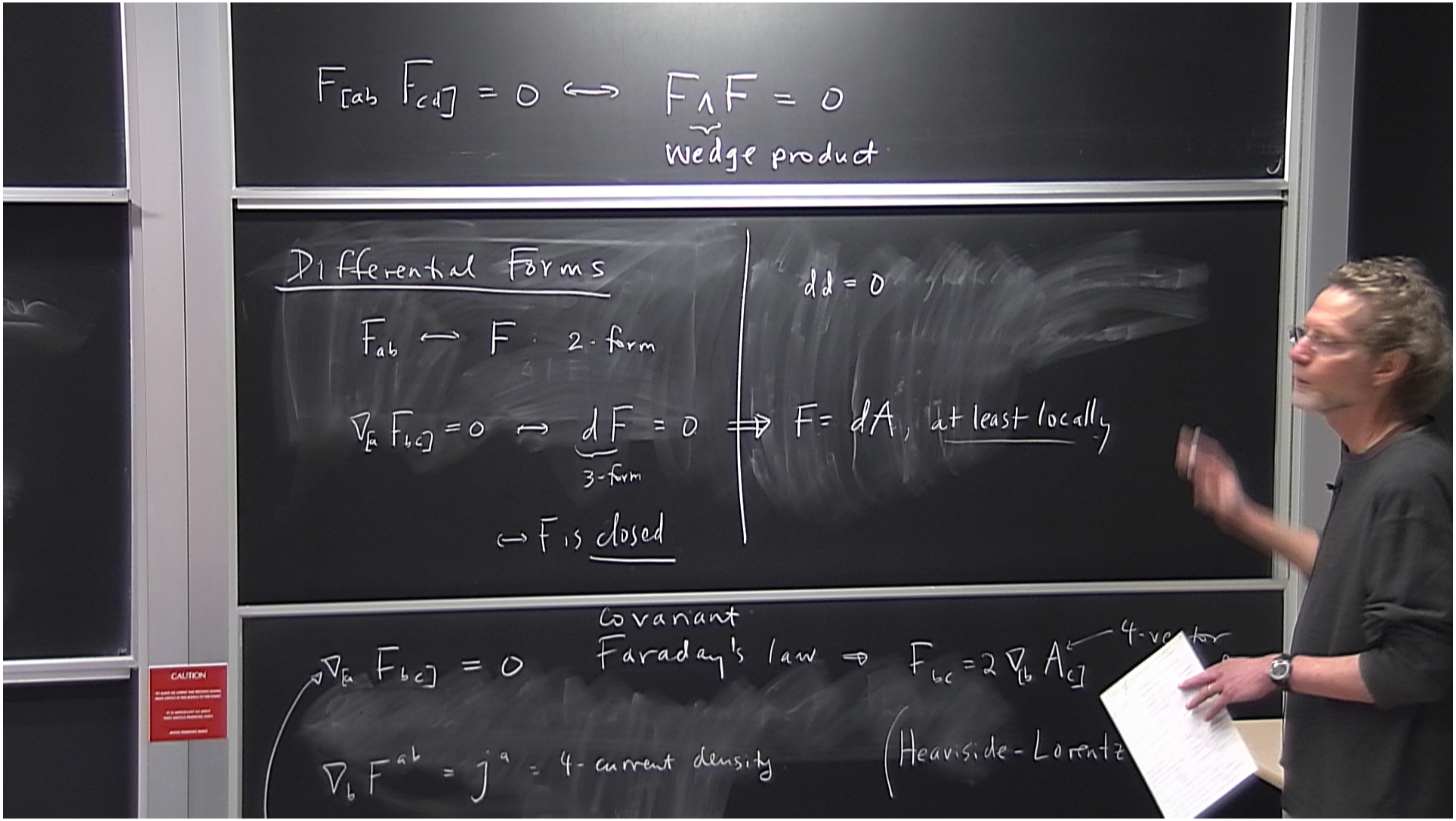
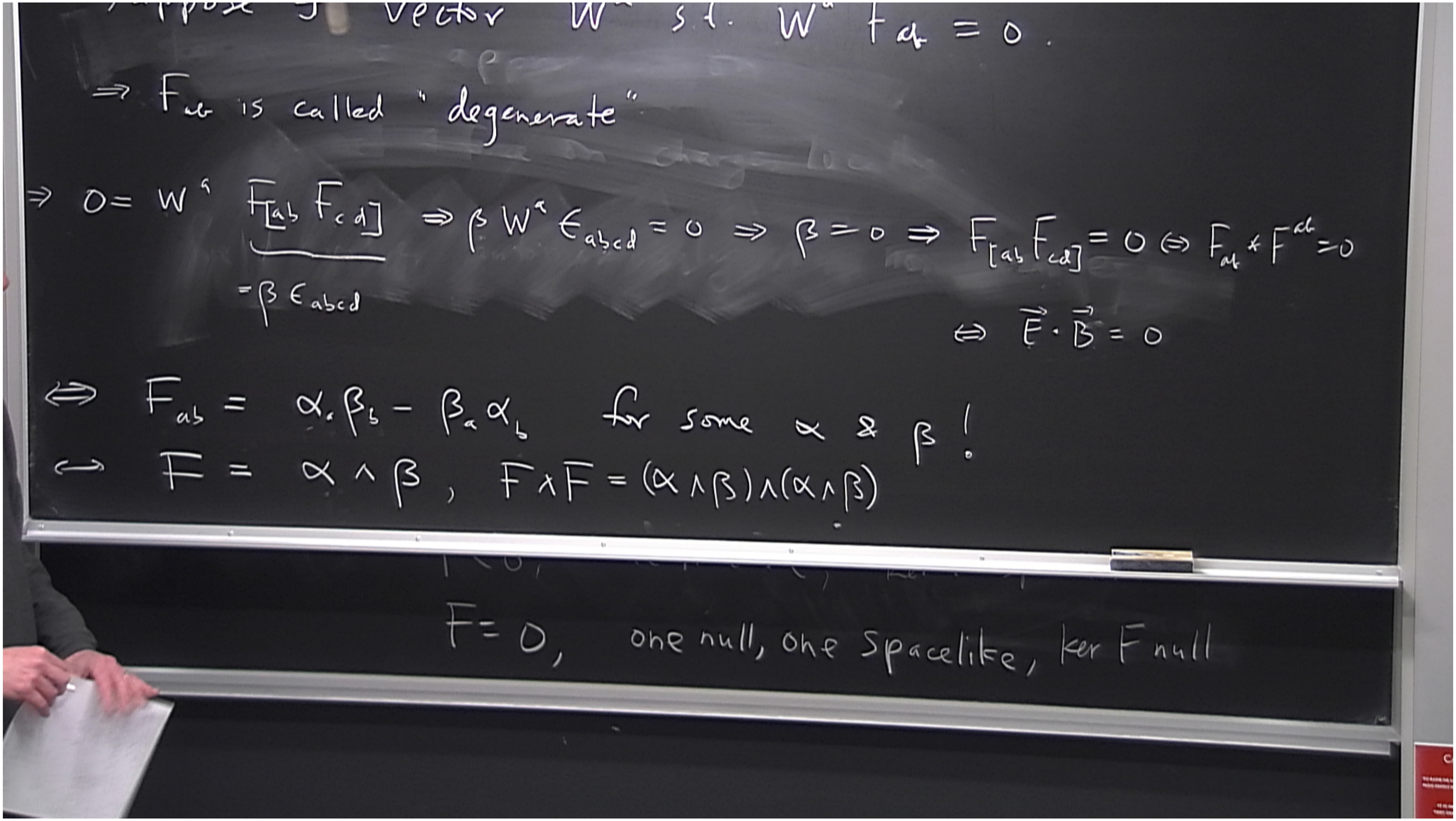
Title: Spacetime approach to force-free magnetospheres - Lecture 2

