

Title: Spacetime approach to force-free magnetospheres - Lecture 3

Date: Feb 26, 2014 09:30 AM

URL: <http://pirsa.org/14020157>

Abstract:

A 1-form

$F = dA$ 2-form field strength

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A 1-form

$F = dA$ 2-form field strength

$dF = 0$ covariant Faraday law

$d * F$

A 1-form

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$d * F = J$ ← current 3-form

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$F = dA$ 2-form field strength

$dF = 0$ covariant Faraday

$d * F = J$ ← current 3-form

$$F_{ab}F^{ab} = 2(B^2 - E^2)$$

$$4 \vec{E} \cdot \vec{B}$$

$$\epsilon^{abcd} \underbrace{F_{ab}F_{cd}}$$

$$F \wedge F$$

A 1-form

$F = dA$ 2-form field strength

$dF = 0$ covariant Faraday law

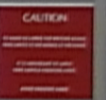
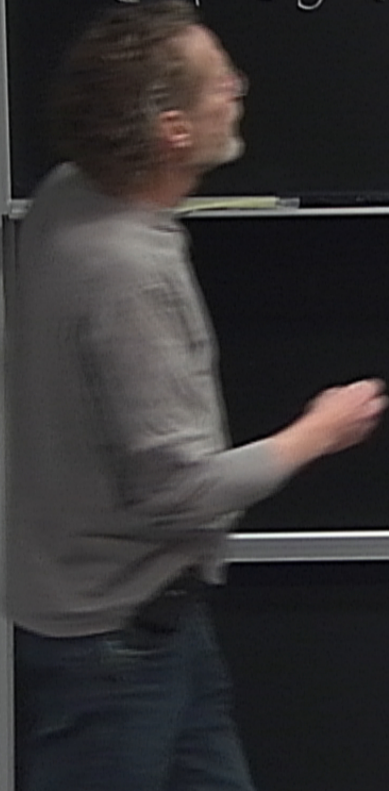
$d * F = J$ ← current 3-form

$$F_{ab} * F^{ab} = 4 \vec{E} \cdot \vec{B}$$

$$\frac{1}{2} \epsilon^{abcd} \underbrace{F_{ab} F_{cd}}_{F \wedge F}$$

$$F \wedge F = 0 \leftrightarrow \vec{E} \cdot \vec{B} = 0$$

"degenerate field"



A 1-form

$$F_{ab}F^{ab} = 2(\vec{B}^2 - E^2)$$

$F = dA$ 2-form field strength

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$dF = 0$ covariant Faraday law

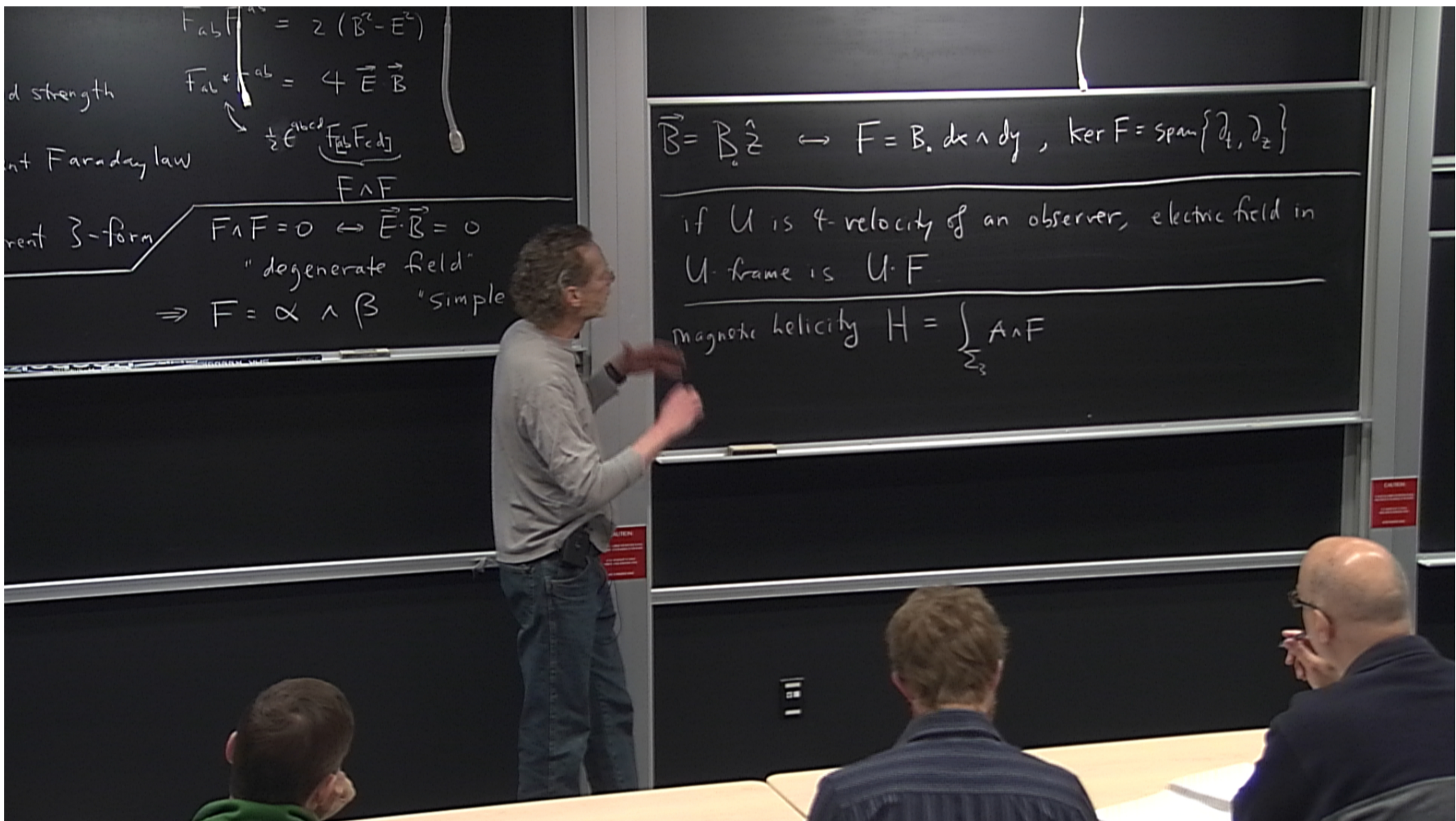
$$\frac{1}{2} \epsilon^{abcd} F_{ab} F_{cd}$$

$d * F = J$ ← current 3-form

$F \wedge F$
 $\vec{E} \cdot \vec{B} = 0$
"generate field"

$\alpha \wedge \beta$ "simple"

$$\vec{B} = B_0 \hat{z} \leftrightarrow F = B_0 dx \wedge dy, \ker F = \text{span}\{\partial_t, \partial_z\}$$



$F_{ab}F^{ab} = 2(\vec{B}^2 - E^2)$
 field strength
 $F_{ab}F^{ab} = 4 \vec{E} \cdot \vec{B}$
 + Faraday law
 $\frac{1}{2} \epsilon^{abcd} F_{ab} F_{cd}$
 $F \wedge F$
 invariant 3-form
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$\vec{B} = B_z \hat{z} \leftrightarrow F = B_z dx \wedge dy, \ker F = \text{span}\{\partial_t, \partial_z\}$

 if U is 4-velocity of an observer, electric field in U -frame is $U \cdot F$

 magnetic helicity $H = \int_{\Sigma_3} A \wedge F$

$F_{ab}F^{ab} = 2(\vec{B}^2 - E^2)$
 d strength $F_{ab}F^{ab} = 4 \vec{E} \cdot \vec{B}$
 + Faraday law $\frac{1}{2} \epsilon^{abcd} [F_{ab}F_{cd}]$
 $F \wedge F$
 rent 3-form $F \wedge F = 0 \leftrightarrow \vec{E} \cdot \vec{B} = 0$
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$$t_{ab} = 2(B^2 - E^2)$$

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law $\frac{1}{2} \epsilon^{abcd} F_{ab} F_{cd}$

