

Title: Spacetime approach to force-free magnetospheres - Lecture 2

Date: Feb 25, 2014 02:30 PM

URL: <http://pirsa.org/14020155>

Abstract:

$$F_{[ab} F_{cd]} = 0 \iff F \underset{\text{wedge product}}{\wedge} F = 0$$

Differential Forms

$$F_{ab} \leftrightarrow F \text{ 2-form}$$

$$\nabla_{[a} F_{bc]} = 0 \iff \underbrace{dF}_{\text{3-form}} = 0 \iff F = dA, \text{ at least locally}$$

$\iff F \text{ is closed}$

$$dd = 0$$

Covariant Faraday's law $\Rightarrow F_{bc} = 2 \nabla_{[b} A_{c]}$ ← 4-vector

$$\nabla_{[a} F_{bc]} = 0$$

$$\nabla_b F^{ab} = j^a = \text{4-current density} \quad (\text{Heaviside-Lorentz})$$

Propose a vector W^a s.t. $W^a F_{ab} = 0$.

$\Rightarrow F_{ab}$ is called "degenerate"

$$\begin{aligned} \Rightarrow 0 = W^a \underbrace{F_{[ab} F_{cd]}}_{= \beta \epsilon_{abcd}} &\Rightarrow \beta W^a \epsilon_{abcd} = 0 \Rightarrow \beta = 0 \Rightarrow F_{[ab} F_{cd]} = 0 \Leftrightarrow F_{ab} * F^{cd} = 0 \\ &\Leftrightarrow \vec{E} \cdot \vec{B} = 0 \end{aligned}$$

$$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b \quad \text{for some } \alpha \text{ \& } \beta !$$

$$\Leftrightarrow F = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta)$$

$F = 0$, one null, one spacelike, $\ker F$ null

Suppose \exists vector W^a s.t. $W^a F_{at} = 0$.

$\Rightarrow F_{at}$ is called "degenerate"

$$\Rightarrow 0 = W^a \underbrace{F_{[ab} F_{cd]}}_{= \beta \epsilon_{abcd}} \Rightarrow \beta W^a \epsilon_{abcd} = 0 \Rightarrow \beta = 0 \Rightarrow F_{[ab} F_{cd]} = 0 \Leftrightarrow F_{at} + F^{at} = 0$$
$$\Leftrightarrow \vec{E} \cdot \vec{B} = 0$$

$$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b \quad \text{for some } \alpha \text{ \& } \beta!$$

$$\Leftrightarrow F = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta$$

$F = 0$, one null, one spacelike, $\ker F$ null

$$F_{[ab} F_{cd]} = 0 \iff \underbrace{F \wedge F}_{\text{Wedge product}} = 0 \iff F = \alpha \wedge \beta \text{ for some } \alpha, \beta$$

F is "simple"

Differential Forms

$$F_{ab} \iff F: \text{2-form}$$

$$\nabla_{[a} F_{bc]} = 0 \iff \underbrace{dF}_{\text{3-form}} = 0 \iff F = dA, \text{ at least locally}$$

$\iff F$ is closed

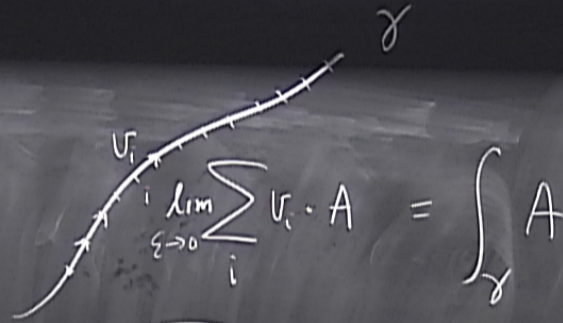
$$dd = 0$$

$$\nabla_{[a} F_{bc]} = 0 \quad \text{Covariant Faraday's law} \implies F_{bc} = 2 \nabla_{[b} A_{c]} \quad \leftarrow \begin{array}{l} \text{4-vector} \\ \text{potential} \end{array}$$

$$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b \quad \text{for some } \alpha \text{ \& } \beta!$$