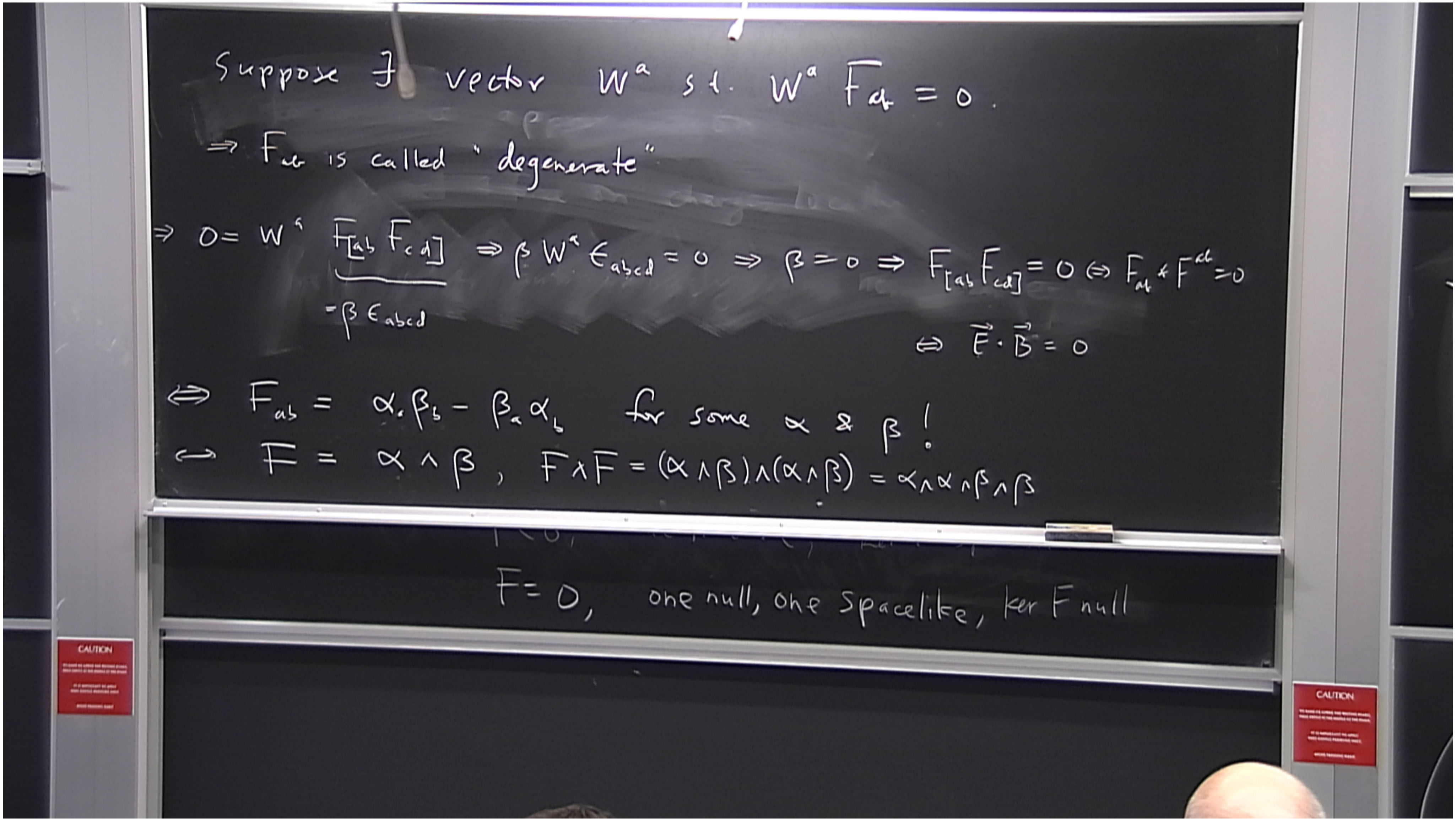
Date: Feb 25, 2014 02:30 PM

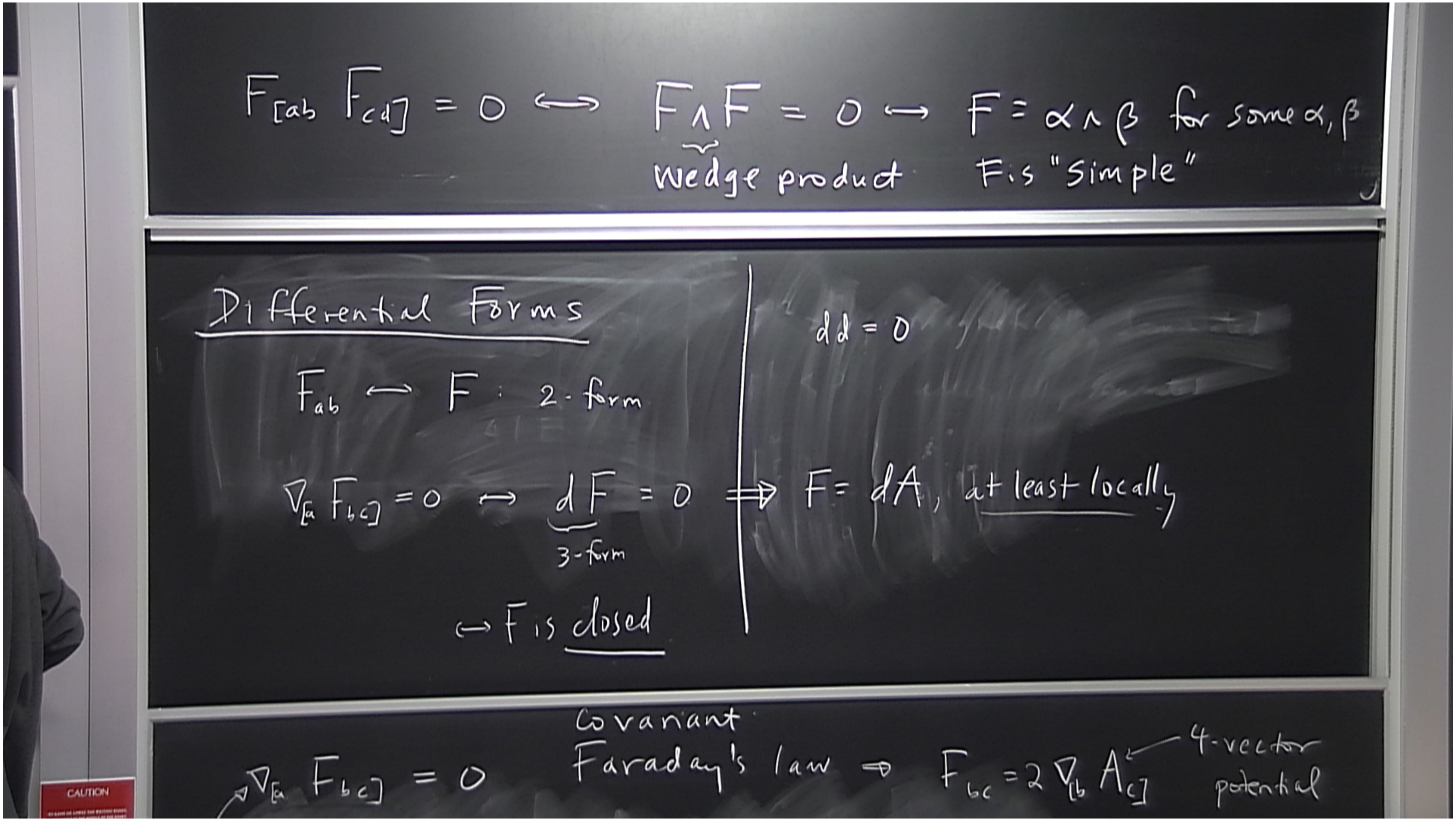
URL: http://pirsa.org/14020155

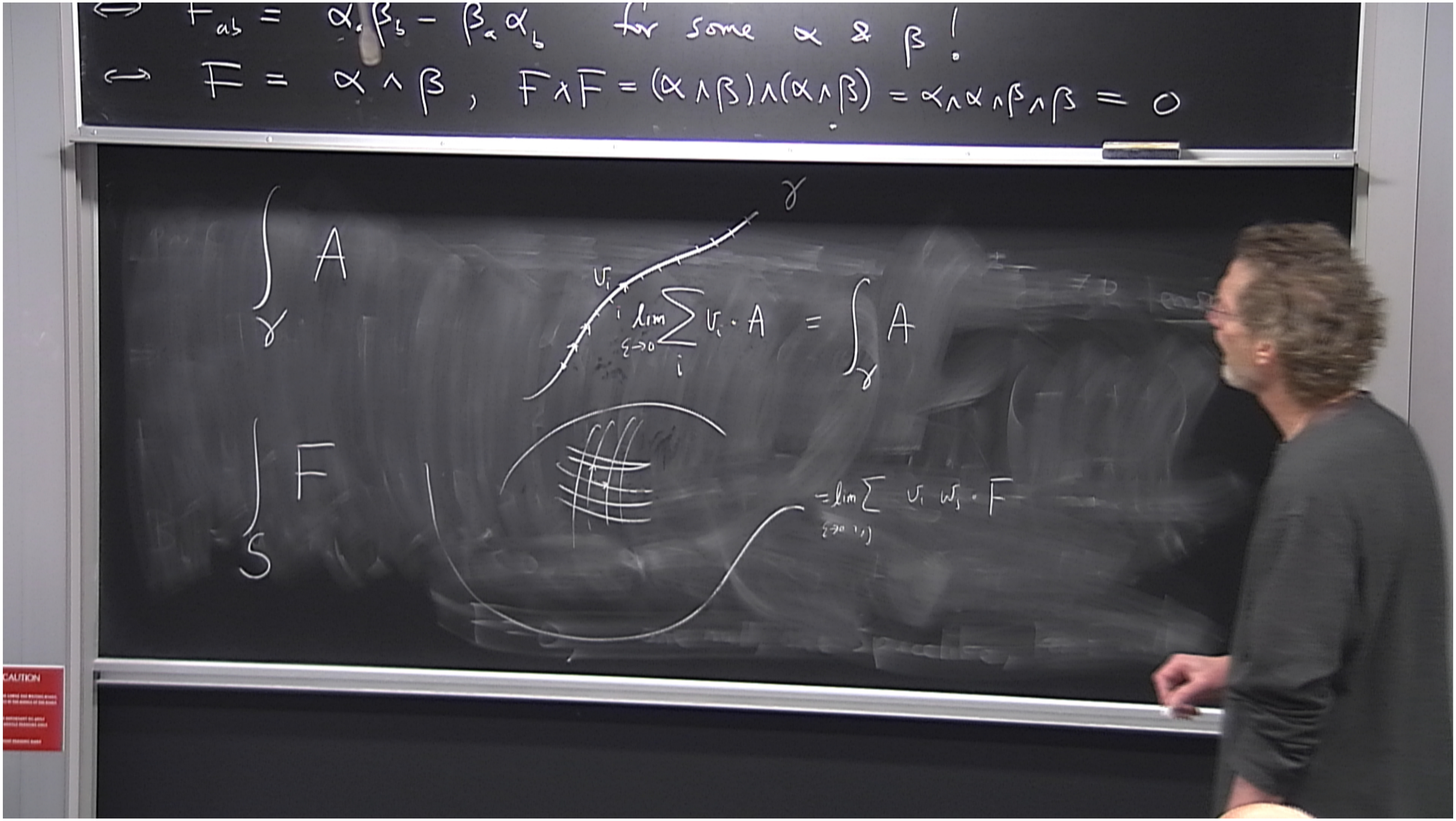
Abstract:

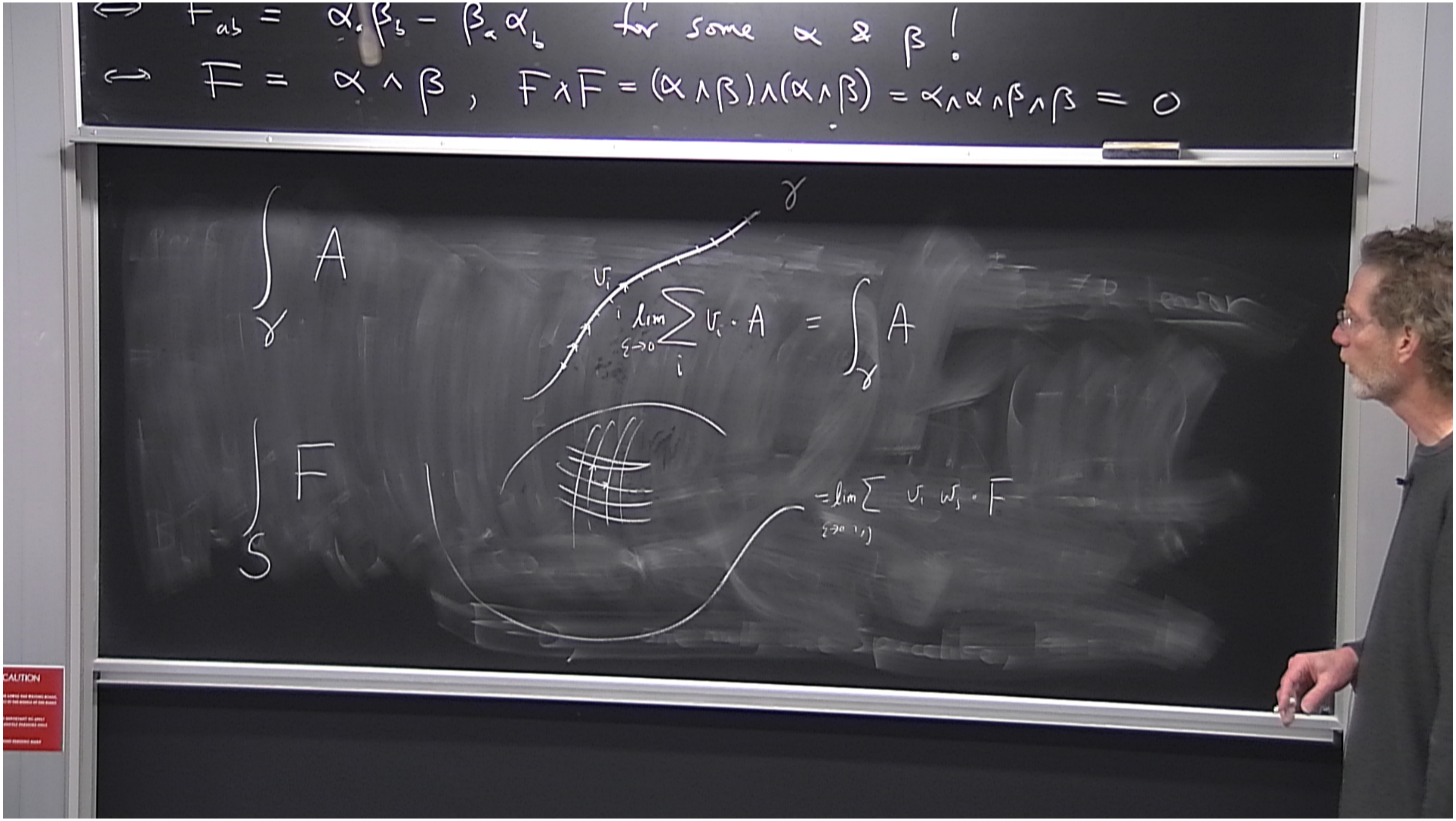

F[ab Fcd] = 0 ⟷ F∧F = 0
wedge product
Differential Forms
Fab ⟷ F : 2-form
∇[a Fbc] = 0 ⟷ dF = 0 ⟹ F = dA, at least locally
3-form
⟷ F is closed
dd = 0
Covariant
∇[a Fbc] = 0 Faraday's law ⟹ Fbc = 2 ∇[b Ac]
4-vector
∇b F^ab = j^a = 4-current density
(Heaviside-Lorentz

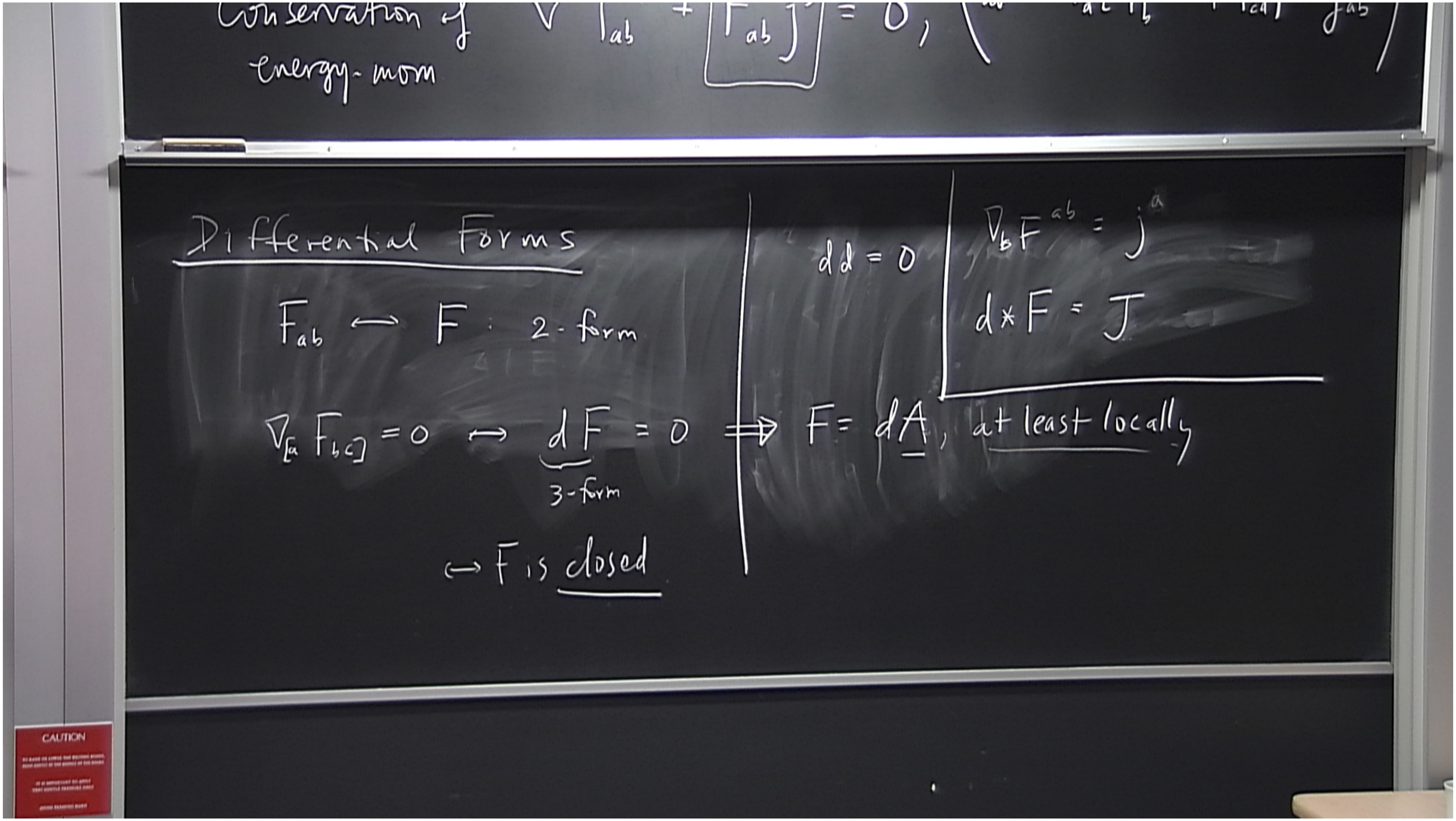

vector W^a s.t. $W^a F_{ab} = 0$.
$\Rightarrow F_{ab}$ is called "degenerate"
$\Rightarrow 0 = W^a F_{[ab} F_{cd]} \Rightarrow \beta W^a \epsilon_{abcd} = 0 \Rightarrow \beta = 0 \Rightarrow F_{[ab} F_{cd]} = 0 \Leftrightarrow F_{ab} {*F}^{ab} = 0$
$= \beta \epsilon_{abcd}$
$\Leftrightarrow \vec{E} \cdot \vec{B} = 0$
$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b$ for some α & β!
$\Leftrightarrow F = \alpha \wedge \beta$, $F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta)$
$F = 0$, one null, one spacelike, ker F null

