Title: Introduction to Quantum Field Theory for Cosmology - Lecture 19

Date: Mar 18, 2014 04:00 PM

URL: http://pirsa.org/14030001

Abstract:

Application to specific cosmological models

The standard model of cosmology holds that the very early universe underwent a short period of almost exponential expansion, "inflation".

→ Begin by studying QFT in de Sitter spacetime:

The deSitter FRW spacetime can be defined through

$$\boxed{a(t) := e^{Ht} \quad \text{for all } t \in \mathbb{R}}$$

Notes:
* t is the time on a comoving observer's wrist watch
* large H $\Leftrightarrow$ large acceleration

Here: $H > 0$ is a constant, the "Hubble constant".

File Edit View Insert Actions Tools Help

The de Sitter horizon

Proposition: (in particle picture) (Note: large $H \Leftrightarrow$ small horizon d_H)

Objects (or any observers) who are further apart than a proper distance of $d_H = 2/H$ can never meet, and cannot communicate.

Proof: * Consider an observer in a galaxy A. Let us choose the origin of the comoving (and proper) coordinate system(s) to be where this observer sits.

* Now suppose that, at some arbitrary time, t_S, this observer sends a radio signal towards another galaxy, B.

2 / 26

2

File Edit View Insert Actions Tools Help

Proof : * Consider an observer in a galaxy A. Let us choose the origin of the comoving (and proper) coordinate system(s) to be where this observer sits.

* Now suppose that, at some arbitrary time, t_s, this observer sends a radio signal towards another galaxy, B.

* The signal travels in a small time Δt the small comoving distance Δx:

proper distance

our unit convention here

$$\frac{a(t)\,\Delta x}{\Delta t} = c = 1$$

speed of light

$$\Rightarrow \quad \frac{dx}{\ } = a^{-1}(t) \quad \text{i.e.:} \quad \frac{dx}{\ } = e^{-Ht}$$

2 / 26

2

File Edit View Insert Actions Tools Help

* The signal travels in a small time Δt the small comoving distance Δx:

$$\frac{\overbrace{a(t)\,\Delta x}^{\text{proper distance}}}{\Delta t} = \underbrace{c}_{\text{speed of light}} \overset{\text{our unit convention here}}{=} 1$$

$$\Rightarrow \quad \frac{dx}{dt} = a^{-1}(t) \quad \text{i.e.:} \quad \frac{dx}{dt} = e^{-Ht}$$

$$\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + C_1'$$

Fix the integration constant C_1' so that $x(t_s) = 0 \Rightarrow C_1' = \frac{1}{H} e^{-Ht_s}$

$$\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

(Recall: The proper distance traveled is: $d(t) = a(t)\,x(t)$

Clearly: $d(t) \to \infty$ as $t \to \infty$)

$-Ht_s$ (} Terminal comoving

2 / 26

2

File Edit View Insert Actions Tools Help

$$\Rightarrow \qquad \frac{dx}{dt} = a^{-1}(t) \quad \text{i.e.:} \quad \frac{dx}{dt} = e^{-Ht}$$

$$\Rightarrow \qquad x(t) = -\frac{1}{H} e^{-Ht} + C_1$$

Fix the integration constant C_1 so that $x(t_s) = 0 \Rightarrow C_1 = \frac{1}{H} e^{-Ht_s}$

(Recall: The proper distance traveled is: $d(t) = a(t)\, x(t)$
Clearly: $d(t) \to \infty$ as $t \to \infty$)

$$\Rightarrow \qquad x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

$\Rightarrow$ As $t \to \infty$ we have $x(t) \to \frac{e^{-Ht_s}}{H}$.

Terminal comoving distance traveled.

Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$.

Q: Proper distance d_p of such B from A at t_s?

A: $d_s = a(t)\, d_c \Rightarrow d_s = e^{+Ht_s} \frac{e^{-Ht_s}}{H} = \frac{1}{H}$

3 / 26

3

File Edit View Insert Actions Tools Help

Recall: This holds for arbitrary t_s.

$\Rightarrow$ A signal sent by A at any time t_s can only ever reach B if at the time of sending, t_s, the proper distance between A and B is at most $\frac{1}{H}$.

$\Rightarrow$ Any two observers further apart than a proper distance of $2/H$ can never meet or communicate!