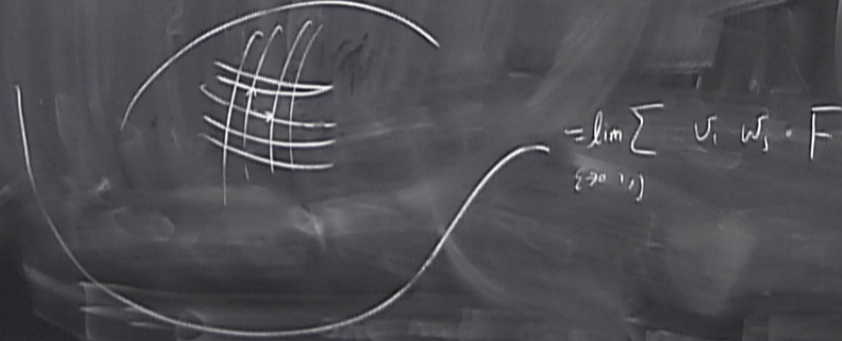
$$\Leftrightarrow F = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$$

$$\int_{\gamma} A$$



A diagram showing a curve γ with a tangent vector v_i at a point. A small area element A is shown near the curve. The equation $\lim_{\xi \rightarrow 0} \sum_i v_i \cdot A = \int_{\gamma} A$ is written next to it.

$$\int_S F$$



A diagram showing a surface S with a grid pattern. A normal vector w_i is shown at a point on the surface. The equation $\lim_{\xi \rightarrow 0} \sum v_i w_i \cdot F$ is written next to it.

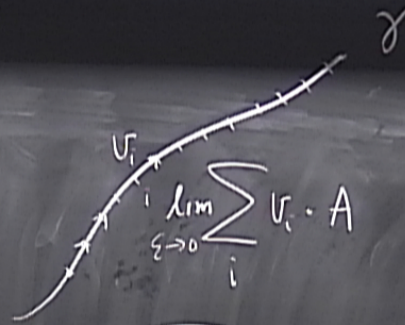
$$= \lim_{\xi \rightarrow 0} \sum v_i w_i \cdot F$$

CAUTION
Do not touch the blackboard
Do not touch the whiteboard
Do not touch the chalk
Do not touch the eraser

$$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b \quad \text{for some } \alpha \text{ \& } \beta!$$

$$\Leftrightarrow F = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$$

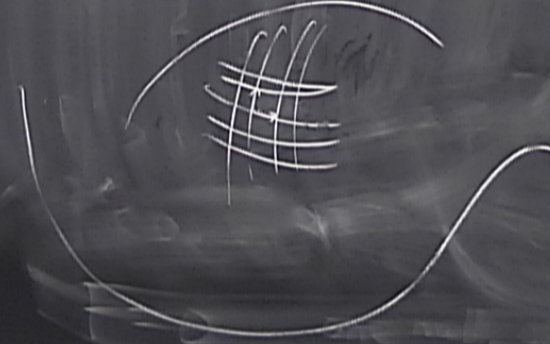
$$\int_{\gamma} A$$



A diagram showing a curve γ with a tangent vector v_i at a point. A small area element A is shown near the curve. The equation $\lim_{\xi \rightarrow 0} \sum_i v_i \cdot A = \int_{\gamma} A$ is written next to it.

$$\lim_{\xi \rightarrow 0} \sum_i v_i \cdot A = \int_{\gamma} A$$

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A diagram showing a surface S with a grid pattern. A normal vector w_j is shown at a point on the surface. The equation $\lim_{\xi \rightarrow 0} \sum v_i w_j \cdot F$ is written next to it.

