Title: Introduction to Quantum Field Theory for Cosmology - Lecture 21

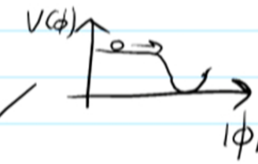
Date: Mar 25, 2014 04:00 PM

URL: http://pirsa.org/14030002

Abstract:

Quantum fluctuations during cosmic inflation

e.g.



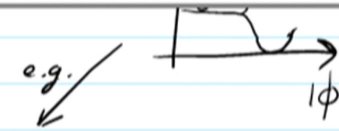
Strategy:

- We assume a suitable potential $V(\phi)$ and suitable initial conditions, as discussed before.

$\Rightarrow$ Solutions $\phi_0(t)$ and $a_0(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

- We consider the case of small inhomogeneities in

File Edit View Insert Actions Tools Help



Strategy:

- We assume a suitable potential $V(\phi)$ and suitable initial conditions, as discussed before.

$\Rightarrow$ Solutions $\phi_0(t)$ and $a_0(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

- We consider the case of small inhomogeneities in the inflaton field:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

1/23

1

File Edit View Insert Actions Tools Help

- We consider the case of small inhomogeneities in the inflaton field:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

- This means that we must also consider small fluctuations in the metric, because:

In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

1 / 23

1

File Edit View Insert Actions Tools Help

This means that we must also consider small fluctuations in the metric, because:

In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{\text{infl.}}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right]$$

$\Rightarrow$ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

2 / 23

2

File Edit View Insert Actions Tools Help

In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{\text{infl.}}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\tfrac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}m^2\phi^2\right]$$

$\Rightarrow$ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

inhomogenities of $\phi(x)$ induce inhomogenities of $g_{\mu\nu}(x)$:

2 / 23

2

File Edit View Insert Actions Tools Help

$\rightsquigarrow$ Consider also small inhomogeneities in the metric, i.e., in the spacetime

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with } |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.
- In lowest order perturbation theory we first find the classical solutions $g_{0\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

3 / 23

3

File Edit View Insert Actions Tools Help

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with } |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.

- In lowest order perturbation theory we first find the classical solutions $g_{0\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

- Then, we quantize only the $\hat{\varepsilon}(x,\eta)$ and $\hat{\gamma}_{\mu\nu}(x,\eta)$.

- Why does this approximation help?

3 / 23

3

File Edit View Insert Actions Tools Help

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.

- In lowest order perturbation theory we first find the classical solutions $g_{0\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

- Then, we quantize only the $\hat{\epsilon}(x,\eta)$ and $\hat{\gamma}_{\mu\nu}(x,\eta)$.

- Why does this approximation help?

  * For fields $\hat{\epsilon}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the

3 / 23

3

File Edit View Insert Actions Tools Help

- In lowest order perturbation theory we first find the classical solutions $g_{0\mu\nu}(\eta) = a(\eta)\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

- Then, we quantize only the $\hat{\varepsilon}(x,\eta)$ and $\hat{\gamma}_{\mu\nu}(x,\eta)$.

- Why does this approximation help?

  * For fields $\hat{\varepsilon}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\varepsilon}, \hat{\gamma}_{\mu\nu}$.

  * This is because we can assume that in their

3 / 23

3

File Edit View Insert Actions Tools Help

□ Why does this approximation help?

* For fields $\hat{\epsilon}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\epsilon}$, $\hat{\gamma}_{\mu\nu}$.

* This is because we can assume that in their equations of motion all terms that are quadratic or of higher power are negligible.

* This means that the quantum fields $\hat{\epsilon}$ and $\hat{\gamma}_{\mu\nu}$ have no potential terms, nor any mass terms.

⇒ We will obtain a free, i.e., noninteracting quantum field theory whose nontriviality only

4/23

4

File Edit View Insert Actions Tools Help

* This is because we can assume that in their equations of motion all terms that are quadratic or of higher power are negligible.

* This means that the quantum fields $\hat{\epsilon}$ and $\hat{h}_{\mu\nu}$ have no potential terms, nor any mass terms.