Interpretation: In the case where a deSitter exponential expansion lasts forever, between any objects of proper distance $> 2/H$, space is being created faster than

(Remark: Notice that the

4

4 / 26

File Edit View Insert Actions Tools Help

Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obeys $\lambda \ll \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e., when $\lambda \gg \frac{1}{H}$, assuming that their mass is small: $m \ll H$.

Proof:

1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

- Recall: $\eta(t) := \int^{t} \frac{1}{a(t')} dt'$

The choices of the integration constant C merely mean different fixed shifts

$\int^{t} e^{-Ht'} dt'$

5

5 / 26

File Edit View Insert Actions Tools Help

Proof:

1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

- Recall: $\eta(t) := \int^t \frac{1}{a(t')} dt'$

here: $\eta(t) = \int^t e^{-Ht'} dt'$

$= -\frac{1}{H} e^{-Ht} + C$

The choices of the integration constant C merely mean different fixed shifts in the time coordinate η relative to the time coordinate t.

- **Notice:**

- As $t \to -\infty$ we have $\eta \to -\infty$.

- But as $t \to +\infty$ we have $\eta \to C$.

5

5 / 26

File Edit View Insert Actions Tools Help

here: $\eta(t) = \int e^{-Ht'} dt'$

$$= -\frac{1}{H} e^{-Ht} + C_1$$

different fixed shifts in the time coordinate η relative to the time coordinate t.

Notice:

- As $t \to -\infty$ we have $\eta \to -\infty$.
- But as $t \to +\infty$ we have $\eta \to C_1$.

Choose $C_1 = 0$:

$\Rightarrow$

$$\eta(t) = -\frac{1}{H}\frac{1}{a(t)}$$

5 / 26

5

File Edit View Insert Actions Tools Help

- As $t \to -\infty$ we have $\eta \to -\infty$.
- But as $t \to +\infty$ we have $\eta \to C'$.
- Choose $C' = 0$:

$\Rightarrow$

$$\eta(t) = -\frac{1}{H}\frac{1}{a(t)}$$

$$a(t) = -\frac{1}{H\eta(t)}$$

i.e.:

$$\boxed{a(\eta) = -\frac{1}{H\eta}}$$

6 / 26

6

File Edit View Insert Actions Tools Help

2) Introduce $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$:

- We have: $\hat{\chi}_k(\eta) = -\frac{1}{H\eta}\,\hat{\phi}_k(\eta)$

- $\hat{\chi}_k$ obeys this Klein Gordon equation

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\,\hat{\chi}_k(\eta) = 0$$

with:

$$\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

- Exercise: Show that in the de Sitter case this yields:

7 / 26

7

- We have: $\hat{\chi}_k(\eta) = -\frac{[illegible]}{H\eta}\,\varphi_k(\eta)$

- $\hat{\chi}_k$ obeys this Klein Gordon equation

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\,\hat{\chi}_k(\eta) = 0$$

with:

$$\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

- Exercise: Show that in the de Sitter case this yields:

$$\boxed{\omega_k^2(\eta) = k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}$$

File Edit View Insert Actions Tools Help

$$\boxed{\omega_k^2(\eta) = k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}$$

3.) Check for imaginary frequencies.

- Recall:

  I.e. Compton wavelength $1/m \gg$ Hubble horizon $1/H$

  We are assuming $m \ll H$.

  (This will be the case in the analogous calculation for realistic inflation)

- Thus: $\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} < 0$

7 / 26

7

File Edit View Insert Actions Tools Help

3.) Check for imaginary frequencies.

- Recall:

  I.e. Compton wavelength $1/m \gg$ Hubble horizon $1/H$

  We are assuming $m \ll H$.

  (This will be the case in the analogous calculation for realistic inflation)

- Thus: $\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} < 0$

- Therefore: For each mode k there comes a time when ω_k^2 becomes negative!

- The time when a mode k crosses the horizon is given b

8/26

8

File Edit View Insert Actions Tools Help

Recall:

$1/m \gg$ Hubble horizon $1/H$

We are assuming $m \ll H$.

(This will be the case in the analogous calculation for realistic inflation)

Thus: $\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} < 0$

Therefore: For each mode k there comes a time when ω_k^2 becomes negative!