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"degenerate field"

$$\Rightarrow F = \alpha \wedge \beta \text{ "simple"}$$

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if U is 4-velocity of an observer, electric field in U -frame is $U \cdot F$

magnetic helicity $H = \int \underbrace{A \wedge F}_{\text{helicity current}} \quad d(A \wedge F) = F \wedge F = 0$

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$$d(A \wedge F) = F \wedge F = 0$$

For static B-field in Minkowski space, minimizing energy at fixed helicity $\Rightarrow \vec{\nabla} \times \vec{B}$



CAUTION

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$$F_{ab}F^{ab} = 2(\vec{B}^2 - \vec{E}^2)$$

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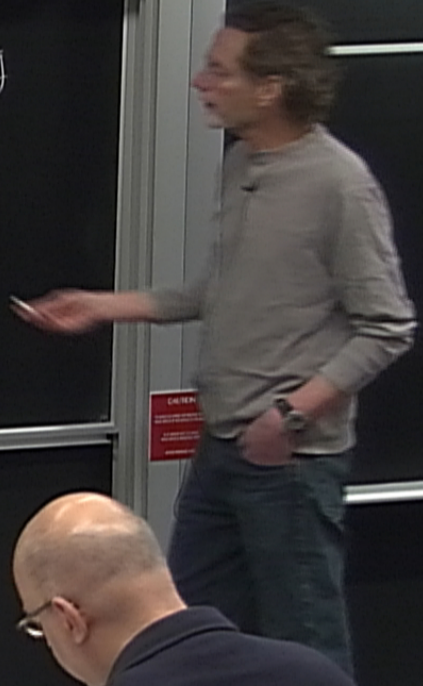
Magnetic helicity $H = \int_{\Sigma_3} A \wedge F$

helicity current

$$d(A \wedge F) = F \wedge F = 0$$

For static B-field in Minkowski space, minimizing energy at fixed helicity $\Rightarrow \vec{\nabla} \times \vec{B} = \text{const} \cdot \vec{B}$

$$\vec{j} = \text{const} \cdot \vec{B}$$

$$\vec{E} = \vec{j} \times \vec{B}$$


For static B-field in Minkowski space, minimizing energy at fixed helicity \Rightarrow

$$\vec{\nabla} \times \vec{B} = \text{const} \cdot \vec{B}$$

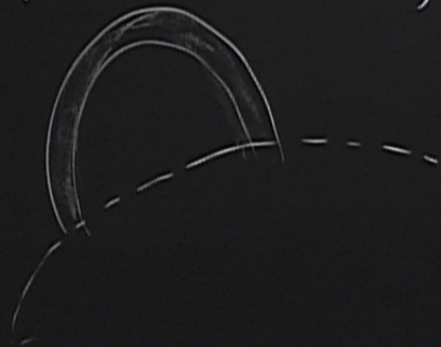
$$\vec{j} = \text{const} \cdot \vec{B}$$

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CAUTION

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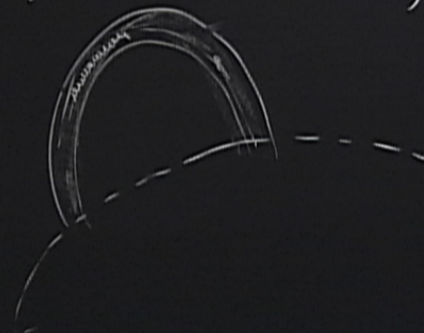
$$\vec{j} = \text{const} \cdot \vec{B}$$
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CAUTION

CAUTION

0
d"
imple"

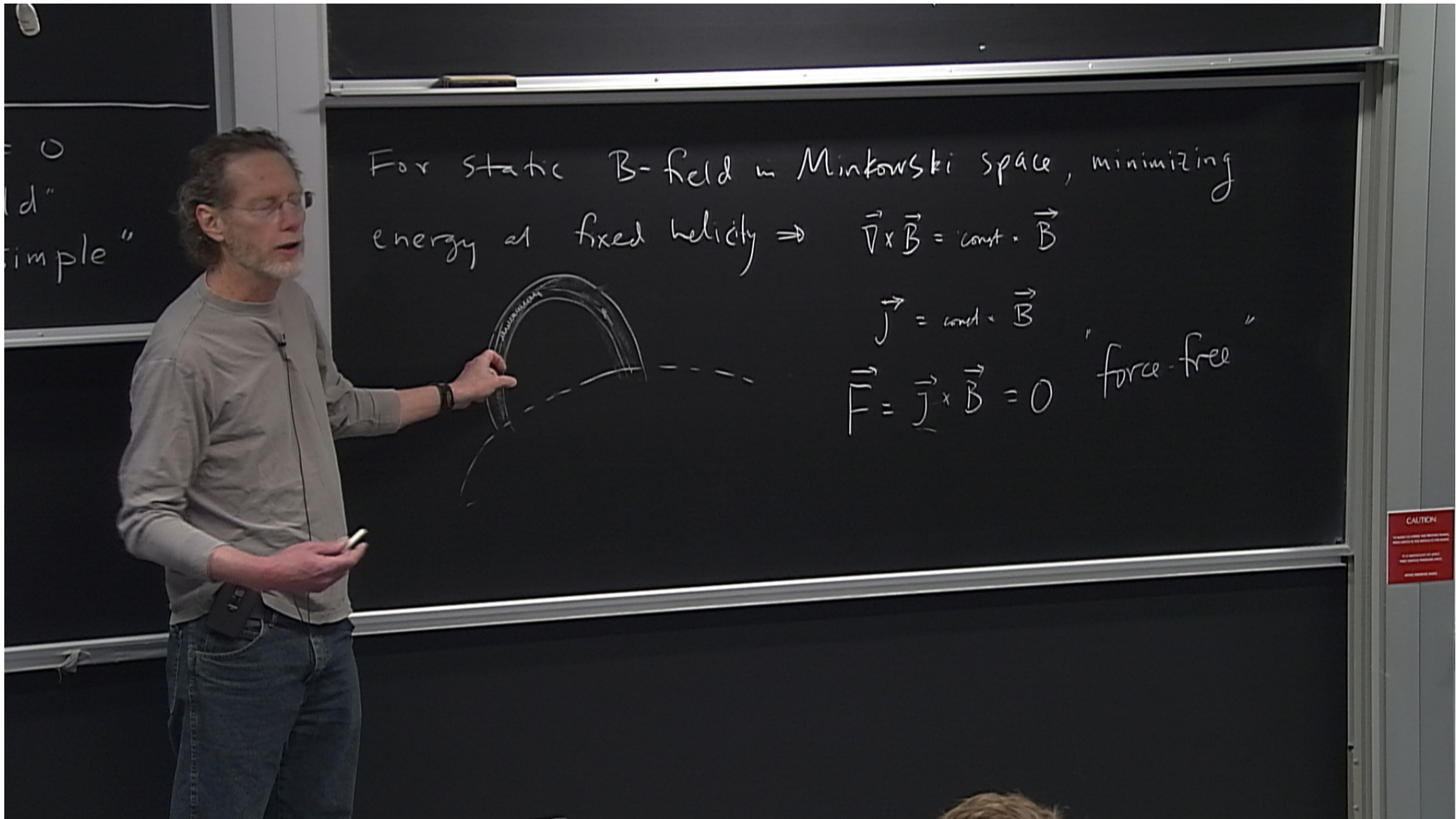
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CAUTION
DO NOT TOUCH THE SURFACE OF THE BOARD
OR THE BOARD ITSELF
WHEN THE BOARD IS HOT