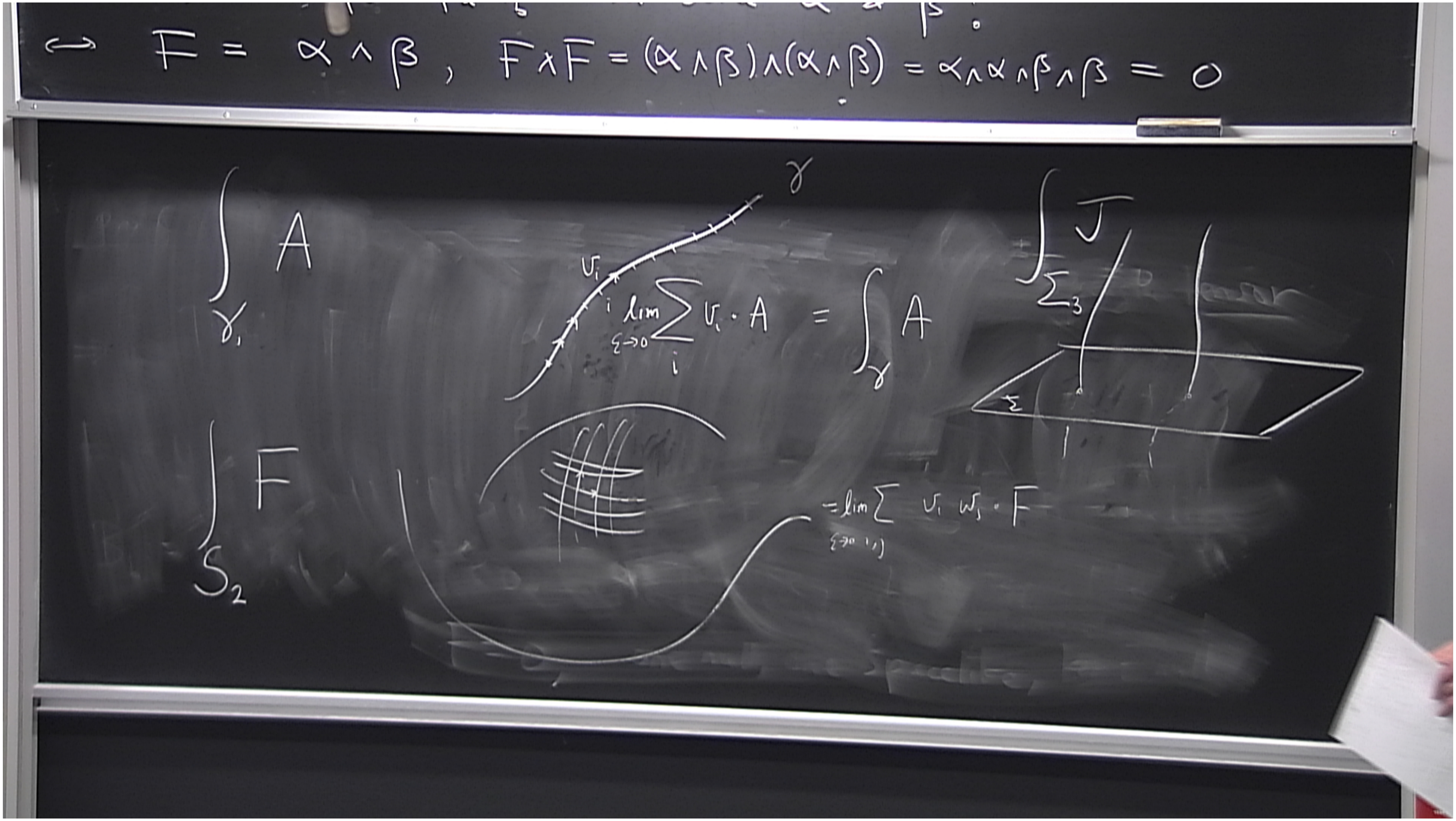

Suppose $\exists$ vector W^a s.t. $W^a F_{ab} = 0$.
$\Rightarrow F_{ab}$ is called "degenerate"
$\Rightarrow 0 = W^a F_{[ab} F_{cd]} \Rightarrow \beta W^a \epsilon_{abcd} = 0 \Rightarrow \beta = 0 \Rightarrow F_{[ab} F_{cd]} = 0 \Leftrightarrow F_{ab} * F^{ab} = 0$
$= \beta \epsilon_{abcd}$
$\Leftrightarrow \vec{E} \cdot \vec{B} = 0$
$\Leftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b$ for some α & β!
$\Leftrightarrow F = \alpha \wedge \beta$, $F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta$
$F = 0$, one null, one spacelike, ker F null


$F_{[ab} F_{cd]} = 0 \iff F \wedge F = 0 \iff F = \alpha \wedge \beta$ for some α, β
wedge product
F is "simple"
Differential Forms
$F_{ab} \leftrightarrow F$: 2-form
$\nabla_{[a} F_{bc]} = 0 \iff dF = 0 \Rightarrow F = dA$, at least locally
3-form
$\iff$ F is closed
$dd = 0$
Covariant
$\nabla_{[a} F_{bc]} = 0$ Faraday's law $\Rightarrow F_{bc} = 2 \nabla_{[b} A_{c]}$
4-vector potential

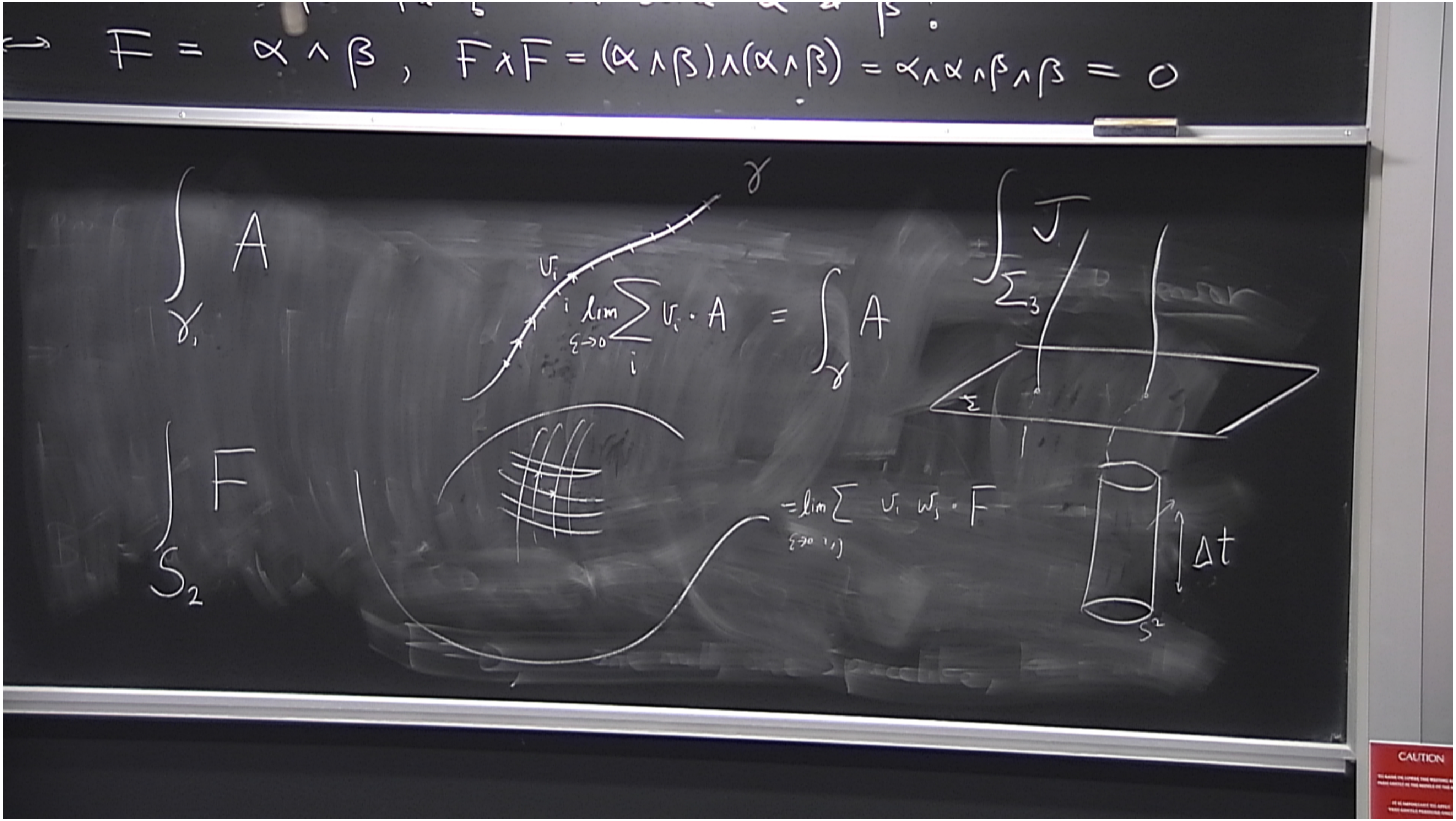

$\Longleftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b$ for some α & β !
$\Longleftrightarrow F = \alpha \wedge \beta$, $F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$
$\int_\gamma A$
γ
v_i
$\lim_{\varepsilon \to 0} \sum_i v_i \cdot A = \int_\gamma A$
$\int_S F$
$= \lim \sum v_i w_j \cdot F$