The time when a mode k crosses the horizon is given by:

$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}$ (here and henceforth neglecting the mass term)

8 / 26

8

File Edit View Insert Actions Tools Help

4) Conclusion:

- A mode oscillates as long as:

Recall: $\eta \in (-\infty, 0)$ i.e. $|\eta| \gg 1/k$ means early times.

$$|\eta| \gg \frac{1}{k} \quad \text{i.e., while } |\eta| k \gg 1 \qquad (a)$$

(Used that $\sqrt{2}$ and 1 are of same order of magnitude)

- A mode has imaginary frequency from when

This is late times, i.e. when $\eta \approx 0$.

$$|\eta| \ll \frac{1}{k} \quad \text{i.e., from when } |\eta| k \ll 1 \qquad (b)$$

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

9 / 26

9

File Edit View Insert Actions Tools Help

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

comoving wavelength

$$|\eta| k = \frac{1}{Ha} \frac{2\pi}{L}$$

Transforming to the proper wavelength, $\lambda = a(\eta) L$, we obtain:

$$|\eta| k = \frac{2\pi}{H\lambda}$$

(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys:
$$\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$$
)

Thus, finally, the two cases, (a) and (b) become:

9 / 26

9

i.e. $|\eta| \gg 1/k$ means early times.

$$|\eta| \gg \frac{1}{k} \quad \text{i.e., while } |\eta| k \gg 1 \qquad \text{ⓐ}$$

(Used that $\sqrt{2}$ and 1 are of same order of magnitude)

A mode has imaginary frequency from when

This is late times, i.e. when $\eta \approx 0$.

$$|\eta| \ll \frac{1}{k} \quad \text{i.e., from when } |\eta| k \ll 1 \qquad \text{ⓑ}$$

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

comoving wavelength

$$|\eta| k = \frac{1}{Ha} \frac{2\pi}{L}$$

File Edit View Insert Actions Tools Help

└ comoving wavelength

$$|\eta| k = \frac{1}{Ha} \frac{2\pi}{L}$$

Transforming to the proper wavelength, $\lambda = a(\eta) L$, we obtain:

$$|\eta| k = \frac{2\pi}{H \lambda}$$

(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys: $\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$)

Thus, finally, the two cases, a and b become:

a A mode oscillates as long as $|\eta| k \gg 1$

i.e., as long as $\frac{2\pi}{\dots} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{\dots}}$

9

9 / 26

File Edit View Insert Actions Tools Help

Transforming to the proper wavelength, $\lambda = a(\eta)L$, we obtain:

$$|\eta| k = \frac{2\pi}{H\lambda}$$

(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys:
$$\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$$
)

Thus, finally, the two cases, ⓐ and ⓑ become:

- A mode oscillates as long as: $|\eta| k \gg 1$

  i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{H}}$ ⓐ'

- A mode has imaginary frequency from when:

10 / 26

10

File Edit View Insert Actions Tools Help

- A mode oscillates as long as: $|\eta| k \gg 1$

  i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{H}}$ (a')

- A mode has imaginary frequency from when:

  $|\eta| k \ll 1$, i.e., from when $\boxed{\lambda \gg \frac{1}{H}}$ (b')

This is what we had set out to show.

The more realistic case of a de Sitter expansion of a finite duration

10 / 26

10

File Edit View Insert Actions Tools Help

The more realistic case of a de Sitter expansion of a finite duration

- Consider the case that spacetime was exponentially expanding only in a finite time interval:

$$\eta_i < \eta < \eta_f$$

and assume that spacetime was expanding slowly or was even Minkowski before η_i, and after η_f.

- **Recall:** The time when a mode, k, crosses the horizon is:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k} \quad \text{i.e.} \quad \eta_{hor}(L) \approx -L\frac{1}{\sqrt{2}\pi}$$

11 / 26

11

File Edit View Insert Actions Tools Help

- Consider the case that spacetime was exponentially expanding only in a finite time interval:

$$\eta_i < \eta < \eta_f$$

and assume that spacetime was expanding slowly or was even Minkowski before η_i, and after η_f.