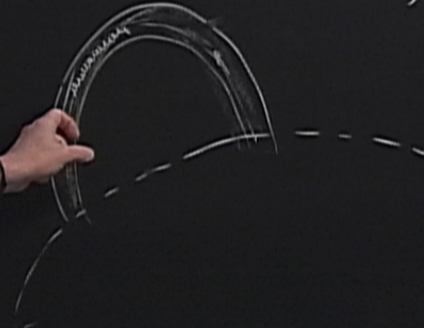
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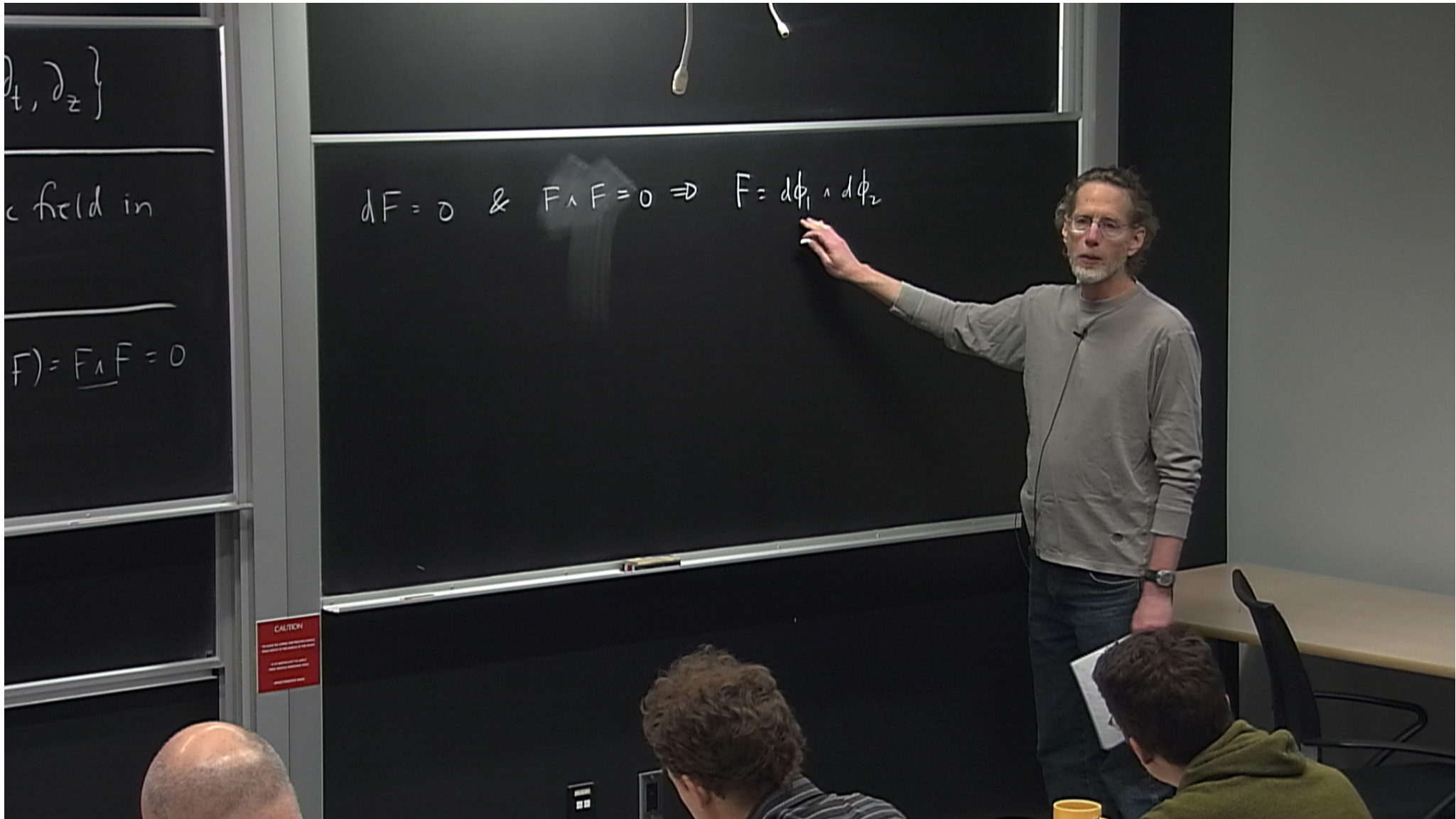