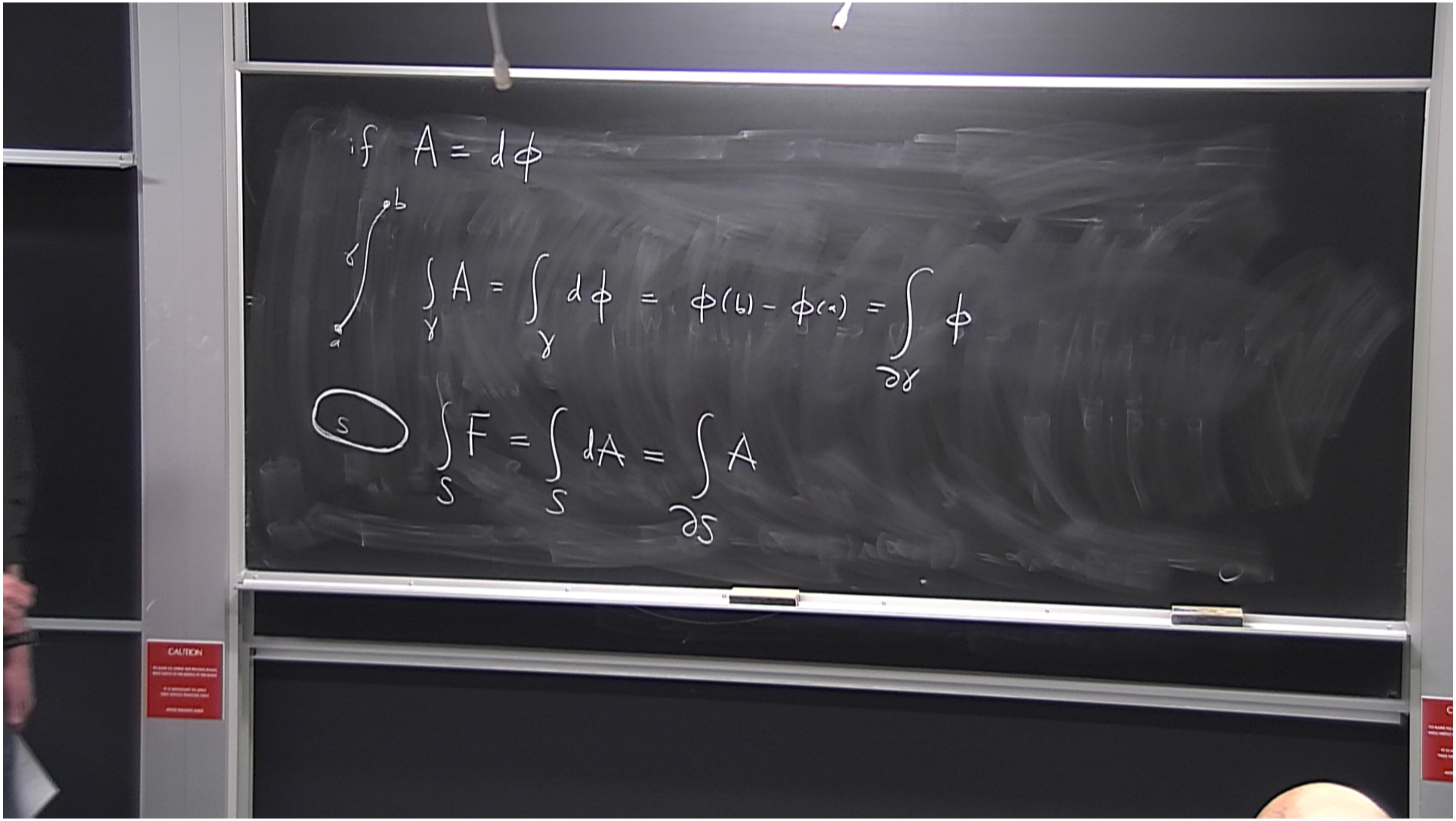

$\Longleftrightarrow F_{ab} = \alpha_a \beta_b - \beta_a \alpha_b$ for some α & β !
$\Longleftrightarrow F = \alpha \wedge \beta$, $F \wedge F = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = \alpha \wedge \alpha \wedge \beta \wedge \beta = 0$
$\int_\gamma A$
γ
v_i
$\lim_{\varepsilon \to 0} \sum_i v_i \cdot A = \int_\gamma A$
$\int_S F$
$= \lim \sum v_i w_j \cdot F$
CAUTION

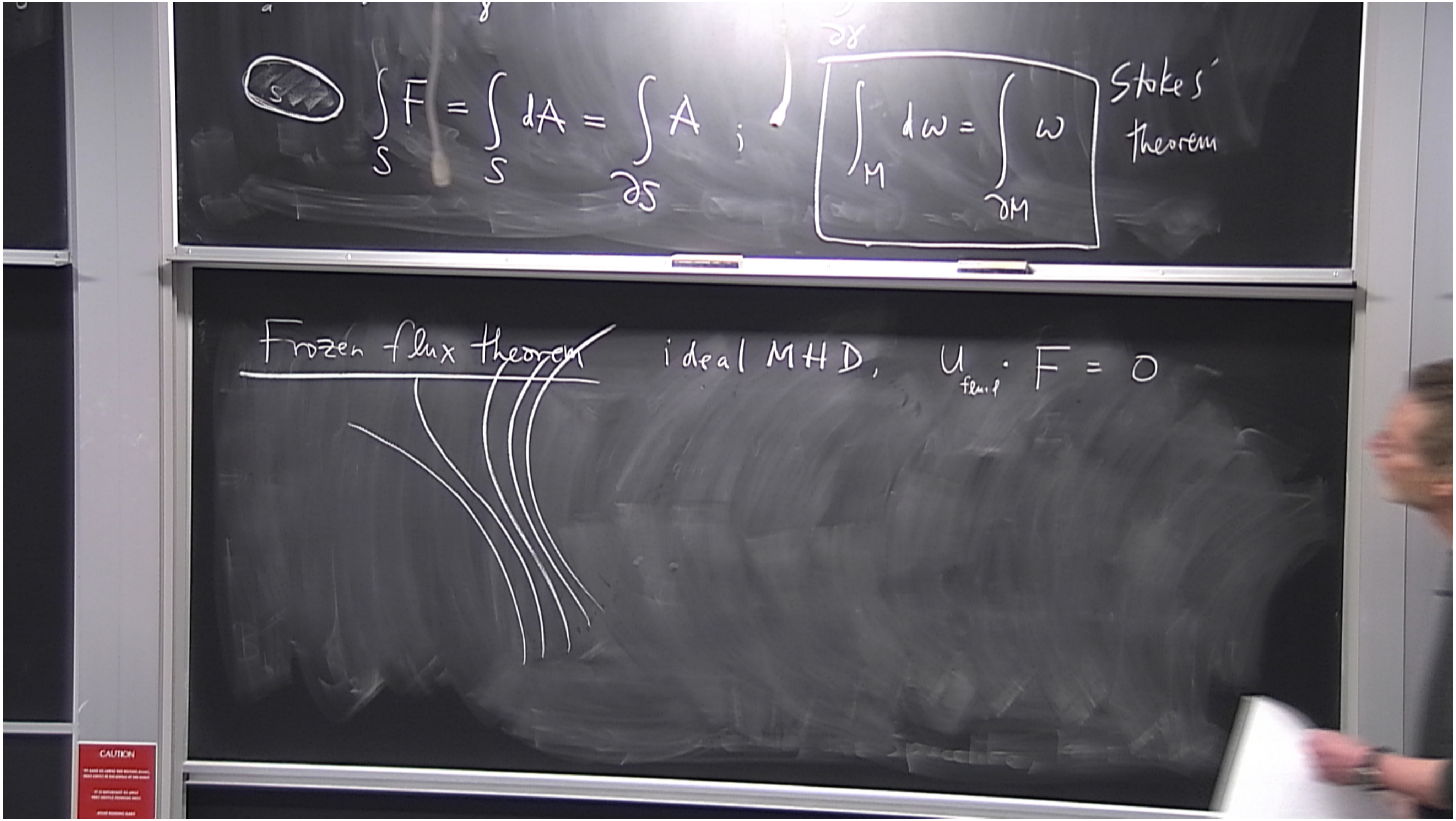

Conservation of energy-mom
Differential Forms
F_ab ⟷ F : 2-form
∇_[a F_bc] = 0 ⟷ dF = 0 ⟹ F = dA, at least locally
3-form
⟷ F is closed
dd = 0
∇_b F^ab = j^a
d*F = J



