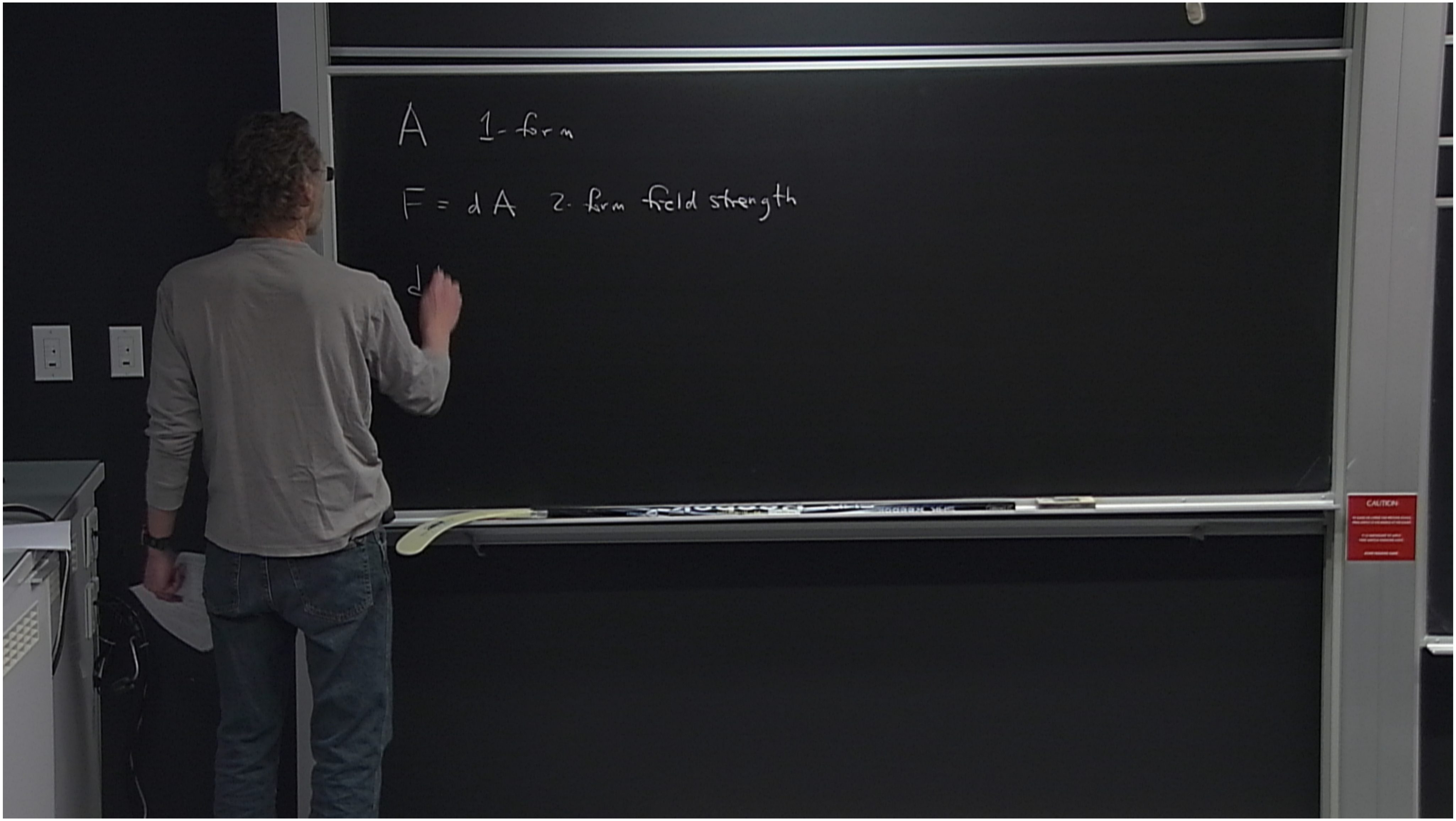
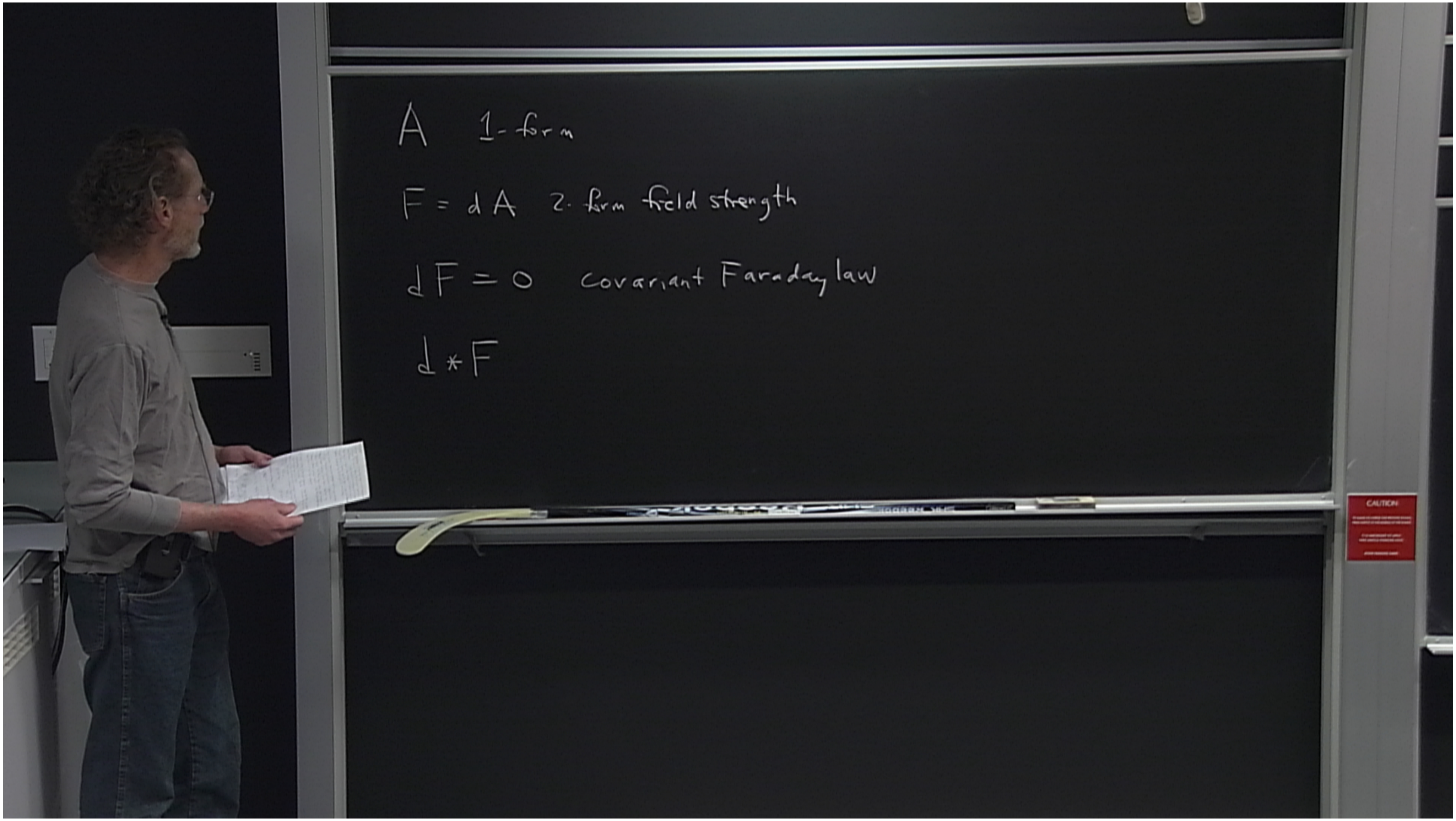
Title: Spacetime approach to force-free magnetospheres - Lecture 3

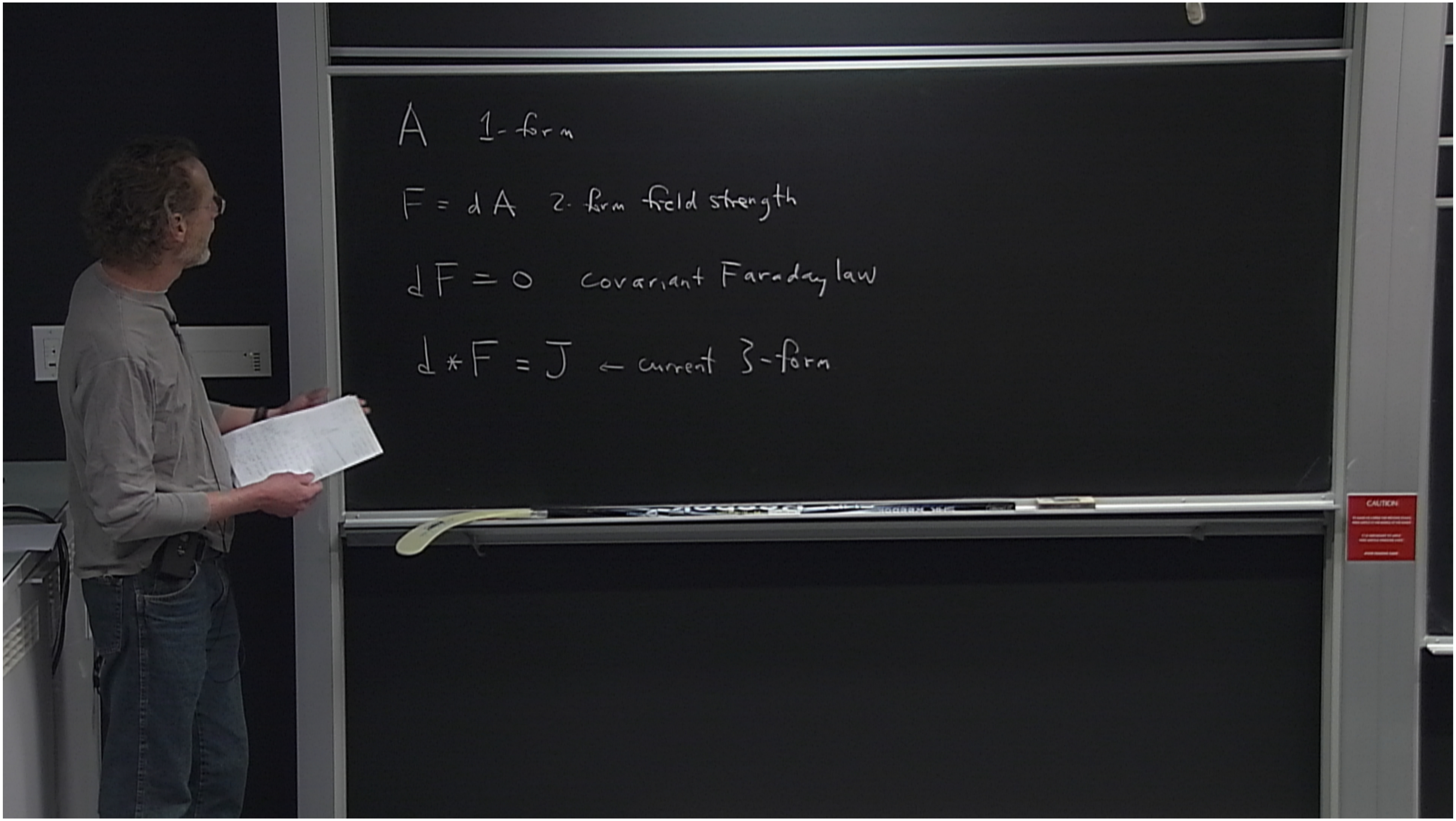
Date: Feb 26, 2014 09:30 AM

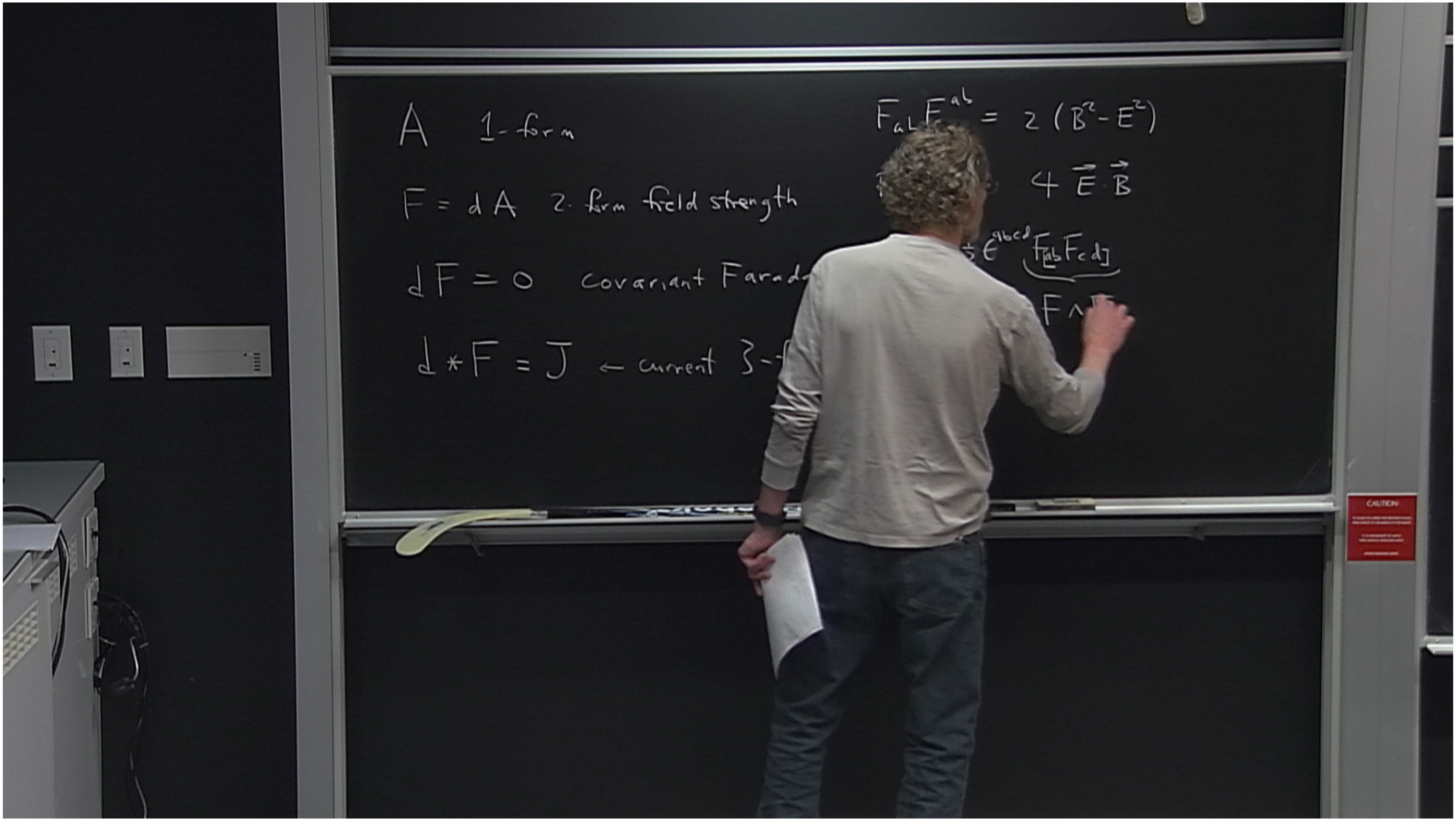
URL: http://pirsa.org/14020157

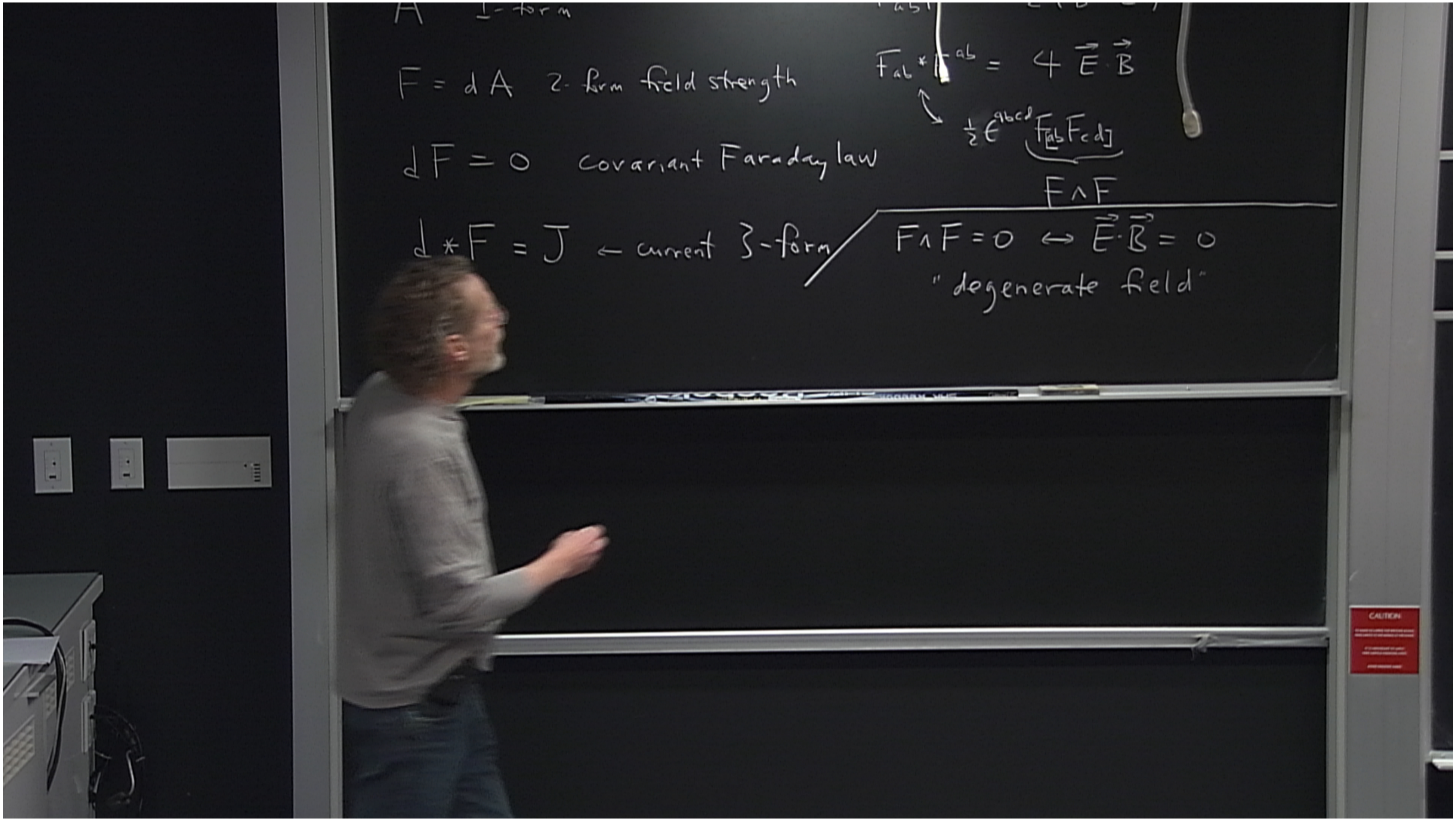
Abstract:

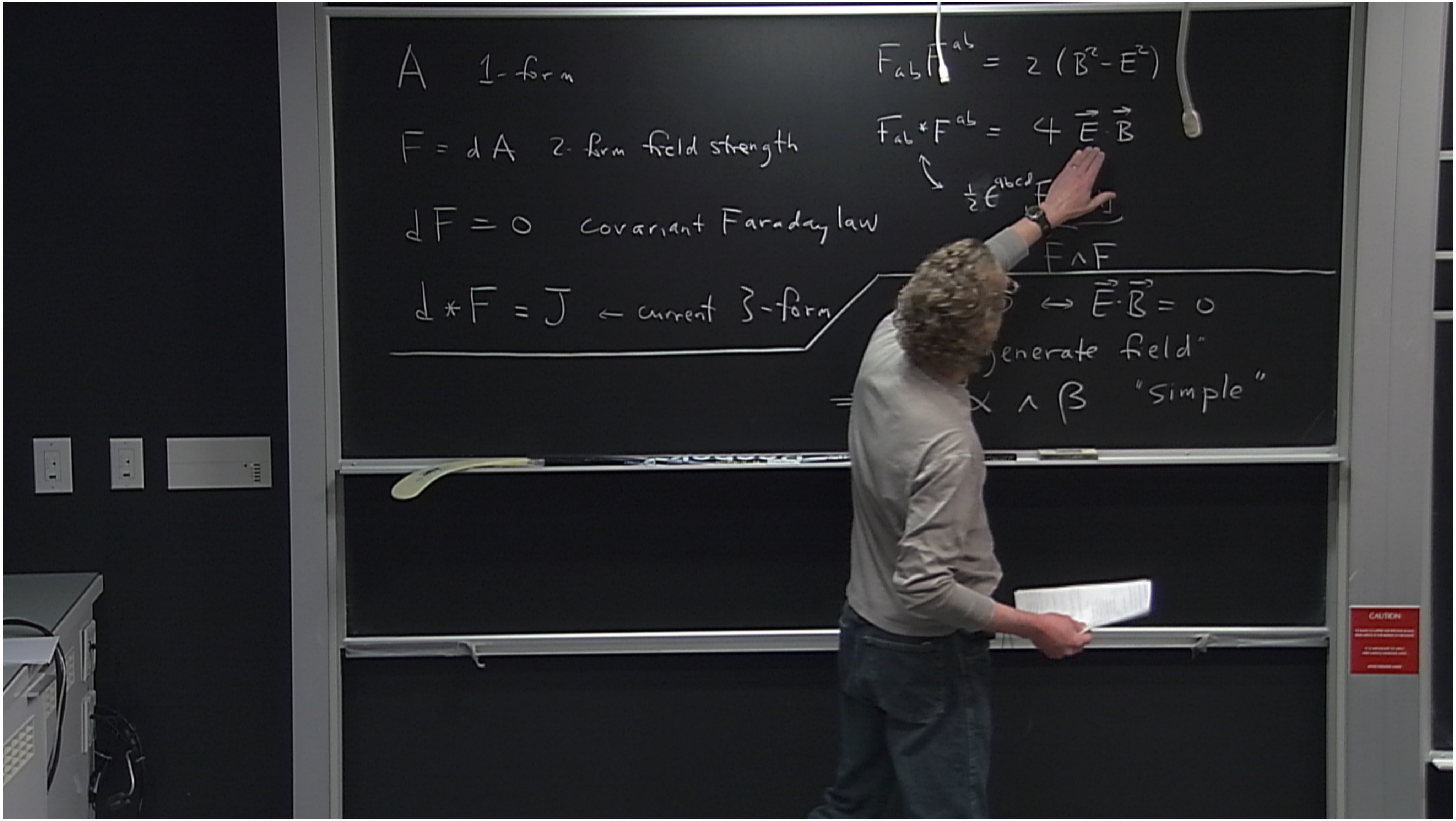

A 1-form
F = dA 2-form field strength
d

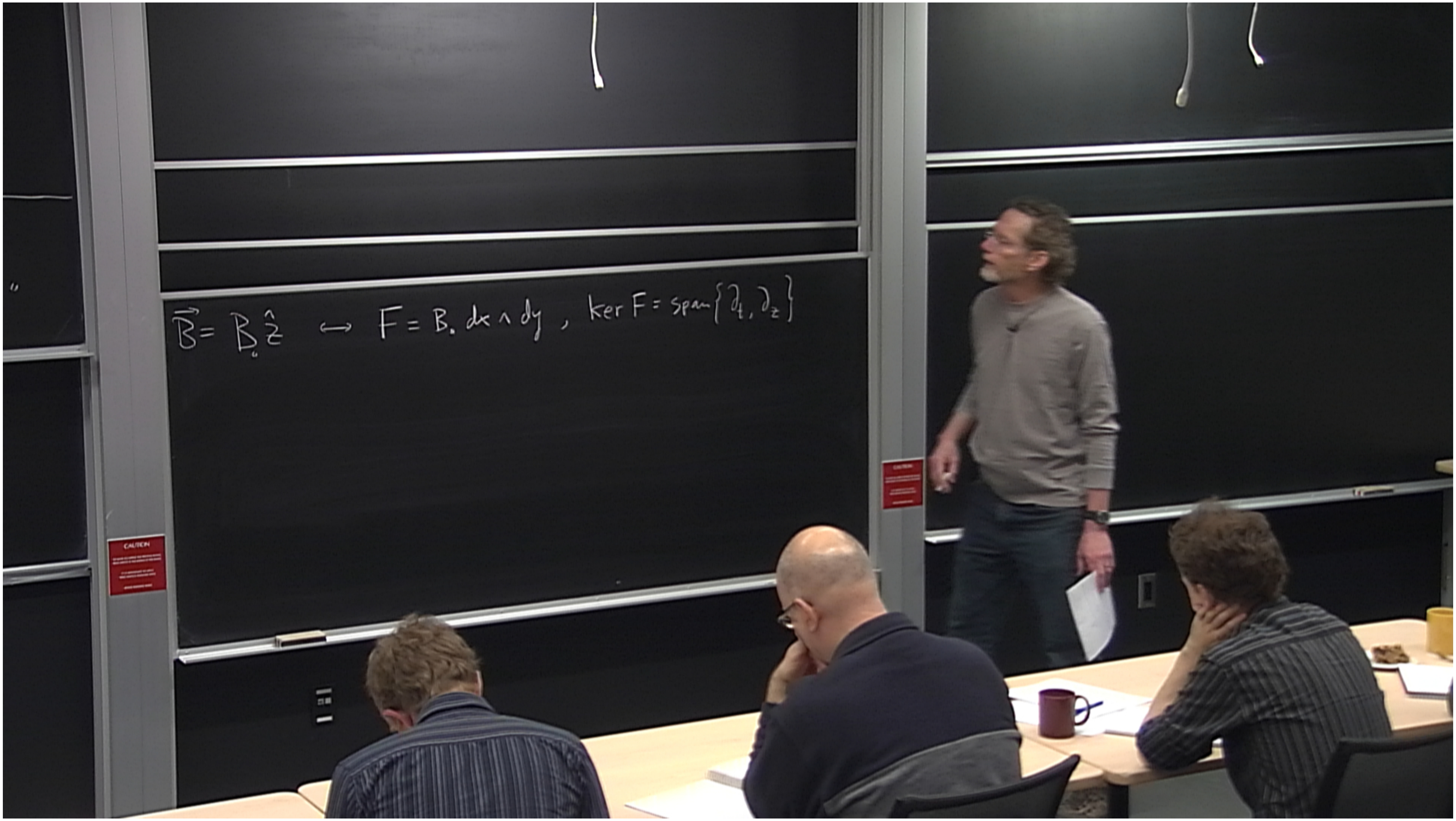

A 1-form
F = dA 2-form field strength
dF = 0 covariant Faraday law
d*F

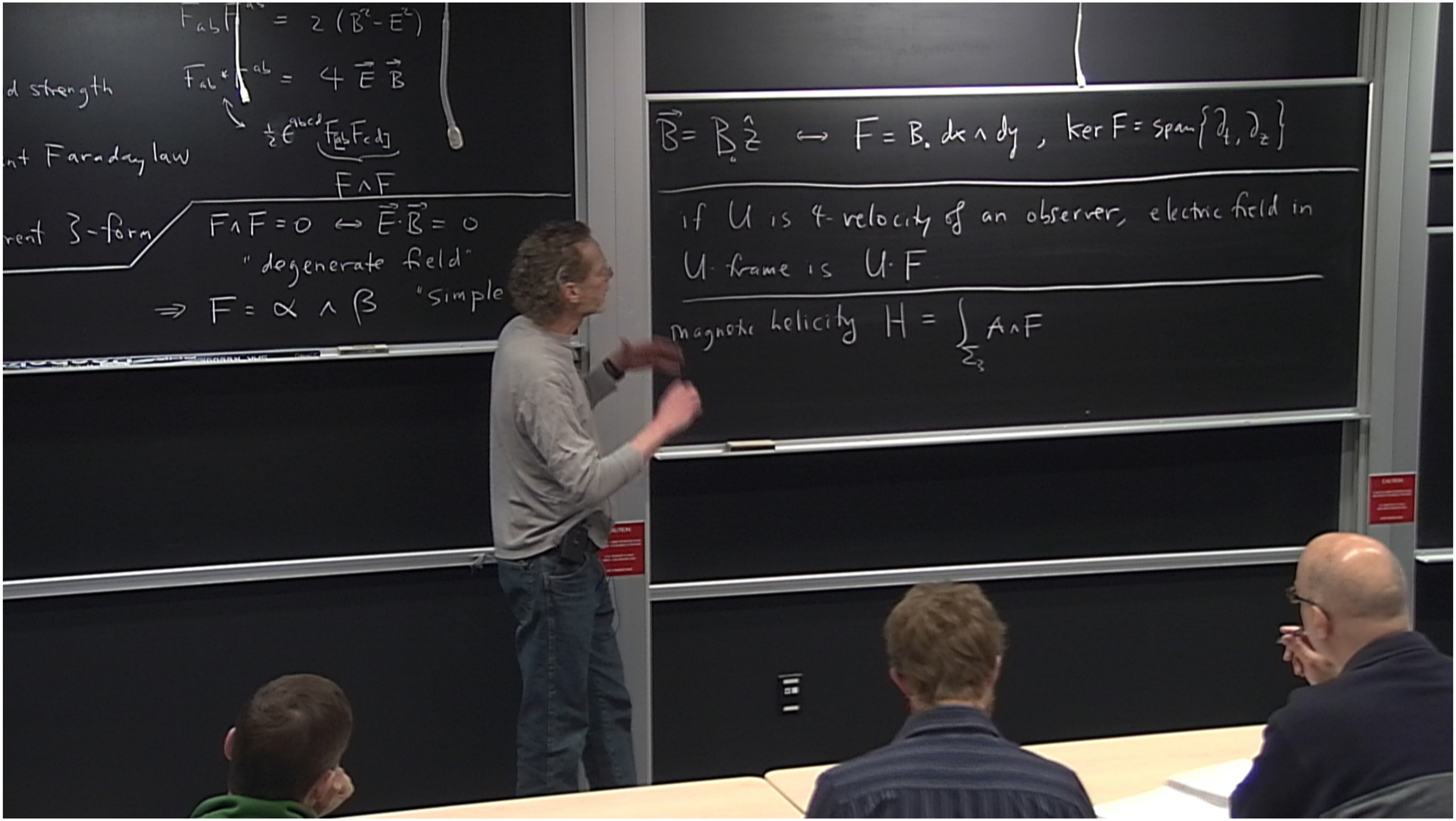

A 1-form
F = dA 2-form field strength
dF = 0 covariant Faraday law
d*F = J ← current 3-form


A 1-form
F = dA 2-form field strength
dF = 0 covariant
d*F = J ← current
F_ab F^ab = 2 (B^2 - E^2)
4 E · B
CAUTION

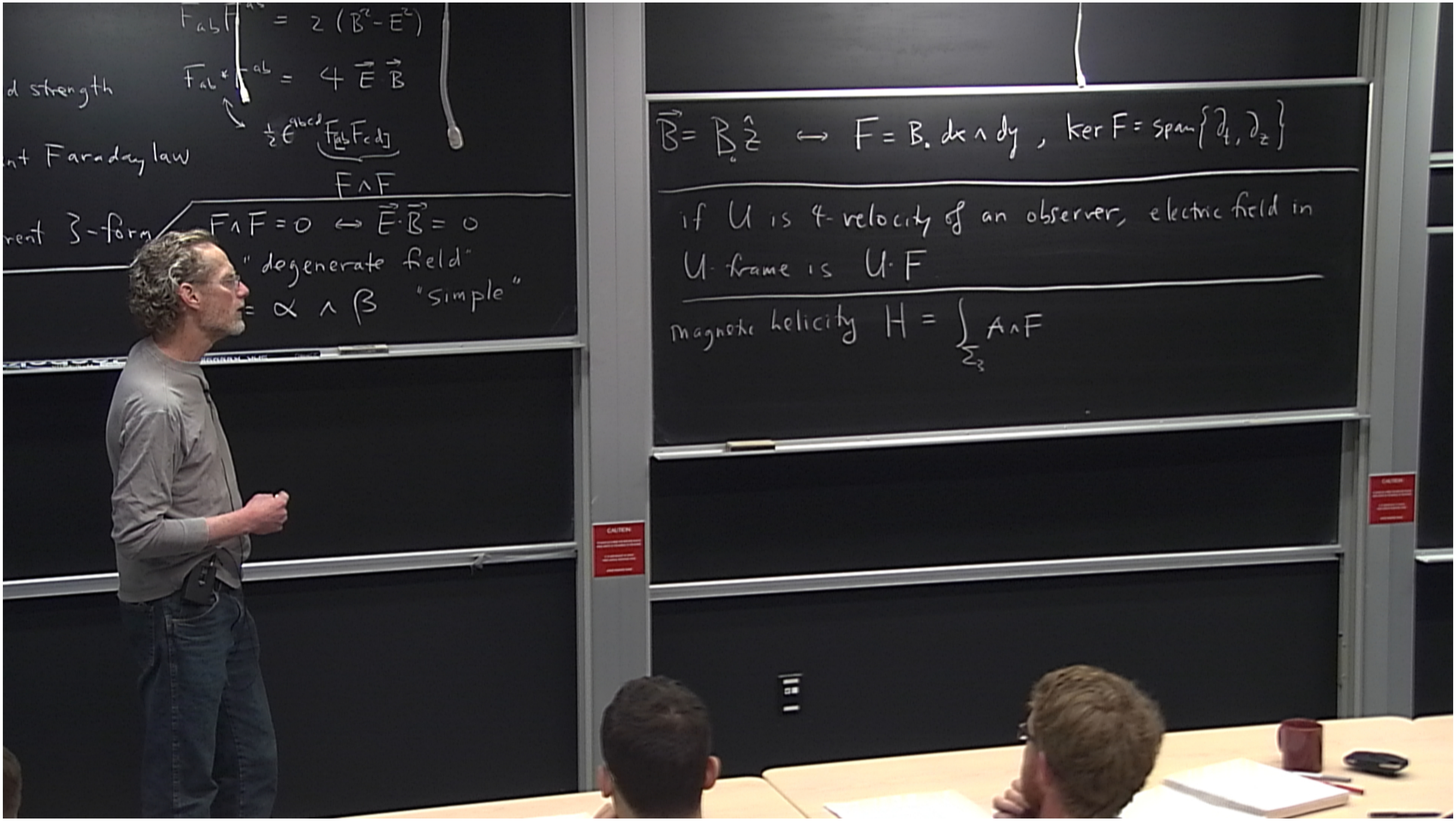

F = d A 2-form field strength
d F = 0 covariant Faraday law
d * F = J ← current 3-form
F_ab *F^ab = 4 E · B
½ ε^abcd F[ab Fcd]
F ∧ F
F ∧ F = 0 ⇔ E · B = 0
"degenerate field"

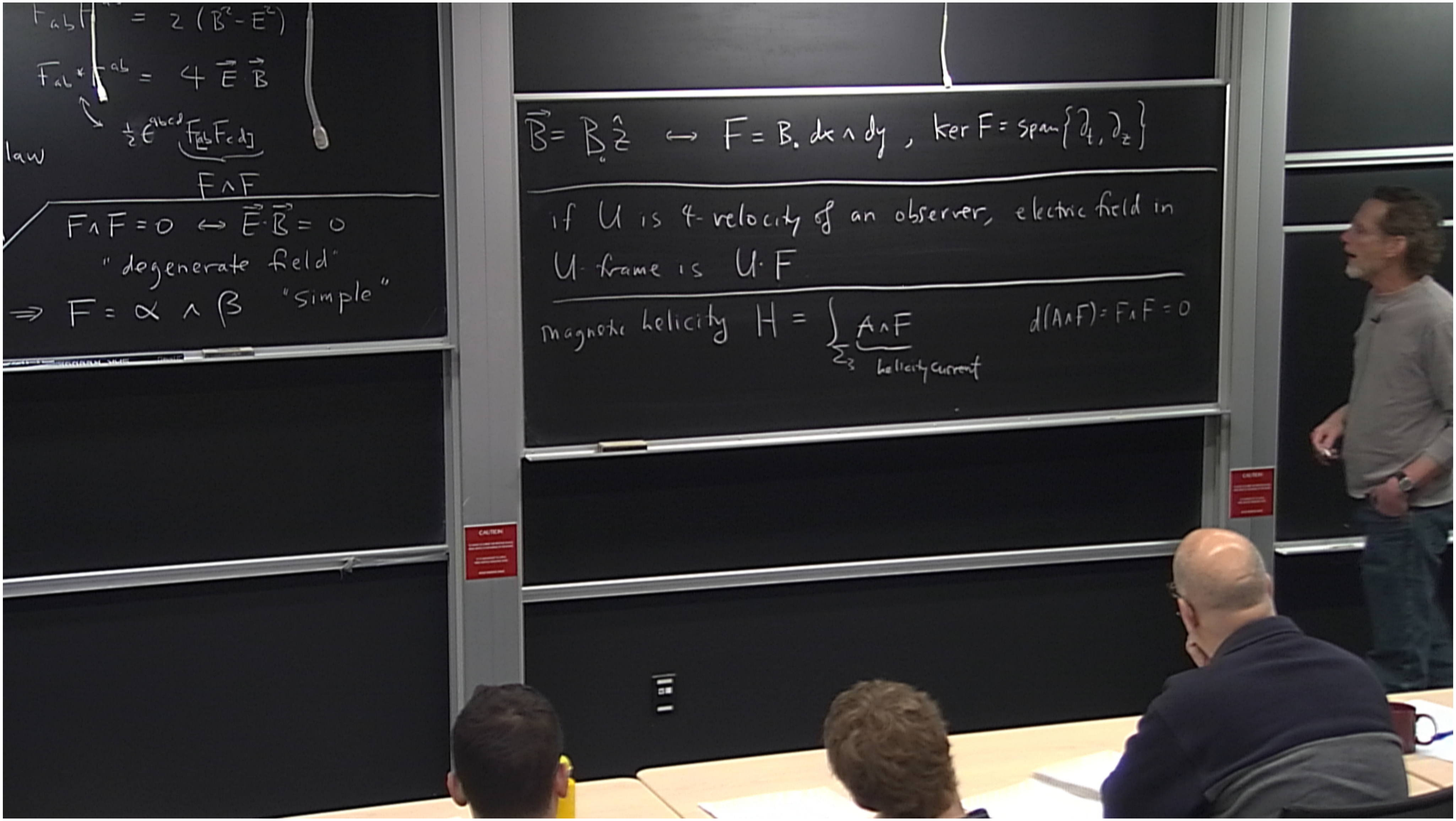

A 1-form
F = dA 2-form field strength
dF = 0 covariant Faraday law
d*F = J ← current 3-form
$F_{ab}F^{ab} = 2(B^2 - E^2)$
$F_{ab}{*F}^{ab} = 4\vec{E}\cdot\vec{B}$
F∧F
$\leftrightarrow \vec{E}\cdot\vec{B} = 0$
enerate field"
"simple"

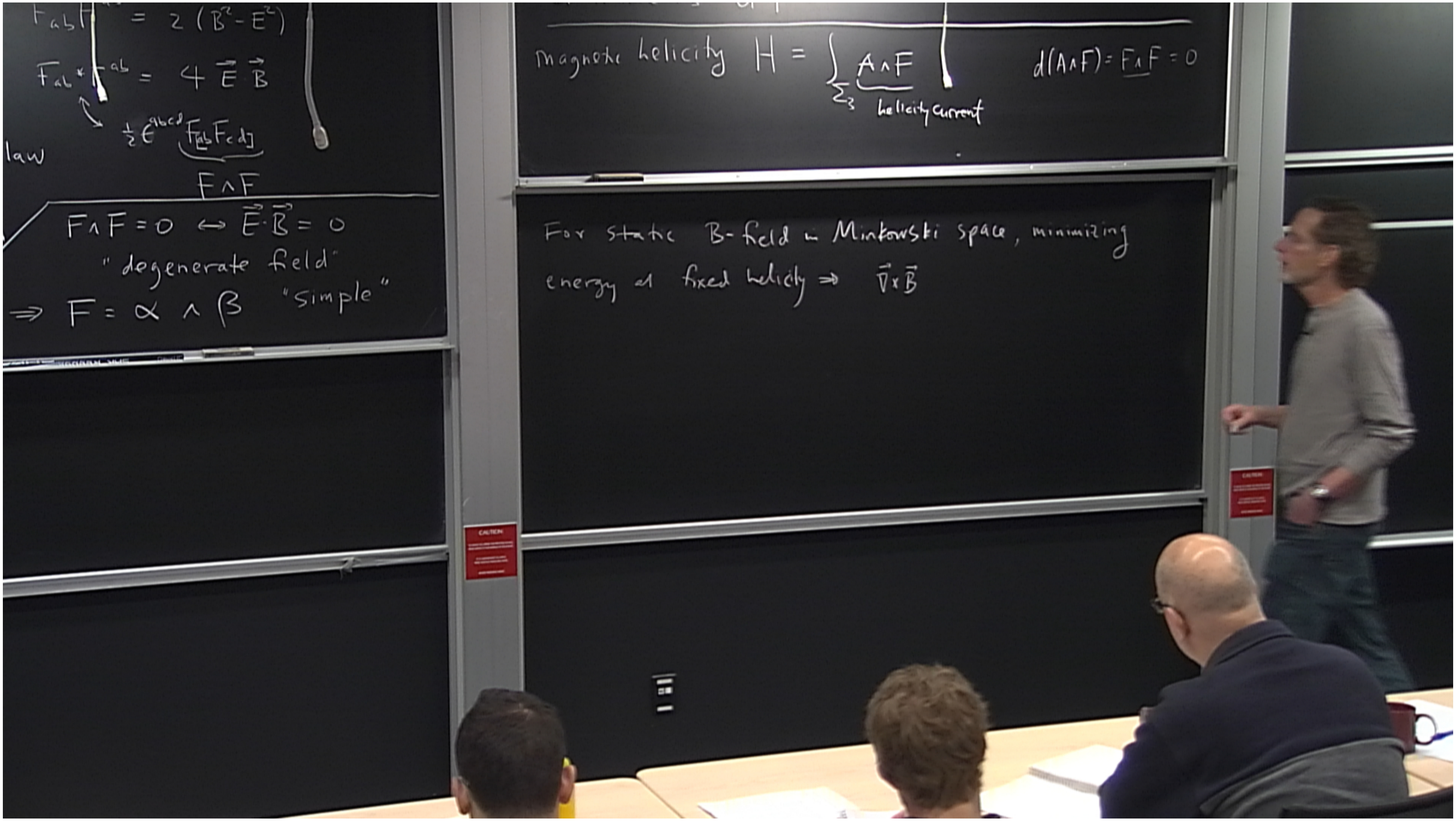

$\vec{B} = B_0 \hat{z} \longleftrightarrow F = B_0\, dx \wedge dy$, $\ker F = \mathrm{span}\{\partial_t, \partial_z\}$



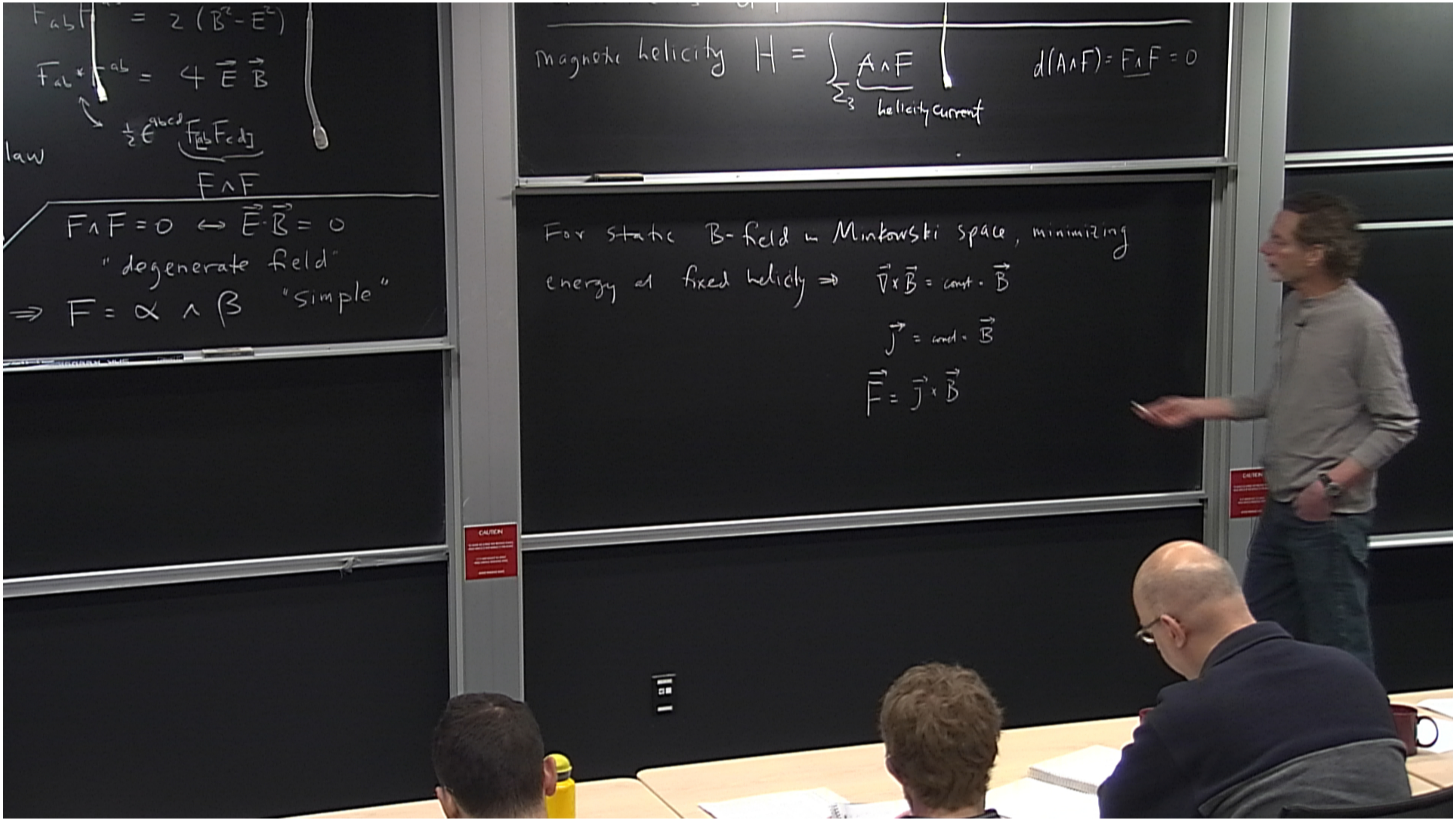
F∧F = 0 ⟺ E·B = 0
"degenerate field"
⇒ F = α ∧ β "simple"
B = B₀ ẑ ⟷ F = B₀ dx ∧ dy , ker F = span{∂t, ∂z}
if U is 4-velocity of an observer, electric field in
U-frame is U·F
magnetic helicity H = ∫Σ3 A∧F