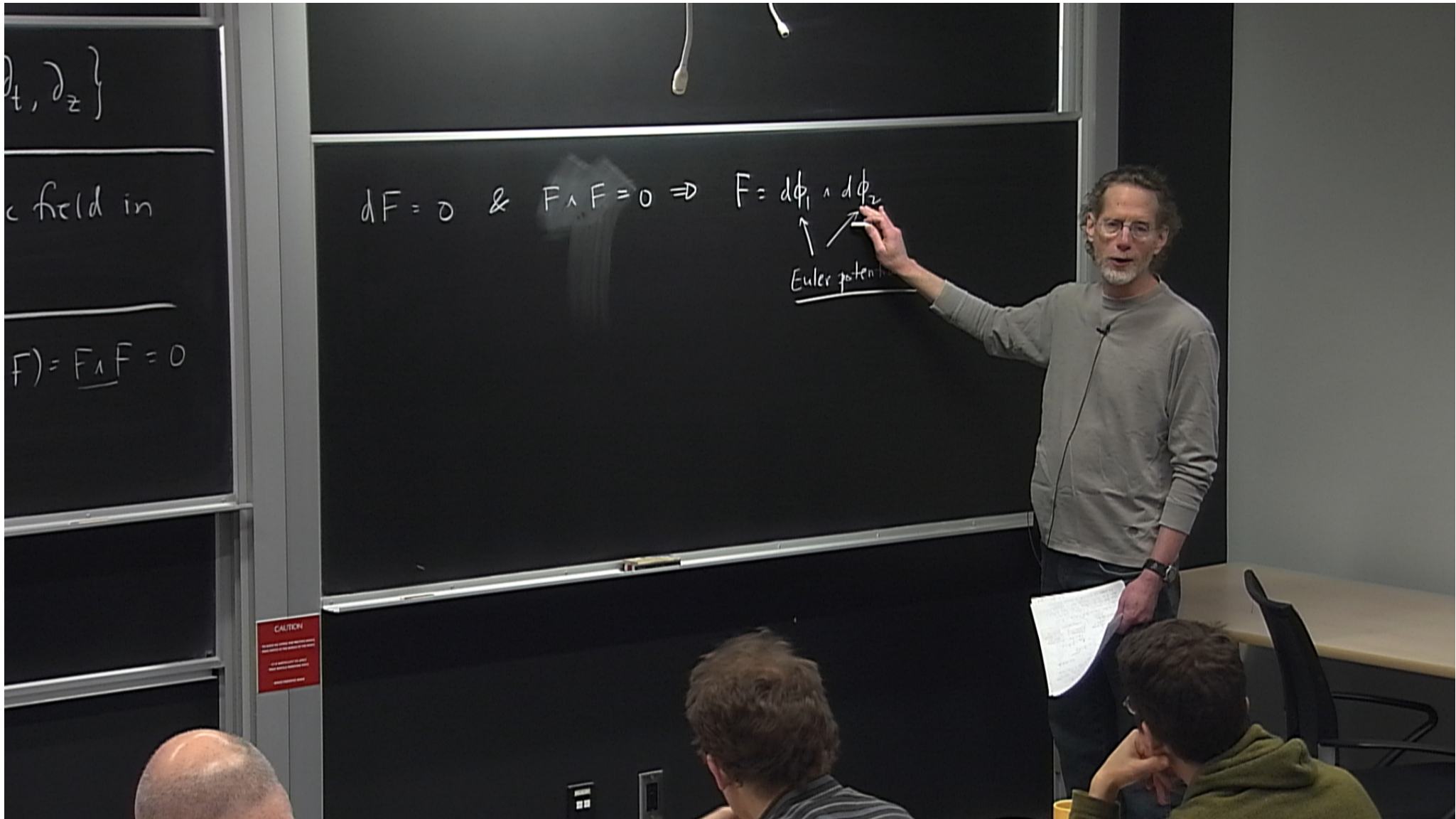
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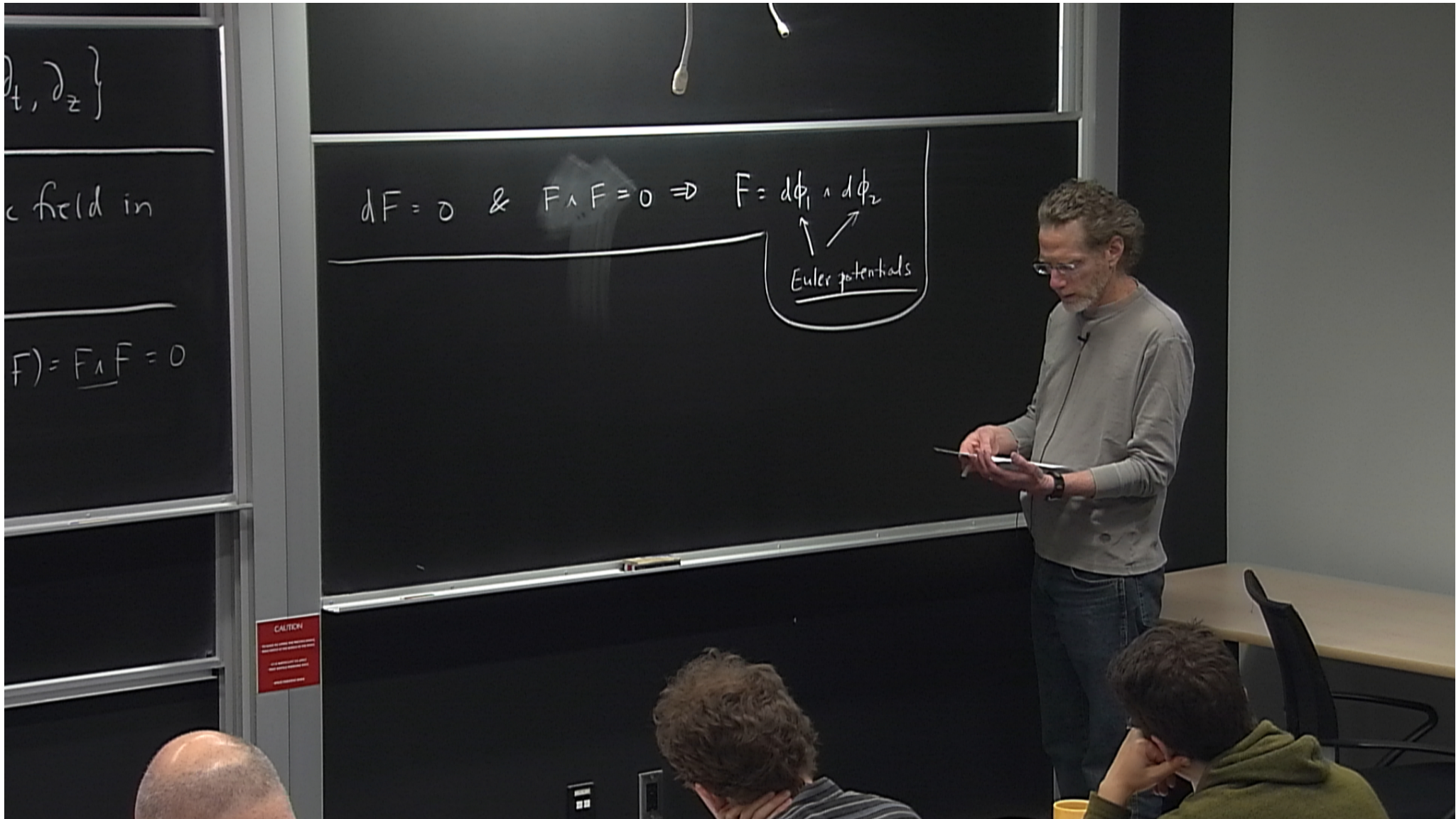
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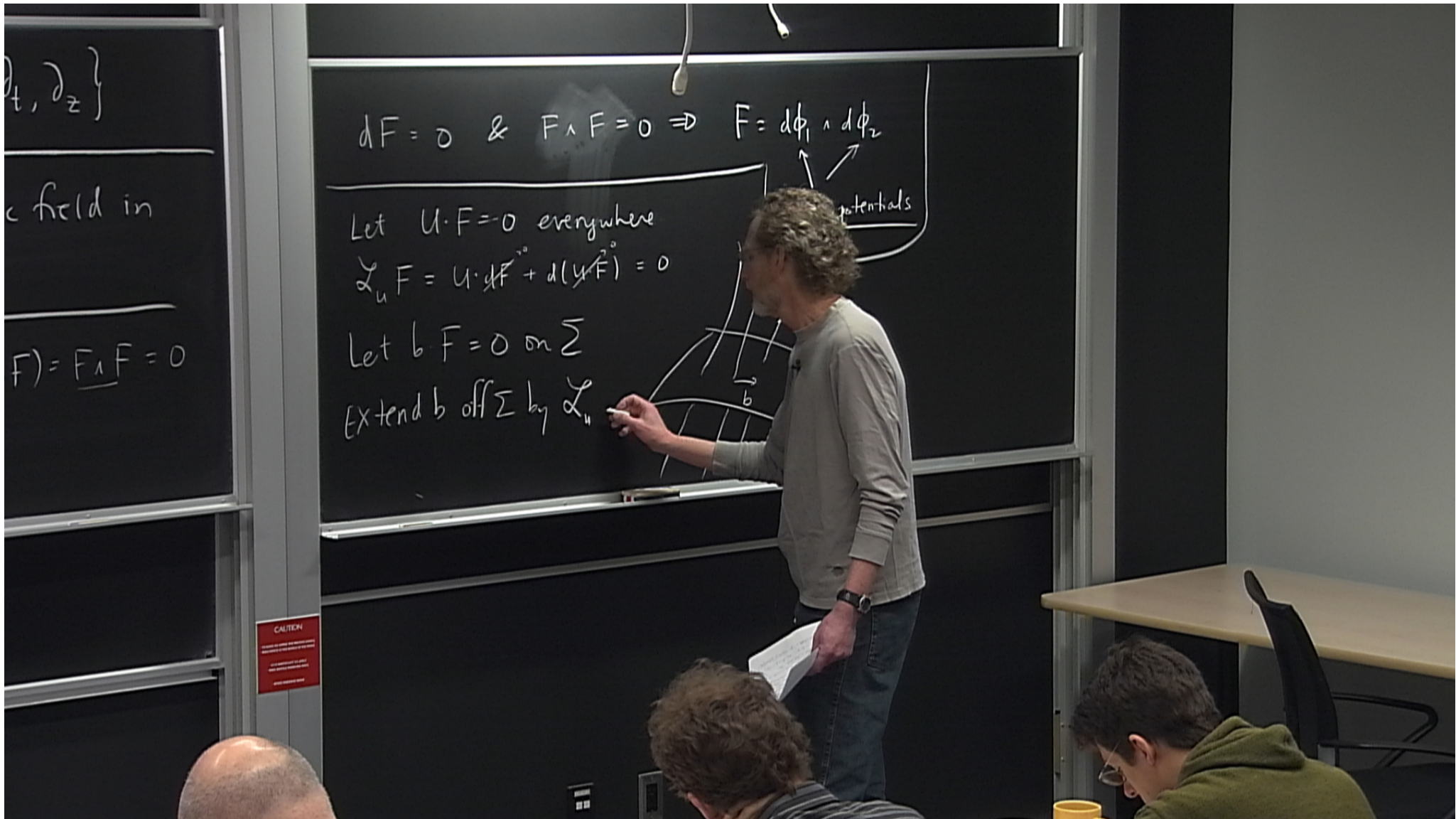
$$\vec{F} = \vec{j} \times \vec{B} = 0 \quad \text{"force-free"}$$