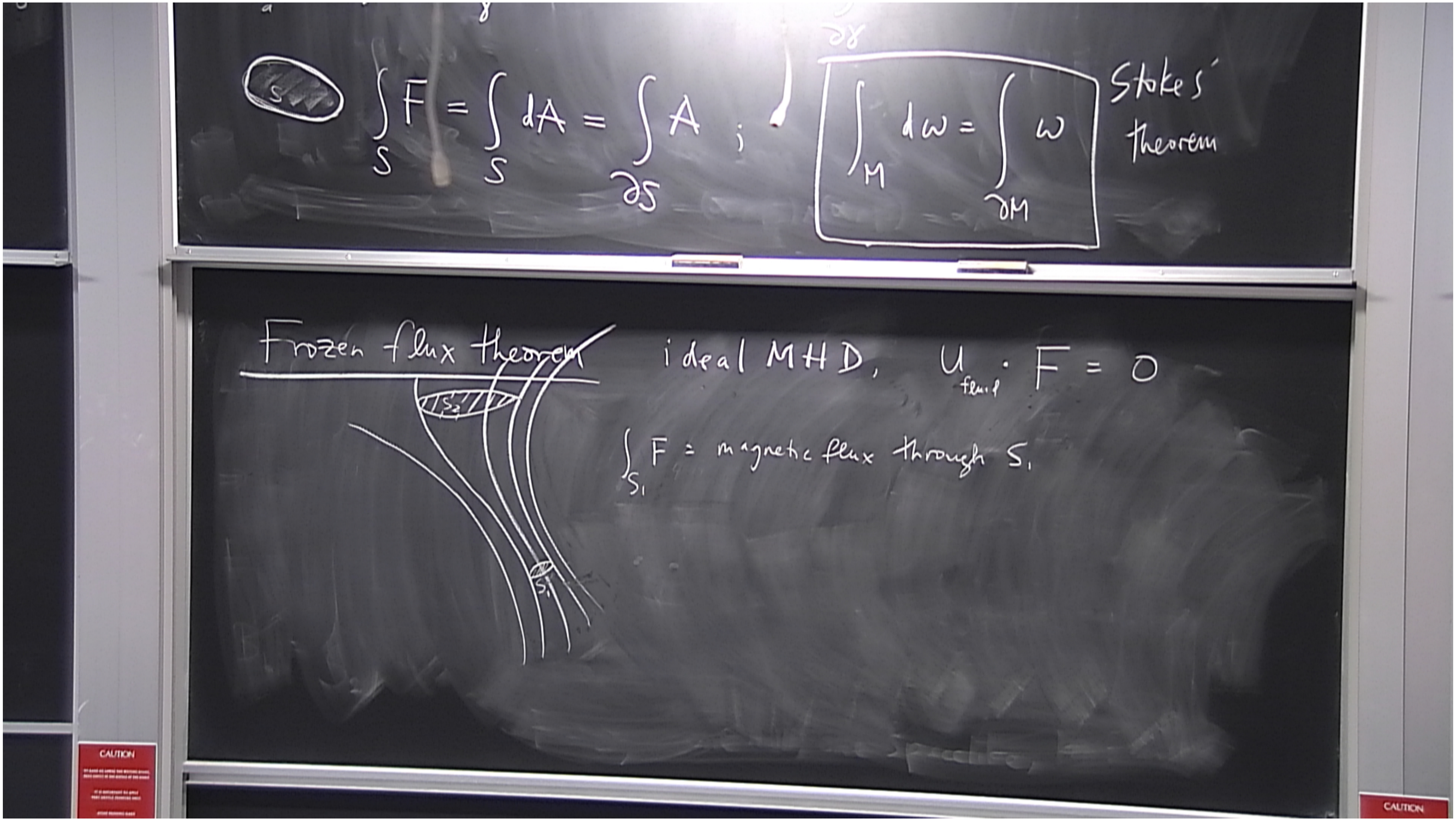
F = α ∧ β , F∧F = (α∧β)∧(α∧β) = α∧α∧β∧β = 0
∫_{γ_1} A
γ
v_i
lim_{ε→0} Σ_i v_i · A = ∫_γ A
∫_{Σ_3} J
Σ
∫_{S_2} F
= lim Σ v_i w_j · F

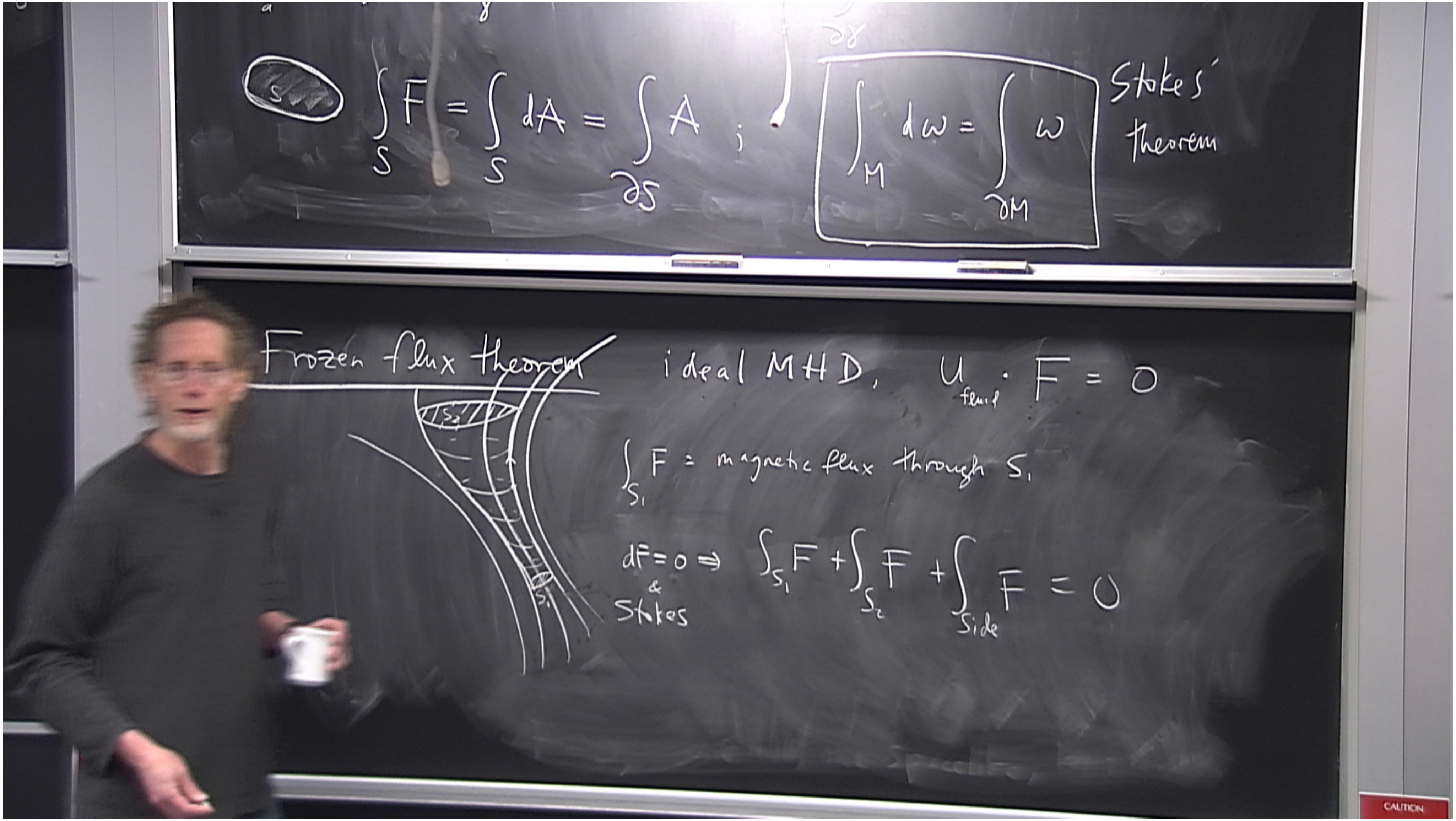

F = α ∧ β , F∧F = (α∧β)∧(α∧β) = α∧α∧β∧β = 0
∫γ₁ A
lim ε→0 Σi vi · A = ∫γ A
∫S₂ F
= lim Σ vi wj · F
∫Σ₃ J
Σ
Δt
S²