$$= \lim_{\xi \rightarrow 0} \sum v_i w_j \cdot F$$

CAUTION
Do not touch the board
Do not touch the board
Do not touch the board

Conservation of energy-mom $\nabla_{[a} T_{ab]} = 0$, $\nabla_b T^{ab} = j^a$

Differential Forms

$$F_{ab} \leftrightarrow F: \text{2-form}$$

$$\nabla_{[a} F_{bc]} = 0 \rightarrow \underbrace{dF}_{\text{3-form}} = 0$$

$\Leftrightarrow F$ is closed

$$dd = 0$$

$$\nabla_b F^{ab} = j^a$$

$$d * F = J$$

$$\Rightarrow F = \underline{dA}, \text{ at least locally}$$

CAUTION

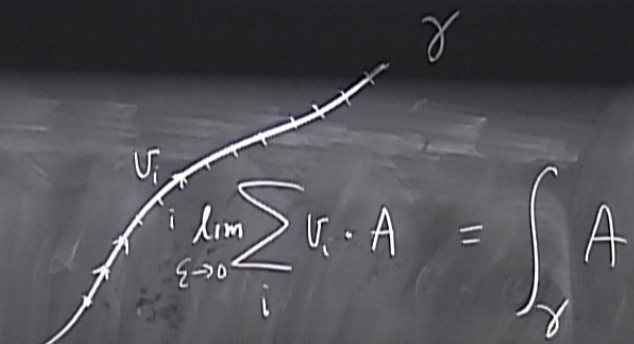
DO NOT USE LIFELINE AND ESCAPE ROUTES
WHEN ENTERING OR LEAVING THE BUILDING

DO NOT SMOKING OR DRINKING
WHEN ENTERING OR LEAVING THE BUILDING

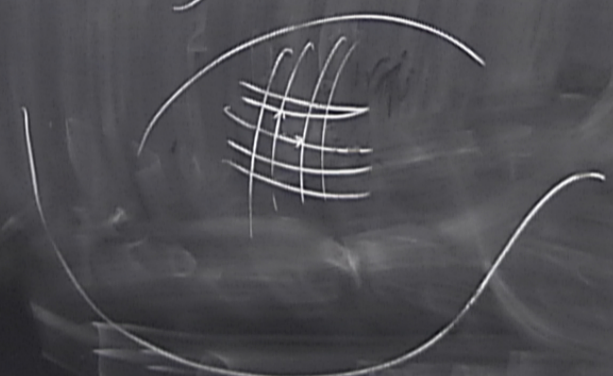
PLEASE RECYCLE WASTE

$$\hookrightarrow \Pi = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$$

$$\int_{\gamma_1} A$$



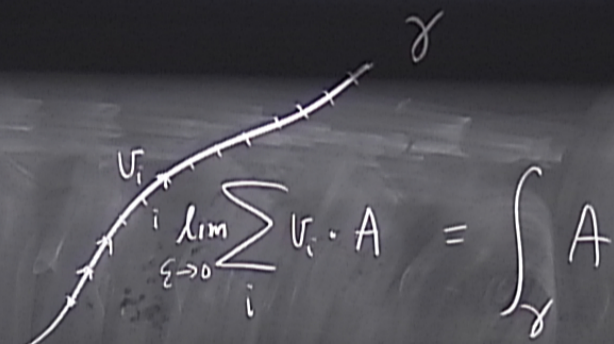
$$\int_{S_2} F$$



$$= \lim_{\xi \rightarrow 0} \sum v_i \cdot w_i = F$$

$$\rightarrow F = \alpha \wedge \beta, \quad F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$$

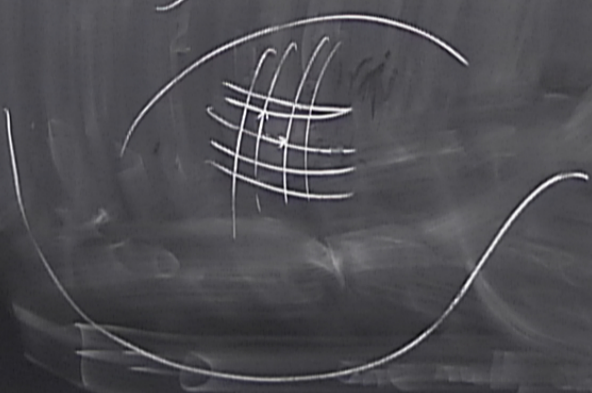
$$\int_{\gamma_1} A$$



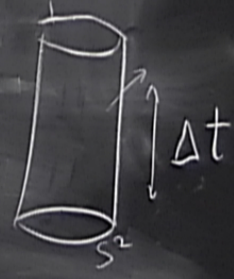
$$\int_{\Sigma_3} J$$



$$\int_{S_2} F$$



$$= \lim_{\xi \rightarrow 0} \sum v_i \omega_i \cdot F$$



if $A = d\phi$



$$\int_{\gamma} A = \int_{\gamma} d\phi = \phi(b) - \phi(a) = \int_{\partial\gamma} \phi$$