∂_t, ∂_z

c field in

$F) = \underline{F} \wedge F = 0$

$dF = 0 \ \& \ F \wedge F = 0 \Rightarrow F = d\phi_1 \wedge d\phi_2$

potentials

Let $U \cdot F = 0$ everywhere

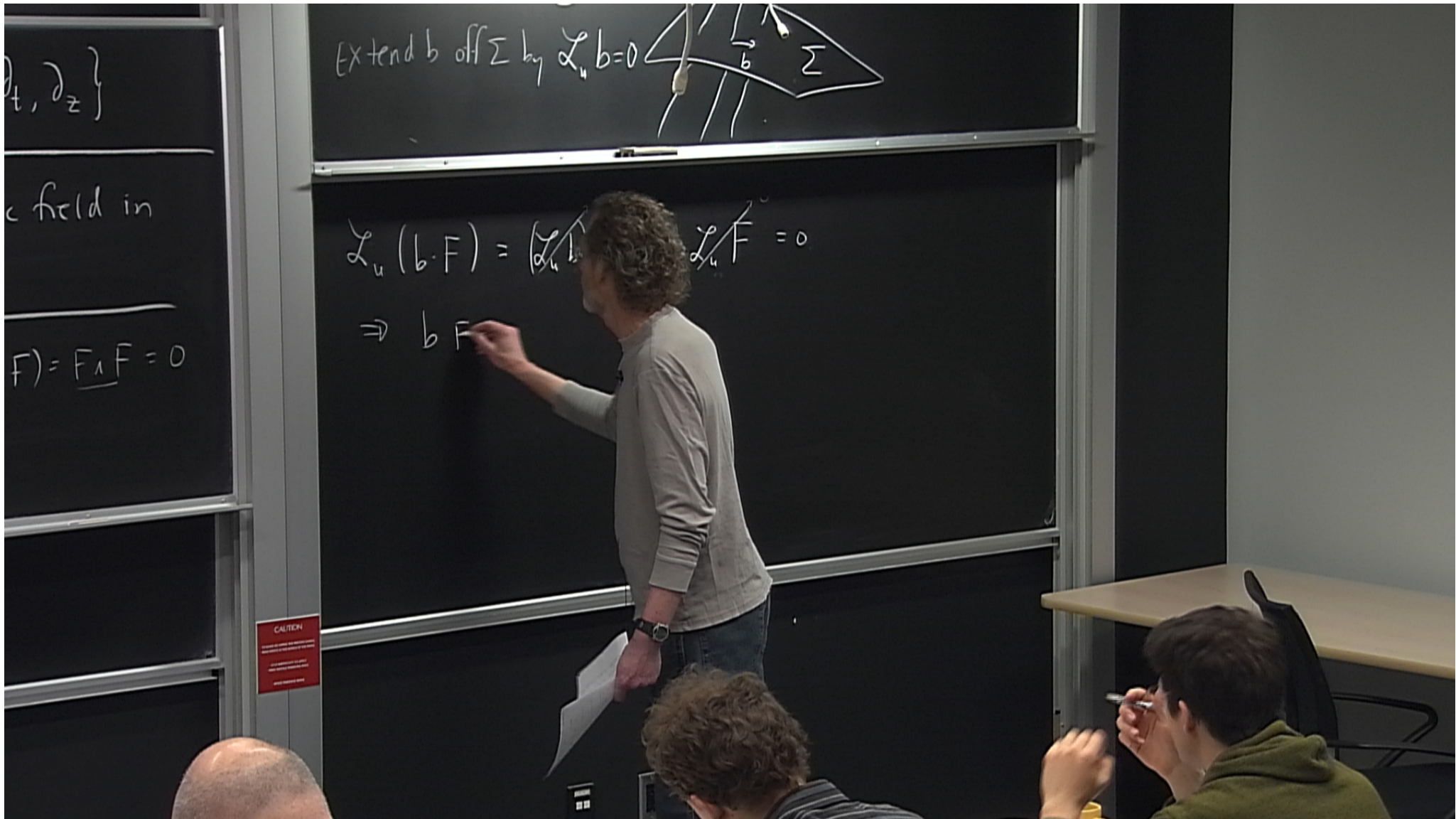
$\mathcal{L}_U F = U \cdot dF + d(U \cdot F) = 0$

Let $b \cdot F = 0$ on Σ

Extend b off Σ by \mathcal{L}_U



CAUTION



∂_t, ∂_z

c field in

$$F) = \underline{F}_1 F = 0$$

Extend b off Σ by $\mathcal{L}_u b = 0$



$$\mathcal{L}_u (b \cdot F) = (\mathcal{L}_u b) \cdot F + b \cdot \mathcal{L}_u F = 0$$

$$\Rightarrow b \cdot F = 0 \text{ everywhere}$$

$$\mathcal{L}_u b = 0 \Rightarrow [u, b] = 0 \Rightarrow$$

CAUTION

∂_t, ∂_z

c field in

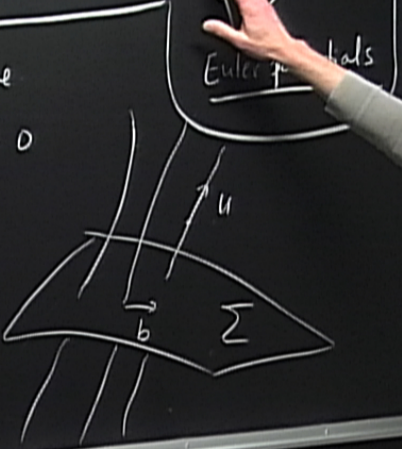
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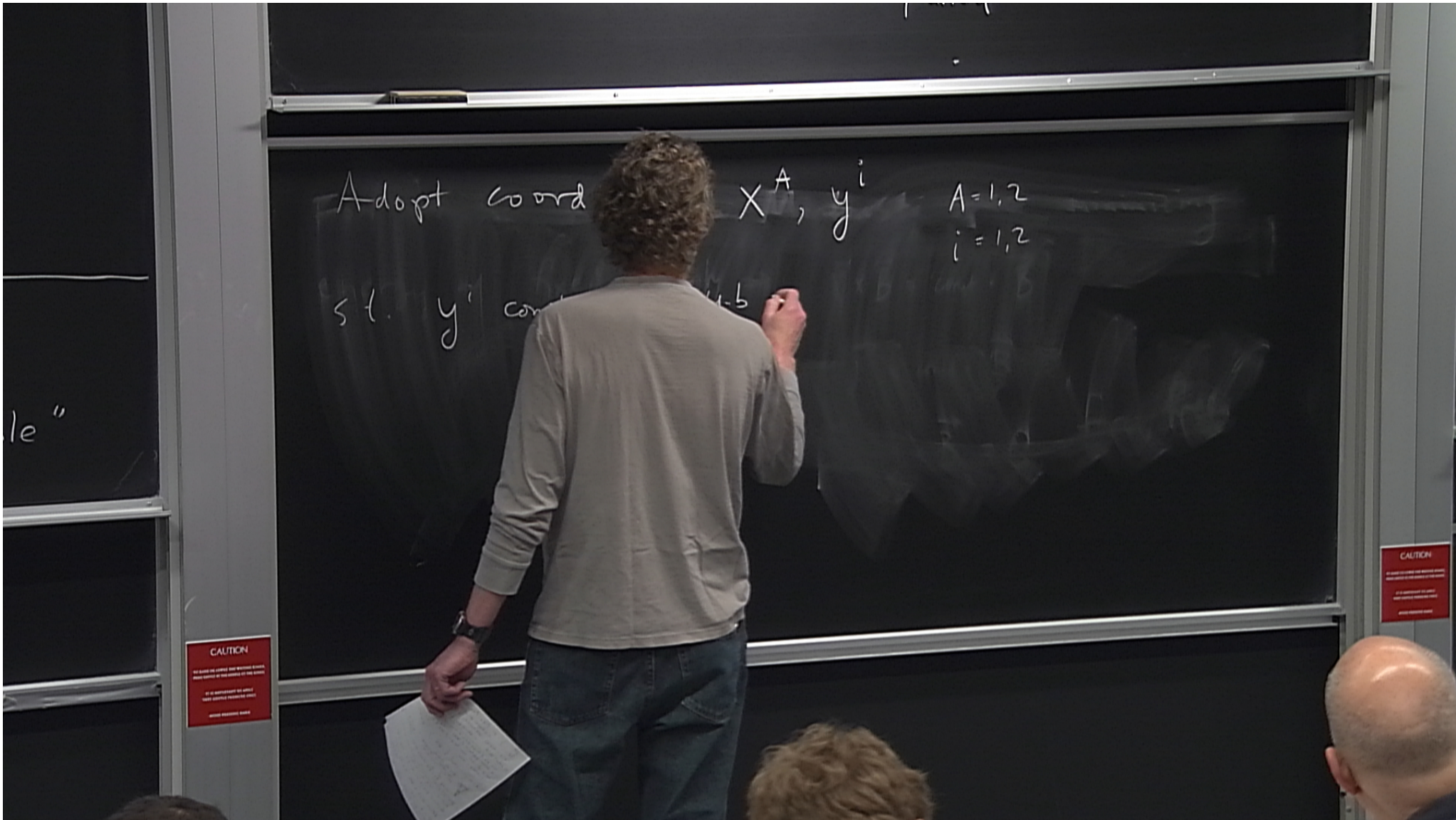
Let $b \cdot F = 0$ on Σ