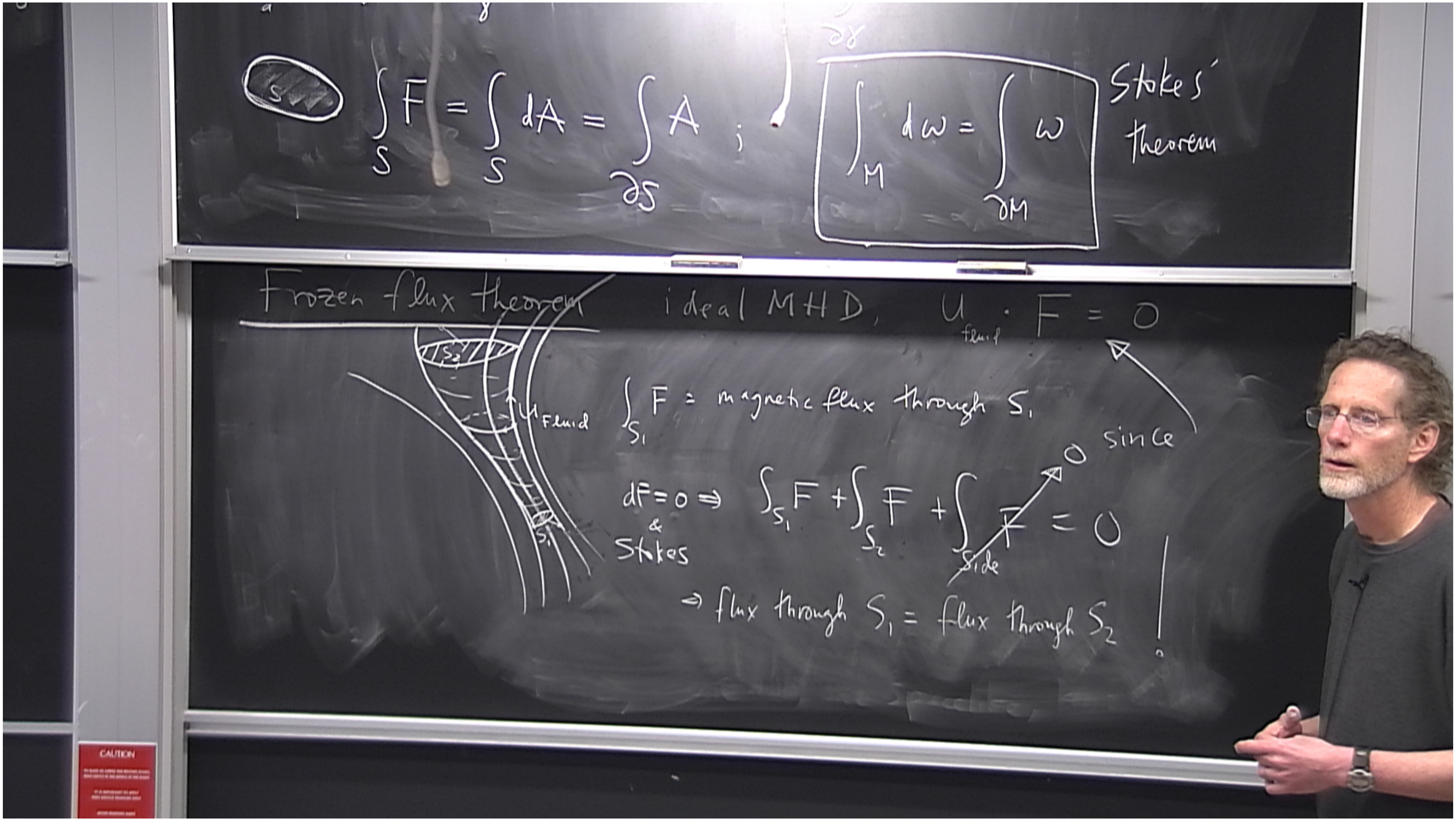

if $A = d\phi$
b
γ
a
$\int_\gamma A = \int_\gamma d\phi = \phi(b) - \phi(a) = \int_{\partial\gamma} \phi$
S
$\int_S F = \int_S dA = \int_{\partial S} A$
CAUTION

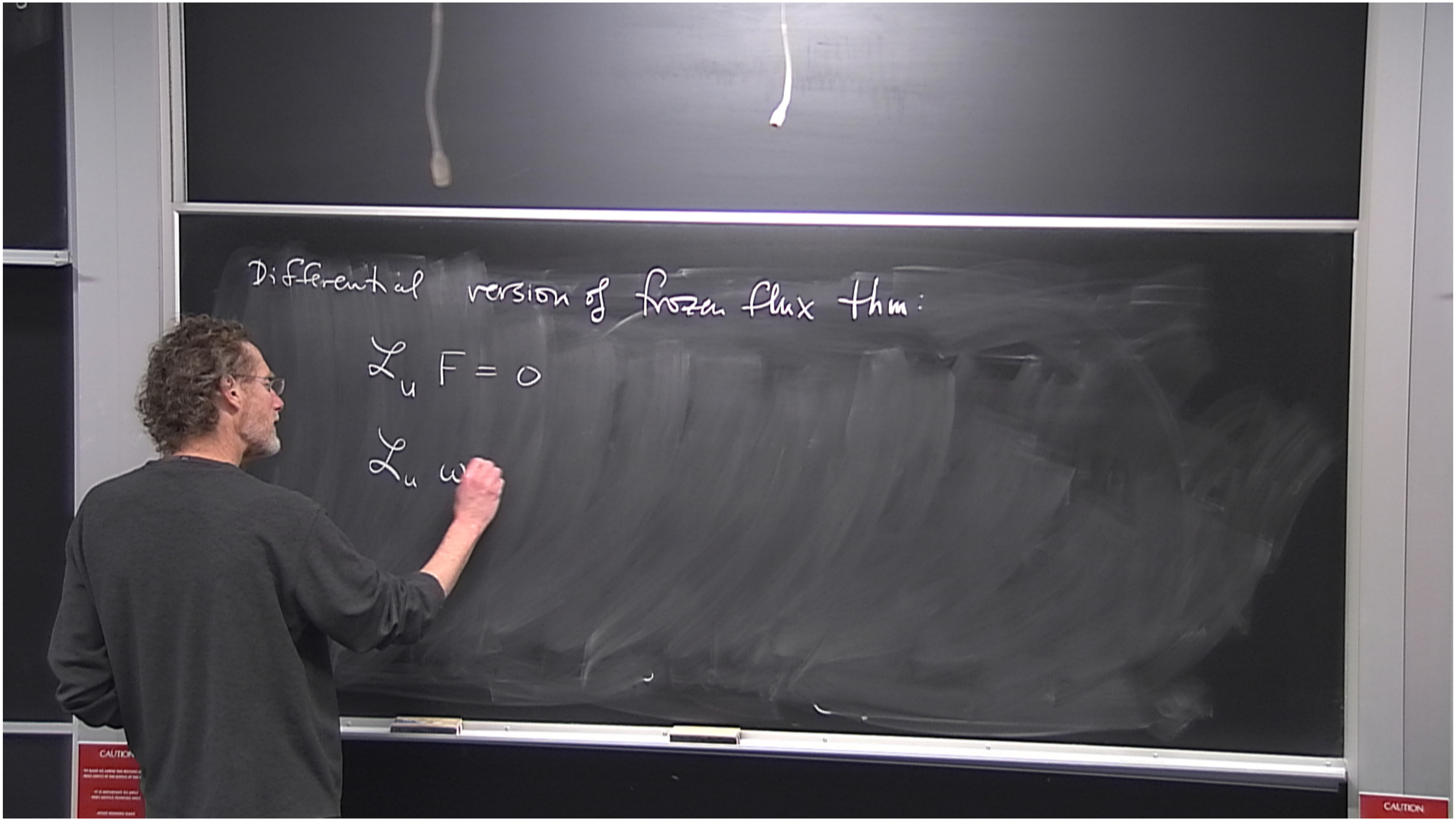


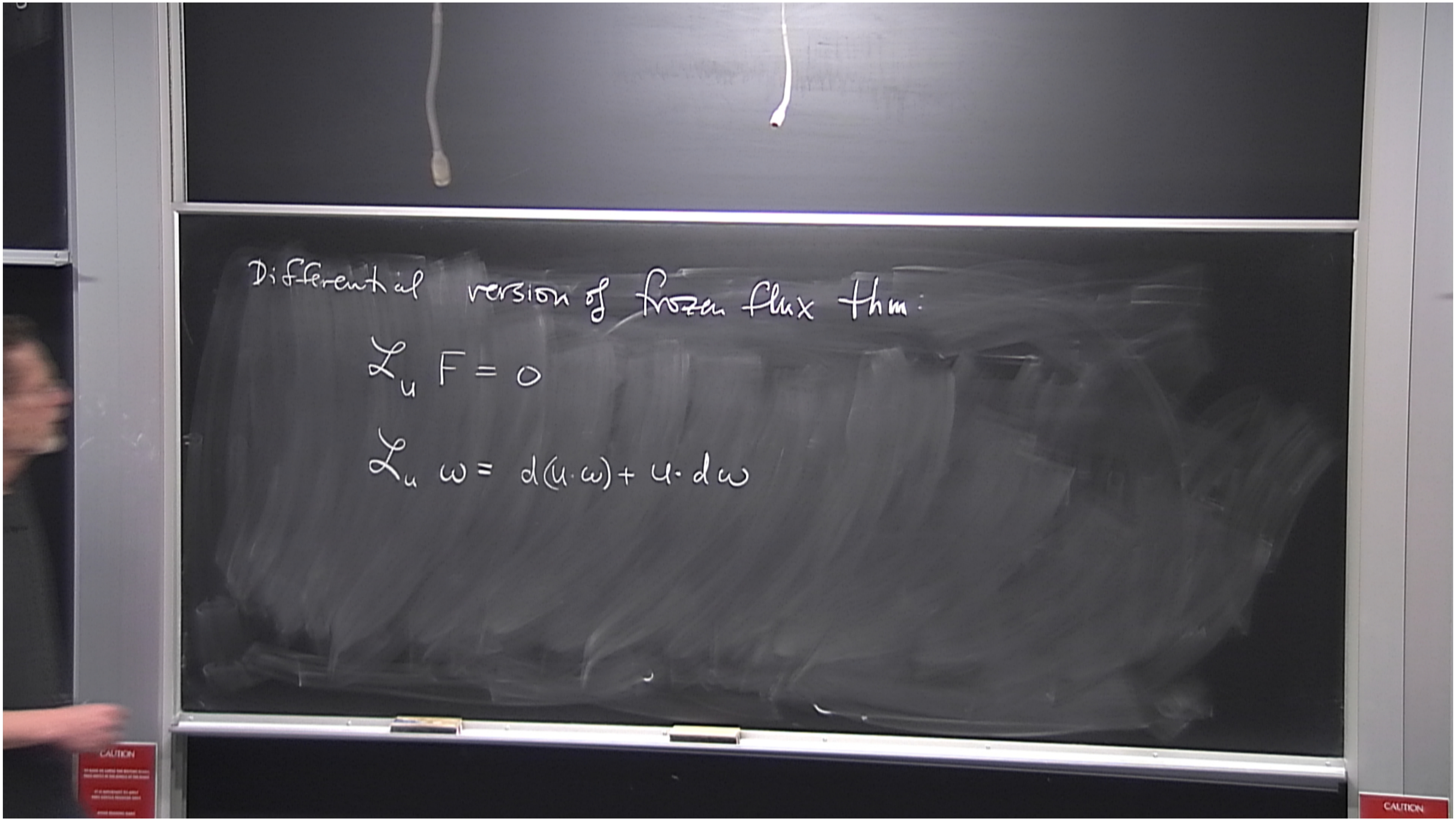
$\int_S F = \int_S dA = \int_{\partial S} A$;
$\int_M d\omega = \int_{\partial M} \omega$
Stokes' theorem
Frozen flux theorem
ideal MHD, $U_{fluid} \cdot F = 0$
CAUTION