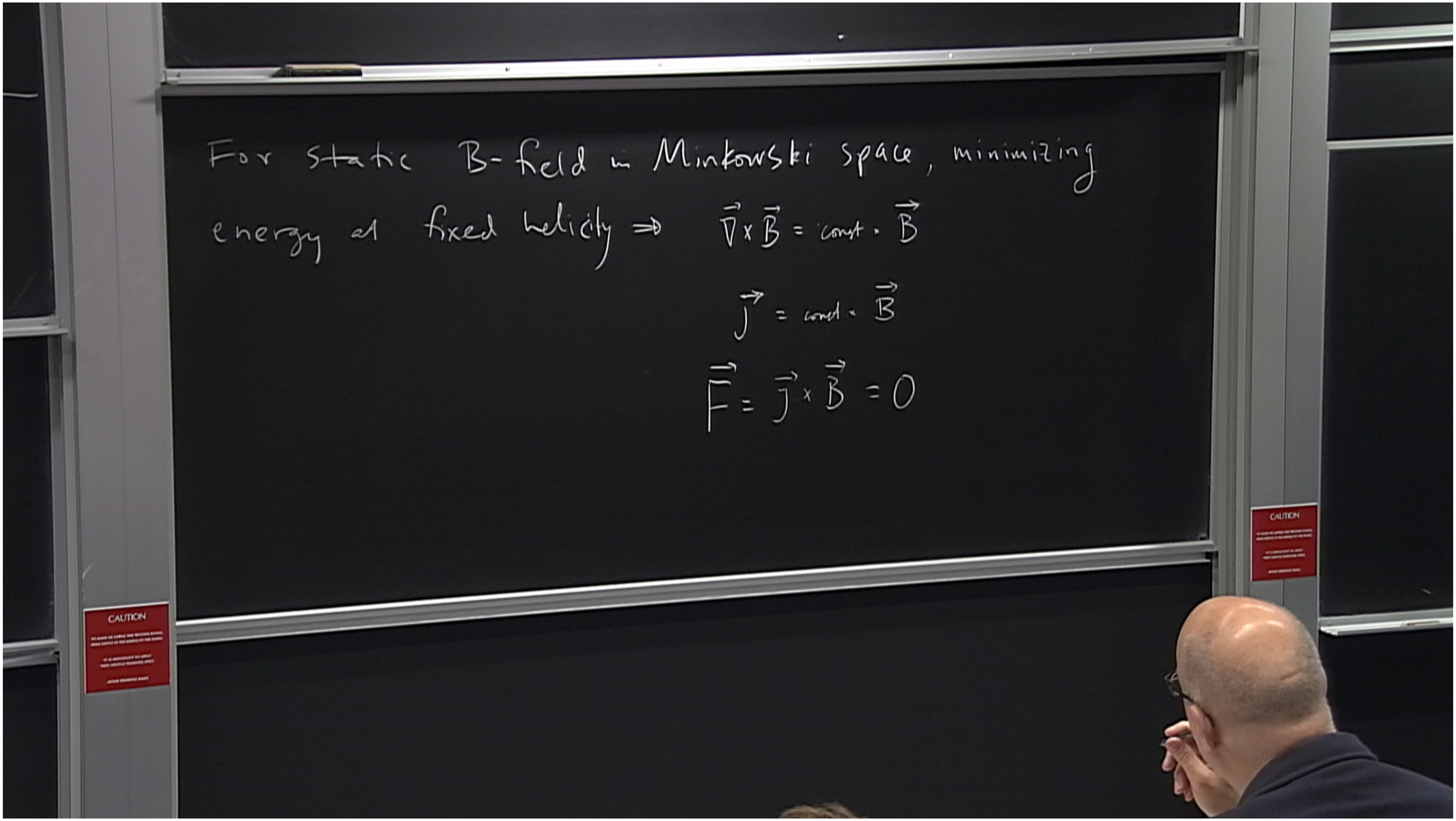

F ∧ F = 0 ⟺ E⃗ · B⃗ = 0
"degenerate field"
"simple"
B⃗ = B₀ ẑ ⟷ F = B₀ dx ∧ dy , ker F = span{∂t, ∂z}
if U is 4-velocity of an observer, electric field in
U-frame is U·F
magnetic helicity H = ∫Σ₃ A ∧ F

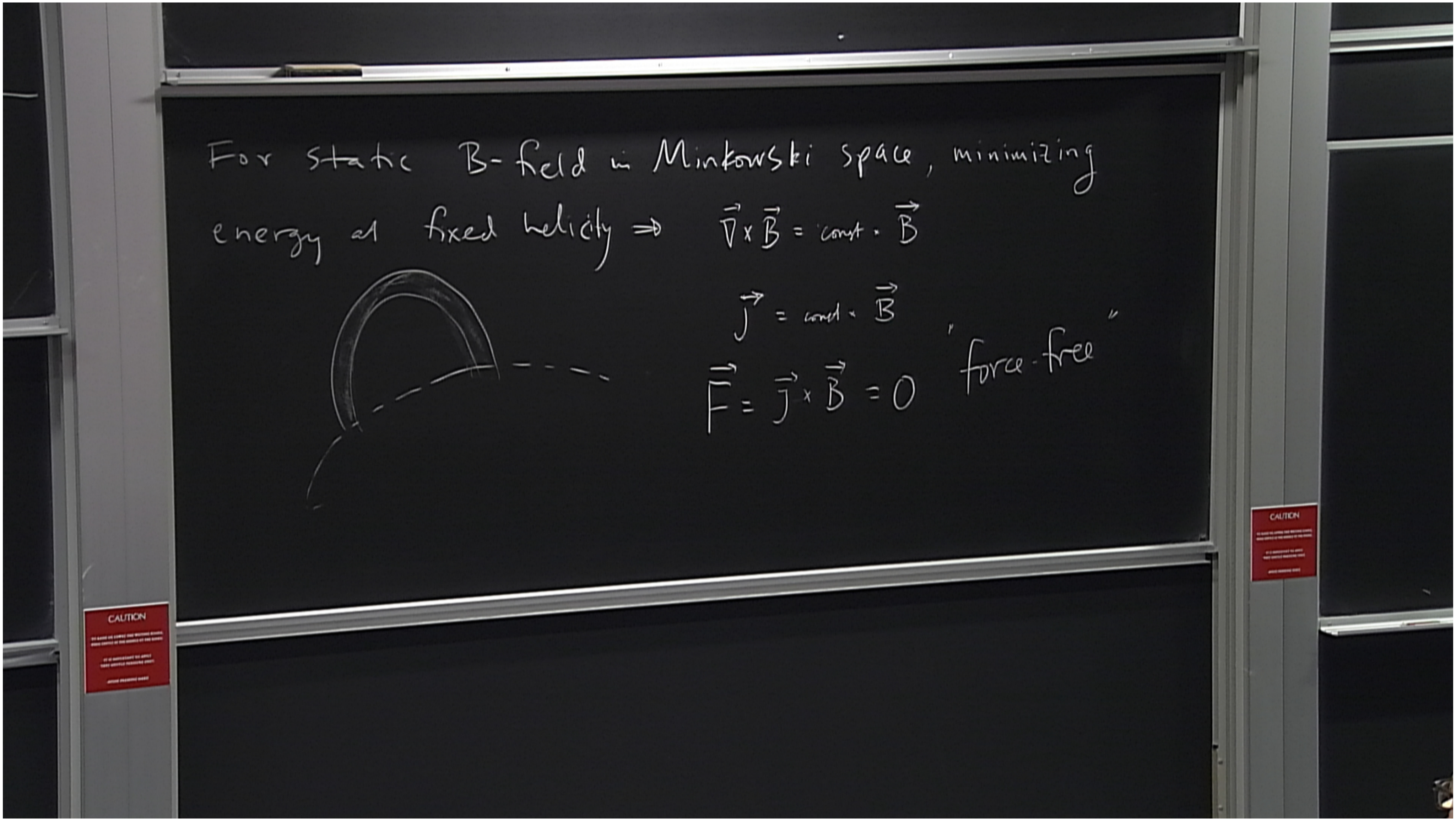

B = B₀ ẑ ⟷ F = B₀ dx ∧ dy , ker F = span{∂t, ∂z}
if U is 4-velocity of an observer, electric field in
U-frame is U·F
magnetic helicity H = ∫Σ3 A∧F
helicity current
d(A∧F) = F∧F = 0
F∧F = 0 ⟷ E·B = 0
"degenerate field"
⇒ F = α ∧ β "simple"
law

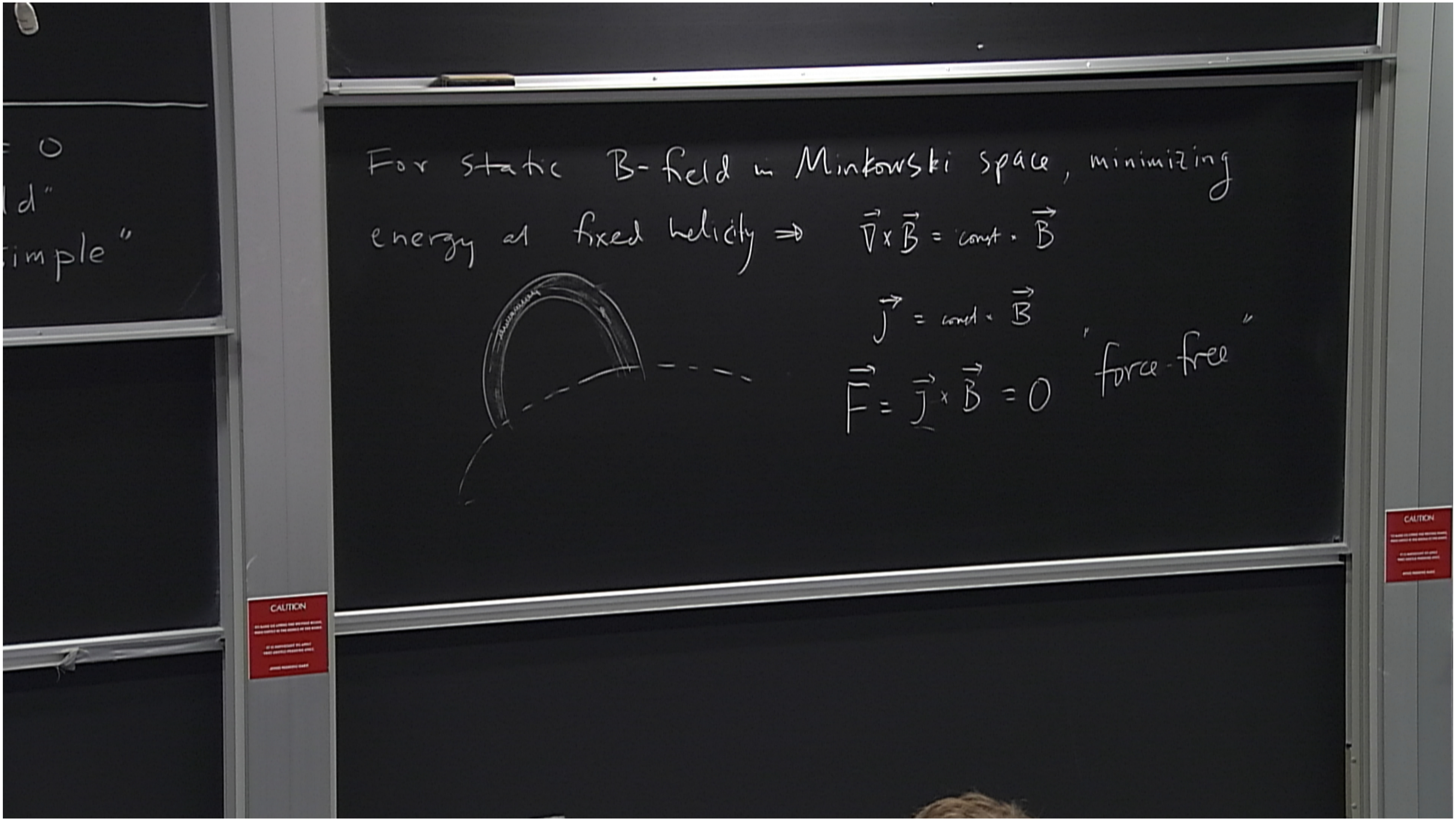


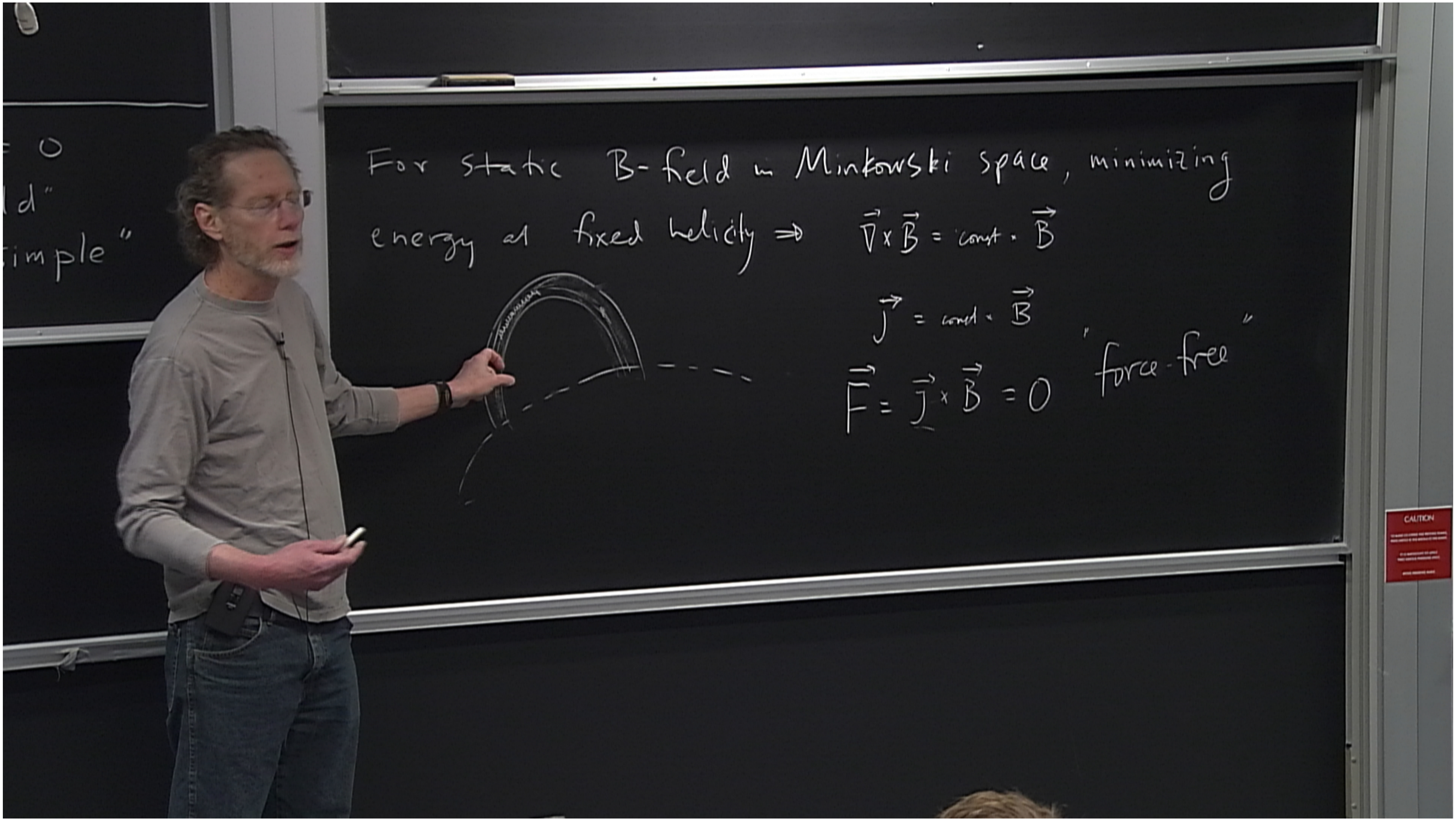
magnetic helicity H = ∫ A∧F
Σ3
helicity current
d(A∧F) = F∧F = 0
For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ ∇×B
F∧F = 0 ⟺ E·B = 0
"degenerate field"
⇒ F = α ∧ β "simple"
law

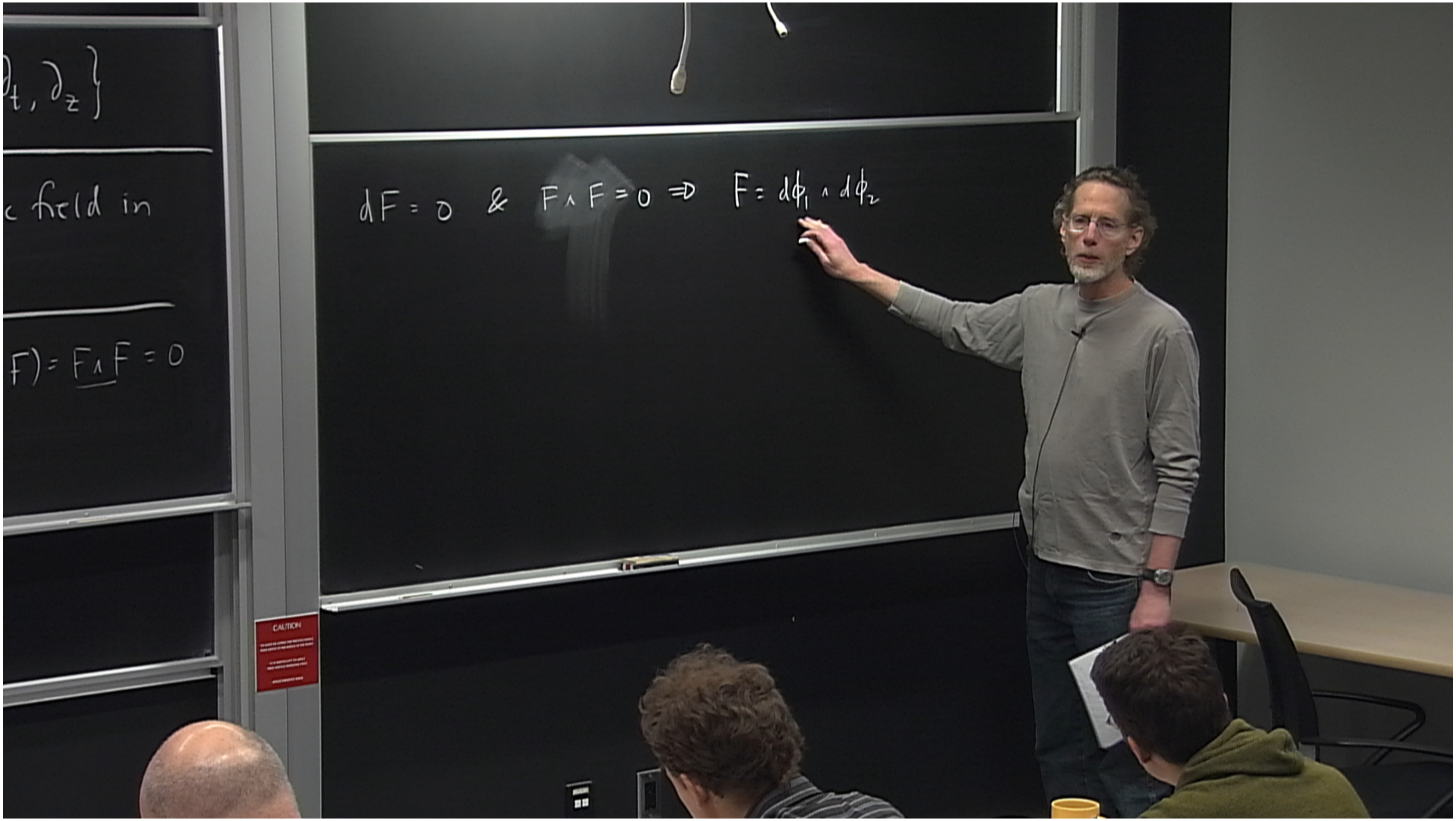

magnetic helicity H = ∫Σ3 A∧F
helicity current
d(A∧F) = F∧F = 0
For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ ∇×B = const · B
j = const · B
F = j × B
law
F∧F
F∧F = 0 ⇔ E·B = 0
"degenerate field"
⇒ F = α ∧ β "simple"
CAUTION

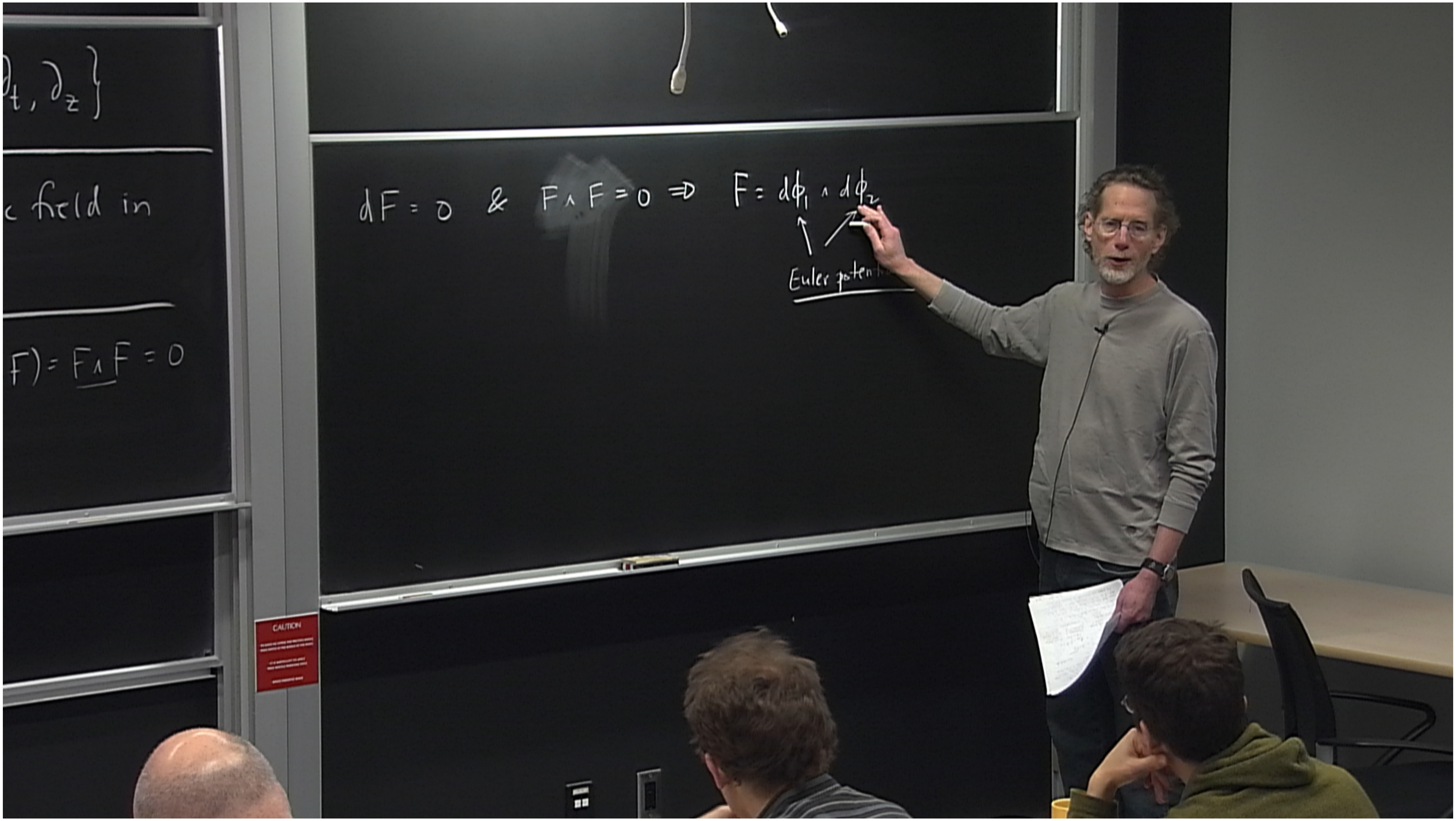

For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ $\vec{\nabla} \times \vec{B} = \text{const} \cdot \vec{B}$
$\vec{j} = \text{const} \cdot \vec{B}$
$\vec{F} = \vec{j} \times \vec{B} = 0$
CAUTION
CAUTION