⇒ We will obtain a free, i.e., noninteracting quantum field theory whose nontriviality only stems from the time-varying parameters $\phi_0(\eta)$, $a_0(\eta)$.

Intuition:

4 / 23

4

File Edit View Insert Actions Tools Help

Intuition:

We should expect two more sources of nontriviality:

1) Interdependence of φ and $h_{\mu\nu}$ inhomogeneities:

* Much of the inhomogeneities of $\hat{h}_{\mu\nu}(x,\eta)$ will be induced by the inhomogeneities of the inflaton, $\hat{\varphi}(x,\eta)$.

* Vice versa: we can also read the Einstein eqn from left to right $\Rightarrow$ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics

5 / 23

5

1) Interdependence of φ and $g_{\mu\nu}$ inhomogeneities:

* Much of the inhomogeneities of $\hat{g}_{\mu\nu}(x,\eta)$ will be induced by the inhomogeneities of the inflaton, $\hat{\varphi}(x,\eta)$.

* Vice versa: we can also read the Einstein eqn from left to right $\Rightarrow$ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

2) Some of $g_{\mu\nu}(x,\eta)$ is independent of φ!

5 / 23

5

File Edit View Insert Actions Tools Help

Recall:

Gravity is a force with some similarity to electromagnetism:

- Some electromagnetic fields only exist because there are charges or currents.

Similarly: Some part of the metric will depend on ϕ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of ϕ. $\rightarrow$

- But: also, some electromagnetic fields are self-sustaining, i.e., they exist independently, with their own dynamics.

Exercise: show this $\longrightarrow$

- Namely:

$\vec{E} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$\vec{E}(x) = \vec{E}(x) + \vec{E}(x)$

6 / 23

6

File Edit View Insert Actions Tools Help

Exercise: show this → **Namely:**

$\vec{E}: \mathbb{R}^3 \to \mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_s(x) + \vec{E}_v(x)$$

"gradient part" or "scalar part"

"curl part" or "vector part"

Here, $\vec{E}_s$ and $\vec{E}_v$ derive from a scalar function Λ and a vector field $\vec{A}$ respectively:

$$\vec{E}_s = \vec{\nabla}\Lambda \quad \text{and} \quad \vec{E}_v = \vec{\nabla}\times\vec{A}$$

They obey: $\vec{\nabla}\times\vec{E}_s = \vec{0}$ and $\vec{\nabla}\cdot\vec{E}_v = 0$ **(A)**

6 / 23

6

File Edit View Insert Actions Tools Help

Exercise for physics students: verify →

According to the Maxwell equations, the scalar part, e.g., of the electric field, $\vec{E}$, is caused by (or causes) the electric charge density

$$\nabla \cdot \vec{E}_s = \rho$$

[* An unusual but mathematically equivalent viewpoint.
* E.g. D-branes in string theory are charges that are defined from this viewpoint.

while the vector part is charge independent

$$\nabla \times \vec{E}_v = -\frac{\partial \vec{B}}{\partial t}$$

Thus, $\vec{E}$ and $\vec{B}$ fields can sustain each other, which makes possible nonzero electromagnetic fields (namely traveling waves) even where there are no charges. ←

and similarly for the magnetic field $\vec{B}(x)$.

Similarly, some curvature exists only where there is energy-momentum.

7 / 23

7

File Edit View Insert Actions Tools Help

- Similarly, some curvature exists only where there is energy-momentum.
- But, also, some curvature is self-sustaining, with dynamics, e.g., gravitational waves.

⇒ We should expect that $\hat{\mathcal{J}}_{\mu\nu}(x,\eta)$ contains:

- * some curvature that is induced by (or induces) the inflaton inhomogeneities.
- * some curvature inhomogeneities that are self-sustaining, i.e. that possess

8

8 / 23

File Edit View Insert Actions Tools Help

$\Rightarrow$ We should expect that $\hat{\mathcal{H}}_{\mu\nu}(x,y)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.

* some curvature inhomogeneities that are self-sustaining, i.e., that possess their own dynamics - and therefore also their own quantum fluctuations.

How to seperate these inhomogeneities of $\mathcal{H}_{\mu\nu}(x,y)$?