Extend b off Σ by $\mathcal{L}_U b = 0$



Euler-Lagrange

CAUTION



Adopt coordinates x^A, y^i $A=1,2$
 $i=1,2$

s.t. y^i constant on u-b surfaces

$$F = f(x^A, y^i) dy^1 \wedge dy^2 = f(y^i) dy^1 \wedge dy^2$$

$$dF = \frac{\partial f}{\partial x^A} dx^A \wedge dy^1 \wedge dy^2 = 0 \Rightarrow f = f(y^i)$$

Adopt coordinates x^A, y^i $A=1,2$
 $i=1,2$

s.t. y^i constant on u-b surfaces

$$F = f(x^A, y^i) dy^1 \wedge dy^2 = \underline{f(y^i)} dy^1 \wedge dy^2$$

$$dF = \frac{\partial f}{\partial x^A} dx^A \wedge dy^1 \wedge dy^2 = 0 \Rightarrow f = f(y^i) = \underline{d\tilde{y}^1 \wedge dy^2}$$

$$\tilde{y}^1(y, y^i) = \int_0^{y_1} f(s, y_2) ds$$

$$d\tilde{y}^1 = f(y, y^i) dy^1 + (\quad) dy^2$$

Let $U \cdot F =$
 $\mathcal{L}_U F = U \cdot \nabla F =$
 Let $b \cdot F =$
 Extend b off

CAUTION

CAUTION

$$d(A \wedge F) = \underline{F} \wedge F = 0$$

$\mathcal{L}_u b = 0 \Rightarrow [u, b] = 0 \Rightarrow u \& b$ surface forming

$$dF = 0 \ \& \ F \wedge F = 0 \Rightarrow F = d\phi_1 \wedge d\phi_2$$

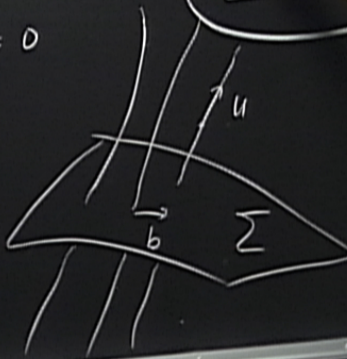
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Euler-Lagrange



$$\tilde{y}^i(y^1, y^2) = \int_0^{y^i} f(s, y^2) ds$$

$$d\tilde{y}^i = f(y^1, y^2) dy^1 + () dy^2$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = d\tilde{y}^i \wedge dy^j$$

CAUTION

$$\Rightarrow F = \alpha \wedge \beta \quad \text{"SIIP"}$$

$$0 = d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$$

$$\Rightarrow d\alpha \wedge \alpha \wedge \beta = 0$$

$$\downarrow \beta \wedge \alpha \wedge \beta = 0$$

$$\Rightarrow F = \alpha \wedge \beta \quad \text{simple}$$

$\ker F$

"flux surfaces"

Carter,
Uchida

$$\Rightarrow F = \alpha \wedge \beta \quad \text{simple}$$

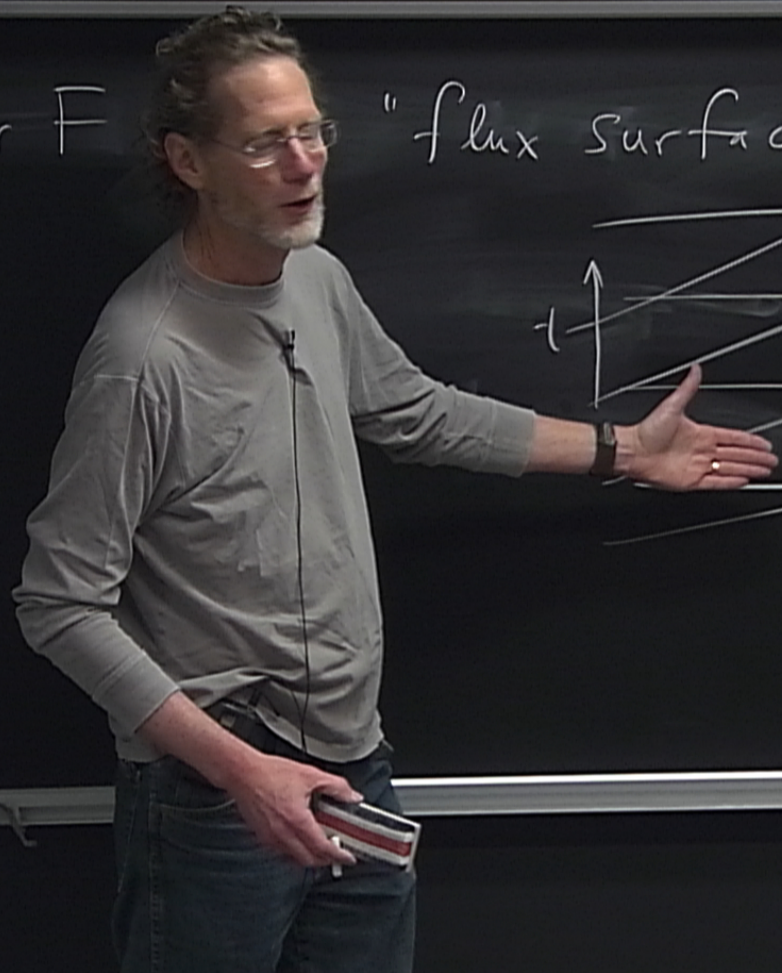
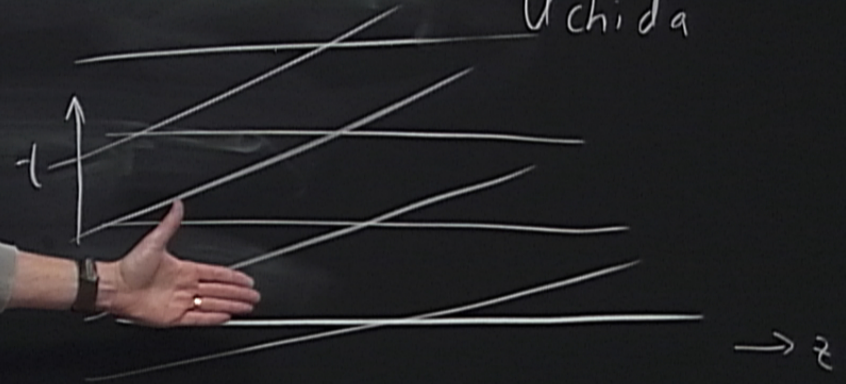
$\ker F$ are "flux surfaces" Carter,
Uchida

$$\Rightarrow F = \alpha \wedge \beta \quad \text{simple}$$

$\ker F$

"flux surfaces"