- **Recall:** The time when a mode, k, crosses the horizon is:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k} \quad \text{i.e.} \quad \eta_{hor}(L) \approx -L\frac{1}{\sqrt{2}\pi}$$

Thus, modes k with $\eta_{hor}(k) \notin [\eta_i, \eta_f]$ never cross

11 / 26

11

File Edit View Insert Actions Tools Help

⇒ Three classes of modes:

1. "Small" modes:

These are modes which are so small that by the time their proper wavelength would reach the Hubble horizon length the de Sitter period is already over:

$$\eta_{hor}(k) \gg \eta_f$$

Recall: Both sides are negative

i.e.:

$$\frac{\sqrt{2}}{k} \ll |\eta_f|$$

Recall: $\eta_{hor} \approx \frac{\sqrt{2}}{k}$

$$L \ll |\eta_f|$$

12 / 26

12

File Edit View Insert Actions Tools Help

i.e.: $\frac{\sqrt{I}}{k} \ll |\eta_f|$ Recall: $\eta_{hor} \approx \frac{\sqrt{I}}{k}$

$$L \ll |\eta_f|$$

Their quantum fluctuations do not get "amplified", as we will see.

2. "Medium" size modes:

These are the modes which do cross the horizon because

$$\eta_i < \eta_{hor}(k) < \eta_f$$

12 / 26

12

File Edit View Insert Actions Tools Help

2. "Medium" size modes:

These are the modes which do cross the horizon because

$$\eta_i < \eta_{hor}(k) < \eta_f$$

The quantum fluctuations of those modes are of cosmological interest.

3. "Large" modes:

These are modes which were larger than the horizon already at η_i. In the inflationary scenario they are today

13 / 26

13

File Edit View Insert Actions Tools Help

3. "Large" modes:

These are modes which were larger than the horizon already at η_i. In the inflationary scenario they are today very much larger than the visible universe. They may only contribute, effectively, like a cosmological constant - and may even be the origin of Λ.

Quantum fluctuations in de Sitter space.

- The usual ansatz

13 / 26

13

File Edit View Insert Actions Tools Help

Quantum fluctuations in de Sitter space.

- The usual ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)$$

succeeds, as always, for any function v_k which obeys:

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0 \qquad (a)$$

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k'^*(\eta) = 2i \qquad (b)$$

14 / 26

14

File Edit View Insert Actions Tools Help

succeeds, as always, for any function v_k which obeys:

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0 \qquad (a)$$

$$v_k'(\eta)\, v_k'^{*}(\eta) - v_k(\eta)\, v_k'^{*}(\eta) = 2i \qquad (b)$$

- The solution space of (a) can be shown to be spanned, for example, by these two real-valued Bessel functions

$$u_k(\eta) := \sqrt{k|\eta|}\; J_n(k|\eta|)$$

14 / 26

14

File Edit View Insert Actions Tools Help

The solution space of (a) can be shown to be spanned, for example, by these two real-valued Bessel functions

$$u_k(\eta) := \sqrt{k|\eta|}\; J_n(k|\eta|)$$

(not complex conjugation, just another symbol) → $$\bar{u}_k(\eta) := \sqrt{k|\eta|}\; Y_n(k|\eta|)$$

← generalizations of sine and cosine

where:

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{3}{2}$$

(by our assumption of very small mass)

14 / 26

14

File Edit View Insert Actions Tools Help

$$u_k(\eta) := \sqrt{k|\eta|}\, J_n(k|\eta|)$$

(not complex conjugation, just another symbol) → $$\bar{u}_k(\eta) := \sqrt{k|\eta|}\, Y_n(k|\eta|)$$

← generalizations of sine and cosine

where:

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{3}{2}$$ (by our assumption of very small mass)

Thus: every mode function v_k is a linear combination

$$v_k(\eta) = A_k\, u_k(\eta) + B_k\, \bar{u}_k(\eta) \qquad (*)$$

with complex coefficients A_k, B_k.

15 / 26

15

File Edit View Insert Actions Tools Help

Thus: every mode function v_k is a linear combination

$$v_k(\eta) = A_k u_k(\eta) + B_k \bar{u}_k(\eta) \qquad (*)$$

with complex coefficients A_k, B_k.

How to identify the state of the system?

Strategy:

a. Check if modes start out in an adiabatic

15 / 26

15

File Edit View Insert Actions Tools Help

How to identify the state of the system?

- Strategy:

a. Check if modes start out in an adiabatic regime (the small and medium ones do).

b. Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then.

c. Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

d. Calculate $\delta\phi$ at the end of the exponential

16 / 26

16

File Edit View Insert Actions Tools Help

b. Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then.

c. Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

d. Calculate $\delta\phi_k$ at the end of the exponential expansion, η_f, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3\, |v_k(\eta_f)|^2$$

Important: We know that v_k is a linear

16 / 26

16

File Edit View Insert Actions Tools Help

a. Check if modes start out in an adiabatic regime.