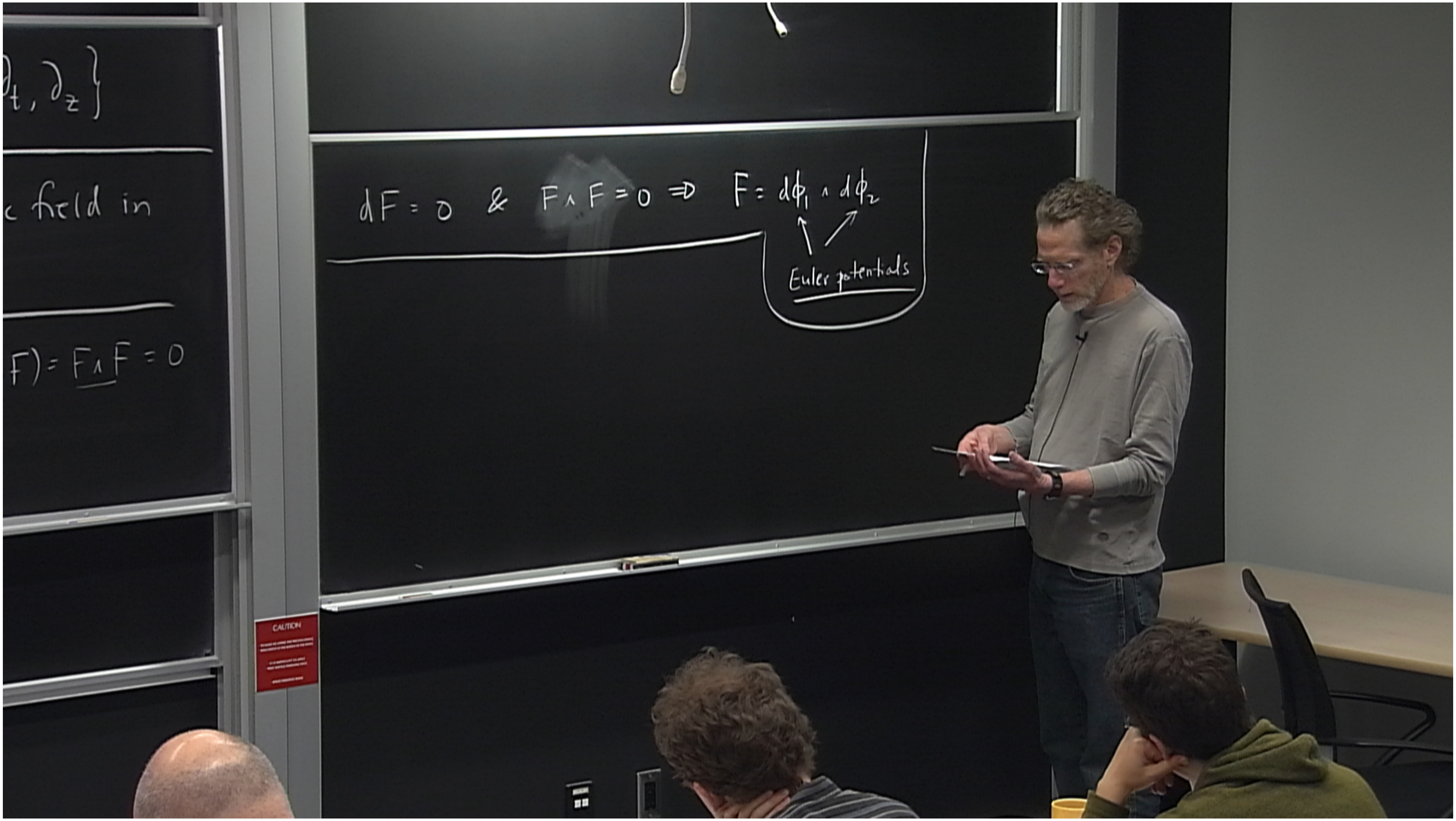

For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ $\vec{\nabla} \times \vec{B} = \text{const} \cdot \vec{B}$
$\vec{j} = \text{const} \cdot \vec{B}$
$\vec{F} = \vec{j} \times \vec{B} = 0$ "force-free"

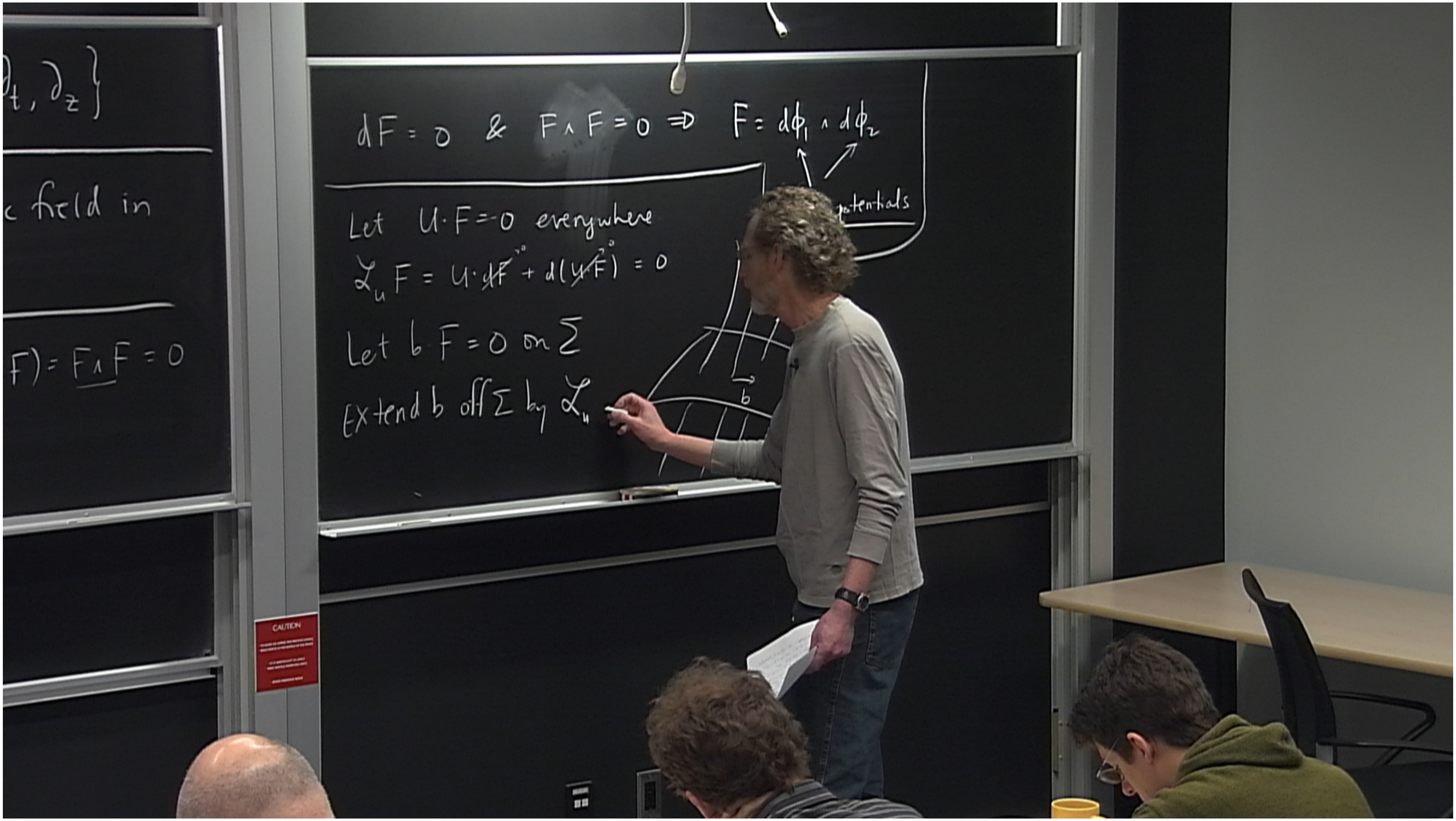

For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ $\vec{\nabla} \times \vec{B} = \text{const} \cdot \vec{B}$
$\vec{j} = \text{const} \cdot \vec{B}$
$\vec{F} = \vec{j} \times \vec{B} = 0$ "force-free"
CAUTION
CAUTION

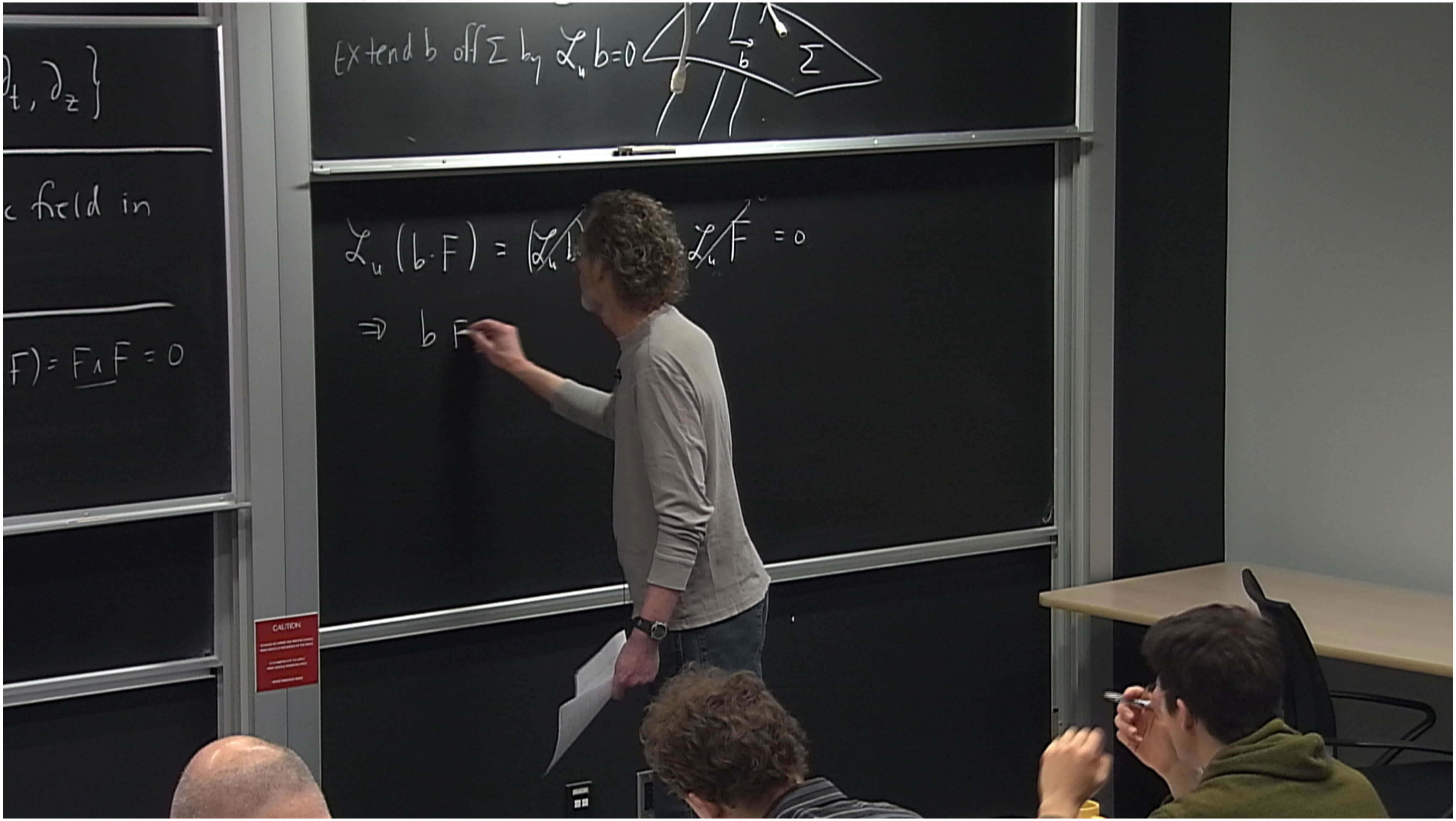

For static B-field in Minkowski space, minimizing
energy at fixed helicity ⇒ ∇ × B = const · B
j = const · B
F = j × B = 0
"force-free"

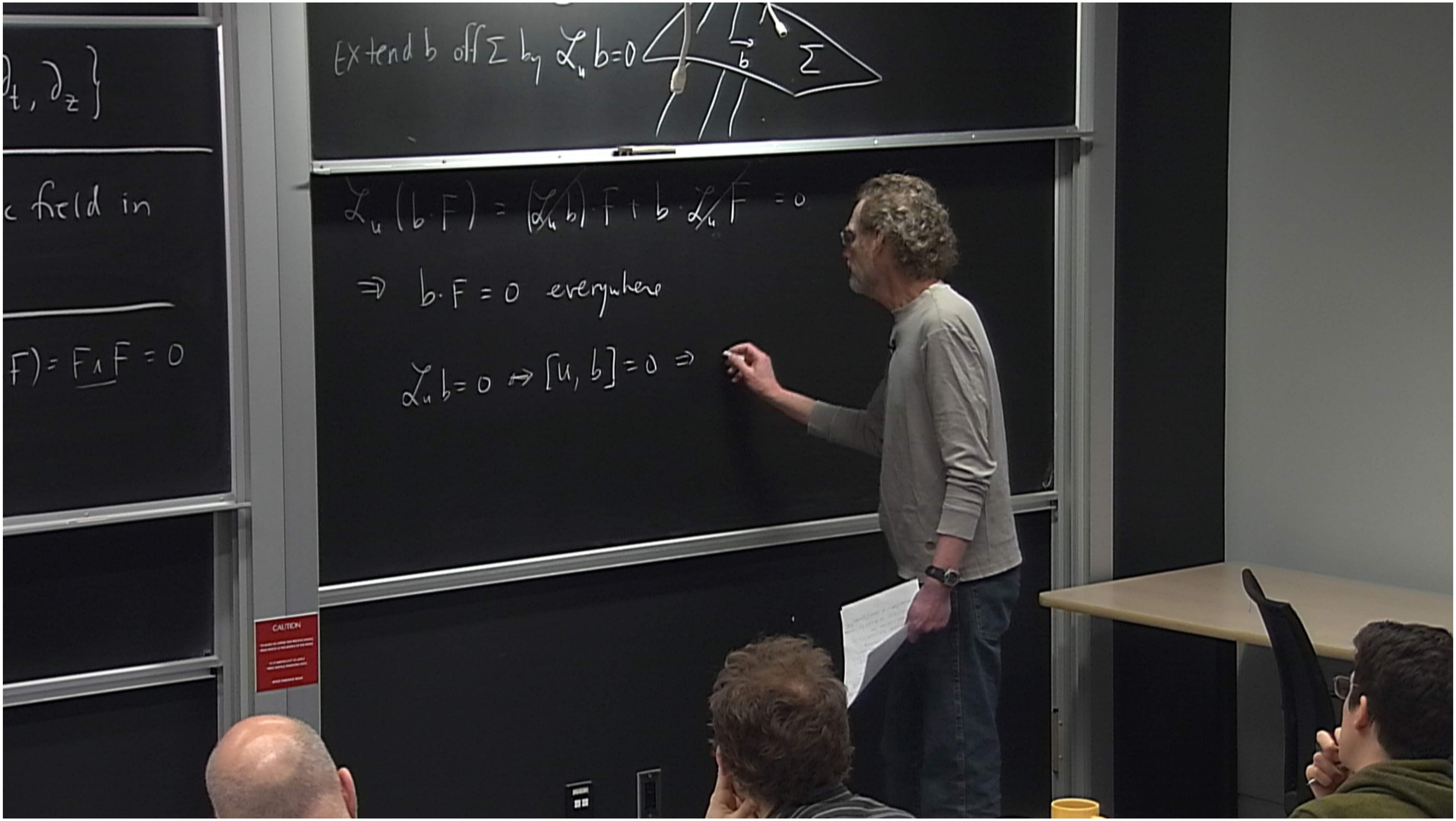

$\partial_t, \partial_z \}$
field in
$F) = F \wedge F = 0$
$dF = 0$ & $F \wedge F = 0 \Rightarrow F = d\phi_1 \wedge d\phi_2$

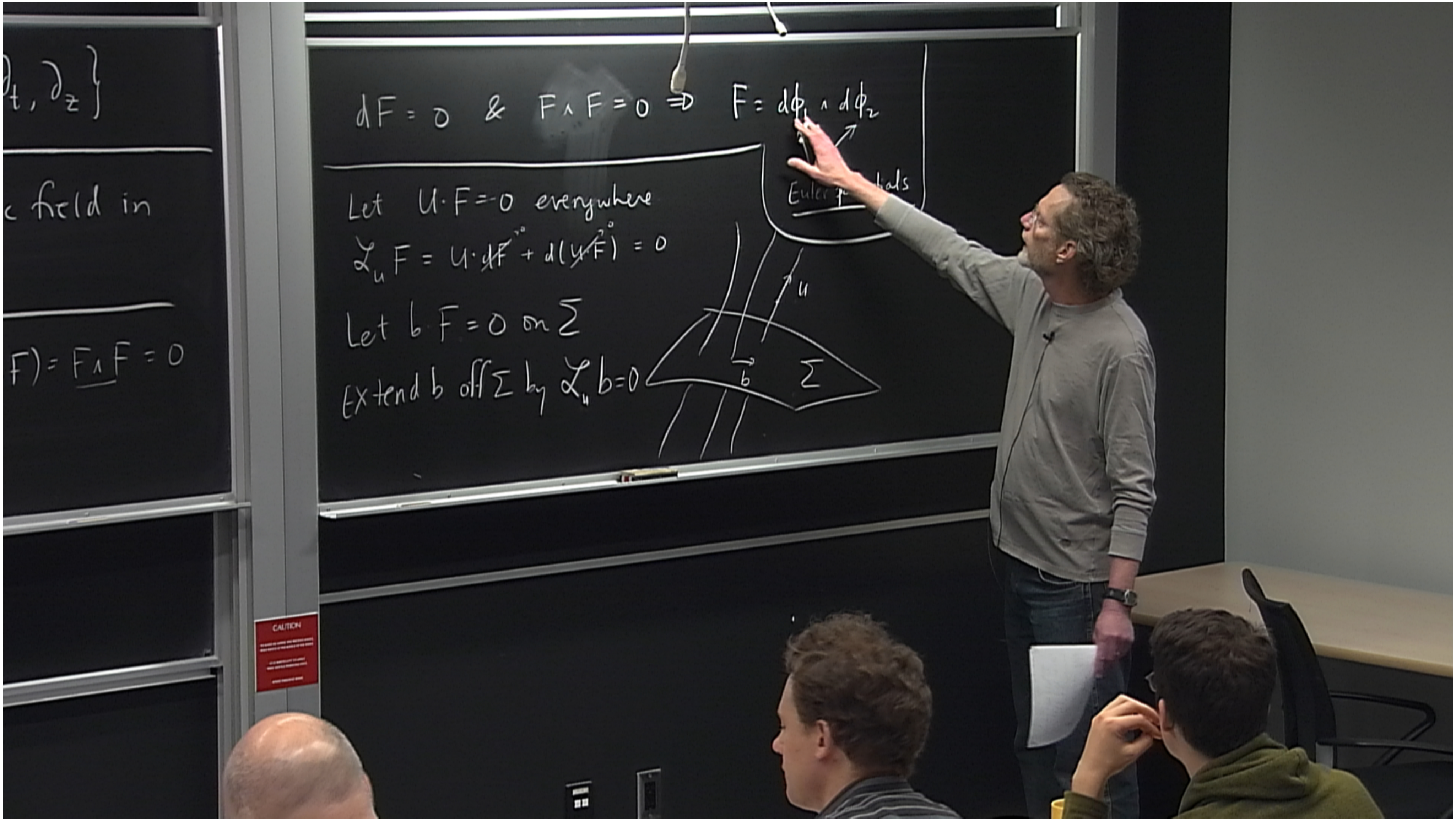

dF = 0 & F ∧ F = 0 ⇒ F = dφ1 ∧ dφ2
Euler poten
field in
F) = F ∧ F = 0
CAUTION

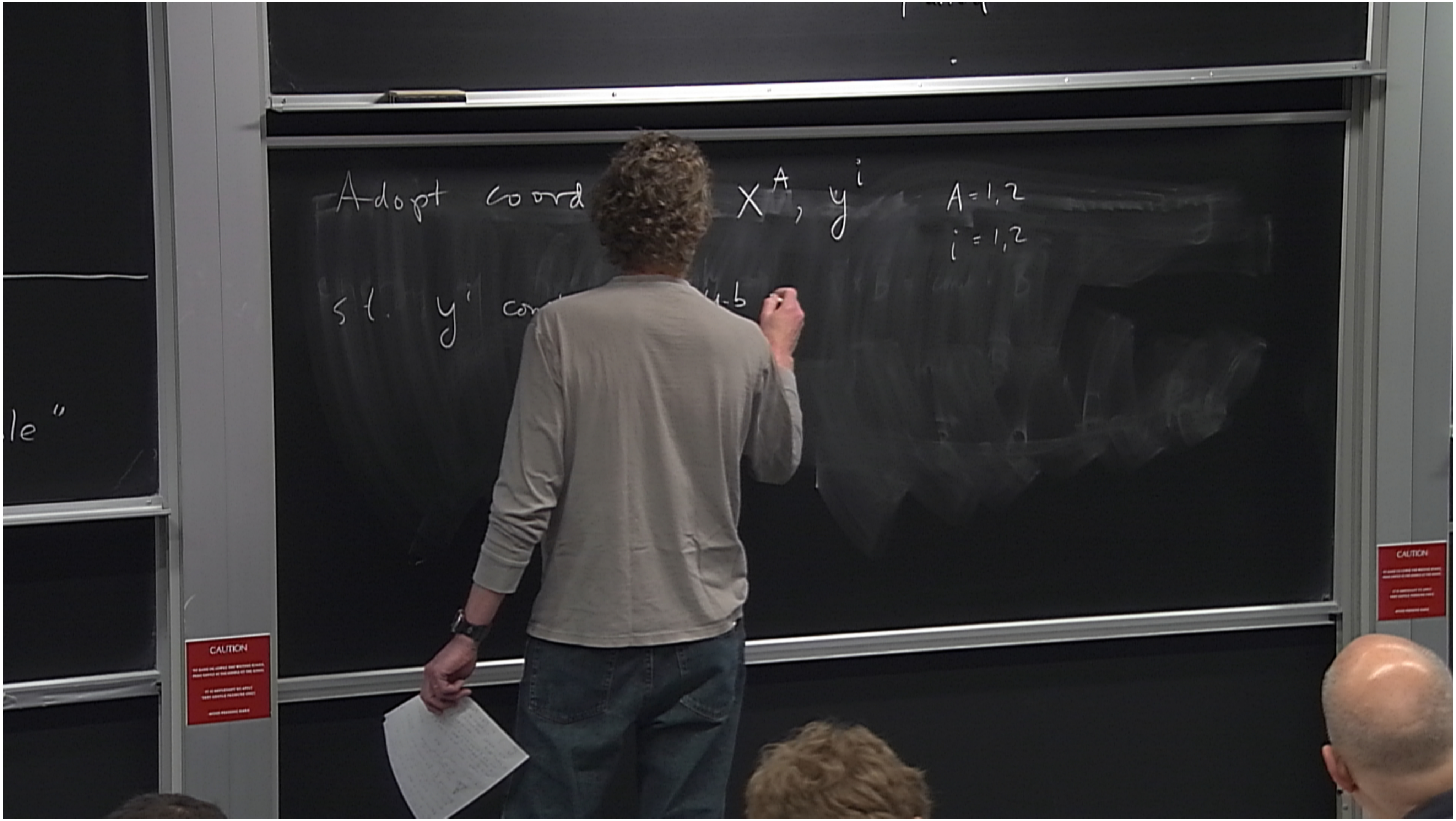

dF = 0 & F ∧ F = 0 ⇒ F = dφ1 ∧ dφ2
Euler potentials
field in
F) = F ∧ F = 0
CAUTION