8 / 23

8

File Edit View Insert Actions Tools Help

How to seperate these inhomogeneities of $\gamma_{\mu\nu}(x,\eta)$?

Similar to vector fields $\vec{E}: \mathbb{R}^3 \to \mathbb{R}^3$ we have for the tensor field γ:

□ The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.

b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.

9 / 23

9

File Edit View Insert Actions Tools Help

□ The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.

b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.

c) The part of $\gamma_{\mu\nu}$ which is purely tensor.

9 / 23

9

File Edit View Insert Actions Tools Help

Decomposition of $g_{\mu\nu}(x,\eta)$, with respect to its spatial structure:

- One usually writes the "line element" ds^2, i.e., the infinitesimal proper distance (squared) from x to $x+dx$ as

$$ds^2 = g_{\mu\nu}(x,\eta)\, dx^\mu dx^\nu \quad \text{with} \quad dx^\mu = (d\eta, dx^1, dx^2, dx^3)$$

- Then, the decomposition takes the form:

$$ds^2 = \underbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}_{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

- Here, the spatially "scalar" part of the inhomogeneities reads

10

File Edit View Insert Actions Tools Help

infinitesimal proper distance (squared) from x to $x+dx$ as

$$ds^2 = g_{\mu\nu}(x,\eta)\,dx^\mu dx^\nu \quad \text{with} \quad dx^\mu = (d\eta, dx^1, dx^2, dx^3)$$

- Then, the decomposition takes the form:

$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_s^2}_{\text{scalar}} + \underbrace{ds_v^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

- Here, the spatially "scalar" part of the inhomogeneities reads

$$ds_s^2 = a^2(\eta)\Big[2\Phi(x,\eta)\,d\eta^2 - 2\sum_{i=1}^{3}\frac{\partial}{\partial x^i}B(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\Big(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}E(x,\eta)\Big)dx^i dx^j\Big]$$

10 / 23

10

File Edit View Insert Actions Tools Help

$$ds_s^2 = a^2(\eta)\Big[2\Phi(x,\eta)\,d\eta^2 - 2\sum_{i=1}^{3}\frac{\partial}{\partial x^i}B(x,\eta)\,dx^i d\eta$$
$$-\sum_{i,j=1}^{3}\Big(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}E(x,\eta)\Big)dx^i dx^j\Big]$$

where Φ, Ψ, B and E are scalar functions.

- The spatially "vector" part of the metric reads:

$$ds_v^2 = a^2(\eta)\Big[2\sum_{i=1}^{3}V_i(x,\eta)\,dx^i d\eta$$
$$-\sum_{i,j=1}^{3}\Big(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\Big)dx^i dx^j\Big]$$

10 / 23

10

File Edit View Insert Actions Tools Help

- The spatially "vector" part of the metric reads:

$$ds_V^2 = a^2(\eta)\left[2\sum_{i=1}^{3} V_i(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\right)dx^i dx^j\right]$$

where V_i and W_i are 3-vector fields.

- The spatially "tensor" part of the metric is the remainder, i.e., is what cannot be derived from a scalar or vector fields:

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3} h_{ij}(x,\eta)\,dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

11

11 / 23

File Edit View Insert Actions Tools Help

- The spatially "tensor" part of the metric is the remainder, i.e., is what cannot be derived from a scalar or vector fields:

$$ds_T^2 = a^2(\eta) \sum_{i,j=1}^{3} h_{ij}(x,\eta)\, dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

- Remark regarding the fields V^i, W^i and h_{ij}:

Analogous to the equations (A) above in electromagnetism,

$$\vec{\nabla} \times \vec{E}_s = \vec{0} \quad \text{and} \quad \vec{\nabla} \cdot \vec{E}_v = 0,$$

we now have:

11 / 23

11

File Edit View Insert Actions Tools Help

□ Remark regarding the fields V^i, W^i and h_{ij}:

Analogous to the equations (A) above in electromagnetism,

$$\vec{\nabla}\times\vec{E}_s = \vec{0} \text{ and } \vec{\nabla}\cdot\vec{E}_v = 0,$$

we now have:

* V_i, W_j obey:

$$\vec{\nabla}\cdot\vec{V} = 0, \quad \vec{\nabla}\cdot\vec{W} = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{3} \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.}\right)$$