Indeed, we see from the K.G. eqn.

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

that at very early times, $\eta \ll 0$, we have roughly Minkowski:

$$v_k''(\eta) + k^2 v_k(\eta) = 0$$

(except if k is very small, i.e., for very large modes.)

b. Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then - namely the Minkowski vacuum.

17 / 26

17

File Edit View Insert Actions Tools Help

b. Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then – namely the Minkowski vacuum.

Note: we could also use the adiabatic vacuum criterion, with little difference.

c. Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Thus, v_k is the usual Minkowski mode function at early times:

$$v_k = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta + i\alpha} \quad \text{for} \quad \eta \ll 0$$

18 / 26

18

File Edit View Insert Actions Tools Help

Note: we could also use the adiabatic vacuum criterion, with little difference.

c. Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Thus, v_k is the usual Minkowski mode function at early times:

$$v_k = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta + i\alpha} \quad \text{for} \quad \eta \ll 0$$

(we are neglecting the mass term for simplicity, and because it is realistic) → i.e.

$$v_k = \frac{1}{\sqrt{k}} e^{ik\eta + i\alpha} \quad \text{for} \quad \eta \ll 0$$

18 / 26

18

File Edit View Insert Actions Tools Help

Technical observation: At early times, $\eta \ll 0$:

$$u_k(\eta) \approx \sqrt{\frac{2}{\pi}} \cos(k|\eta| + \text{const})$$

same constant

$$\bar{u}_k(\eta) \approx \sqrt{\frac{2}{\pi}} \sin(k|\eta| + \text{const})$$

$\Rightarrow$ **Proposition:**

In terms of u_k, $\bar{u}_k$ the mode function v_k reads:

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}}_{=A_k} u_k(\eta) - i \underbrace{\sqrt{\frac{\pi}{2k}}}_{=B_k} \bar{u}_k(\eta)$$

i.e.:

$$\boxed{v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(J_n(k|\eta|) - i\, Y_n(k|\eta|) \right)}$$

File Edit View Insert Actions Tools Help

same constant

$$\bar{u}_k(\eta) \simeq \sqrt{\frac{2}{\pi}} \sin(k|\eta| + \text{const})$$

⇒ **Proposition:**

In terms of u_k, $\bar{u}_k$ the mode function v_k reads:

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}}_{=A_k} u_k(\eta) - i \underbrace{\sqrt{\frac{\pi}{2k}}}_{=B_k} \bar{u}_k(\eta)$$

i.e.: $$\boxed{v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(J_m(k|\eta|) - i\, Y_m(k|\eta|) \right)}$$

Proof: Exercise.

19 / 26

19

File Edit View Insert Actions Tools Help

d. Now we can calculate $\delta\phi_k$ at the end of the exponential expansion, η_f, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3 \, |v_k(\eta_f)|^2$$

Case 1: Very small modes

They are those with k large enough, so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

20 / 26

20

File Edit View Insert Actions Tools Help

I.e., the Bessel functions in the mode function stay sine and cosine in good approximation for all times η up to η_f.

These modes never cross the horizon and we have, approximately:

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta} \quad \text{for all } \eta$$

Thus:

The vacuum fluctuations at the end of the exponential expansion are still as in Minkowski case:

Recall:

$$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$$

$$\delta\phi_\lambda(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} \frac{1}{\sqrt{k}} \Big|_{k \approx L^{-1}}$$

21 / 26

21

File Edit View Insert Actions Tools Help

I.e., the Bessel functions in the mode function stay sine and cosine in good approximation for all times η up to η_f.

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta} \quad \text{for all } \eta$$

Thus:

The vacuum fluctuations at the end of the exponential expansion are still as in Minkowski case:

Recall:

$$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$$

$$\delta\phi_\lambda(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} \frac{1}{\sqrt{k}}\Big|_{k \approx L^{-1}}$$

$$= \frac{1}{a(\eta_f) L}$$

$$= 1/\lambda(\eta_f)$$

proper wavelength at time η_f. (neglecting factors of 2π)

21 / 26

21

File Edit View Insert Actions Tools Help

at the time η_f, i.e., when the exponential expansion ends:

Then, the K.G. eqn. is to a good approximation:

Recall:

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

$$v_k''(\eta) - \frac{2}{\eta^2} v_k(\eta) = 0$$

and a basis of solutions is easy to find, e.g.:

$$w_k^{(1)}(\eta) = (k|\eta|)^2$$ decaying for $\eta \to 0$

$$w_k^{(2)}(\eta) = \frac{1}{k|\eta|}$$ growing for $\eta \to 0$

23 / 26

23

File Edit View Insert Actions Tools Help

Recall:

$$u_k(\eta) := \sqrt{k|\eta|}\, J_n(k|\eta|)$$

$$\bar{u}_k(\eta) := \sqrt{k|\eta|}\, Y_n(k|\eta|)$$

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \simeq \frac{3}{2}$$

Indeed: use this property of the Bessel functions:

$$\left.\begin{aligned} u_k(\eta) &\to \frac{2^{-n}}{\Gamma(n+1)}\,(k|\eta|)^{n+\frac{1}{2}} \to 0 \\ \bar{u}_k(\eta) &\to \frac{-\Gamma(n)}{\pi}\,2^n\,(k|\eta|)^{\frac{1}{2}-n} \to \infty \end{aligned}\right\} \text{as } \eta \to 0 \quad (\text{i.e. as } t \to \infty)$$

Recall:

$$V_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}\left(J_n(k|\eta|) - i\,Y_n(k|\eta|)\right)$$

Therefore, for late η:

$$V_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}\,\frac{\Gamma(n)}{\pi}\,2^n}_{A_k}(k|\eta|)^{\frac{1}{2}-n} + \text{negligible}$$

Recall:

$$\underbrace{}_{a^{-1}(\eta_1)}$$

24 / 26

24

File Edit View Insert Actions Tools Help

Recall:

$u_k(\eta) := \sqrt{k|\eta|}\, J_n(k|\eta|)$

$\bar{u}_k(\eta) := \sqrt{k|\eta|}\, Y_n(k|\eta|)$

$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \simeq \frac{3}{2}$

$$\left.\begin{aligned} u_k(\eta) &\to \frac{2^{-n}}{\Gamma(n+1)} (k|\eta|)^{n+\frac{1}{2}} \to 0 \\ \bar{u}_k(\eta) &\to \frac{-\Gamma(n)}{\pi} 2^n (k|\eta|)^{\frac{1}{2}-n} \to \infty \end{aligned}\right\} \text{as } \eta \to 0 \ (\text{i.e. as } t \to \infty)$$

Recall:

$V_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(J_n(k|\eta|) - i Y_n(k|\eta|) \right)$

Therefore, for late η:

$$V_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}} \frac{\Gamma(n)}{\pi} 2^n}_{A_k} (k|\eta|)^{\frac{1}{2}-n} + \text{negligible}$$

Recall:

$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$

$$\delta\phi_k(\eta_f) \approx \underbrace{H \eta_f}_{a^{-1}(\eta_f)} k^{3/2} \sqrt{\frac{\pi}{2k}} \frac{\Gamma(n)}{\pi} 2^n (k|\eta_f|)^{\frac{1}{2}-n} \Big/ h$$

24 / 26

24

File Edit View Insert Actions Tools Help

Recall:

$$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$$

$$\delta\phi_L(\eta_f) \approx \overbrace{H\eta_f}^{a^{-1}(\eta_f)} k^{3/2} \sqrt{\frac{\pi}{2k}}\, \frac{\Gamma(n)}{\pi}\, 2^n (k|\eta_f|)^{\frac{1}{2}-n} \Big|_{k=L^{-1}}$$

$$\Rightarrow \delta\phi_L(\eta_f) \approx H\left(\frac{|\eta|}{L}\right)^{\overbrace{\frac{3}{2}-n}^{\approx 0}} \cdot \Gamma(n)\, \frac{2^n}{\pi}$$

independent of η_f! $\Rightarrow$ May as well evaluate right after horizon crossing.

For comparison, recall case 1, small modes, whose fluctuation amplitudes are as on Minkowski space:

$$\delta\phi_\lambda \approx 1/\lambda$$

$$\boxed{\delta\phi_L(\eta_f) \approx \underbrace{H\cdot 2^{3/2}\Gamma(3/2)/\pi}_{}} \quad \text{for } n = 3/2$$

independent of $L \Rightarrow$ indep. also of λ!

$\Rightarrow$ The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is

24 / 26

24

File Edit View Insert Actions Tools Help

$$\Rightarrow \delta\phi_L(\eta_f) \approx H\left(\frac{|\eta|}{L}\right)^{\overbrace{\frac{3}{2}-n}} \cdot \Gamma(n)\,\frac{2^n}{\pi}$$

independent of η_f! $\Rightarrow$ May as well evaluate right after horizon crossing.