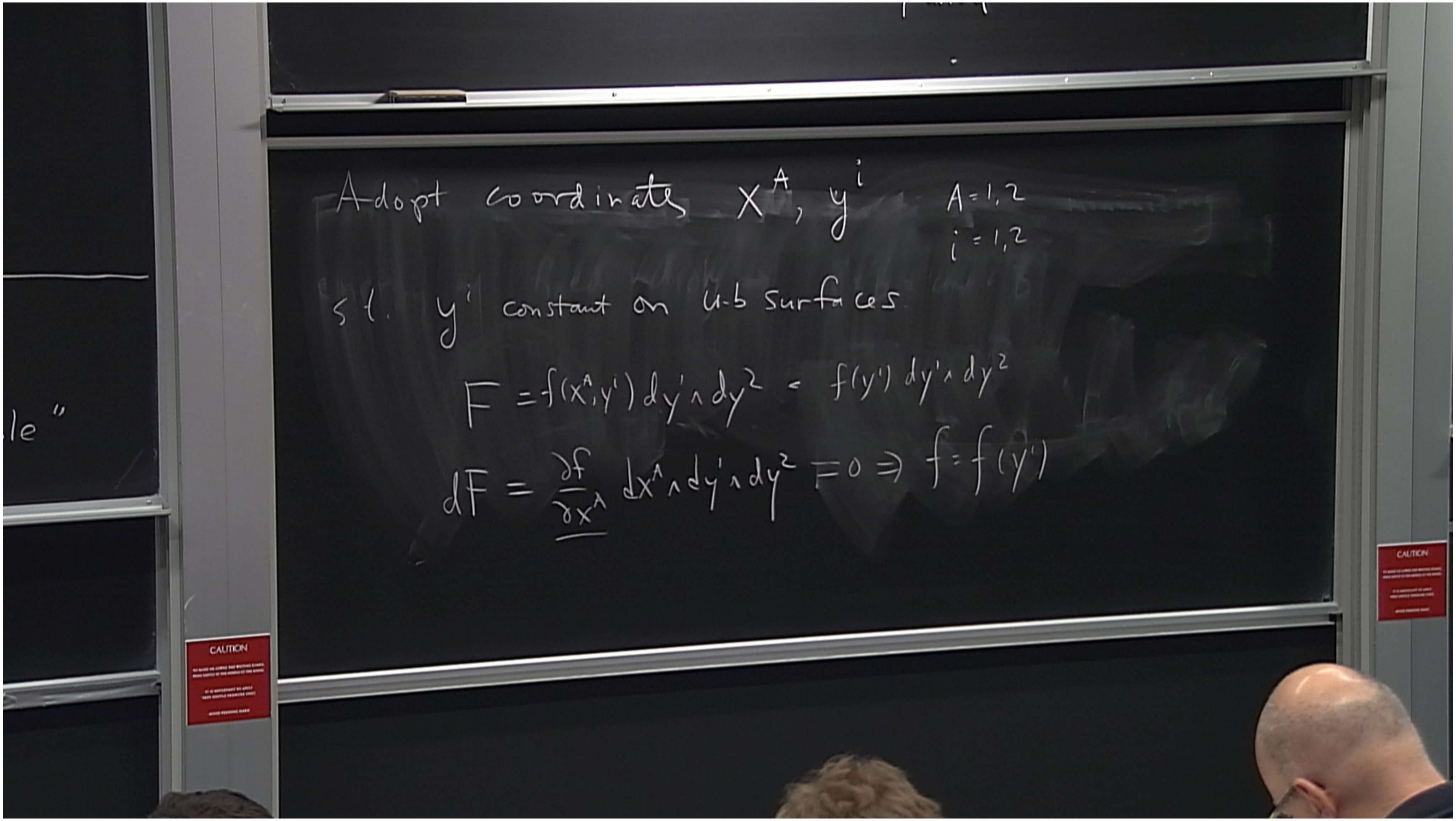

∂t, ∂z }
field in
F) = F∧F = 0
dF = 0 & F∧F = 0 ⇒ F = dφ1 ∧ dφ2
potentials
Let U·F = 0 everywhere
Lu F = U·dF + d(U·F) = 0
Let b·F = 0 on Σ
Extend b off Σ by Lu
b

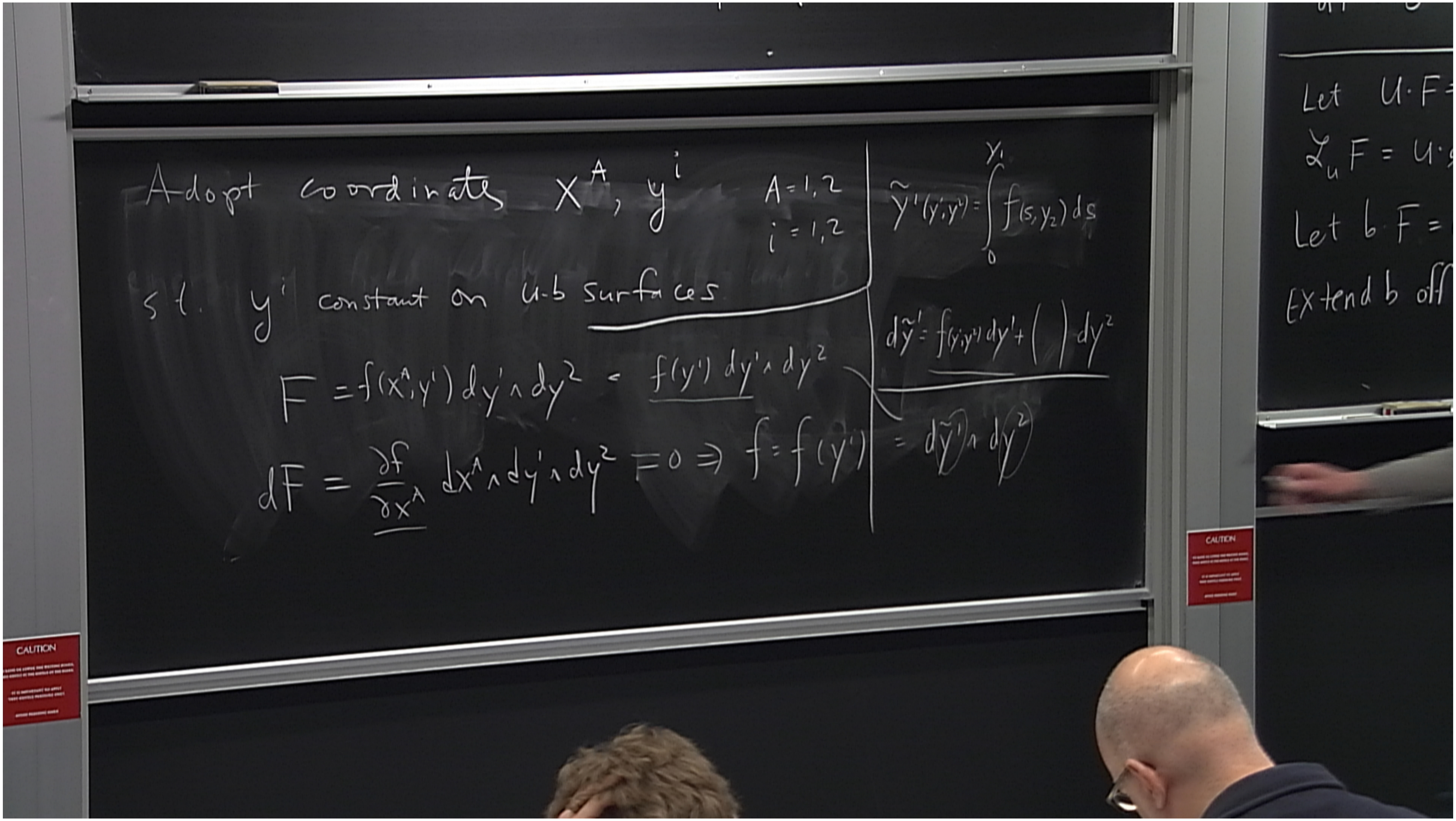

Extend b off Σ by $\mathcal{L}_u b = 0$
Σ
$\mathcal{L}_u (b \cdot F) =$
$= 0$
$\Rightarrow$ b F
c field in
$F) = F \wedge F = 0$
CAUTION

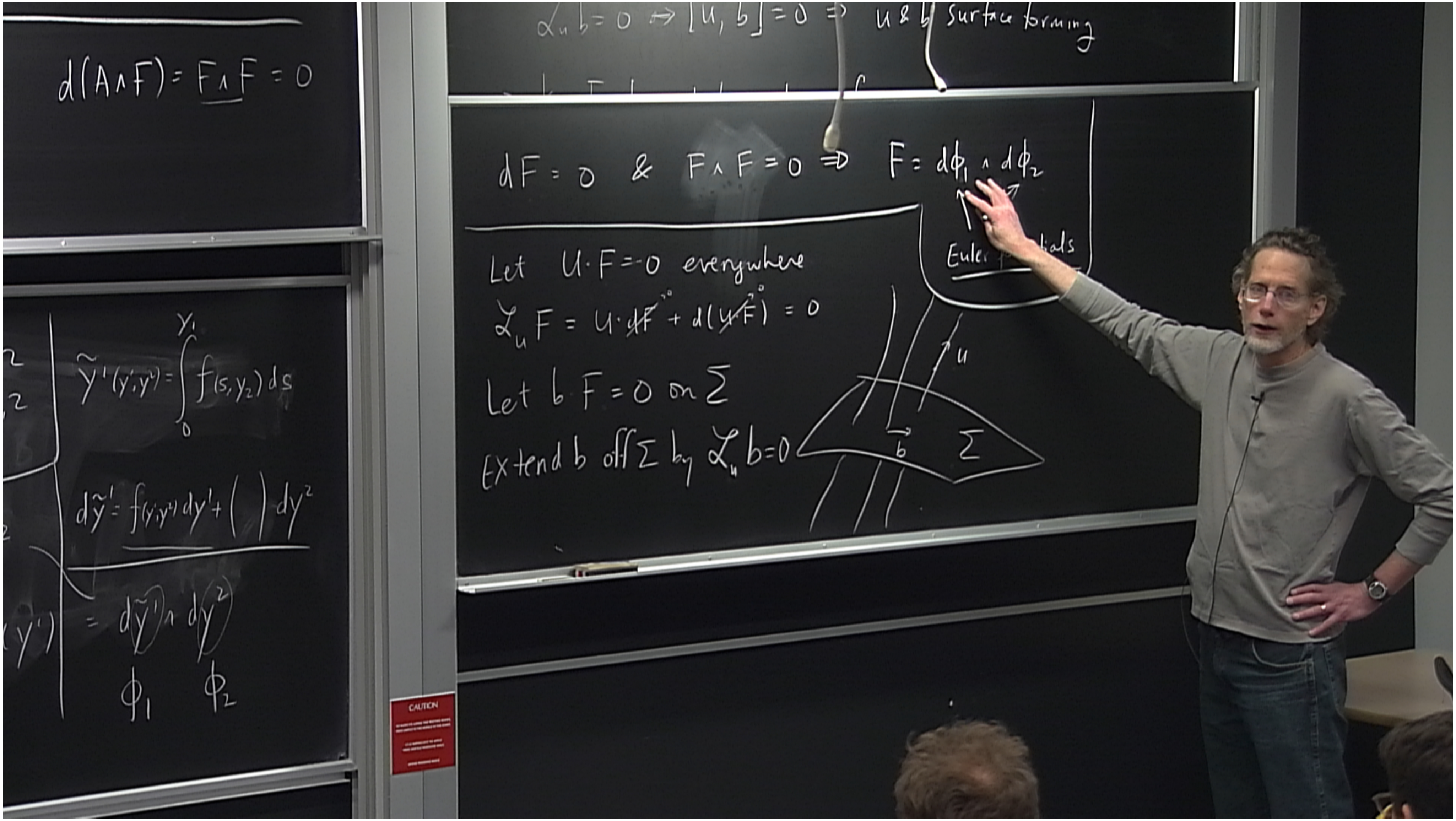

Extend b off Σ by $\mathcal{L}_u b = 0$
b
Σ
$\mathcal{L}_u (b \cdot F) = (\mathcal{L}_u b) \cdot F + b \cdot \mathcal{L}_u F = 0$
⇒ b · F = 0 everywhere
$\mathcal{L}_u b = 0 \Leftrightarrow [u, b] = 0 \Rightarrow$
field in
F) = F ∧ F = 0

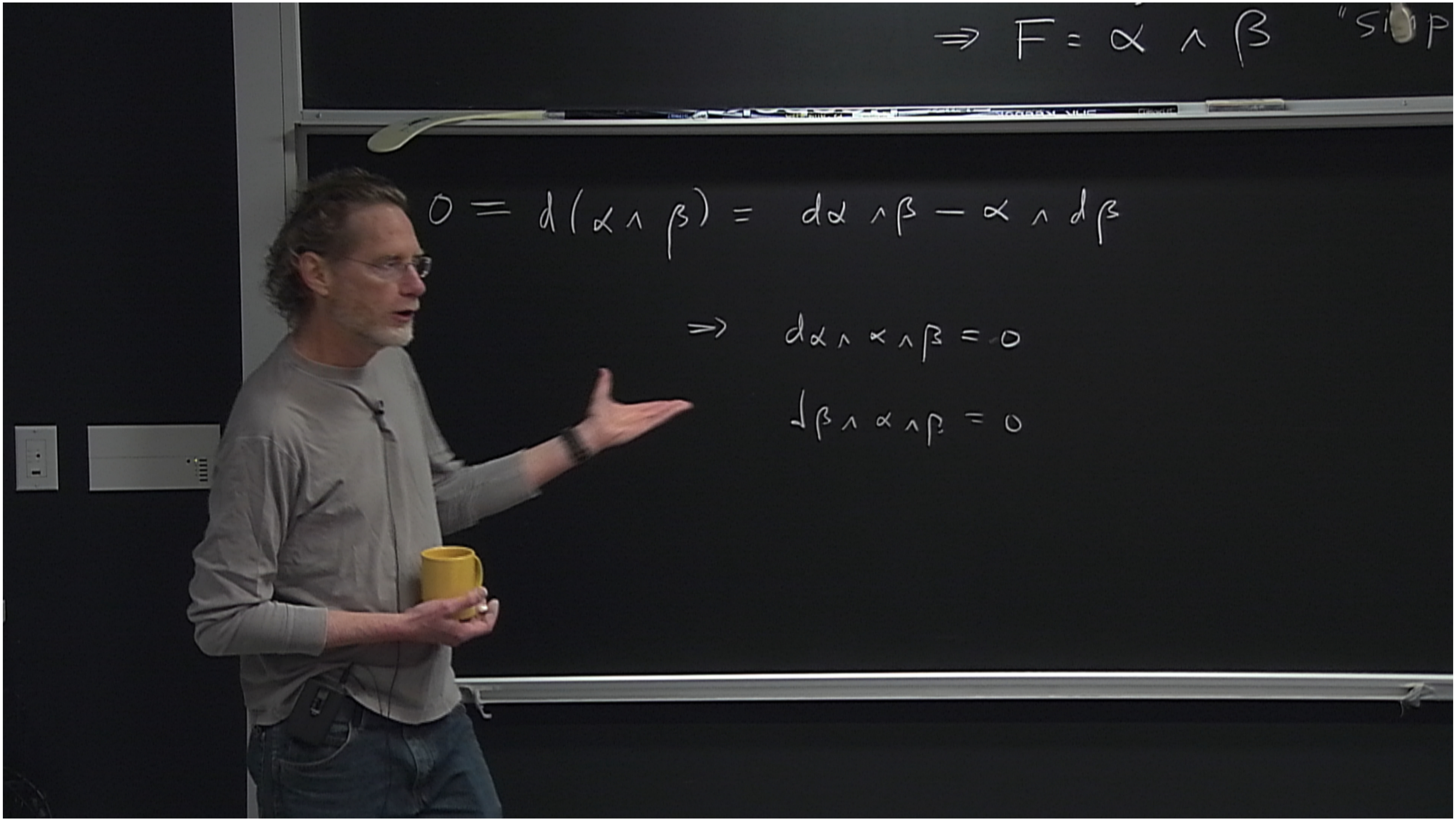

dF = 0 & F ∧ F = 0 ⇒ F = dφ1 ∧ dφ2
Euler potentials
Let U·F = 0 everywhere
Let b·F = 0 on Σ
Extend b off Σ by 𝓛u b = 0
u
b
Σ
field in
F ∧ F = 0

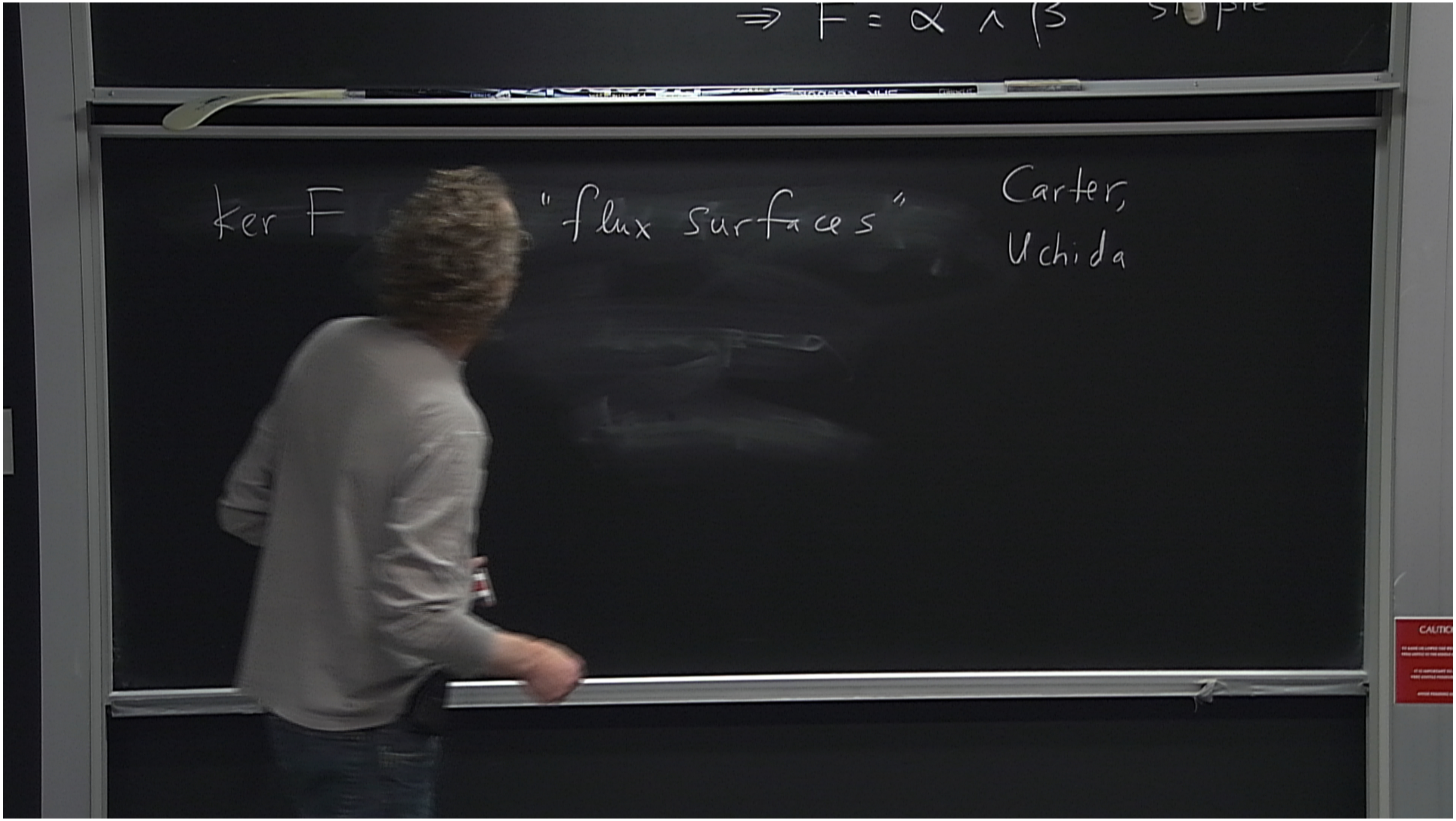

Adopt coord
X^A, y^i
A = 1,2
i = 1,2
s.t. y^i

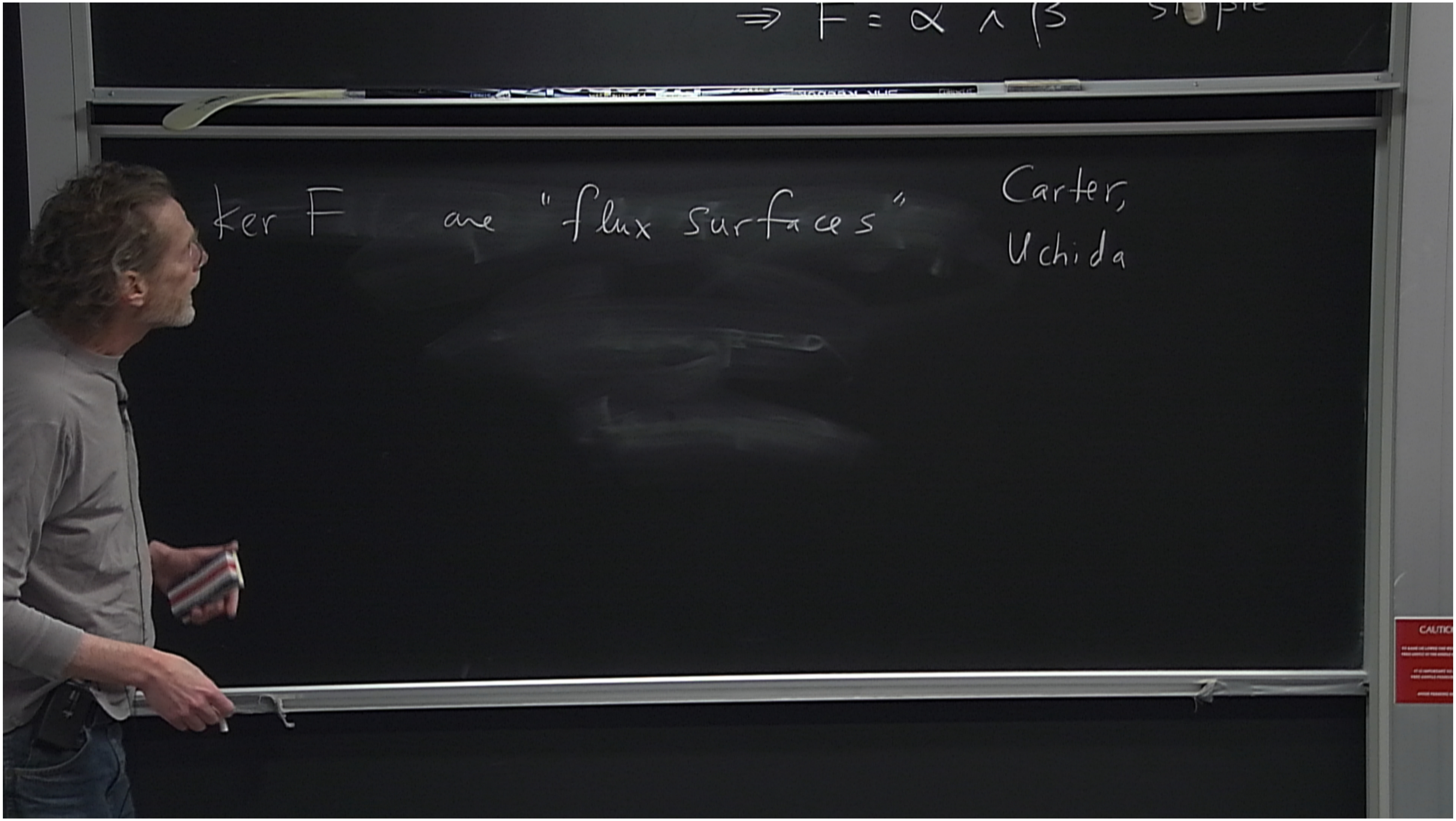

Adopt coordinates x^A, y^i
$A=1,2$
$i=1,2$
s.t. y^i constant on w-b surfaces
$F = f(x^A, y^i)\, dy^1 \wedge dy^2 = f(y^i)\, dy^1 \wedge dy^2$
$dF = \frac{\partial f}{\partial x^A} dx^A \wedge dy^1 \wedge dy^2 = 0 \Rightarrow f = f(y^i)$

