Title: Introduction to Quantum Field Theory for Cosmology - Lecture 22

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Abstract:

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QFT for Cosmology, Achim Kempf, Winter 14, Lecture 22

Note Title

Inflationary generating of scalar and tensor fluctuations

Recall:

- We decompose the inflaton field $\phi(x,\eta)$:

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta)$$

where:

* $\phi_0(\eta)$ is assumed large and is treated classically.

* $\varphi(x,\eta) =: \delta\phi(x,\eta)$ describes a field of small

1 / 26

1

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- $\phi_0(\eta)$ is assumed large and is treated classically.
- $\varphi(x,\eta) =: \delta\phi(x,\eta)$ describes a field of small inhomogeneities and is to be quantized: $\hat{\varphi}(x,\eta)$

1 / 26

1

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- We decompose the metric $g_{\mu\nu}(x,\eta)$:

$$g_{\mu\nu}(x,\eta) = a_0^2(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta)$$

$a_0^2(\eta)$: treated classically

$\gamma_{\mu\nu}(x,\eta)$: assumed small, to be quantized

- Here, $\gamma_{\mu\nu}(x,\eta)$ can be decomposed into scalar, vector and tensor-type inhomogeneities, using functions
$E, B, \Psi, \Phi, V_i, W_i, h_{ij}$.

- Slicing of spacetime into spacelike hypersurfaces can be done so that all these fctns vanish, except for $\Psi(x,\eta)$, $h_{ij}(x,\eta)$.

2 / 26

2

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- We noticed that $\delta\phi(x,\eta) = \varphi(x,t)$ combines with the scalar part of the metric inhomogeneities $\Psi(x,\eta)$, due to the Einstein eqn, to yield one dynamical entity, named

2

2 / 26

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- We noticed that $\delta\phi(x,\eta) = \varphi(x,t)$ combines with the scalar part of the metric inhomogeneities $\Psi(x,\eta)$, due to the Einstein eqn, to yield one dynamical entity, namely:

$$r(x,\eta) = -\Psi(x,\eta) - \frac{a'(\eta)}{a(\eta)} \frac{\varphi(x,\eta)}{\phi_0'(\eta)}$$

Physically, what is $r(x,\eta)$?

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2 / 26

2

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* First term: $\Psi(x,\eta)$ is the (scalar) metric's fluctuation.

* Second term: In $\frac{a'}{a}\frac{1}{\phi_0'}\varphi$, the $\varphi(x,\eta)$ is the scalar field's fluctuation.

Consider now: 2 Useful choices for foliations of spacetime into spacelike hypersurfaces of equal time:

a.) Foliate so that on surfaces of equal time, η, one has: $\varphi \equiv 0$.

$\rightsquigarrow$ Equal time hypersurfaces chosen so that all

3 / 26

3

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⇝ Equal time hypersurfaces chosen so that all points of equal value of ϕ have equal value of time.

Note: Only possible if ϕ decays over time (e.g. slow roll inflation, but not de Sitter).

⇝ We see that $r(x,\eta)$ expresses now purely metric fluctuations

⇝ Technically, these are fluctuations in the

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2 / 26

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3 / 26

3

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⇝ We see that $\tau(x,\eta)$ expresses now purely metric fluctuations

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b.) Foliate so that on surfaces of equal time, η, one has: $\Psi \equiv 0$

In this case, along each equal time surface there is no local bloating - but instead the inflaton field fluctuates.

Question:

Why does the contribution of the inflaton

in $\tau(x,\eta)$ take this particular form:

4

4 / 26

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Question:

Why does the contribution of the inflaton in $\mathcal{R}(x,\eta)$ take this particular form:

$$\frac{a'(\eta)}{a(\eta)} \frac{\varphi(x,\eta)}{\phi_0'(\eta)} \quad ?$$

Answer:

4 / 26

4

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Answer:

* The inflaton's inhomogeneities imply locally varying expansion rates.

⇒ some regions are ahead, others lag behind in their expansion.

* Changing the spacetime slicing from a) to b) has to turn pure intrinsic curvature, namely local bloating

$$\frac{\delta a(x,\eta)}{a(\eta)}$$

into pure inflaton fluctuations $\varphi(x,\eta)$.

