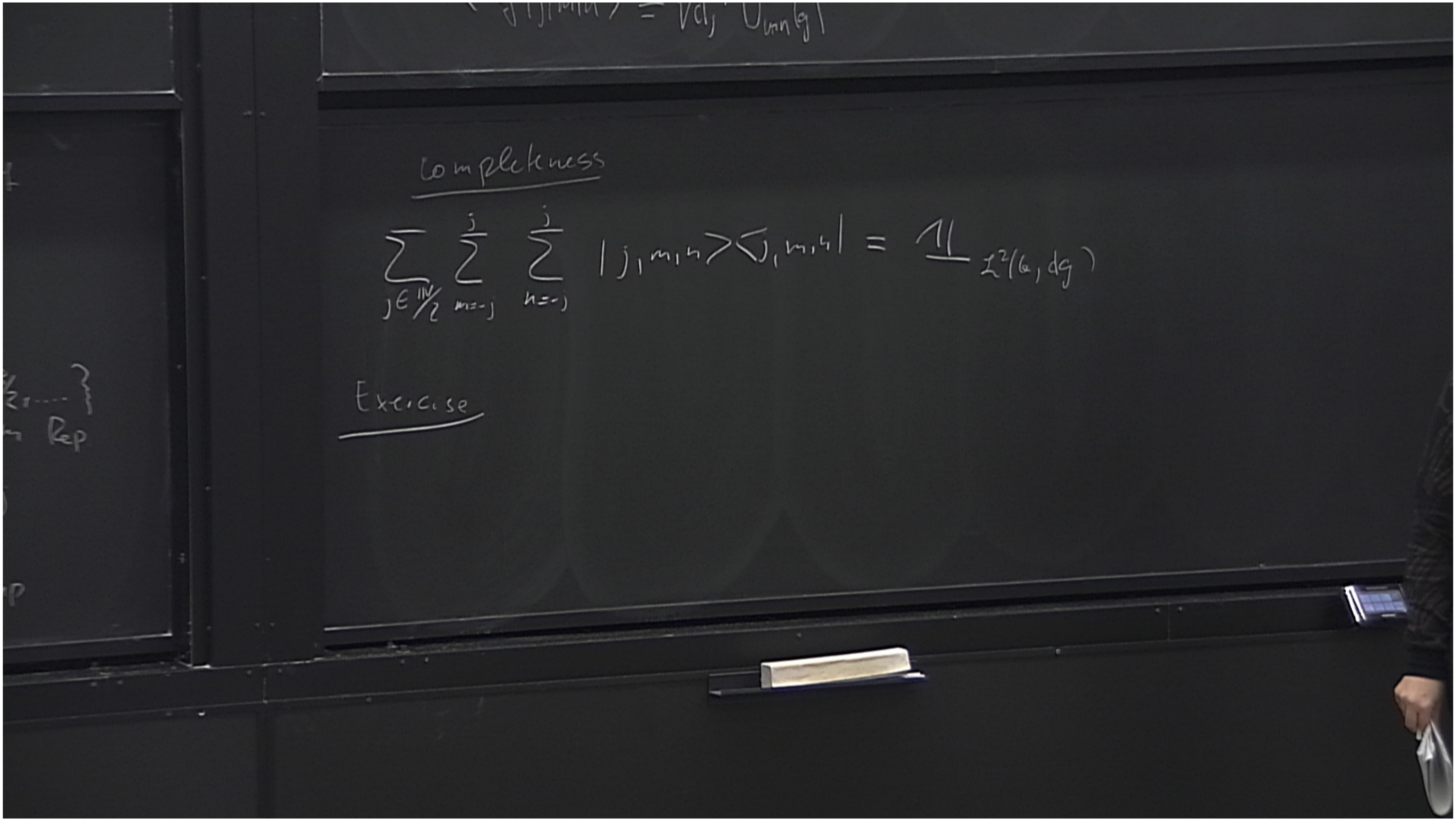

completeness
$\sum_{j \in \mathbb{N}/2} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |j,m,n\rangle\langle j,m,n| =$
Rep