S

$$\int_S F = \int_S dA = \int_{\partial S} A$$

CAUTION
PLEASE DO NOT TOUCH THE SURFACE OF THE MIRROR
IT IS EXTREMELY HOT AND MAY CAUSE BURNS
IF YOU TOUCH IT
PLEASE DO NOT TOUCH THE MIRROR



$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_S dA = \int_{\partial S} A$$

$$\int_M d\omega = \int_{\partial M} \omega$$

Stokes' Theorem

Frozen flux theorem

ideal MHD,

$$\mathbf{u}_{\text{fluid}} \cdot \mathbf{F} = 0$$



CAUTION
BE CAREFUL NOT TO TOUCH THE HOT SURFACE OF THE BOARD OR THE BOARD ITSELF.
IF YOU ARE NOT SURE, ASK FOR HELP.
PLEASE DO NOT TOUCH THE BOARD.



$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_S dA = \int_{\partial S} A$$

$$\int_M d\omega = \int_{\partial M} \omega$$

Stokes' Theorem

Frozen flux theorem



ideal MHD, $\mathbf{u}_{\text{fluid}} \cdot \mathbf{F} = 0$

$\int_{S_1} \mathbf{F} =$ magnetic flux through S_1 .

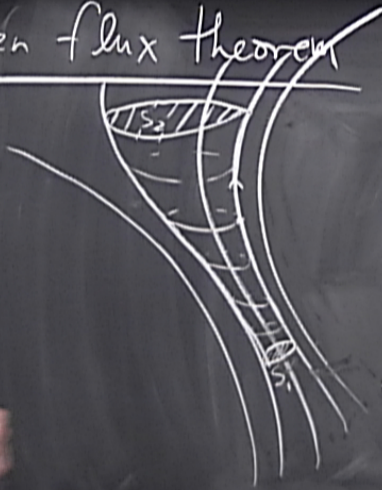


$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_S dA = \int_{\partial S} A$$

$$\int_M d\omega = \int_{\partial M} \omega$$

Stokes' Theorem

Frozen flux theorem



ideal MHD, $\mathbf{u}_{\text{fluid}} \cdot \mathbf{F} = 0$

$\int_{S_1} \mathbf{F} \cdot d\mathbf{A}$ = magnetic flux through S_1

$$d\mathbf{F} = 0 \Rightarrow \int_{S_1} \mathbf{F} + \int_{S_2} \mathbf{F} + \int_{\text{Side}} \mathbf{F} = 0$$

& Stokes



$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_S dA = \int_{\partial S} A$$

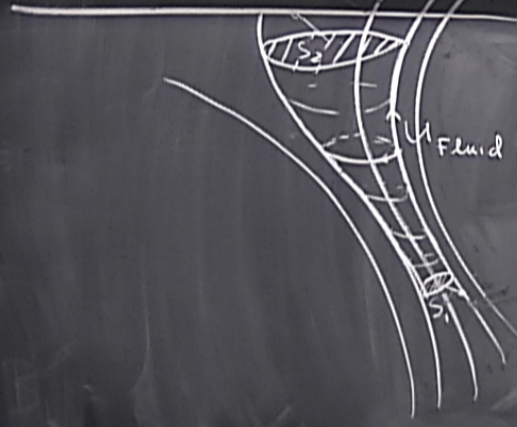
$$\int_M d\omega = \int_{\partial M} \omega$$