$\delta\eta(x)$ is the time "lag" between slicings a) and

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5

5 / 26

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$$\frac{\delta a}{a} = \frac{1}{a}\frac{\delta a}{\delta\phi}\delta\phi = \frac{1}{a}\frac{\delta a}{\delta\eta}\frac{\delta\eta}{\delta\phi}\delta\phi = \frac{a'}{a}\frac{1}{\phi'}\delta\phi$$

(Note: We also have

5 / 26

5

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$$= \frac{a'}{a}\frac{1}{\phi'}\varphi \checkmark$$

(Note: We also have $\frac{\delta a}{a} = \frac{a'}{a}\frac{1}{\phi'}\varphi = \frac{a}{z}\varphi$ where $z = \frac{a^2}{a'}\phi'$ from previous lecture)

Ramifications:

5 / 26

5

Ramifications:

- The intrinsic curvature inhomogeneities

very large when ϕ_0' is very small

$$r = -\Psi - \frac{a'}{a}\frac{1}{\phi_0'}\varphi$$

can become strongly enhanced, namely, as it happens, for close to de Sitter inflation:

i.e., for $a(t) \approx e^{Ht}$

i.e., for $H = \frac{\dot{a}}{a} \approx const$ $\left(\text{recall: } H \sim \sqrt{V(\phi)}\right)$

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6 / 26

6

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6 / 26

6

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□ **Why?** Recall that:

$\frac{\delta a}{a} = \frac{1}{a} \frac{\delta a}{\delta \eta} \delta \eta \ \delta\phi = \frac{a'}{} \frac{1}{} \varphi$

6 / 26

6

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$$\frac{\delta a}{a} = \frac{1}{a}\frac{\delta a}{\delta \eta}\frac{\delta \eta}{\delta \phi}\delta\phi = \frac{a'}{a}\frac{1}{\phi_0'}\,\ell$$

Thus: Assume $\phi' = \frac{\delta\phi}{\delta\eta} \ll 1$

$$\Rightarrow \quad \frac{\delta\eta}{\delta\phi} \gg 1$$

Intuition:

$\frac{\delta\eta}{\delta\phi} \gg 1$ means that the local time-lag $\delta\eta$

7 / 26

7

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7 / 26

7

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Could it be a problem?

Observations: We know the size of $|r|$ from the CMB. The curvature fluctuations r are of order 10^{-5}. Also, there is evidence that the Hubble radius increased during inflation. Namely, the fluctuations of modes that crossed it late are smaller. So inflation was significantly different from de Sitter.

Is there a preferred slicing of spacetime, say a) or b) in nature?

* Not during inflation, but at its end point!

8 / 26

8

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Is there a preferred slicing of spacetime, say a) or b) in nature?

* Not during inflation, but at its end point!

So it is slicing type a.)

* Why? At each point in space, inflation ends the moment the value of ϕ drops to its minimum. Then, $v(x,y)$ is intrinsic curvature.

8

8 / 26

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Quantization:

- As Hawking, Starobinski and others realized in the early 1980s, the quantization of the inhomogeneity functions $\hat{\tau}(x,\eta)$ and $\hat{h}_{ij}(x,\eta)$ yields inhomogeneities as vacuum fluctuations.

- Recall:

  * Defined auxiliary field for scalar inhomogeneities:

"Mukhanov variable"

$$u(x,\eta) := - z(\eta)\,\tau(x,\eta)$$

9 / 26

9

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8 / 26

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$$u(x,\eta) := -z(\eta)\,\tau(x,\eta)$$

* The advantage is that this field has no friction term in its equation of motion:

9 / 26

9

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9

9 / 26

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$$u(x,\eta) := -z(\eta)\,r(x,\eta)$$

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$$u_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right) u_k(\eta) = 0$$

* Similarly, we introduced an auxiliary

9 / 26

9

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$$u_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right) u_k(\eta) = 0$$

* Similarly, we introduced an auxiliary field $p_{ij}(x,\eta)$ for the tensor inhomogeneities:

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h'_{ij}(x,\eta)$$

Note: * The components of p_{ij} are not all independent, because h_{ij} obeys:

$h_{ij} = h_{ji}$ and $\sum^{3} h_{ii} = 0$ and in particular:

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are independent, because h_{ij} obeys:

$h_{ij} = h_{ji}$ and $\sum_{i=1}^{3} h_{ii} = 0$ and in particular:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x^i} h_{ij}(x,\eta) = 0 \quad \text{i.e.} \quad \sum_{i=1}^{3} k_i\, h_{ij}(k,\eta) = 0$$

* But $\vec{k}$ is the vector that points in the direction in which the mode $\vec{k}$ propagates.