Carter,
Uchida



$$dF = 0 \text{ \& \ } F \wedge F = 0 \Rightarrow F = d\phi_1 \wedge d\phi_2$$

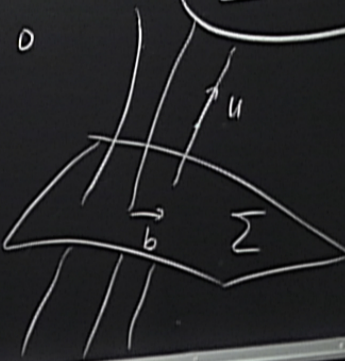
Euler potentials

Let $u \cdot F = 0$ everywhere

$$\mathcal{L}_u F = u \cdot dF + d(u \cdot F) = 0$$

Let $b \cdot F = 0$ on Σ

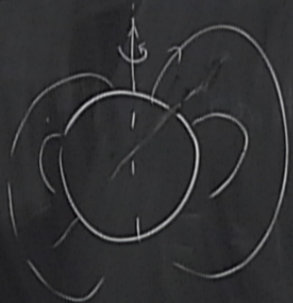
Extend b off Σ by $\mathcal{L}_u b = 0$



$$F \wedge F = 0$$

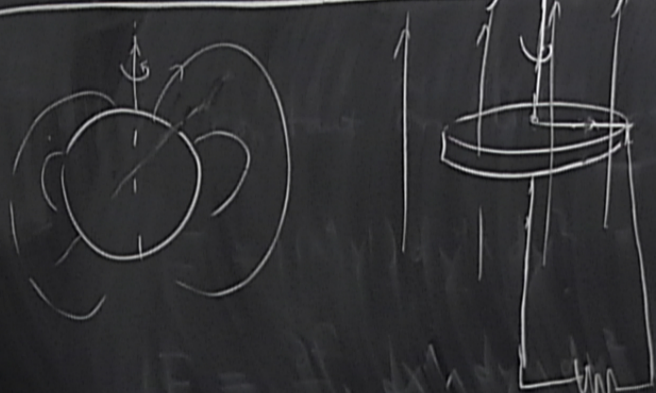
CAUTION

Force-free electrodynamics & pulsars



CAUTION
DO NOT TOUCH THE SURFACE OF THE BOARD
OR THE BOARD ITSELF
WHILE IT IS HOT

Force-free electrodynamics & pulsars



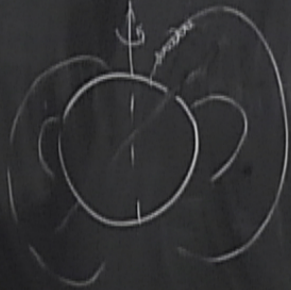
$$\vec{F} = q \vec{v} \times \vec{B}$$



CAUTION
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Force-free electrodynamics & pulsars



$$\vec{F} = q \vec{v} \times \vec{B}$$

$$\Omega$$

Goldreich-Julian (screening)

CAUTION
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WHILE IT IS BEING USED

$$(A \wedge F) = \underline{F} \wedge F = 0$$

$\Rightarrow \ker F$ tangent to u-b surfaces

$$\frac{\text{mag field energy density}}{\text{plasma mass energy density}} \sim \frac{B^2}{\frac{m_e}{e} \Omega B} = \frac{eB/m_e}{\Omega} = \frac{\omega_c}{\Omega}$$

$$\frac{eB}{m_e} = \left(\frac{B}{1T} \right) \times 10^{11} \text{ Hz}$$

(screening)

$$\omega_{GJ} \sim \Omega B$$

(69)

CAUTION
Do not touch the board when it is hot.
Do not touch the board when it is hot.

$$\Rightarrow F = \alpha \lambda \rho$$

Force free $\leftarrow \bar{F}_{ab} j^b = 0.$

$dF = 0$ Covariant Faraday law

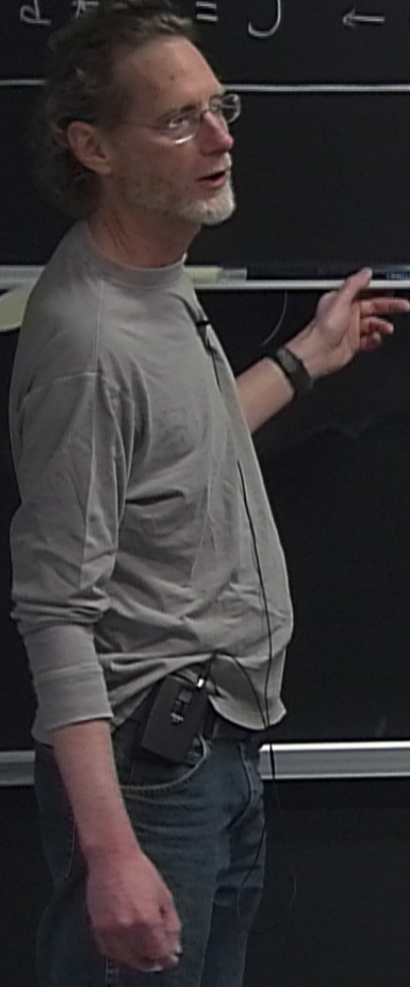
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$F \wedge F$

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"degenerate field"

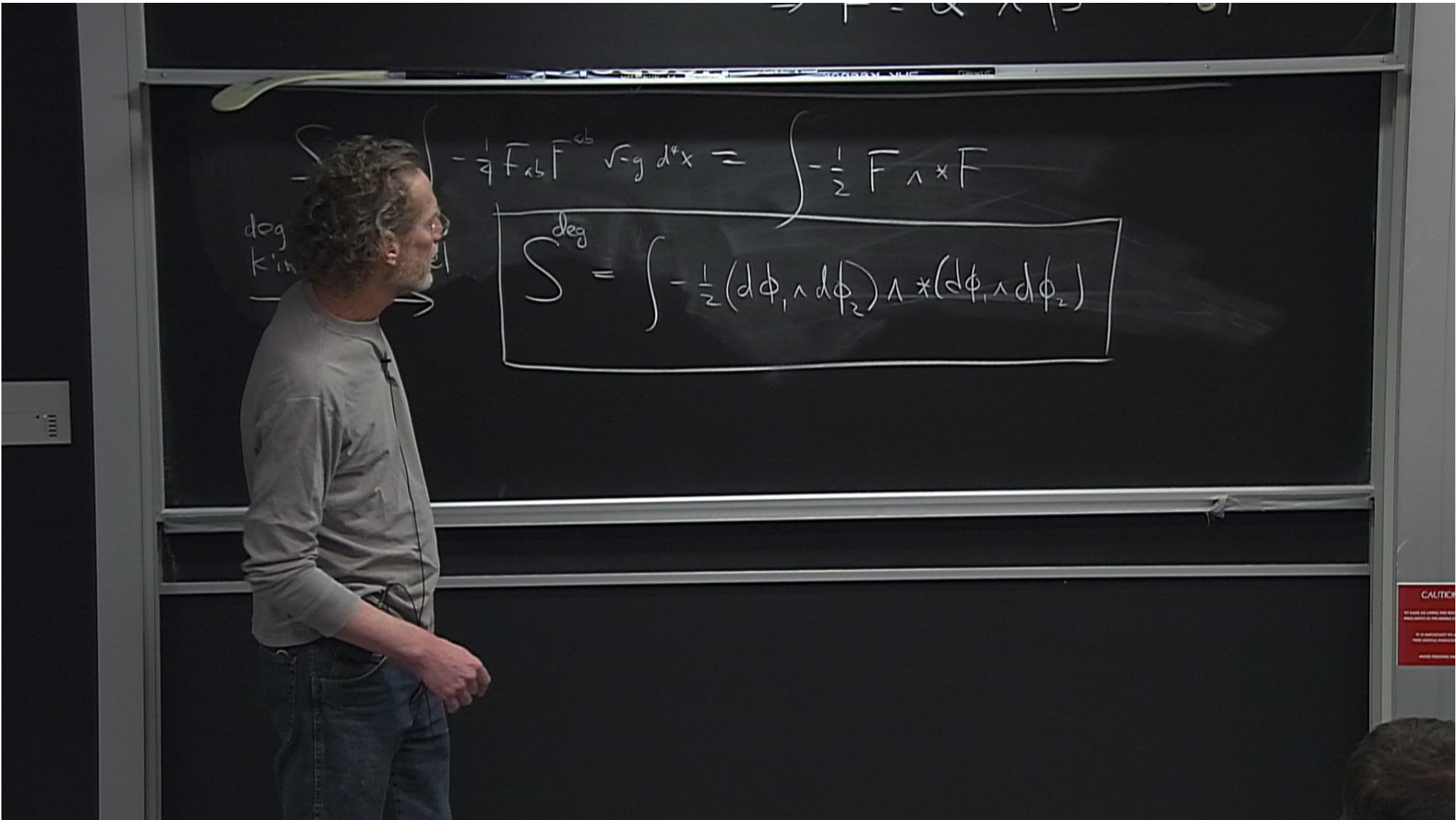
$$\Rightarrow F = \alpha \wedge \beta \quad \text{"simple"}$$