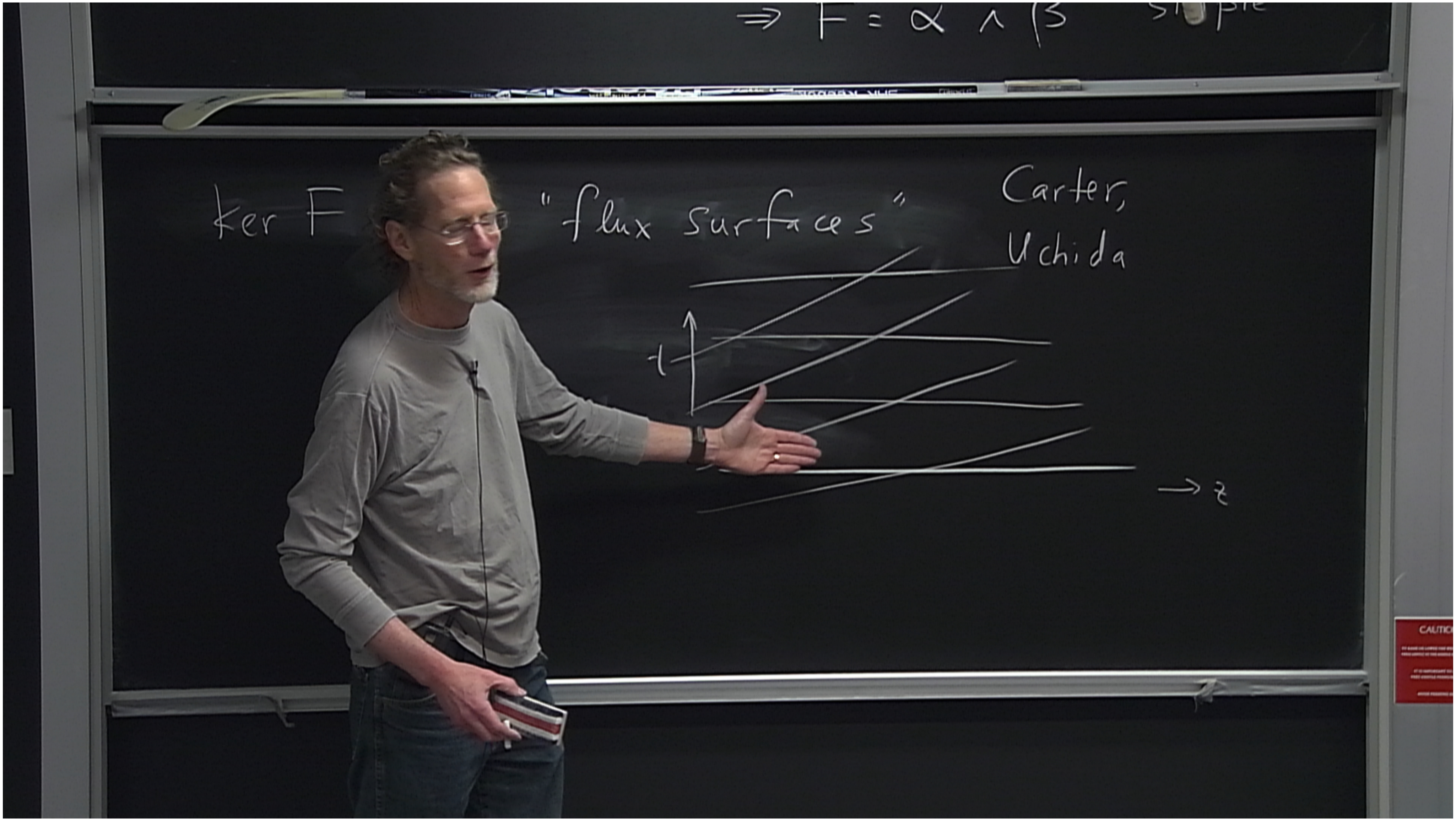

Adopt coordinates X^A, y^i
A = 1, 2
i = 1, 2
s.t. y^i constant on u-b surfaces
$F = f(x^A, y^i)\, dy^1 \wedge dy^2 = f(y^i)\, dy^1 \wedge dy^2$
$dF = \frac{\partial f}{\partial x^A} dx^A \wedge dy^1 \wedge dy^2 = 0 \Rightarrow f = f(y^i)$
$\tilde{y}^1(y^1, y^2) = \int_0^{y_1} f(s, y_2)\, ds$
$d\tilde{y}^1 = f(y^1, y^2)\, dy^1 + (\ )\, dy^2$
$= d\tilde{y}^1 \wedge dy^2$
Let U·F =
$\mathcal{L}_u F$ = U·
Let b·F =
Extend b off
CAUTION

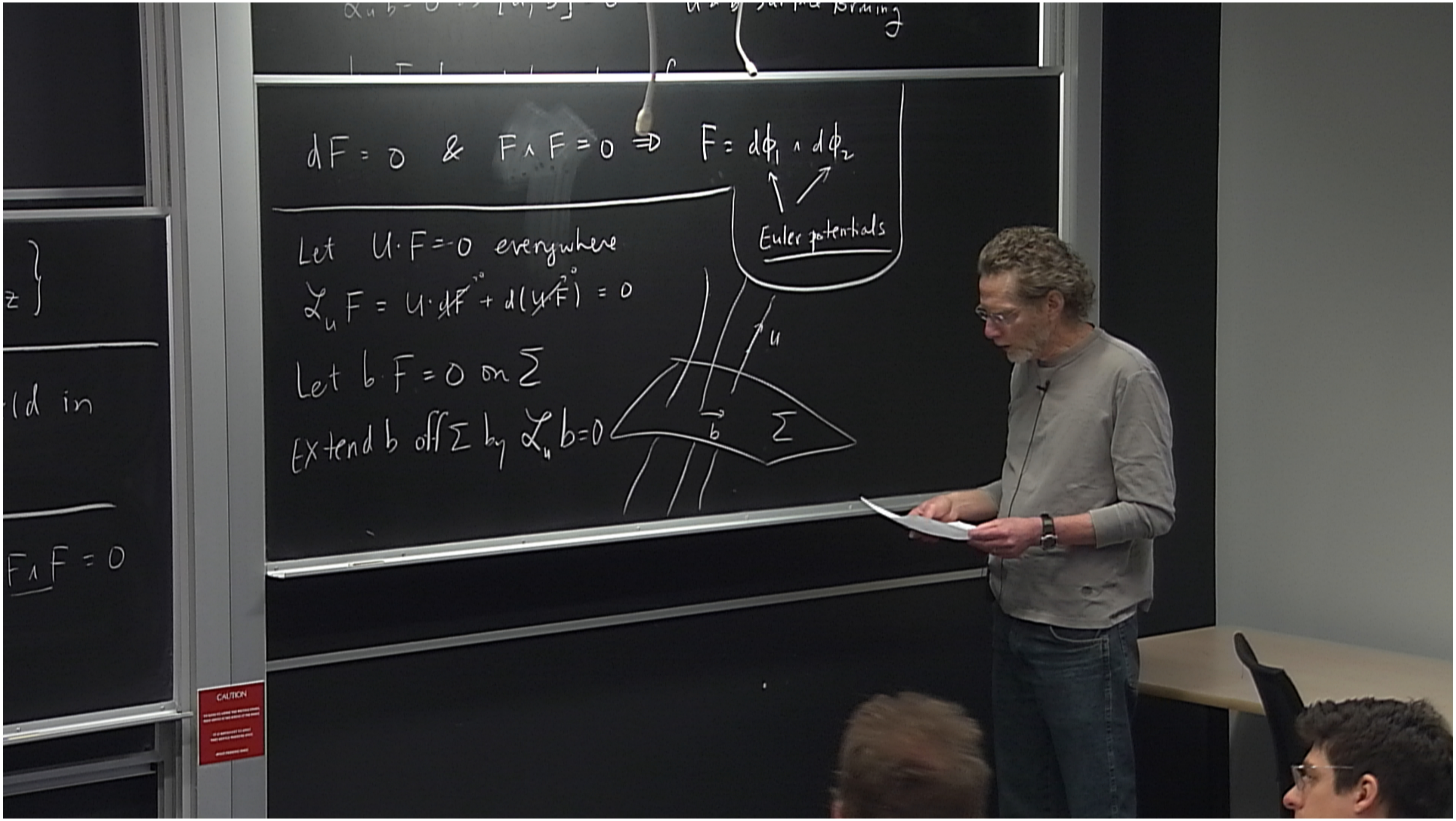

d(A∧F) = F∧F = 0
dF = 0 & F∧F = 0 ⇒ F = dφ1 ∧ dφ2
Let U·F = 0 everywhere
Let b·F = 0 on Σ
Extend b off Σ by ℒu b = 0
Σ
u & b surface forming


⇒ F = α ∧ β "simp
0 = d(α ∧ β) = dα ∧ β − α ∧ dβ
⇒ dα ∧ α ∧ β = 0
dβ ∧ α ∧ β = 0

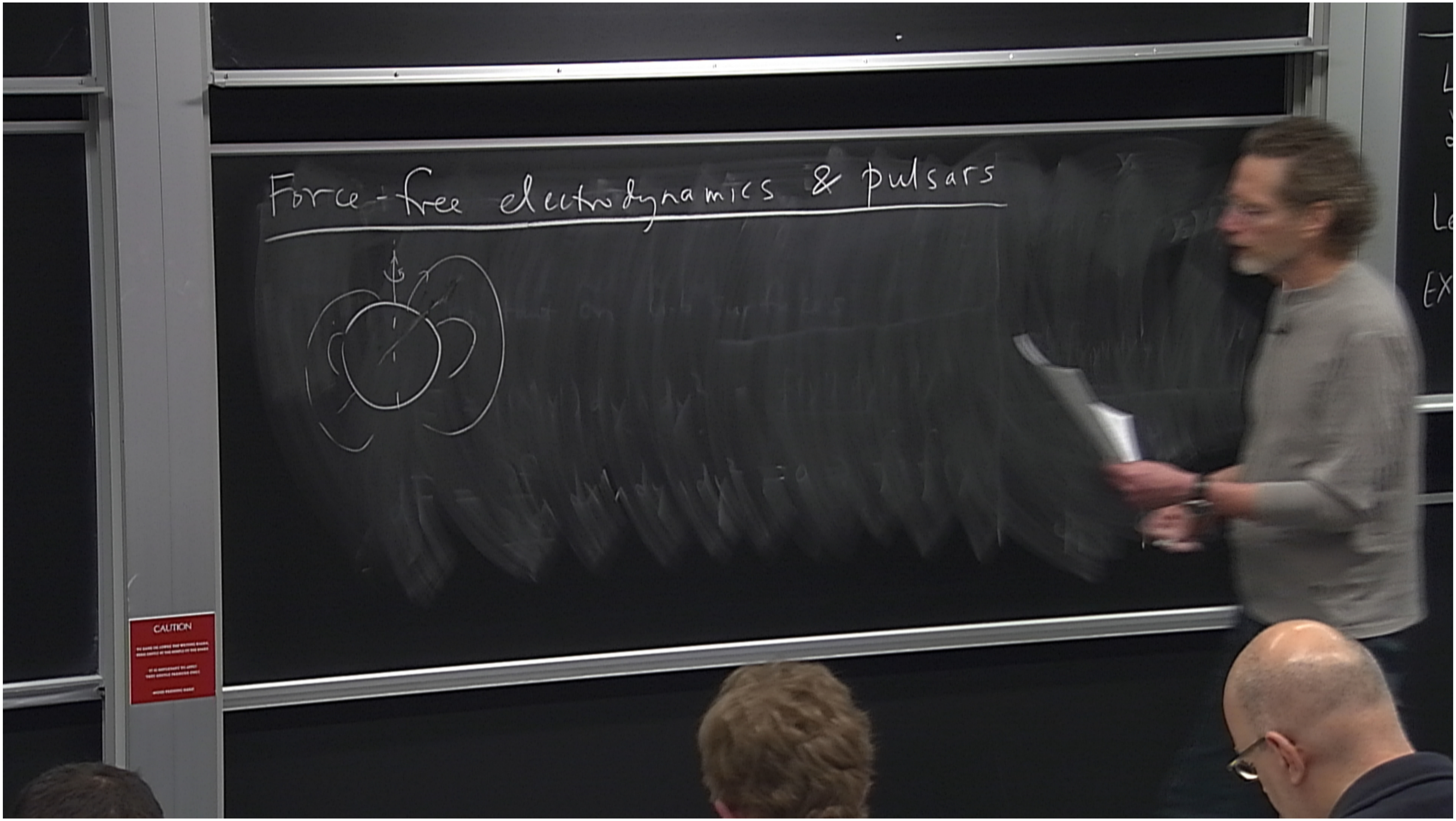

⇒ F = α ∧ β
ker F
"flux surfaces"
Carter,
Uchida

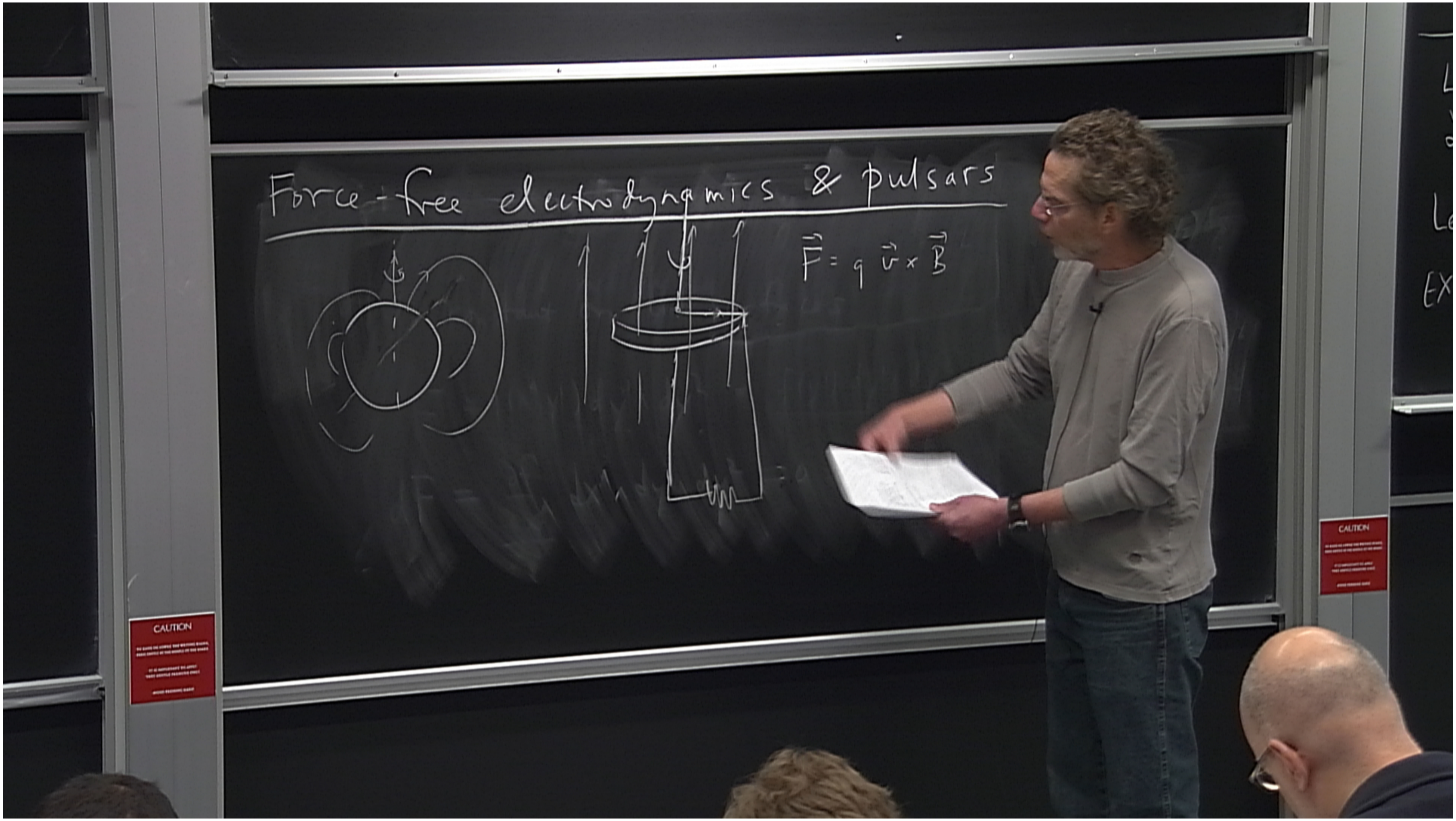

⇒ F = α ∧ β
ker F are "flux surfaces"
Carter,
Uchida

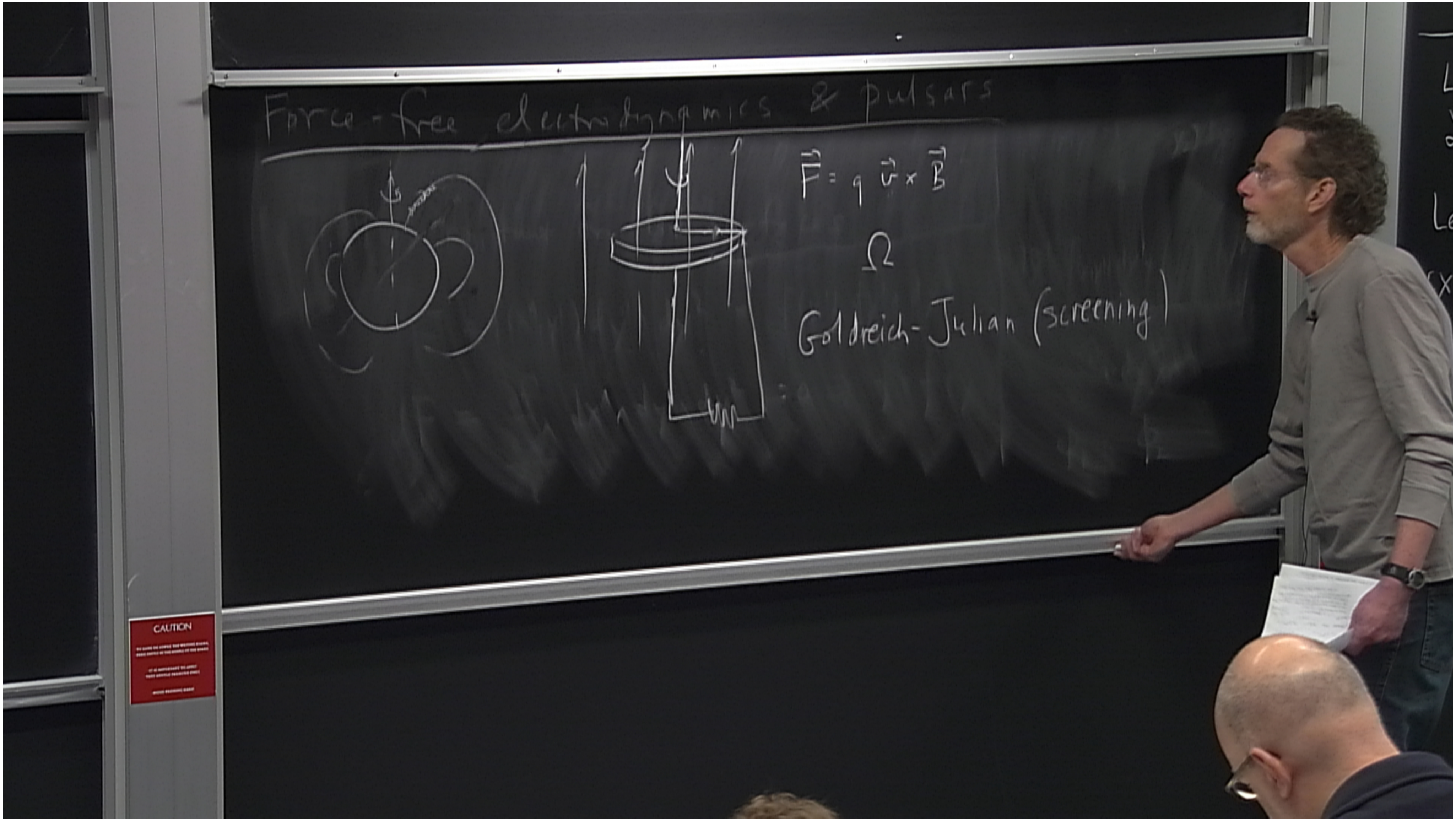

⇒ F = α ∧ β
ker F "flux surfaces"
Carter, Uchida
t
→ z

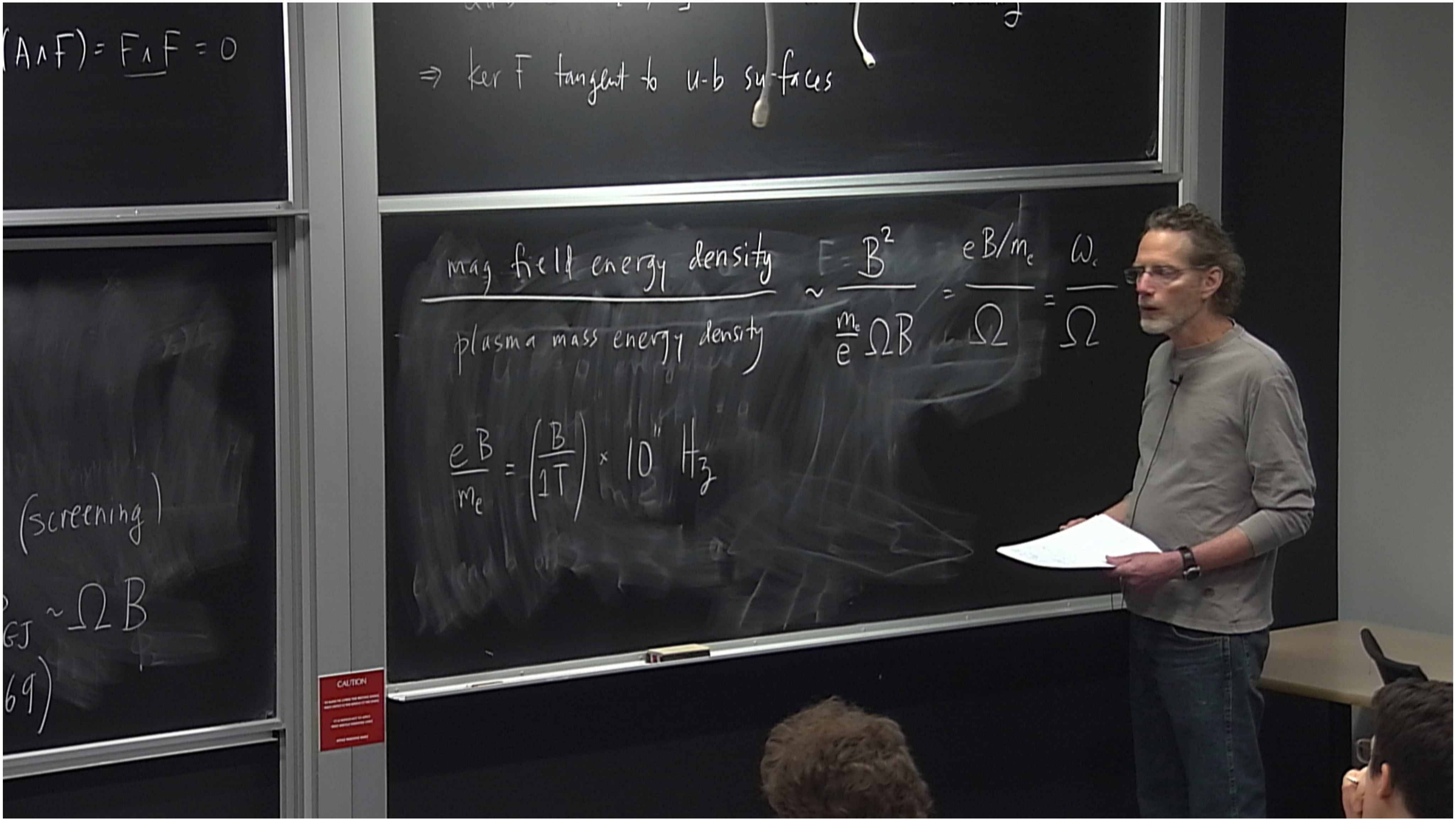


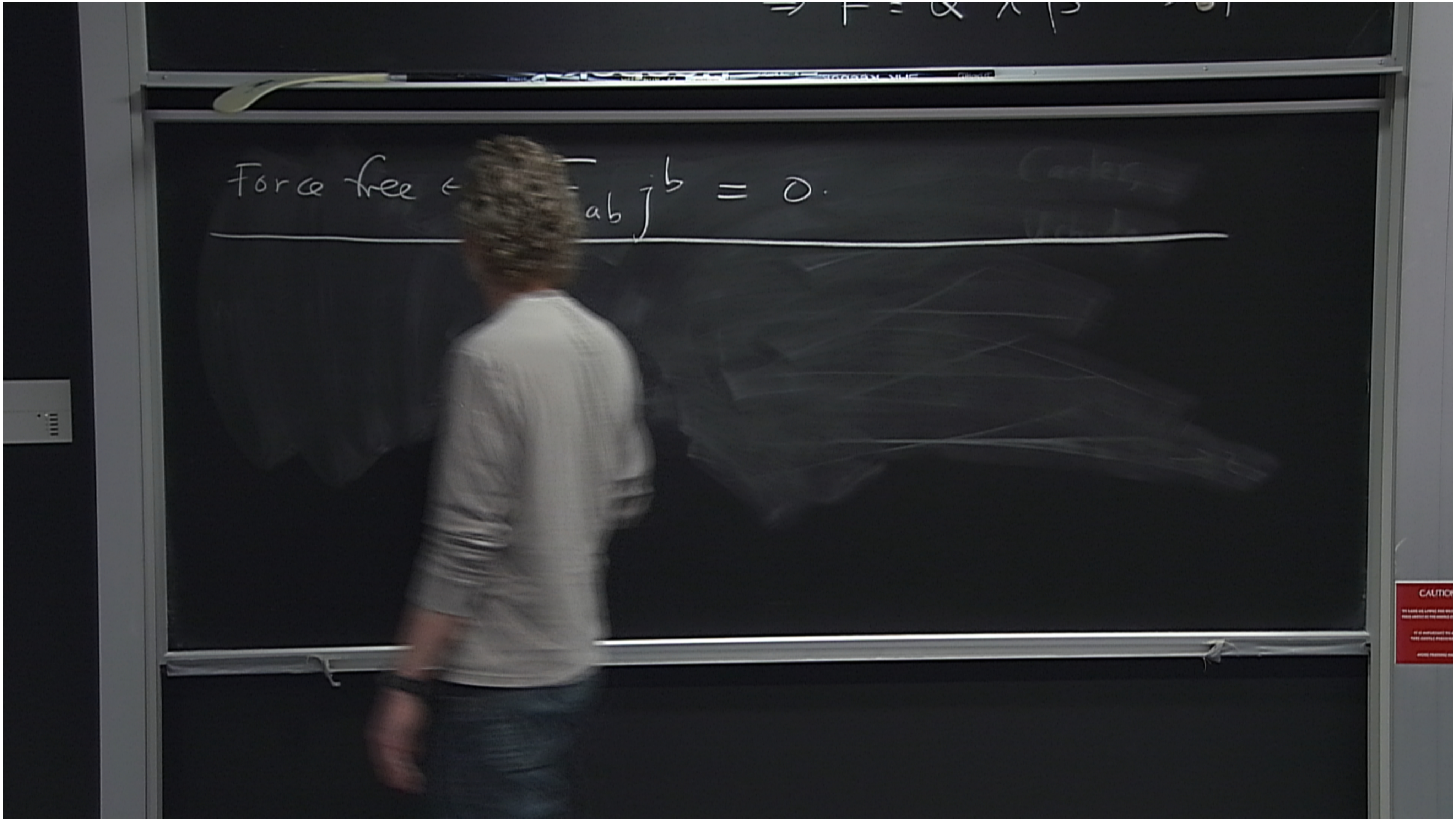
dF = 0 & F∧F = 0 ⇒ F = dφ1 ∧ dφ2
Euler potentials
Let U·F = 0 everywhere
Let b·F = 0 on Σ
Extend b off Σ by ℒu b = 0
u
b
Σ
F∧F = 0
CAUTION