$\Rightarrow$ The equation

$$\sum_{i=1}^{3} k_i\, h_{ij}(k,\eta) = 0$$

(For fixed j, the vectors) $\rightarrow$ means that h_{ij} has no component

10 / 26

10

$$\sum_{i=1} k_i \, h_{ij}(k,\eta) = 0$$

(For fixed j, the vectors h_{ij} and k_i are orthogonal) $\rightarrow$ means that h_{ij} has no component in the propagation direction:

$\Rightarrow$ h_{ij} describes transversal waves (like e.g. tectonic shear waves), not longitudinal waves (such as e.g. sound waves).

$\leadsto$ h_{ij} possesses only 2 degrees of freedom:

$v_{k,\lambda}(\eta)$ with $\lambda = 1, 2$ or $+$ $\times$

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11 / 26

11

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* Polarization decomposition

$$p_{ij}(k,\eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta)\,\varepsilon_{ij}(k,\lambda)$$

Here, $\varepsilon_{ij}(k,\lambda)$ are for each k two arbitrary but fixed matrices, obeying $\sum_{i,j=1}^{3} \varepsilon_{ij}(k,1)\,\varepsilon_{ji}(k,2) = 0$ and:

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad \sum_{i=1}^{3} \varepsilon_{ii} = 0, \quad \sum_{i=1}^{3} k_i\,\varepsilon_{ij} = 0$$

It is convenient to choose

$$\varepsilon_{ij}(-k,\lambda) = \varepsilon^{*}_{ij}(k,\lambda)$$

12 / 26

12

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because then we have (as usual):

$$v_{k,\lambda}(\eta) = v^{*}_{-k,\lambda}(\eta)$$

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12 / 26

12

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* Then, the eqn. of motion for the tensor inhomogeneities reads:

$$v''_{k,\lambda}(\eta) + \left(k^2 - \frac{a''}{a}\right) v_{k,\lambda}(\eta) = 0$$

(As desired, it has no friction term)

Quantum fluctuations

- We need to solve the quantum wave equations

$$\hat{u}''_k(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)\hat{u}_k(\eta) = 0 \qquad \text{scalar}$$

13 / 26

13

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$$\hat{v}_{k,\lambda}''(\eta) + \left(k^2 - \frac{a''}{a}\right)\hat{v}_{k,\lambda}(\eta) = 0 \qquad \text{tensor}$$

along with the commutation relations and hermiticity conditions.

- As before, this reduces to solving the eqns of

13 / 26

13

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- As before, this reduces to solving the eqns of motion for the mode functions, which are complex number-valued, say $\tilde{u}_k(\eta)$, $\tilde{v}_{k,\lambda}(\eta)$:

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$$\tilde{v}_{k,\lambda}''(\eta) + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0$$

along with the Wronskian conditions.

- Initial conditions?

We say we choose the "Bunch Davies vacuum".

At early times: * The k^2 term dominates

14 / 26

14

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13 / 26

13

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- **Initial conditions?**

14 / 26

14

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Initial conditions?

the "Bunch Davies vacuum"!

At early times:

* The k^2 term dominates

$\Rightarrow$ Can choose Minkowski-like init. conds.

The mode fctns at late times?

At late times:

* The mode k crossed the Hubble horizon:

* The terms $\frac{z''}{z}$ and $\frac{a''}{a}$ dominate.

* The harmonic oscillator is inverted

14 / 26

14

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14 / 26

14

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15 / 26

15

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15 / 26

15

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15 / 26

15

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* Soon after horizon crossing the mode function consists of essentially only the growing solution.

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* Soon after horizon crossing the mode function consists of essentially only the growing solution.

Which is the growing solution at late times?