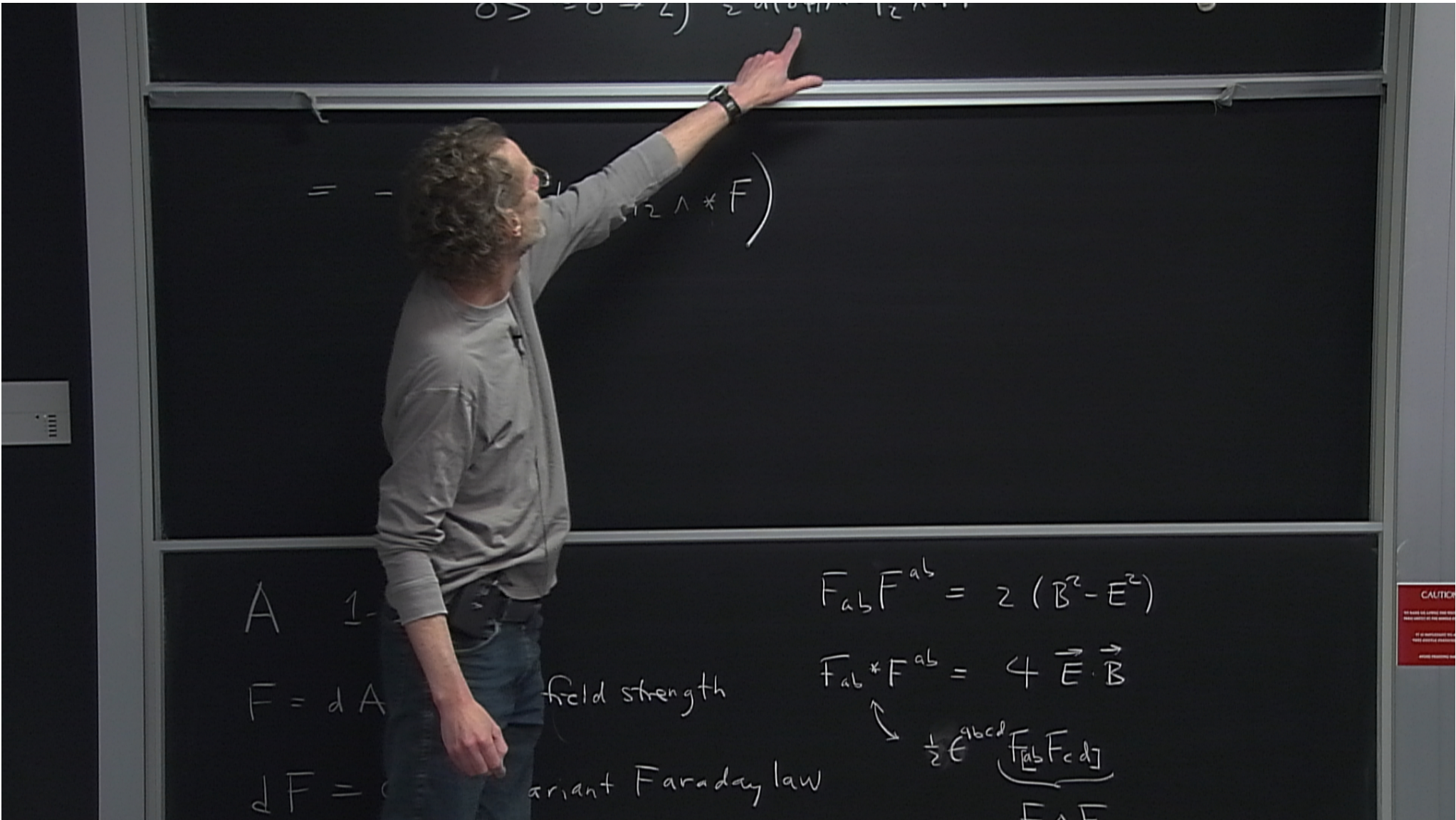


Force $\rightarrow F_{ab} j^b = 0.$

$$-\frac{1}{4} F_{ab} F^{ab} \sqrt{-g} d^4x = \int -\frac{1}{2} F \wedge *F$$

deg





$$= \int (\delta\phi_1 \wedge d\phi_2 \wedge *F) + \int \delta\phi_1 \wedge d(d\phi_2 \wedge *F)$$

$$\boxed{\begin{aligned} d(d\phi_2 \wedge *F) &= 0 \\ d(d\phi_1 \wedge *F) &= 0 \end{aligned}}$$

A

$$F = dA$$

field strength

$$dF =$$

variant Faraday law

$$F_{ab}F^{ab} = 2(B^2 - E^2)$$

$$F_{ab} *F^{ab} = 4 \vec{E} \cdot \vec{B}$$

$$\frac{1}{2} \epsilon^{abcd} \underbrace{[F_{ab} F_{cd}]}_{F \wedge F}$$

$$0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \rightarrow 0$$

$$= - \int d(\delta\phi_1 \wedge d\phi_2 \wedge *F) + \phi_1 \wedge d(d\phi_2 \wedge *F)$$

$$= 0 \quad \forall \delta\phi_1 \Rightarrow$$

| | |
|----------------|----------------------|
| $d(\dots) = 0$ | $= d\phi_1 \wedge J$ |
| $d(\dots)$ | $= d\phi_1 \wedge J$ |

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$$= - \int d(\delta\phi_1 \wedge d\phi_2 \wedge *F) + \delta\phi_1 \wedge d(d\phi_2 \wedge *F)$$

$$= 0 \quad \forall \delta\phi_i \Rightarrow \begin{array}{|l|l|} \hline d(d\phi_2 \wedge *F) = 0 & = d\phi_2 \wedge J \\ \hline d(d\phi_1 \wedge *F) = 0 & = d\phi_1 \wedge J \\ \hline \end{array}$$

A 1-form

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$\downarrow F = 0$ covariant Faraday law

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| | |
|----------------------------|----------------------|
| $d(d\phi_2 \wedge *F) = 0$ | $= d\phi_2 \wedge J$ |
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$$F_{ab} J^b = 0 \iff F_{ab} \epsilon^{bcde} J_{cde}$$

mag. field

plasma mas

$$\frac{eB}{m_e} = \left(\frac{B}{1T} \right)$$

CAUTION

BE CAREFUL NOT TO TOUCH THE BOARD
AS IT IS ELECTRICALLY LIVE
WHEN POWER IS ON

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$$F_{ab} J^b = 0 \iff F_{ab} \epsilon^{bcde} J_{cde} = 0$$

$$\iff F_{a[b} J_{cde]} = 0$$

mag. field

plasma mas

$$\frac{eB}{m_e} = \left(\frac{B}{1T} \right)$$

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$$F_{ab} J = 0 \quad \rightarrow \quad F_{ab} \in \quad J_{cde} = 0$$

$$\rightarrow F_{a[b} J_{cde]} = 0$$

$$\boxed{\alpha_a} \underbrace{\beta_{[b} J_{cde]}} - \underbrace{\beta_a}_{[b} \underbrace{\alpha_{cde]} = 0$$

$$\rightarrow \alpha \wedge J = 0 = \beta \wedge J$$

mag. field

plasma mass

$$\frac{eB}{m_e} = \left(\frac{B}{1T} \right)$$

CAUTION

BE CAREFUL NOT TO TOUCH THE
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