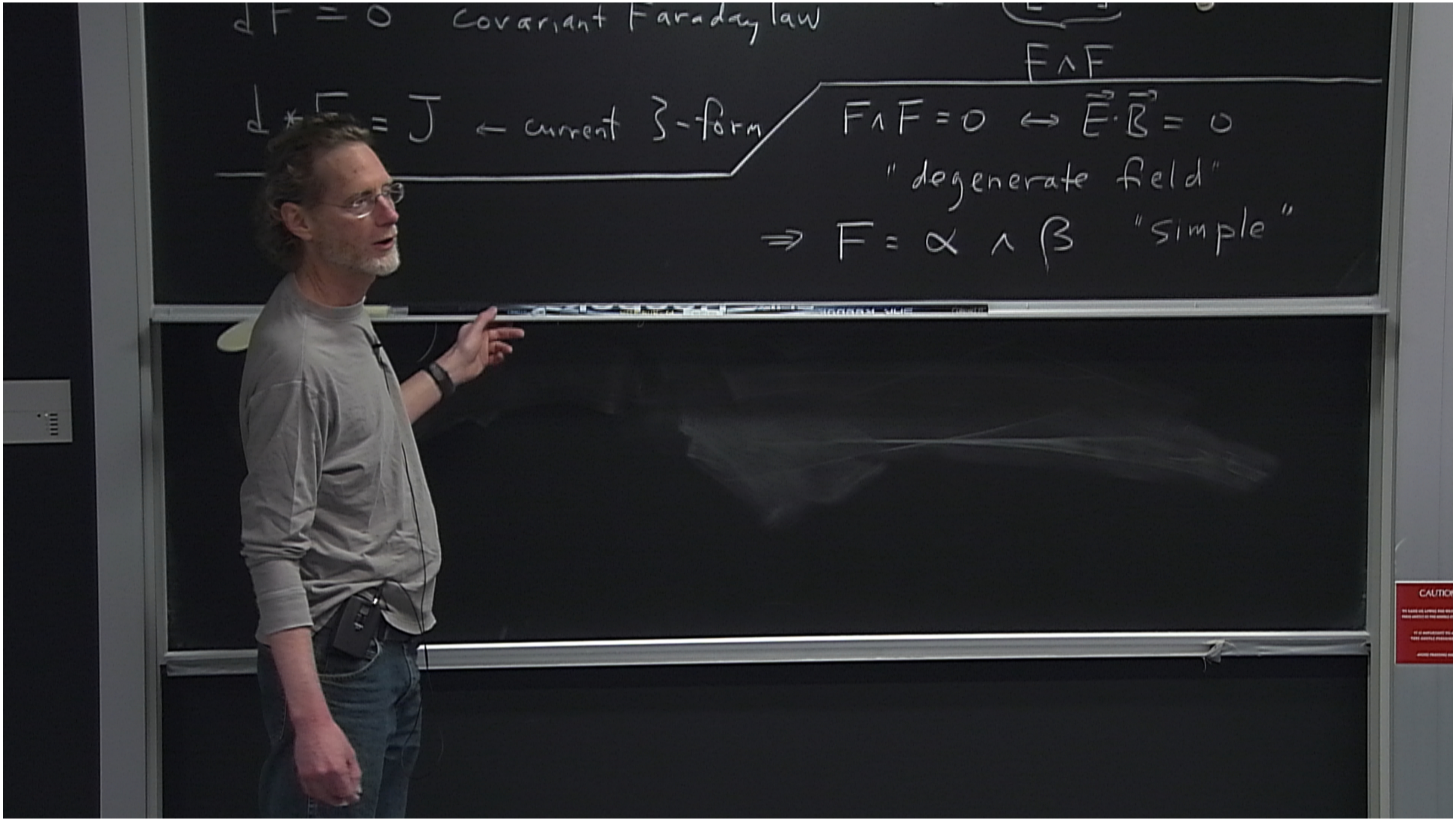

Force-free electrodynamics & pulsars
CAUTION

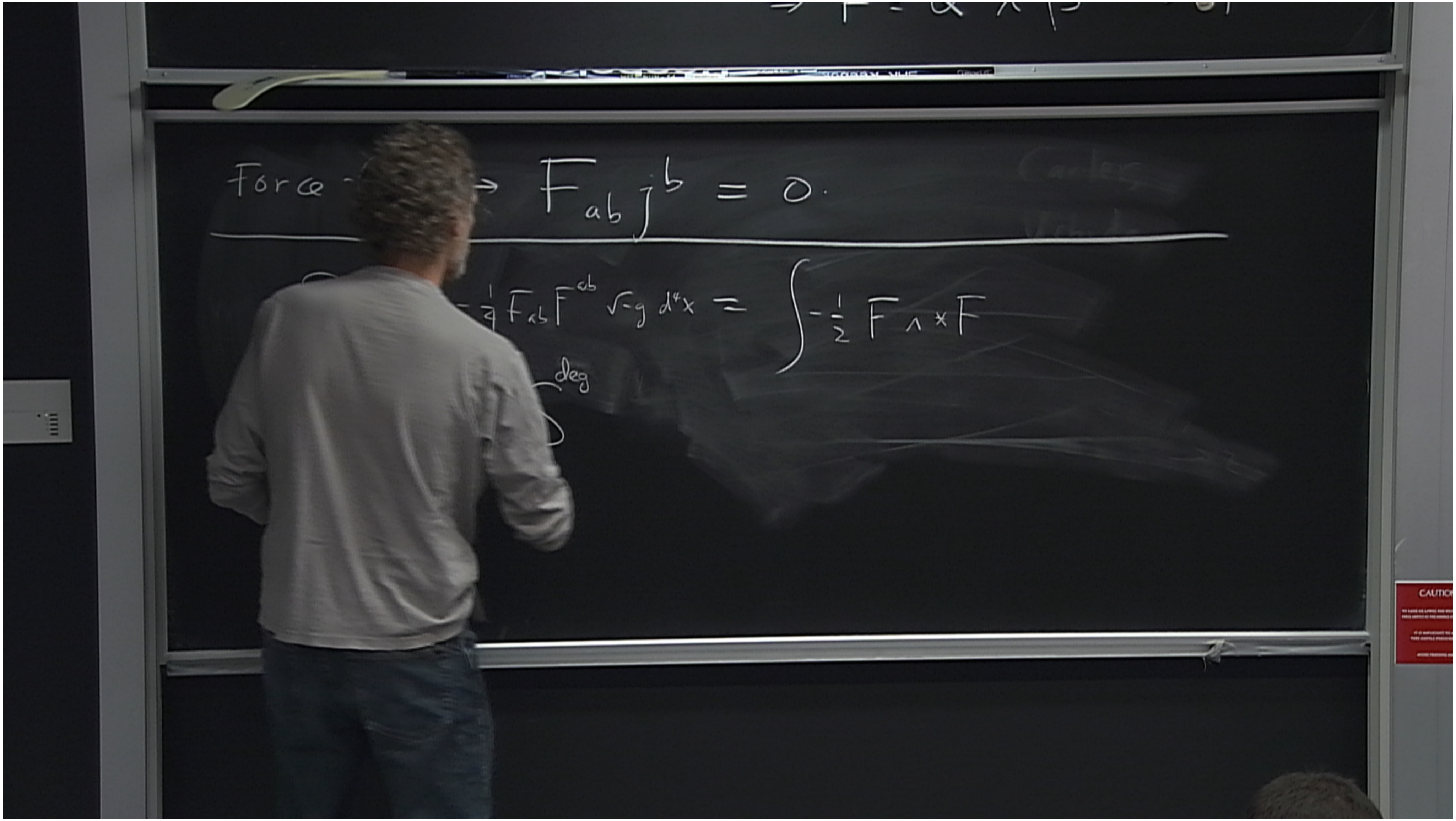

Force-free electrodynamics & pulsars
F = q v × B
CAUTION

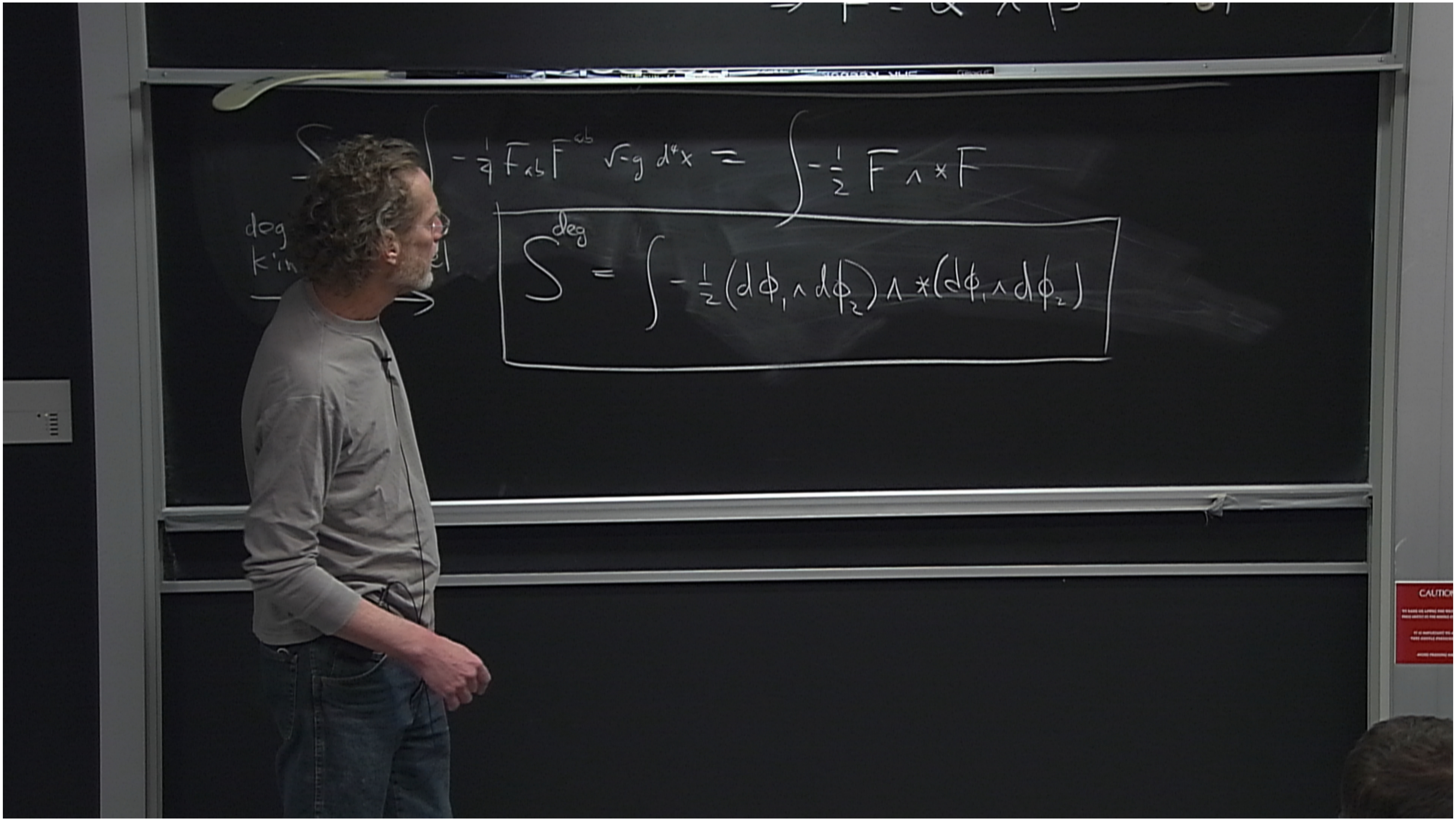

Force-free electrodynamics & pulsars
F = q v × B
Ω
Goldreich-Julian (screening)
CAUTION

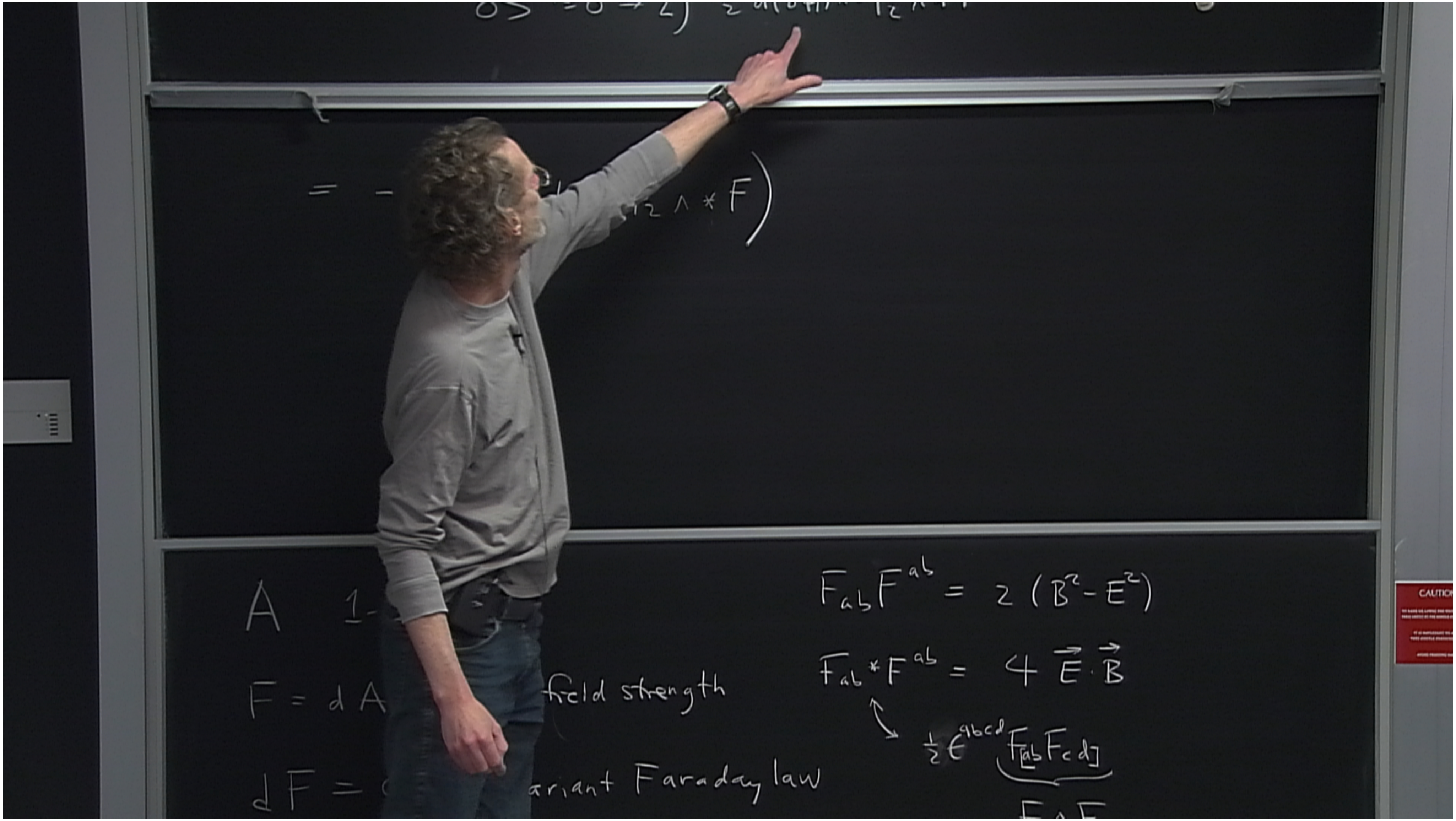

(A∧F) = F∧F = 0
⇒ ker F tangent to u-b surfaces
mag field energy density / plasma mass energy density ~ B² / (mₑ/e ΩB) = (eB/mₑ) / Ω = ωc / Ω
eB/mₑ = (B/1T) × 10¹¹ Hz
(screening)
GJ ~ ΩB
CAUTION

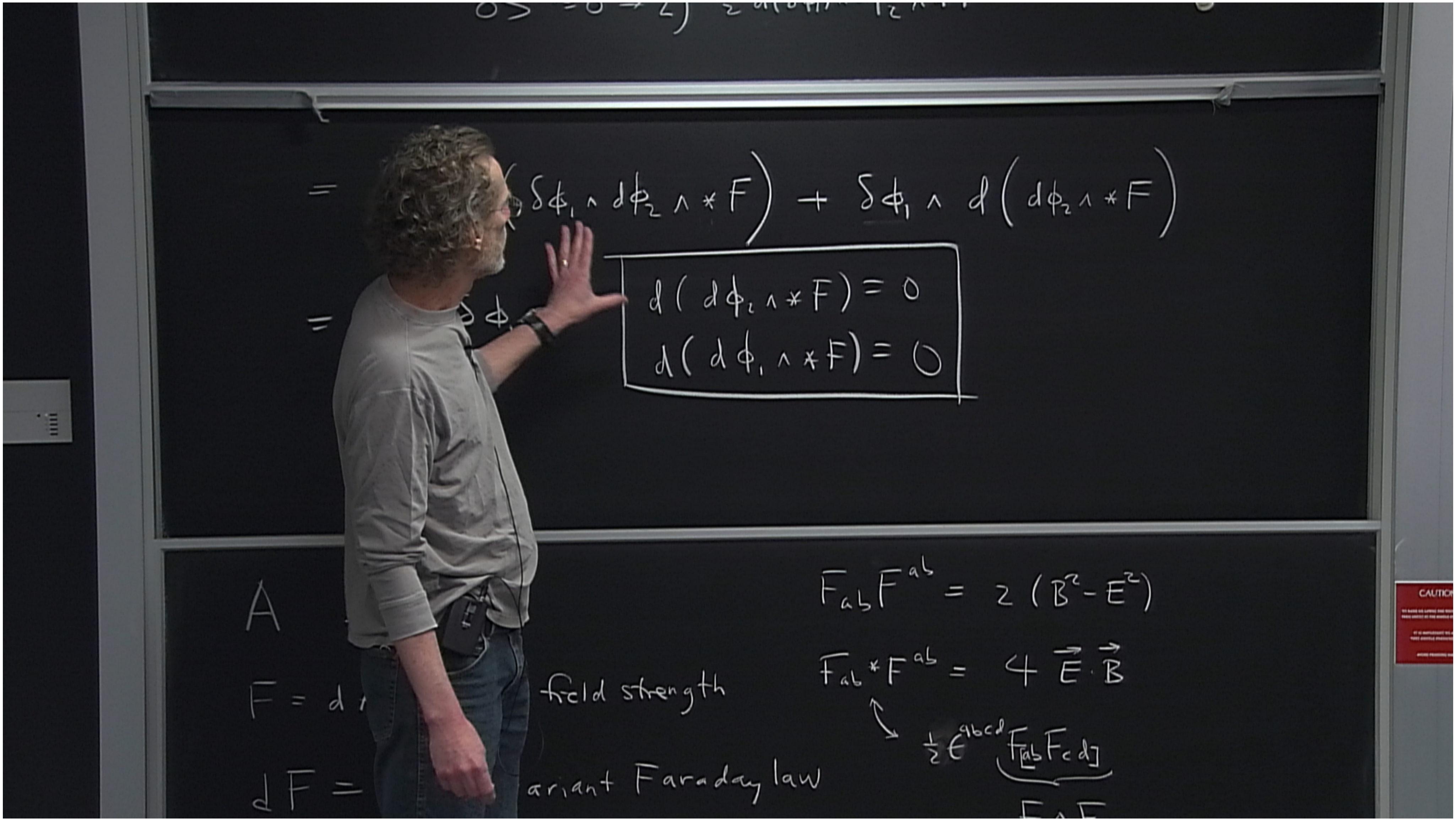

Force free
= 0.