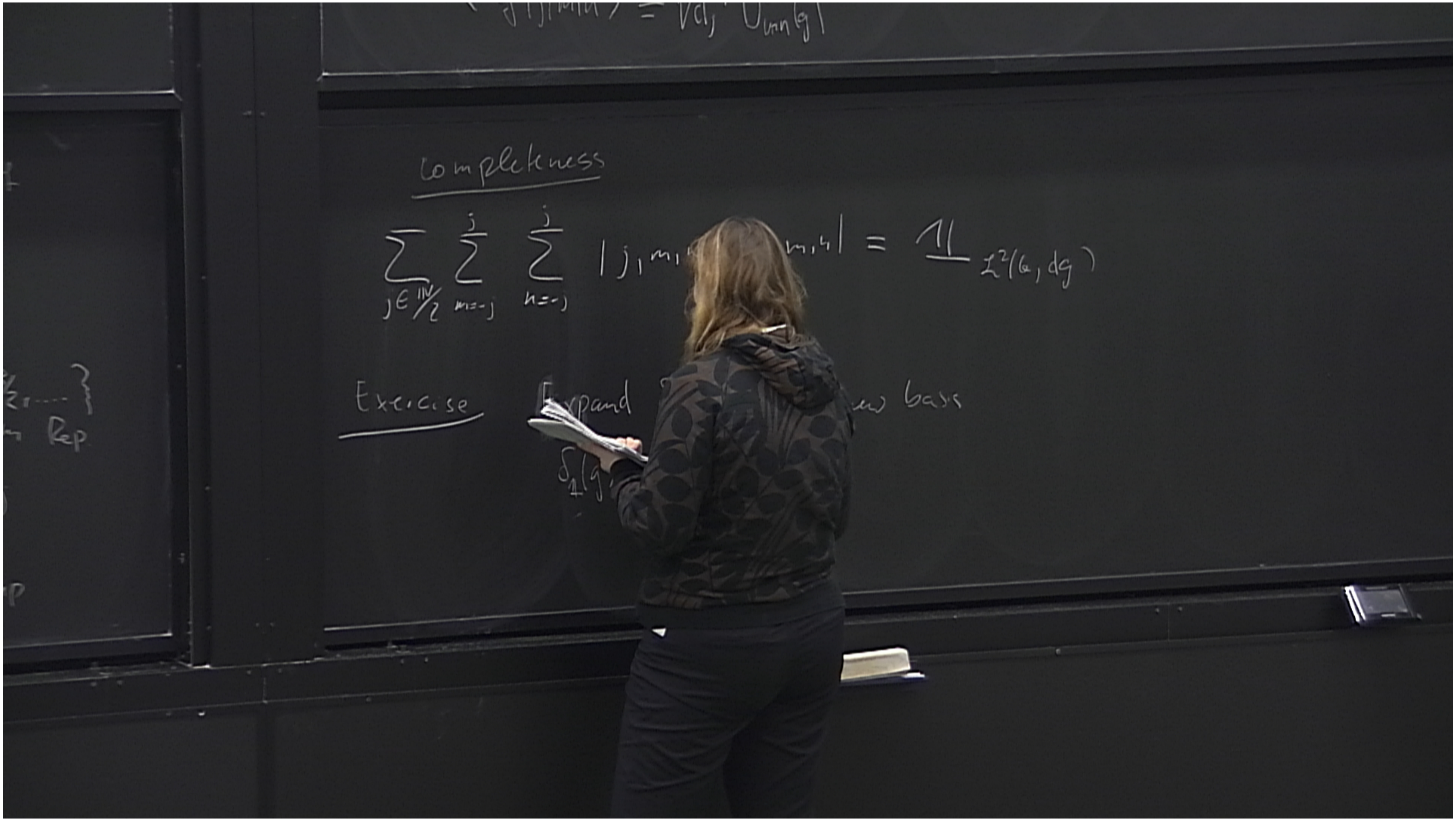

completeness
$\sum_{j\in \mathbb{N}/2}\sum_{m=-j}^{j}\sum_{n=-j}^{j} |j,m,n\rangle\langle j,m,n| = \mathbb{1}_{L^2(G,dg)}$
Rep