Stokes' Theorem

Frozen flux theorem

ideal MHD,

$$\mathbf{u}_{\text{fluid}} \cdot \mathbf{F} = 0$$

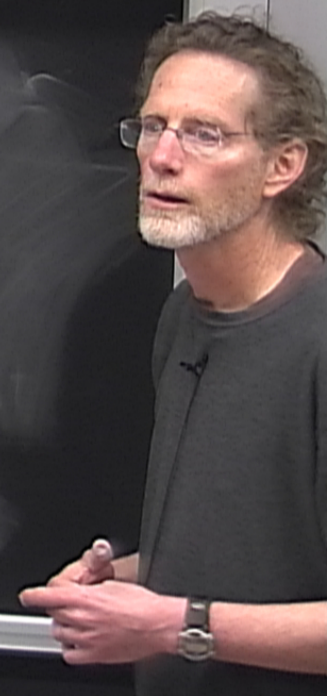


$\int_{S_1} \mathbf{F} =$ magnetic flux through S_1

$$d\mathbf{F} = 0 \Rightarrow \int_{S_1} \mathbf{F} + \int_{S_2} \mathbf{F} + \int_{\text{Side}} \mathbf{F} = 0$$

& Stokes

\Rightarrow flux through $S_1 =$ flux through S_2 !



CAUTION
DO NOT TOUCH THE BOARD OR CHALK
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Differential version of frozen flux thm:

$$\mathcal{L}_u F = 0$$

$$\mathcal{L}_u w$$

Differential version of frozen flux thm:

$$\mathcal{L}_u F = 0$$

$$\mathcal{L}_u \omega = d(u \cdot \omega) + u \cdot d\omega$$

Differential version of frozen flux thm:

$$\mathcal{L}_u F = 0$$

$$\mathcal{L}_u \omega = d(u \cdot \omega) + u \cdot d\omega \quad \text{"Cartan's magic formula"}$$

$$\mathcal{L}_u F = d(\cancel{u \cdot F}) + u \cdot \cancel{dF} = 0$$

Differential version of frozen flux thm:

$$\mathcal{L}_u F = 0$$

$$\mathcal{L}_u \omega = d(u \cdot \omega) + u \cdot d\omega \quad \text{"Cartan's magic formula"}$$

$$\mathcal{L}_u F = d(\cancel{u \cdot F}) + u \cdot \cancel{dF} = 0$$

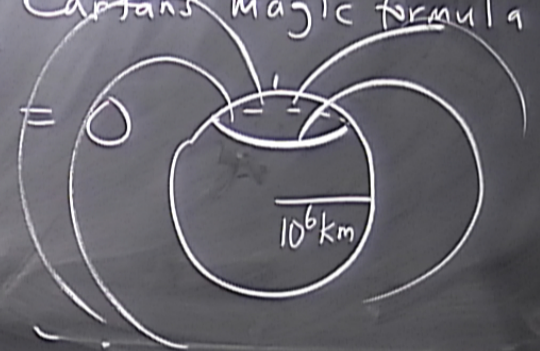
fluid $\tau_{rs} = 0$

version of Noether + flux TAM.

$$\mathcal{L}_u F = 0$$

$$\mathcal{L}_u \omega = d(u \cdot \omega) + u \cdot d\omega \quad \text{"Cartan's magic formula"}$$

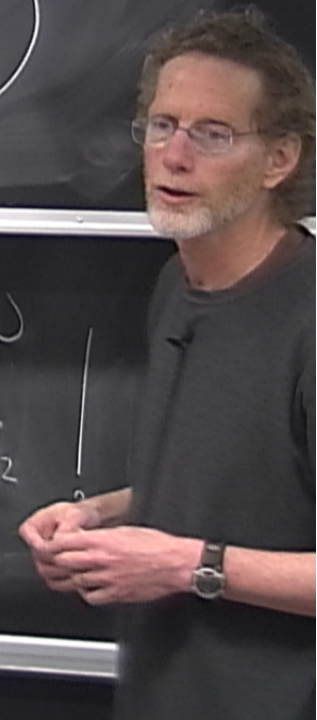
$$\mathcal{L}_u F = d(u \cdot F) + u \cdot dF = 0$$



$$dF = 0 \Rightarrow \int_{S_1} F + \int_{S_2} F + \int_{\text{Side}} F = 0$$

& Stokes

\Rightarrow flux through $S_1 =$ flux through S_2



CAUTION
 Do not touch the blackboard
 as it is very hot and may cause
 injury. Please do not touch the
 blackboard.