* Eqns of motion:

$$\tilde{u}_k''(\eta) + \left(\cancel{k^2} - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0$$

$$\tilde{v}_{k,2}''(\eta) + \left(\cancel{k^2} - \frac{a''}{a}\right)\tilde{v}_{k,2}(\eta) = 0$$

$\Rightarrow$ Growing solution must behave as:

15 / 26

15

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$$\tilde{v}_{k,\lambda}''(\eta) + \left(\cancel{k^2} - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0$$

⇒ Growing solution must behave as:

$\tilde{u}_k(\eta) \sim z(\eta)$ at late η

$\tilde{v}_{k,\lambda}(\eta) \sim a(\eta)$ at late η

⇒ This means that the mode fctns $\tilde{r}_k(\eta)$ and $\tilde{h}_{ij_k}(\eta)$ become

16 / 26

16

* Eqns of motion:

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⇒ Growing solution must behave as:

$$\tilde{u}_k(\eta) \sim z(\eta) \quad \text{at late } \eta$$

$$\tilde{v}_{k,\lambda}(\eta) \sim a(\eta) \quad \text{at late } \eta$$

⇒ This means that the mode fctns $\tilde{r}_k(\eta)$ and $h_{ij\,k}(\eta)$ become constant at late η, i.e., after the mode k crosses the horizon!

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Why?

$$\tilde{v}_k(\eta) = \frac{1}{z(\eta)}\,\tilde{u}(\eta) \sim \frac{z(\eta)}{z(\eta)} \qquad \text{for late } \eta$$

$$\tilde{h}_{ij_k}(\eta) = \frac{1}{a(\eta)}\,\tilde{p}_{ij_k}(\eta) \sim \frac{1}{a(\eta)}\,\tilde{v}_{k,\lambda}(\eta) \sim \frac{a(\eta)}{a(\eta)} \qquad \text{for late } \eta$$

$\Rightarrow$ As expected, the magnitude of the mode k's quantum fluctuations

$$\underbrace{\delta v_k = k^{3/2}|\tilde{v}_k|^2}_{= z^{-1}k^{3/2}|\tilde{u}_k|} \quad \text{and} \quad \underbrace{\delta h_{ij_k} = k^{3/2}|\tilde{h}_{ij_k}|}_{= a^{-1}k^{3/2}|\tilde{v}|}$$

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17 / 26

17

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stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

* Goal now: Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

17 / 26

17

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* **Goal now:** Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

- We need an explicit potential $V(\phi)$ in order to be able to find explicit $a_0(\eta)$, $\phi_0(\eta)$ for which to calculate then the fluctuation spectrum.

- De Sitter is ruled out because:

  * $V(\phi)$, and therefore the temporary "cosmological constant $H \sim \sqrt{V(\phi)}$ must slowly decrease

17 / 26

17

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18 / 26

18

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□ De Sitter is ruled out because:

* $V(\phi)$, and therefore the temporary "cosmological constant" $H \sim \sqrt{V(\phi)}$ must slowly decrease (slow roll).

* In any case, our perturbation assumptions don't allow exact de Sitter, as δv_k would diverge, invalidating the assumption that it is small.

The "slow roll parameters"

Idea:

18 / 26

18

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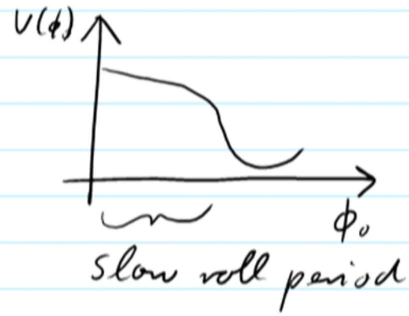
Idea:

* We do not know the exact slow roll potential:

E.g.



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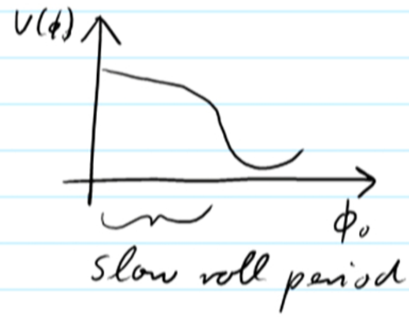

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convenience factor

$$\varepsilon(\phi) := \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \qquad \left(= \frac{\frac{3}{2}\dot{\phi}^2}{V + \frac{1}{2}\dot{\phi}^2} \right)$$

$$\eta(\phi) := \frac{1}{4\pi G}\,\frac{H''(\phi)}{H(\phi)} \qquad \left(= \varepsilon - \frac{\varepsilon'}{\sqrt{16\pi G\varepsilon}} \right)$$

$$\xi(\phi) := \frac{1}{4\pi G}\sqrt{\frac{H'(\phi)\,H'''(\phi)}{H^2(\phi)}}$$

etc...