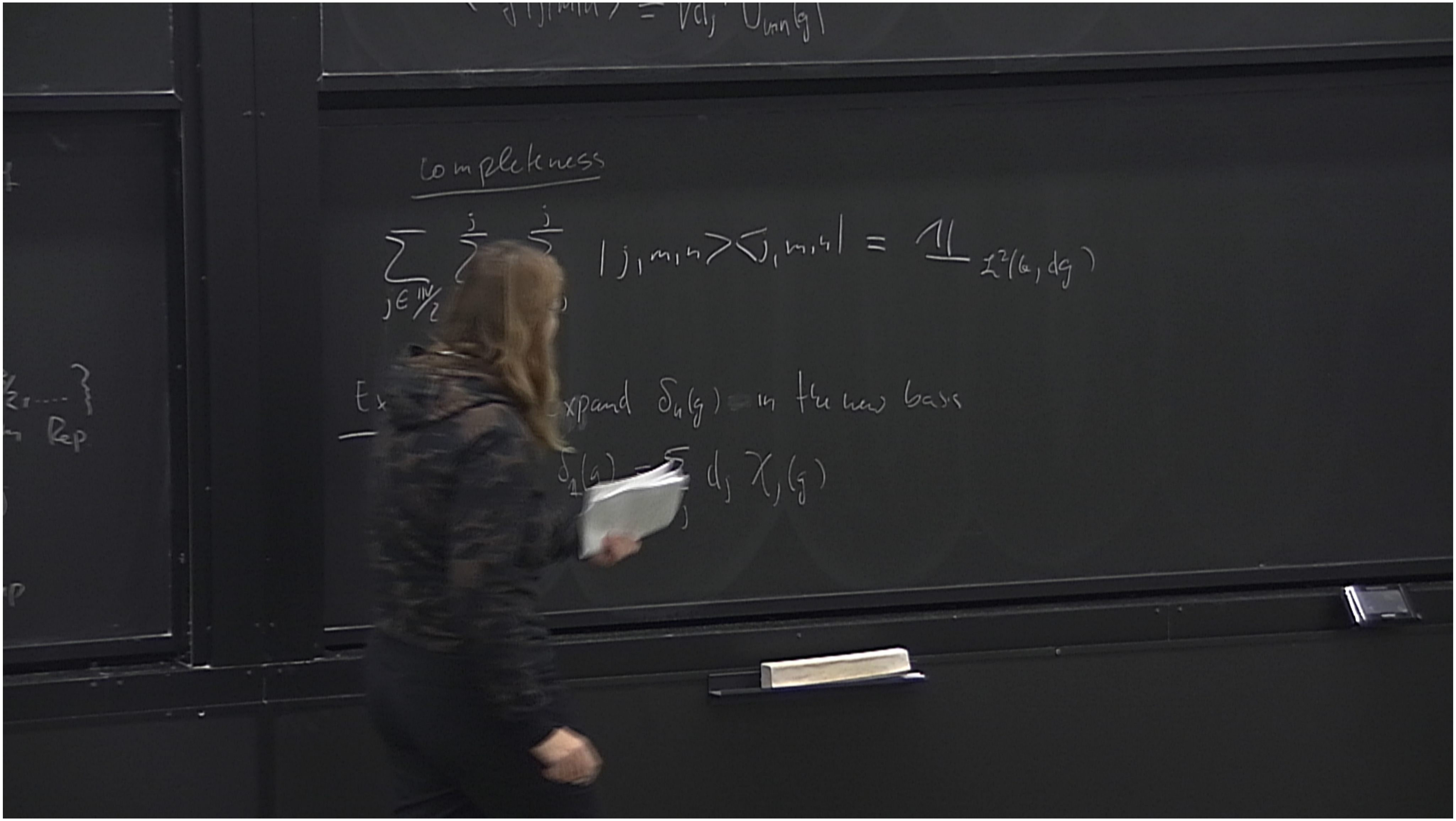
$$\langle g | j, m, n \rangle = \sqrt{d_j}\, U_{mn}(g)$$