* h_{ij} obeys:

$$h_{ij} = h_{ji}, \quad \sum_{i=1}^{3} h^i{}_i = 0, \quad \sum_{i=1}^{3} \frac{\partial}{\partial x_i} h_{ij} = 0$$

12 / 23

File Edit View Insert Actions Tools Help

* V_i, W_j obey:

$$\vec{\nabla}\cdot\vec{V} = 0, \quad \vec{\nabla}\cdot\vec{W} = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{3} \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.}\right)$$

* h_{ij} obeys:

$$h_{ij} = h_{ji}, \quad \sum_{i=1}^{3} h^i{}_i = 0, \quad \sum_{i=1}^{3} \frac{\partial}{\partial x_j} h_{ij} = 0$$

Remark: This implies that h_{ij} describes "Weyl curvature" which is known to describe gravitational waves

The action:

12 / 23

12

File Edit View Insert Actions Tools Help

The action:

We insert the approximation

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \ , \quad g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta),$$

with φ, γ assumed small, into the action:

$$S = \frac{-1}{16\pi G}\int R\sqrt{|g|}\, d^4x$$

$$+ \frac{1}{2}\int (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)\sqrt{|g|}\, d^4x$$

$$+ \text{neglected (other fields)}$$

13 / 23

13

File Edit View Insert Actions Tools Help

$$+ \frac{1}{2} \int (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \sqrt{|g|}\, d^4x$$

$+$ neglected (other fields)

⇝ One obtains many terms with $\Phi, \Psi, B, E, V, w, h$!

□ These terms can be simplified! Why?

Now that space is curved, there is no longer a preferred foliation of spacetime into spacelike hypersurfaces!

⇒ No preferred choice for the coordinate system.

13 / 23

13

File Edit View Insert Actions Tools Help

□ These terms can be simplified! Why?

Now that space is curved, there is no longer a preferred foliation of spacetime into spacelike hypersurfaces!

$\Rightarrow$ No preferred choice for the coordinate system.

(e.g., no prefered conformal time & space cds)

□ But the choice of cds will affect the functions above, i.e. they are in part coordinate system dependent.

$\Rightarrow$ We may choose our $\overbrace{\text{spacelike hypersurfaces}}^{\text{and thus our notion of equal time}}$ so that these functions $\Phi, \Psi, E, B, v, w, h$ vanish or simplify.

14/23

14

File Edit View Insert Actions Tools Help

⇒ We may choose our spacelike hypersurfaces so that these functions $\Phi, \Psi, E, B, v, w, h$ vanish or simplify.

It took on the order of 10 years to clarify this "gauge" question!

□ For detailed references, see e.g.:

* A. Riotto, hep-ph/0210162 (relatively compact)

* R. H. Brandenberger et al, Physics Reports 215, 203 (1992) (long)

14 / 23

14

File Edit View Insert Actions Tools Help

- For detailed references, see e.g.:

  * A. Riotto, hep-ph/0210162 (relatively compact)

  * R.H. Brandenberger et al, Physics Reports 215, 203 (1992) (long)

- Result:

  * For small inhomogeneities (1st order perturbation) nearly all inhomogeneities can be eliminated by suitable coordinate choice.

  * Except, there are two fields, which are coordinate system, i.e.,

15 / 23

15

File Edit View Insert Actions Tools Help

Result:

* For small inhomogeneities (1st order perturbation) nearly all inhomogeneities can be eliminated by suitable coordinate choice.

* Except, there are two fields, which are coordinate system, i.e., "gauge" independent. Namely:

I) A spatially scalar field, r, made of φ and $g_{\mu\nu}$'s scalar part:

recall: $\phi_0(\tau)$ = classical homogeneous inflaton field.

15 / 23

15

File Edit View Insert Actions Tools Help

I) A spatially scalar field, r, made of φ and $\gamma_{\mu\nu}$'s scalar part:

recall: $\phi_0(\tau)$ = classical homogeneous inflaton field.