For comparison, recall case 1, small modes, whose fluctuation amplitudes are as on Minkowski space:

$\delta\phi_\lambda \approx 1/\lambda$

$$\boxed{\delta\phi_L(\eta_f) \approx H\cdot 2^{3/2}\Gamma(3/2)/\pi} \quad \text{for } n = 3/2$$

independent of $L \Rightarrow$ indep. also of λ!

$\Rightarrow$ The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is compensated.

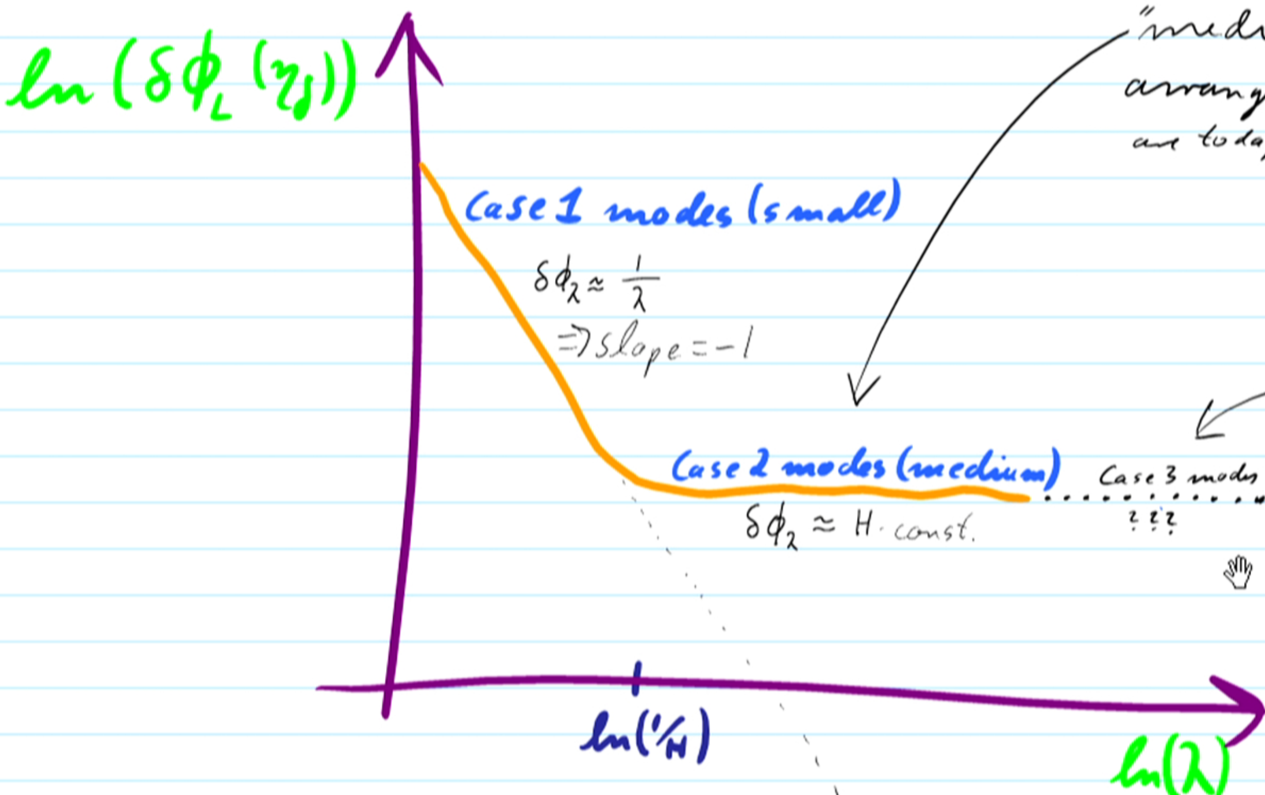
$\Longrightarrow$ The quantum fluctuations of a comoving mode when its proper wavelength λ is getting larger than the Hubble length, i.e., when $\lambda > \lambda_{Hubble} = 1/H$, remain ...

25 / 26

25

⇒ After exponential expansion:

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are today at cosmological scales.



unknown significance (depends on assumption about their initial conditions before the expansion, at η_i: there was no vacuum state for them!)