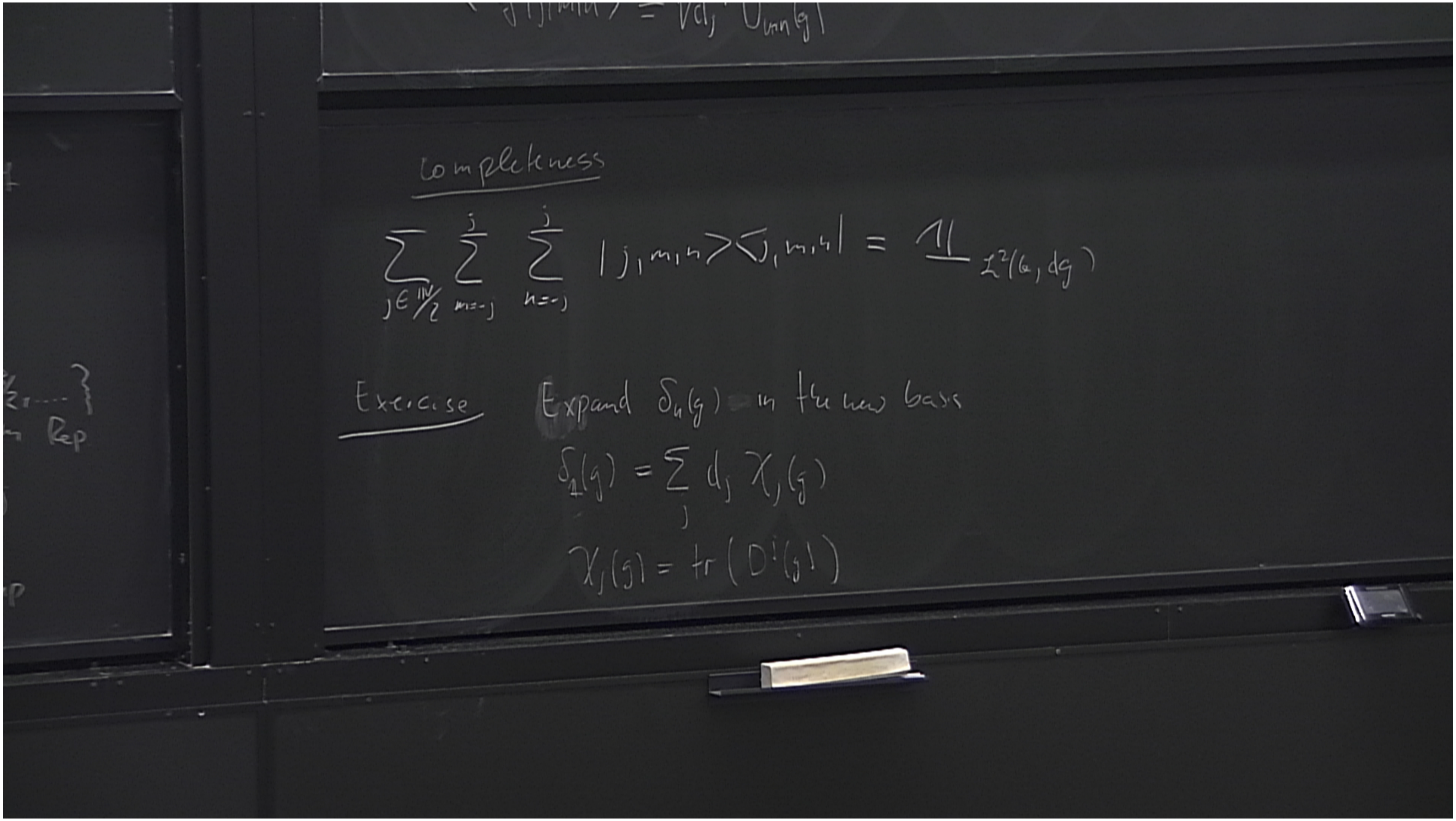
Completeness

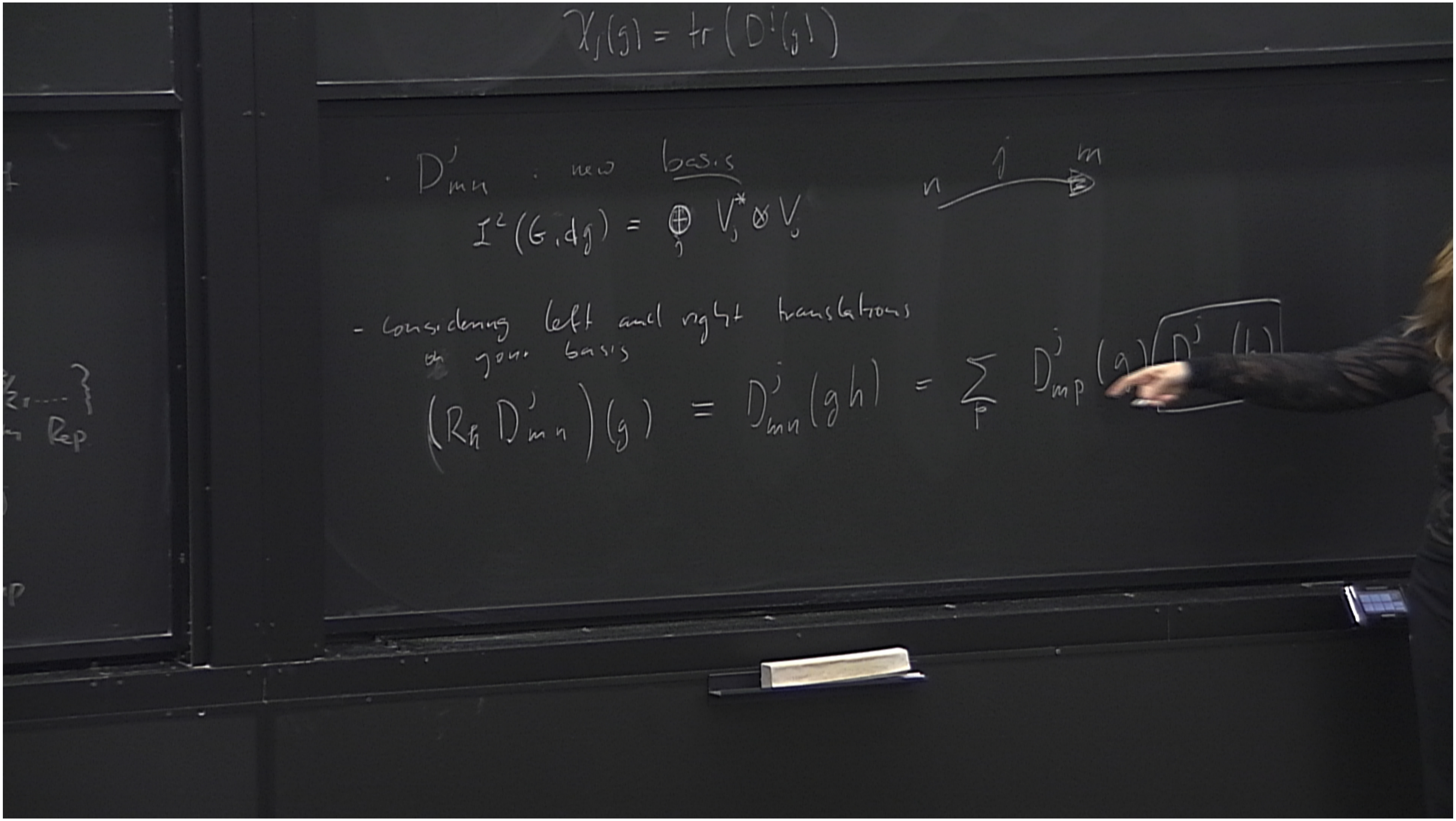
$$\sum_{j \in \mathbb{N}/2} \sum_{m=-j}^{j} \sum_{n=-j}^{j} | j, m, n \rangle \langle j, m, n | = \mathbb{1}_{L^2(G, dg)}$$