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1}\varphi(x,\eta) - \Phi(x,\eta)$$

from inflaton

from "scalar" part of the metric

Recall:

* We predicted that the inflaton's inhomogeneities combine with curvature inhomogeneities to

* form one unified dynamical entity.

It is $r(x,\eta)$.

File Edit View Insert Actions Tools Help

Recall:

* We predicted that the inflaton's inhomogeneities combine with curvature inhomogeneities to
* form one unified dynamical entity.

It is $r(x,\eta)$.

Note: One can show that $r(x,\eta)$ can be viewed as the "intrinsic curvature", which, roughly speaking, measures how much space is locally bloated due to energy momentum.

II) A spatial tensor field:

16 / 23

16

Note: One can show that $r(x,\eta)$ can be viewed as the "intrinsic curvature", which, roughly speaking, measures how much space is locally bloated due to energy momentum.

II) A spatial tensor field:

This is $h_{ij}(x,\eta)$ itself. It represents $T_{\mu\nu}$-independent, so-called Weyl curvature, namely gravitational waves. $h_{ij}(x,\eta)$ measures how much space is locally distorted against itself in different directions.

16 / 23

16

File Edit View Insert Actions Tools Help

II) A spatial tensor field:

This is $h_{ij}(x,\eta)$ itself. It represents $T_{\mu\nu}$-independent, so-called Weyl curvature, namely gravitational waves. $h_{ij}(x,\eta)$ measures how much space is locally distorted against itself in different directions.

Intuition: For cases I, II think of a rubber membrane that is stretched into a flat sheet. It can be locally stretched, say by heating I, or distorted in a rotational way II.

17 / 23

17

File Edit View Insert Actions Tools Help

II) A spatial tensor field:

This is $h_{ij}(x,\eta)$ itself. It represents $T_{\mu\nu}$-independent, so-called Weyl curvature, namely gravitational waves. $h_{ij}(x,\eta)$ measures how much space is locally distorted against itself in different directions.

Intuition: For cases I, II think of a rubber membrane that is stretched into a flat sheet. It can be locally stretched, say by heating I, or distorted in a rotational way II.

17 / 23

17

File Edit View Insert Actions Tools Help

The expanded action

The action $S' = \frac{-1}{16\pi G}\int R\sqrt{|g|}\, d^4x$

$$+ \frac{1}{2}\int (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)\sqrt{|g|}\, d^4x$$

must be expanded to second order in the inhomogeneities in order to obtain their equations of motion to first order:

$$S' = S_s' + S_T'$$

The scalar part:

$$\boxed{S_s' = \frac{1}{2}\int z^2(\eta)\left(\frac{\partial}{\partial x^\mu} r(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu} r(x,\eta)\right) \eta^{\mu\nu}\, d^4x}$$

18 / 23

18

File Edit View Insert Actions Tools Help

$$S' = S'_s + S'_T$$

The scalar part:

$$\boxed{S'_s = \frac{1}{2}\int z^2(\eta)\left(\frac{\partial}{\partial x^\mu} r(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu} r(x,\eta)\right) \eta^{\mu\nu}\, d^4x}$$

Here:

$$\boxed{z(\eta) := \frac{a_0^2(\eta)}{a_0'(\eta)}\,\phi_0'(\eta)} \approx \underbrace{\text{const}\cdot a_0(\eta)}$$

because $a_0'(\eta) \approx \text{const}\, a(\eta)$ and $\phi_0' \approx \text{const}$ during inflation

Remark:

This action is similar to the scalar action which

18 / 23

18

File Edit View Insert Actions Tools Help

Remark:

This action is similar to the scalar action which we considerd so far:

$$S_{KG} = \frac{1}{2}\int a^2(\eta)\left(\frac{\partial}{\partial x^\mu}\phi(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x,\eta)\right)\eta^{\mu\nu}\,d^4x$$

The only difference is that $a(\eta)$ is now replaced by the more complicated (but still classical fixed background function) $z(\eta)$.

- **The tensor part:** Each h_{ij} has exactly our well-known action:

19 / 23

19

File Edit View Insert Actions Tools Help

This action is similar to the scalar action which we considered so far:

$$S_{\kappa G} = \frac{1}{2}\int a^2(\eta)\left(\frac{\partial}{\partial x^\mu}\phi(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x,\eta)\right)\eta^{\mu\nu}\,d^4x$$

The only difference is that $a(\eta)$ is now replaced by the more complicated (but still classical fixed background function) $z(\eta)$.