19 / 26

19

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The simplest solvable case:

* The simplest case is that of

$$\varepsilon(\phi) = c \quad \text{where } c \text{ is a constant.}$$

* In this case:

$$c = \varepsilon(\phi) := \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2$$

Thus,

$$H(\phi) \sim e^{\sqrt{4\pi G}\,\phi}$$

20 / 26

20

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* Exercise: What is the value of s ?

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21 / 26

21

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$$c = \varepsilon = \eta = \xi = \dots$$

* The expansion rate is polynomial:

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Show that we now have

21 / 26

21

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Show that we now have

$$a(t) = a_0\, t^{1/c}$$

(t is proper time)

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22 / 26

22

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22 / 26

22

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The mode equations:

- Scalar: We can now calculate $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi_0'(\eta)$ and

22 / 26

22

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Scalar: We can now calculate $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi_0'(\eta)$ and therefore also the mode equation's term z''/z explicitly, to obtain

A Bessel differential equation

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{(\nu^2 - 1/4)}{\eta^2}\right) \tilde{u}_k(\eta) = 0$$

where: $\nu := \frac{3}{2} + \frac{c}{1-c}$

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \sqrt{\pi}\, e^{i(\nu+1/2)\frac{\pi}{2}} (-\eta)^{1/2} H^{(1)}(-k\eta)$$

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Hankel fctn of 1st kind

23 / 26

23

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Hankel fctn of 1st kind of order ν.

* Behavior after horizon crossing:

23 / 26

23

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* Behavior after horizon crossing:

$$\tilde{u}_k(\eta) \rightarrow e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{-\nu+1/2}$$

* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

Note: Measuring only δr_k does not fix H and H' individually

$$\delta r_k\left(\eta > \eta_{hor}(k)\right) = G\, 2^{\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \left(\nu - \tfrac{1}{2}\right)^{1/2-\nu} \left.\frac{H^2}{|H'|}\right|_{\text{at } k=aH}$$

(i.e. at horizon)

Exercise: verify

* **Notice:** Earlier, for a K.G. field ϕ in a fixed FRW universe, we found: $\delta\phi_k \sim H$

Now: $\delta r_k \sim H^2/|H'|$. Why different? Because r_k is the slicing-independent combination of the scalar part of $\delta g_{\mu\nu}$ and ϕ.

24 / 26

24

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24 / 26

24

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Tensor modes :

* In the power law case, in the mode equation

$$\tilde{v}''_{k,\lambda} + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda} = 0$$

we obtain for the term a''/a

$$\frac{a''}{a} = 2a^2H^2\left(1 - \varepsilon/2\right)$$

which comes out to be (verify) :

$$\frac{a''}{a} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)$$

recall : $\nu = \frac{3}{2} - \frac{c}{1-c}$

25 / 26

25

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$$u''_{k,\lambda} + \left(k^2 - \frac{a''}{a}\right) u_{k,\lambda} = 0$$

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$\Rightarrow$ The mode eqn is again solved by the Hankel function.

$\Rightarrow$ The tensor fluctuation spectrum:

25 / 26

25

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⇒ The mode eqn is again solved by the Hankel function.

⇒ The tensor fluctuation spectrum:

$$\delta h_{ij\,k} = \frac{2}{\sqrt{\pi}} \, 2^{\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu - 1/2)^{1/2-\nu} \sqrt{G} \, H \Big|_{k=aH}$$

propotional to the value of H at horizon crossing.

Observations:

- δT_k is predicted to have seeded oscillations in the hot plasma after re-heating, leading to a calculable fluctuation spectrum in the CMB.

25 / 26

25

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Recall: The slower the roll ($|H'|$ small) the wider away from another fluctuate gauge equivalent and inequivalent slicings.

Tensor modes:

24 / 26

24

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⇒ The tensor fluctuation spectrum:

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25 / 26

25

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Observations:

- δr_k is predicted to have seeded oscillations in the hot plasma after re-heating, leading to a calculable fluctuation spectrum in the CMB.
- The match is very good. The WMAP results show indications that $\varepsilon \neq 0$, namely that $\delta r_k \neq$ const.
- δh_{ij_k} is predicted to have led to polarization fluctuations in the light of the CMB - a pure curl polarization field.
- Observed: BICEP2, March 2014 !

26 / 26

26