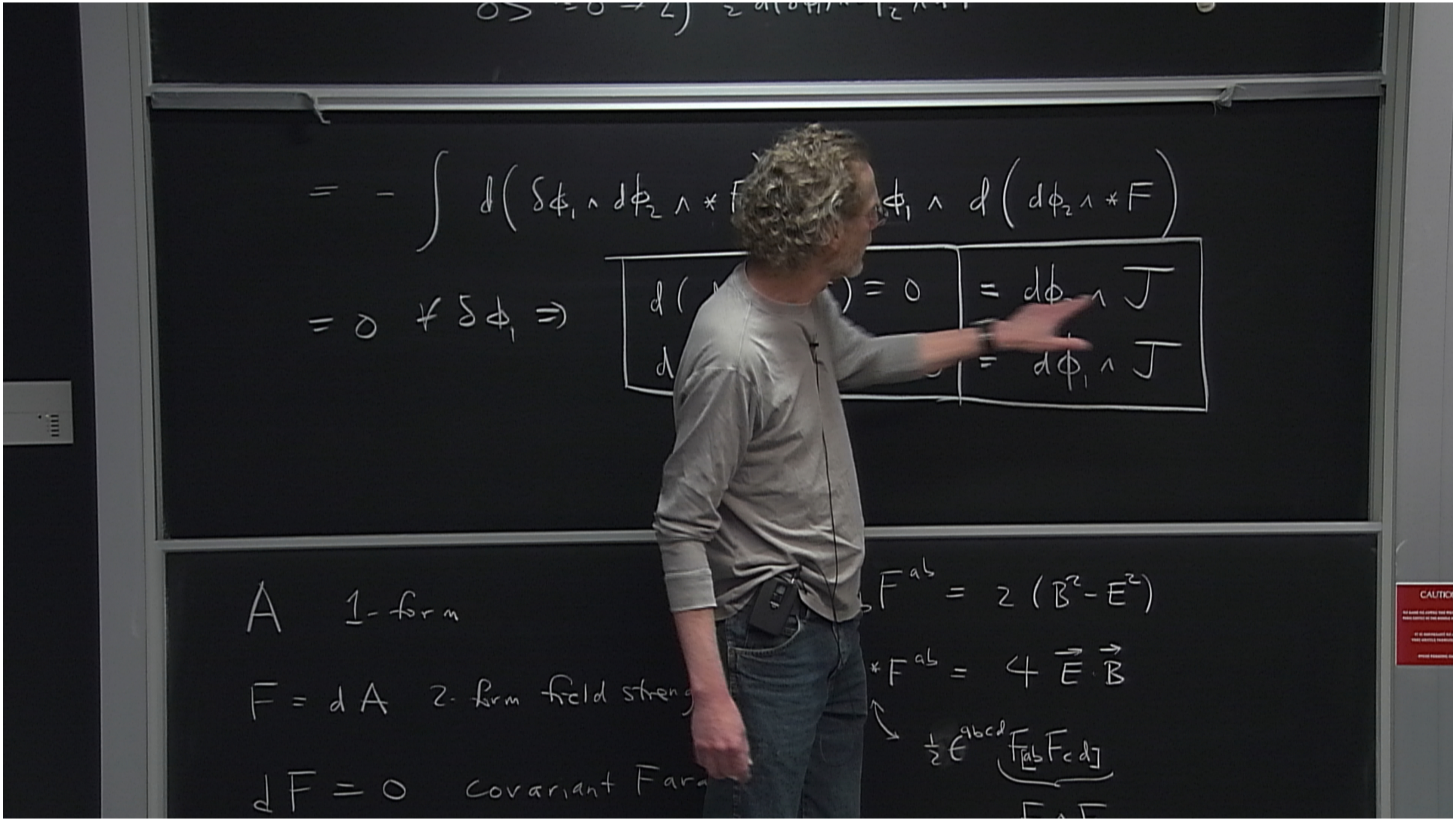

dF = 0 covariant Faraday law
d*F = J ← current 3-form
F∧F
F∧F = 0 ↔ E·B = 0
"degenerate field"
⇒ F = α ∧ β "simple"
CAUTION

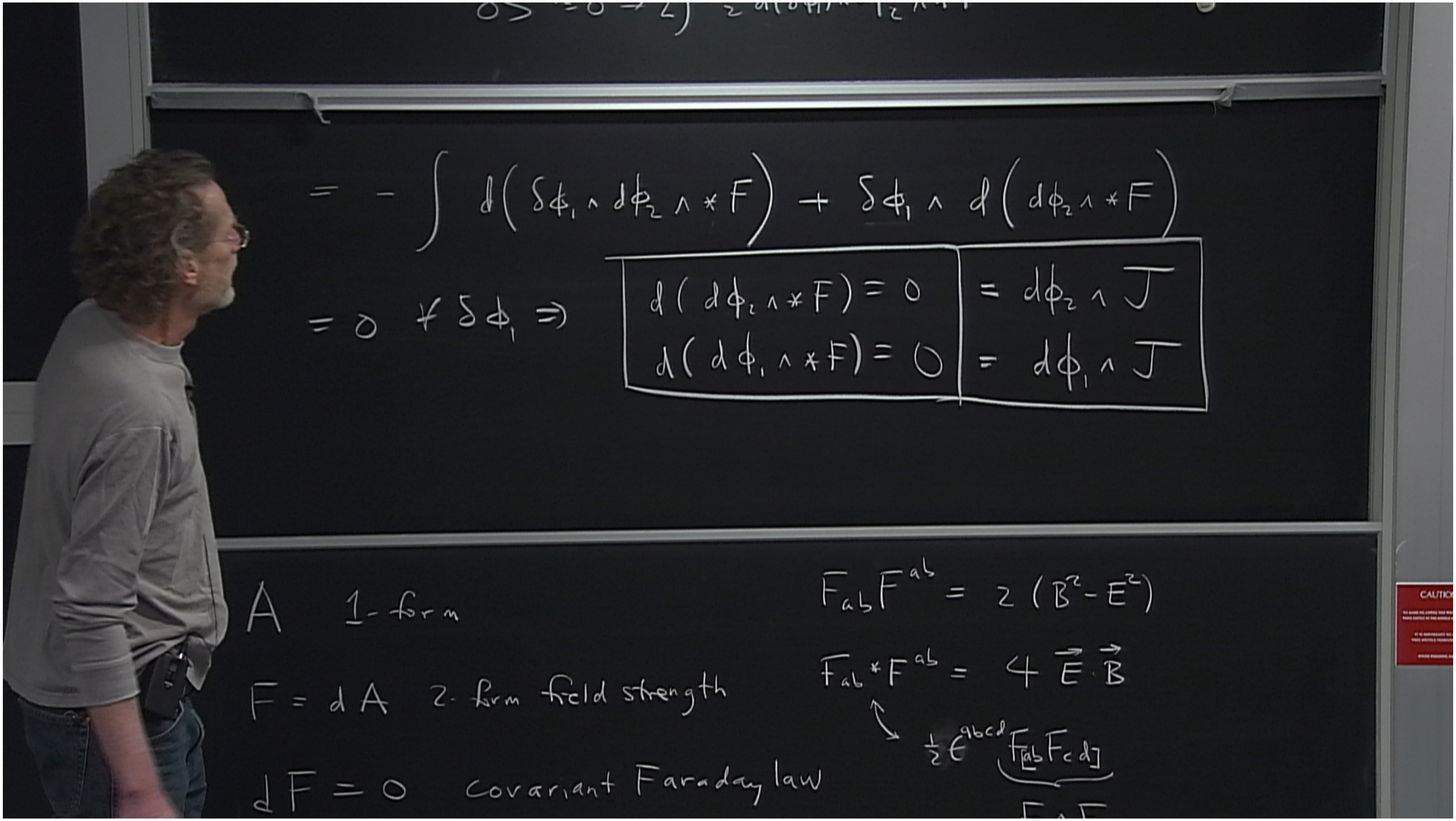

Force
$F_{ab} j^b = 0.$
$\frac{1}{4} F_{ab} F^{ab} \sqrt{-g}\, d^4x = \int -\frac{1}{2} F \wedge *F$
deg

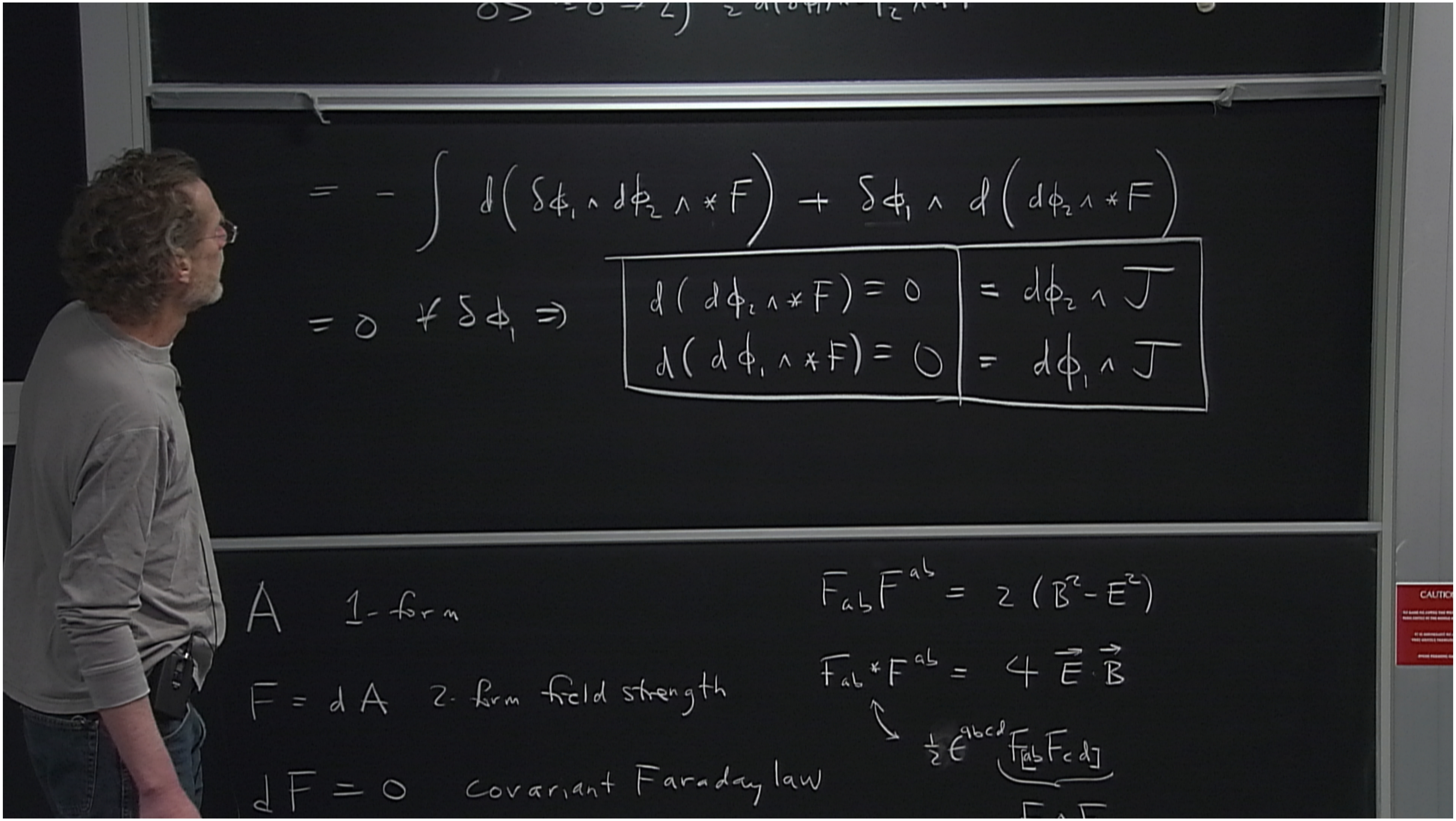

$\int -\frac{1}{4} F_{ab} F^{ab} \sqrt{-g}\, d^4x = \int -\frac{1}{2} F \wedge * F$
$S^{deg} = \int -\frac{1}{2} (d\phi_1 \wedge d\phi_2) \wedge *(d\phi_1 \wedge d\phi_2)$
CAUTION

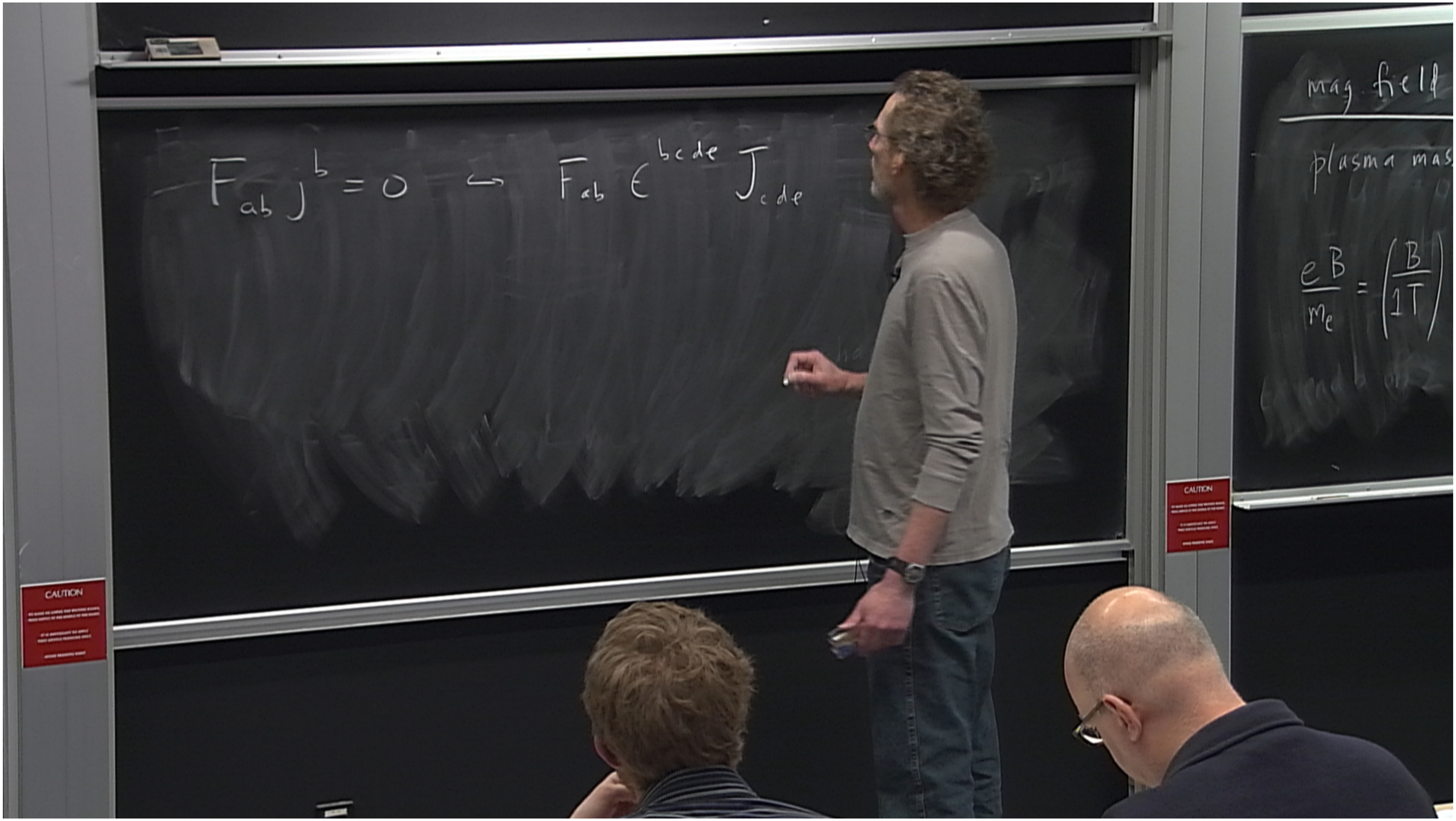

A
F = d A
field strength
d F =
ariant Faraday law
F_ab F^ab = 2 (B² - E²)
F_ab *F^ab = 4 E · B
½ ε^abcd F[ab Fcd]

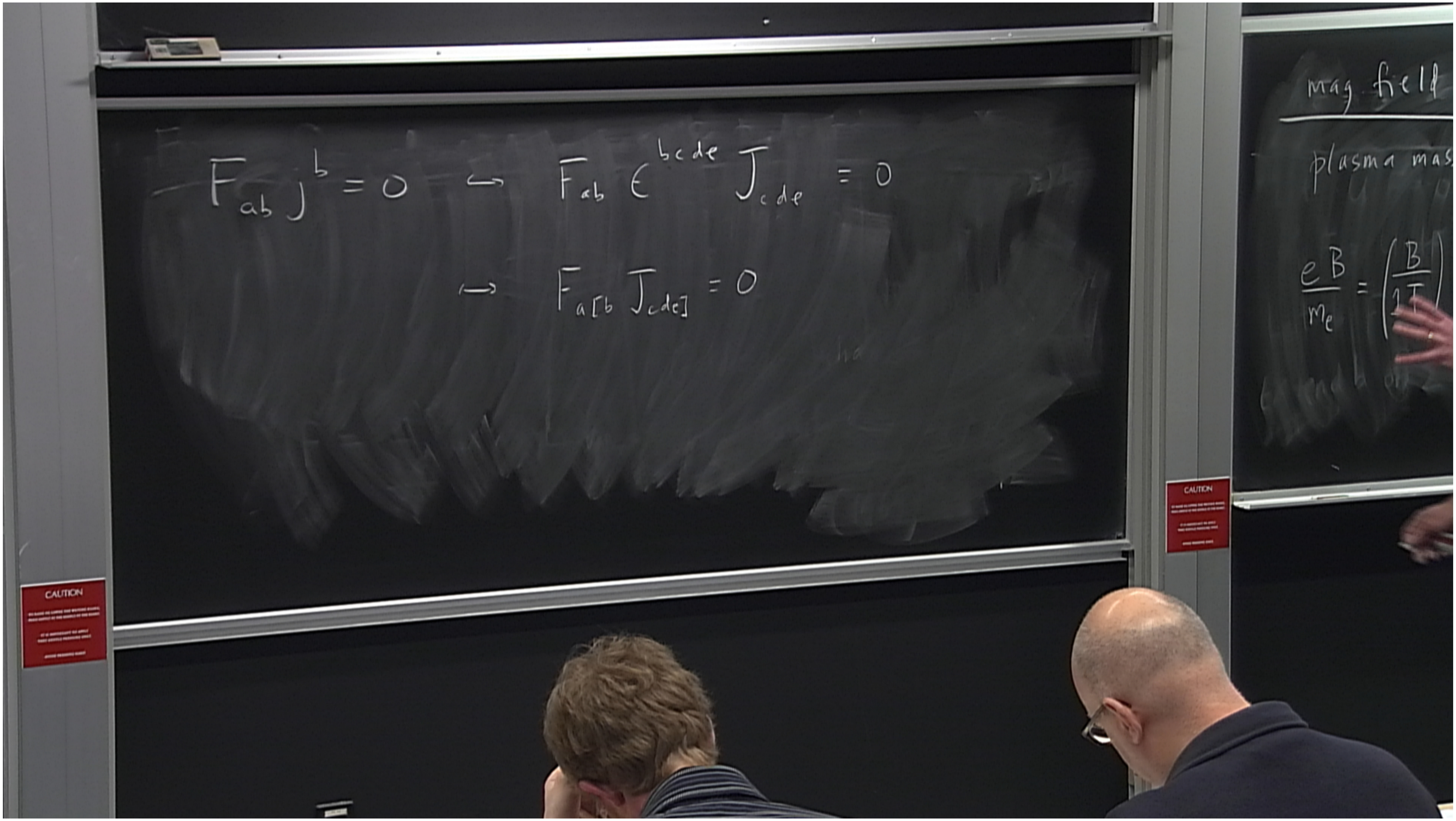

d(dφ₂ ∧ *F) = 0
d(dφ₁ ∧ *F) = 0
field strength
Faraday law
F_ab F^ab = 2(B² − E²)
F_ab *F^ab = 4 E·B
CAUTION

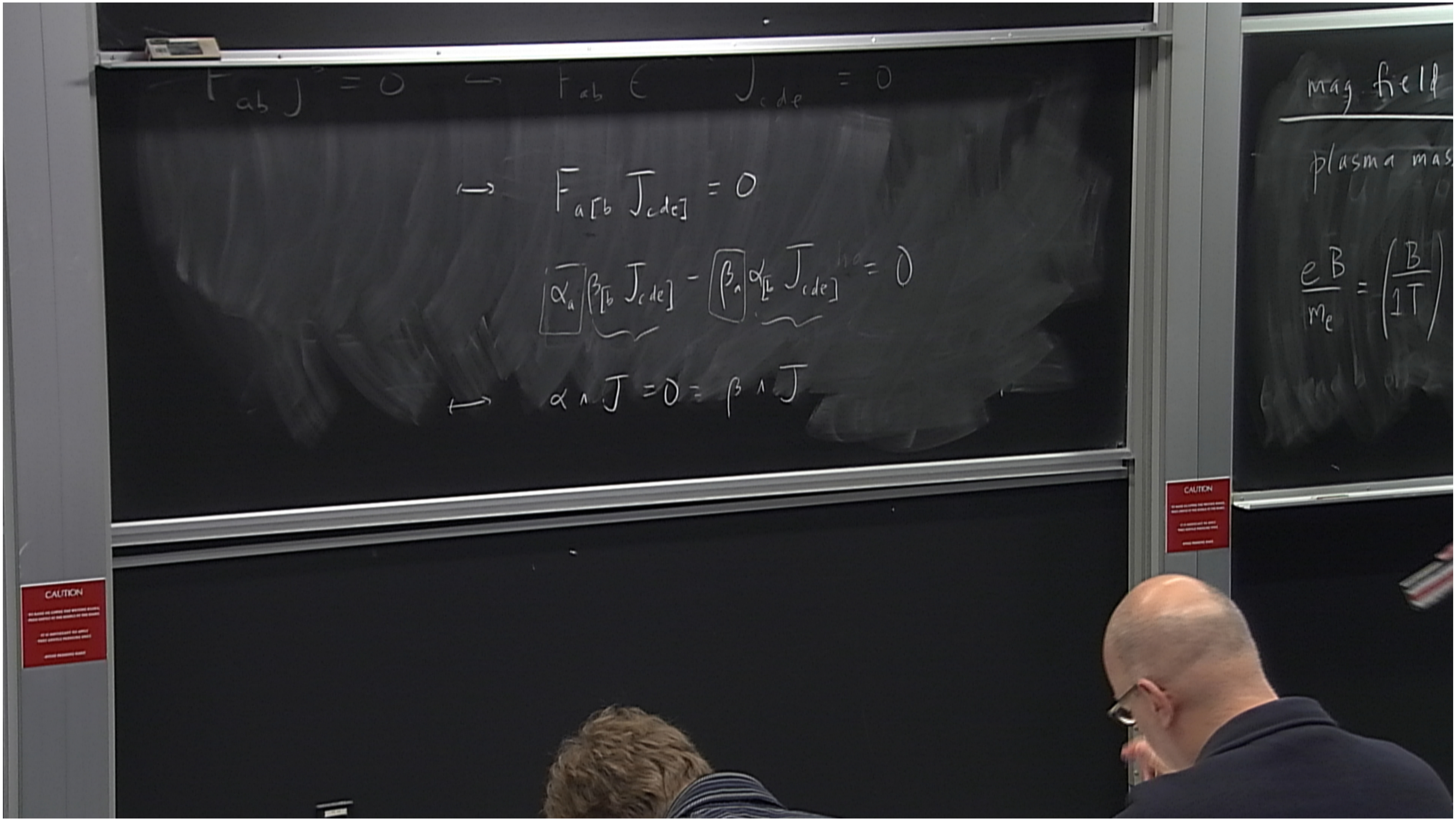

= 0 ∀ δφ₁ ⇒
= dφ₂ ∧ J
= dφ₁ ∧ J
A 1-form
F = dA 2-form field streng
dF = 0 covariant Fara
= 2 (B² − E²)
= 4 E · B
CAUTION


$= -\int d(\delta\phi_1 \wedge d\phi_2 \wedge *F) + \delta\phi_1 \wedge d(d\phi_2 \wedge *F)$
$= 0 \;\forall\, \delta\phi_1 \Rightarrow$
$d(d\phi_2 \wedge *F) = 0$
$= d\phi_2 \wedge J$
$d(d\phi_1 \wedge *F) = 0$
$= d\phi_1 \wedge J$
A 1-form
F = dA 2-form field strength
dF = 0 covariant Faraday law
$F_{ab}F^{ab} = 2(B^2 - E^2)$
$F_{ab} * F^{ab} = 4 \vec{E} \cdot \vec{B}$
$\frac{1}{2}\epsilon^{abcd} F_{[ab}F_{cd]}$


= − ∫ d(δφ₁ ∧ dφ₂ ∧ *F) + δφ₁ ∧ d(dφ₂ ∧ *F)
= 0 ∀ δφ₁ ⇒
d(dφ₂ ∧ *F) = 0
= dφ₂ ∧ J
d(dφ₁ ∧ *F) = 0
= dφ₁ ∧ J
A 1-form
F = dA 2-form field strength
dF = 0 covariant Faraday law
F_ab F^ab = 2 (B² − E²)
F_ab *F^ab = 4 E · B
½ ε^abcd F_[ab F_cd]


$F_{ab} j^b = 0 \leftrightarrow F_{ab} \epsilon^{bcde} J_{cde}$
mag field
plasma mas
$\frac{eB}{m_e} = \left(\frac{B}{1T}\right)$


$F_{ab} j^b = 0 \leftrightarrow F_{ab} \epsilon^{bcde} J_{cde} = 0$
$\leftrightarrow F_{a[b} J_{cde]} = 0$
mag field
plasma mas
$\frac{eB}{m_e} = \left(\frac{B}{1T}\right)$



⟼ F_{a[b} J_{cde]} = 0
⟼ α ∧ J = 0 = β ∧ J
mag field
plasma mass
CAUTION
CAUTION