- **The tensor part:** Each h_{ij} has exactly our well-known action:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}\,d^4x}$$!

19 / 23

19

File Edit View Insert Actions Tools Help

□ The tensor part: Each h_{ij} has exactly our well-known action:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}d^4x} \quad !$$

Quantization of r and h_{ij}:

□ The equations of motion come out to be:

Scalar:

$$r_k''(\eta) + \frac{2z'(\eta)}{z(\eta)}r_k'(\eta) + k^2 r_k(\eta) = 0$$

Tensor:

19 / 23

19

File Edit View Insert Actions Tools Help

Quantization of τ and h_{ij}:

- The equations of motion come out to be:

Scalar:

$$\tau_k''(\eta) + \frac{2z'(\eta)}{z(\eta)}\tau_k'(\eta) + k^2\tau_k(\eta) = 0$$

Tensor:

$$h_{ij_k}''(\eta) + \frac{2a'(\tau)}{a(\tau)}h_{ij_k}'(\eta) + k^2 h_{ij_k}(\eta) = 0$$

- Exercise: verify

- Strategy:

20 / 23

20

File Edit View Insert Actions Tools Help

Recall: Previously in this course, this definition

$$\chi(x,\eta) := a(\eta)\,\phi(x,\eta)$$

achieved an eqn of motion without friction term:

$$\chi_k''(\eta) + \left(k^2 - \frac{a''}{a}\right)\chi_k(\eta) = 0$$

Scalar components:

Since in their action a is replaced by z, we need:

$$u(x,\eta) := -z(\eta)\,\tau(x,\eta)$$

convenient factor

This yields the eqn. of motion without friction:

21 / 23

21

File Edit View Insert Actions Tools Help

Scalar components:

Since in their action a is replaced by z, we need:

$$u(x,\eta) := -z(\eta)\,r(x,\eta)$$

convenient factor

This yields the eqn. of motion without friction:

$$u_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right) u_k(\eta) = 0$$

The tensor components:

21 / 23

21

File Edit View Insert Actions Tools Help

The tensor components:

Here, we can define as previously in the course:

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h_{ij}(x,\eta)$$

convenient factor

to obtain the eqn of motion:

$$p_{ij_k}''(\eta) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right) p_{ij_k}(\eta) = 0$$

The goal:

Quantize $\hat{u}_k(\eta)$, $\hat{p}_{ij}(\eta)$ and calculate $\delta T_k(\eta)$ and $\delta h_{ij}(\eta)$

22 / 23

22

File Edit View Insert Actions Tools Help

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h_{ij}(x,\eta)$$

convenient factor

to obtain the eqn of motion:

$$p''_{ij_k}(\eta) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right) p_{ij_k}(\eta) = 0$$

The goal:

Quantize $\hat{u}_k(\eta)$, $\hat{p}_{ij}(\eta)$ and calculate $\delta\,r_k(\eta)$ and $\delta\,h_{ij}(\eta)$

from them at horizon crossing (after which they are constant).

22 / 23

22

File Edit View Insert Actions Tools Help

to obtain the eqn of motion:

$$p''_{ij_k}(\eta) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right) p_{ij_k}(\eta) = 0$$

The goal:

Quantize $\hat{u}_k(\eta)$, $\hat{p}_{ij}(\eta)$ and calculate $\delta\tau_k(\eta)$ and $\delta h_{ij}(\eta)$

from them at horizon crossing (after which they are constant).

Ramifications: (preview)

* Fluctuations of $\hat{\tau}$ yield local spacetime

22 / 23

22

File Edit View Insert Actions Tools Help

Ramifications: (preview)

* Fluctuations of $\hat{\tau}$ yield local spacetime expansion (and thus eventually cooling) fluctuations

→ temperature spectrum in CMB

* Fluctuations of $\hat{h}$ yield grav. waves background. Should appear in polarization spectrum of CMB.

⇝ BICEP2 experiment found it!

23 / 23

23