10 km

Conservation of energy-mom

$$\nabla^b T_{ab}^{EM} + \boxed{F_{ab} J^b} = 0, \quad T_{ab}^{EM} = F_{ac} F_b^c - \frac{1}{4} F_{cd} F^{cd} g_{ab}$$

Magnetic Helicity

$$H = \int_{\Sigma_3} A \wedge F, \quad \text{under } A \rightarrow A + d\phi$$

$$\uparrow \text{ magnetic helicity} \quad H \rightarrow H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial \Sigma_3} \phi F$$

since

S_2

CAUTION
DO NOT TOUCH THE MIRROR SURFACE
OR THE MIRROR OR THE MIRROR SURFACE
AS IT IS EXTREMELY HOT AND
MAY CAUSE BURNING INJURY

$$\int_{V_f} d(A \cdot F) = \int_{\partial V_f} A \cdot F$$

 Σ_f Σ_i

$$\int_{V_4} \underbrace{d(A \wedge F)}_{F \wedge F = 0} = \int_{\partial V_4} A \wedge F$$

 Σ_f

$\Rightarrow H_f = H_i$ if side
integral vanishes

 Σ_i


$$\int_{V_f} \underbrace{d(A \wedge F)}_{F \wedge F = 0} = \int_{\partial V_f} A \wedge F$$

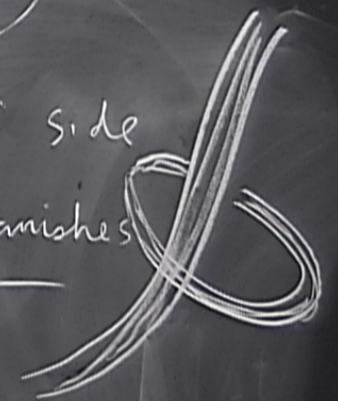
 Σ_f
 $\Rightarrow H_f = H_i$ if side
integral vanishes
 Σ_i

10 km

10 km


$$\int_{V_4} \underbrace{d(A \wedge F)}_{F \wedge F = 0} = \int_{\partial V_4} A \wedge F$$

 Σ_f
 $\Rightarrow H_f = H_i$ if side integral vanishes

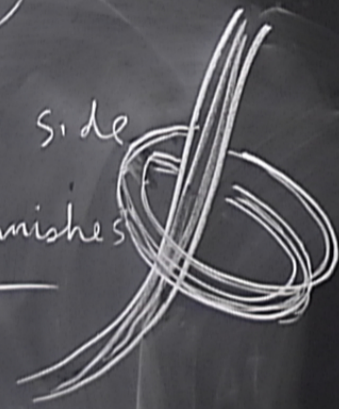
 Σ_i


10 km

10 km



$= H; \text{ if side}$
integral vanishes



Conservation of Energy-mom

lab

lab

Magnetic Helicity

$$H = \int_{\Sigma_3} \mathbf{A} \wedge \mathbf{F}, \quad \text{under } \mathbf{A} \rightarrow \mathbf{A} + d\phi$$

\uparrow
magnetic helicity

$$H \rightarrow H + \int_{\Sigma_3} d(\phi \wedge \mathbf{F})$$

CAUTION
DO NOT TOUCH THE WRITING BOARD.
HANDS OFF AT THE FRONT OF THE BOARD.
IT IS PROHIBITED TO SMILE.
PLEASE REMAIN SEATED.

$$\nabla_b \times F^{ab} = 0$$

Conservation of
energy-mom

$$\nabla^b T_{ab}^{EM} + \boxed{F_{ab} j^b} = 0,$$

4-force density

$$T_{ab}^{EM} = F_{ac} F_b^c - \frac{1}{4} F_{cd} F^{cd} g_{ab}$$

$$H = \int_{\Sigma_3} A \wedge \underline{F}, \quad \text{under } A \rightarrow A + d\phi$$