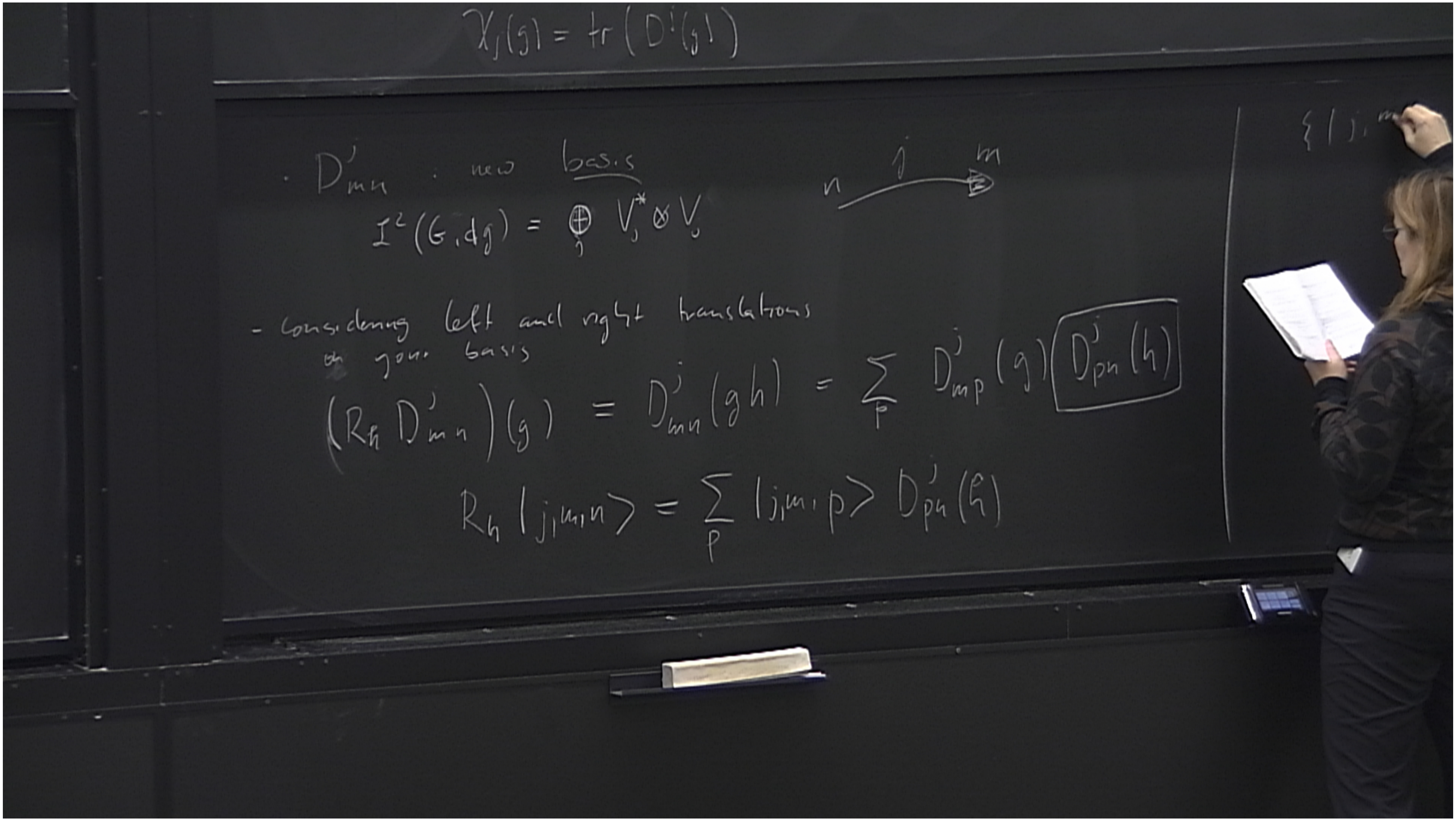

Completeness
$\sum_{j\in\mathbb{N}/2}\sum_{m=-j}^{j}\sum_{n=-j}^{j}|j,m,n\rangle\langle j,m,n| = \mathbb{1}_{L^2(G,dg)}$
Exercise


completeness
Exercise
Expand


completeness
Rep
expand δ1(g) in the new basis


completeness
$\sum_{j\in \mathbb{N}/2}\sum_{m=-j}^{j}\sum_{n=-j}^{j} |j,m,n\rangle\langle j,m,n| = \mathbb{1}_{L^2(G,dg)}$
Exercise
Expand $\delta_1(g)$ in the new basis
$\delta_1(g) = \sum_j d_j \chi_j(g)$
$\chi_j(g) = tr(D^j(g))$


$\chi_j(g) = \mathrm{tr}(D^j(g))$
· D^j_{mn} : new basis
$n \quad j \quad m$
$L^2(G, dg) = \bigoplus_j V_j^* \otimes V_j$
- considering left and right translations on your basis
$(R_h D^j_{mn})(g) = D^j_{mn}(gh) = \sum_p D^j_{mp}(g) \, D^j \; (h)$
Rep


$\chi_j(g) = tr(D^j(g))$
· D^j_{mn} : new basis
$L^2(G, dg) = \bigoplus_j V_j^* \otimes V_j$
n j m
- considering left and right translations on your basis
$(R_h D^j_{mn})(g) = D^j_{mn}(gh) = \sum_p D^j_{mp}(g) \boxed{D^j_{pn}(h)}$
$R_h |j,m,n\rangle = \sum_p |j,m,p\rangle D^j_{pn}(h)$
$\{|j,m$