↑
magnetic
helicity

$$H \rightarrow H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial \Sigma_3}$$

$$\nabla_b \times F^{ab} = 0$$

Conservation of
energy-mom

$$\nabla^b T_{ab}^{\text{EM}} + \boxed{F_{ab} j^b} = 0,$$

4-force density

$$T_{ab}^{\text{EM}} = F_{ac} F_b^c - \frac{1}{4} F_{cd} F^{cd} g_{ab}$$

$$H = \int_{\Sigma_3} A \wedge \underline{F}, \quad \text{under } A \rightarrow A + d\phi$$

↑
magnetic
helicity

$$H \rightarrow H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial \Sigma_3} \phi F$$

Minimum Energy Mag field at fixed helicity

Minimum Energy Mag field at fixed helicity

$$\delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) = 0$$

Minimum Energy Mag. field at fixed helicity

$$\begin{aligned} \delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) &= 0 \\ &= \int \vec{B} \cdot \vec{\nabla} \times \delta \vec{A} + \lambda \int \delta \vec{A} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \delta \vec{A}) \end{aligned}$$

Minimum Energy Mag. field at fixed helicity

$$\delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) = 0$$

$$= \int \vec{B} \cdot \vec{\nabla} \times \delta \vec{A} + \lambda \int \delta \vec{A} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \delta \vec{A})$$

$$= \int (\vec{\nabla} \times \vec{B}) \cdot \delta \vec{A}$$

Minimum Energy Mag. field at fixed helicity

$$\delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) = 0$$

$$= \int \vec{B} \cdot \vec{\nabla} \times \delta \vec{A} + \lambda \left(\delta \vec{A} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \delta \vec{A}) \right)$$

$$= \int (\vec{\nabla} \times \vec{B}) \cdot \delta \vec{A} + \lambda$$

Minimum Energy Mag. field at fixed helicity

$$\delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) = 0$$

$$= \int \vec{B} \cdot \vec{\nabla} \times \delta \vec{A} + \lambda \int \delta \vec{A} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \delta \vec{A})$$

$$= \int (\vec{\nabla} \times \vec{B}) \cdot \delta \vec{A} + 2\lambda \int \vec{B} \cdot \delta \vec{A}$$

Minimum Energy Mag field at fixed helicity

$$\delta \left(\int \frac{1}{2} \mathbf{B}^2 d^3x + \lambda \int \vec{A} \cdot \vec{B} d^3x \right) = 0$$

$$= \int \vec{B} \cdot \vec{\nabla} \times \delta \vec{A} + \lambda \left(\delta \vec{A} \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \delta \vec{A}) \right)$$

$$= \int \underbrace{(\vec{\nabla} \times \vec{B}) \cdot \delta \vec{A}} + 2\lambda \int \vec{B} \cdot \delta \vec{A} = 0 \quad \forall \delta \vec{A} \Rightarrow \vec{\nabla} \times \vec{B}$$

$$= \int (\nabla \times \vec{B}) \cdot \delta \vec{A} + 2\lambda \int \vec{B} \cdot \delta \vec{A} = 0 \quad \forall \delta \vec{A} \Rightarrow \boxed{\nabla \times \vec{B} \propto \vec{B}}$$

\vec{E}, \vec{B}

$$\boxed{\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \partial_t \vec{B} + \nabla \times \vec{E} &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \nabla \cdot \vec{E} &= \rho \\ -\partial_t \vec{E} + \nabla \times \vec{B} &= \vec{j} \end{aligned}}$$

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\partial_t \vec{A} + \nabla \phi \end{aligned}$$

$$\begin{aligned} \vec{F} &= q(\vec{E} + \vec{v} \times \vec{B}) \\ \frac{d\vec{F}}{dt} &= q\vec{E} + \vec{j} \times \vec{B} \\ \frac{dW}{dt} &= \vec{F} \cdot \vec{j} \end{aligned}$$