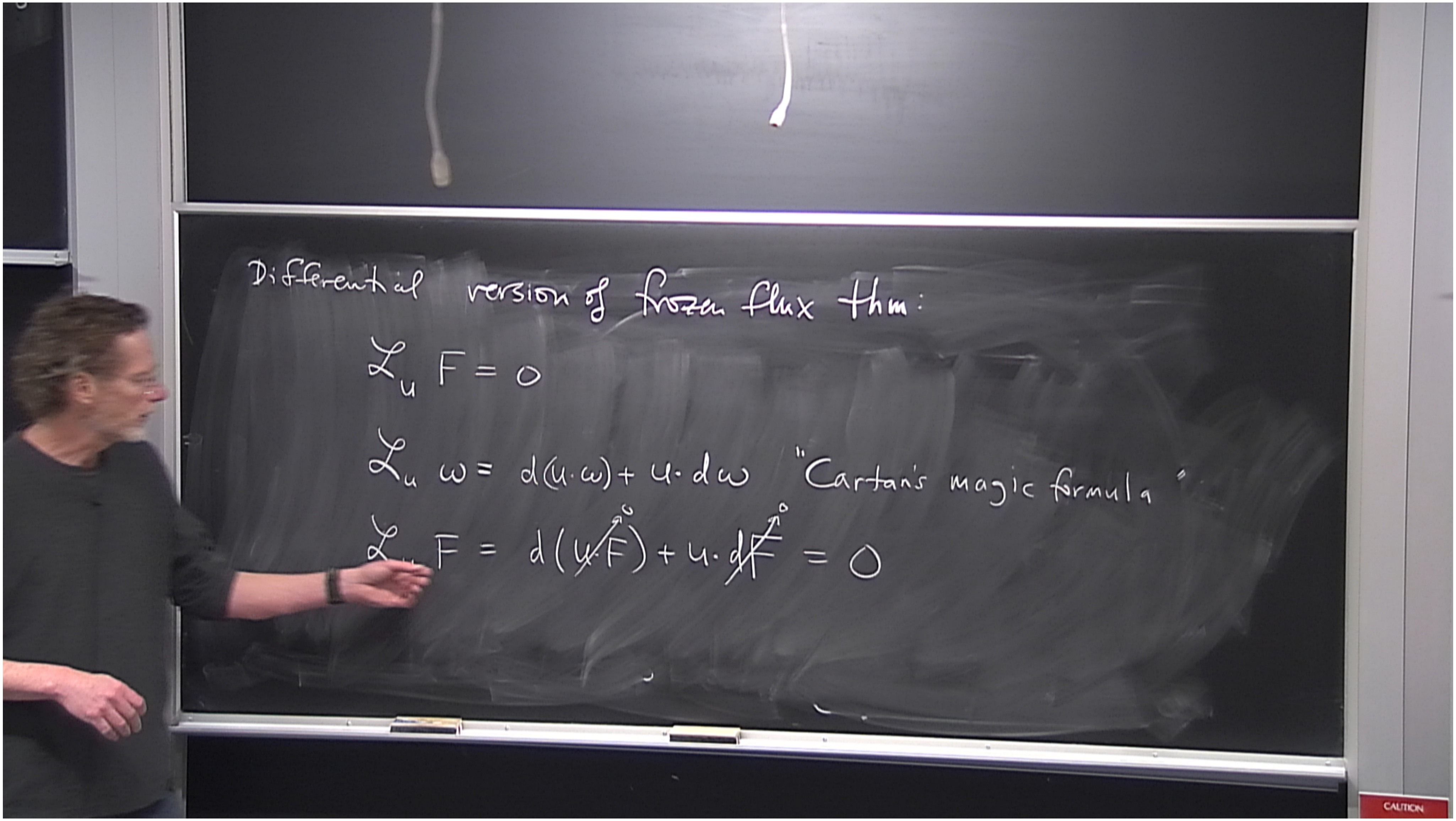

$\int_S F = \int_S dA = \int_{\partial S} A$;
$\int_M d\omega = \int_{\partial M} \omega$
Stokes' theorem
Frozen flux theorem
ideal MHD, $u_{fluid} \cdot F = 0$
$\int_{S_1} F$ = magnetic flux through S_1
S_2
S_1
CAUTION

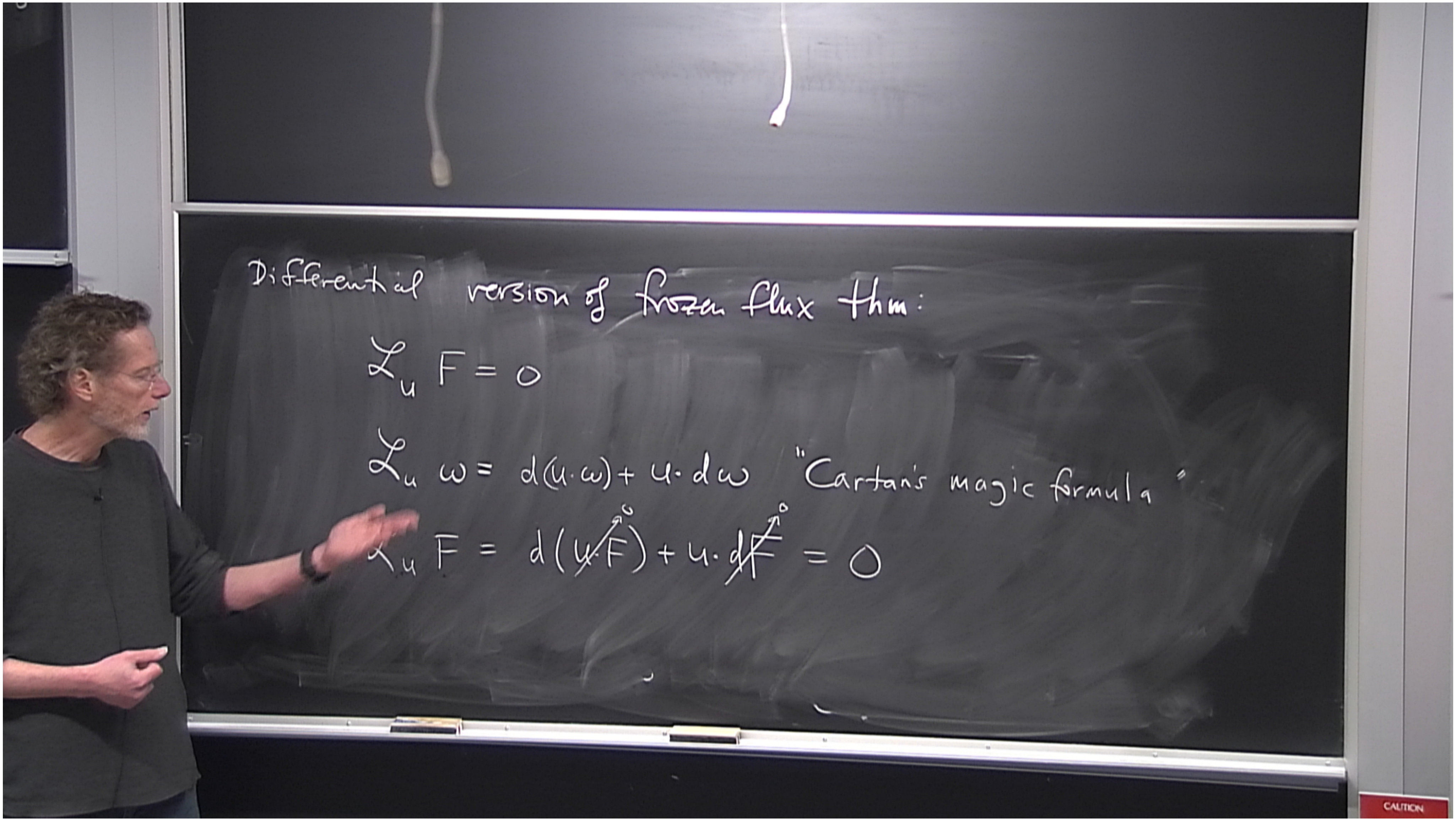

∫S F = ∫S dA = ∫∂S A ;
∫M dω = ∫∂M ω
Stokes' theorem
Frozen flux theorem
ideal MHD, U fluid · F = 0
∫S1 F = magnetic flux through S1
dF = 0 & Stokes ⇒ ∫S1 F + ∫S2 F + ∫Side F = 0
S2
S1


$\int_S F = \int_S dA = \int_{\partial S} A$;
$\int_M d\omega = \int_{\partial M} \omega$
Stokes' theorem
Frozen flux theorem
ideal MHD, $u_{fluid} \cdot F = 0$
u_{Fluid}
S_1
S_2
$\int_{S_1} F$ = magnetic flux through S_1
since
$dF = 0 \Rightarrow \int_{S_1} F + \int_{S_2} F + \int_{Side} F = 0$
& Stokes
$\Rightarrow$ flux through S_1 = flux through S_2 !
CAUTION


Differential version of frozen flux thm:
$\mathcal{L}_u F = 0$
$\mathcal{L}_u \omega$


Differential version of frozen flux thm:
$\mathcal{L}_u F = 0$
$\mathcal{L}_u \omega = d(u \cdot \omega) + u \cdot d\omega$
CAUTION
CAUTION


Differential version of frozen flux thm:
$\mathcal{L}_u F = 0$
$\mathcal{L}_u \omega = d(u\cdot\omega) + u\cdot d\omega$ "Cartan's magic formula"
$\mathcal{L}_u F = d(u\cdot F) + u\cdot dF = 0$
CAUTION


Differential version of frozen flux thm:
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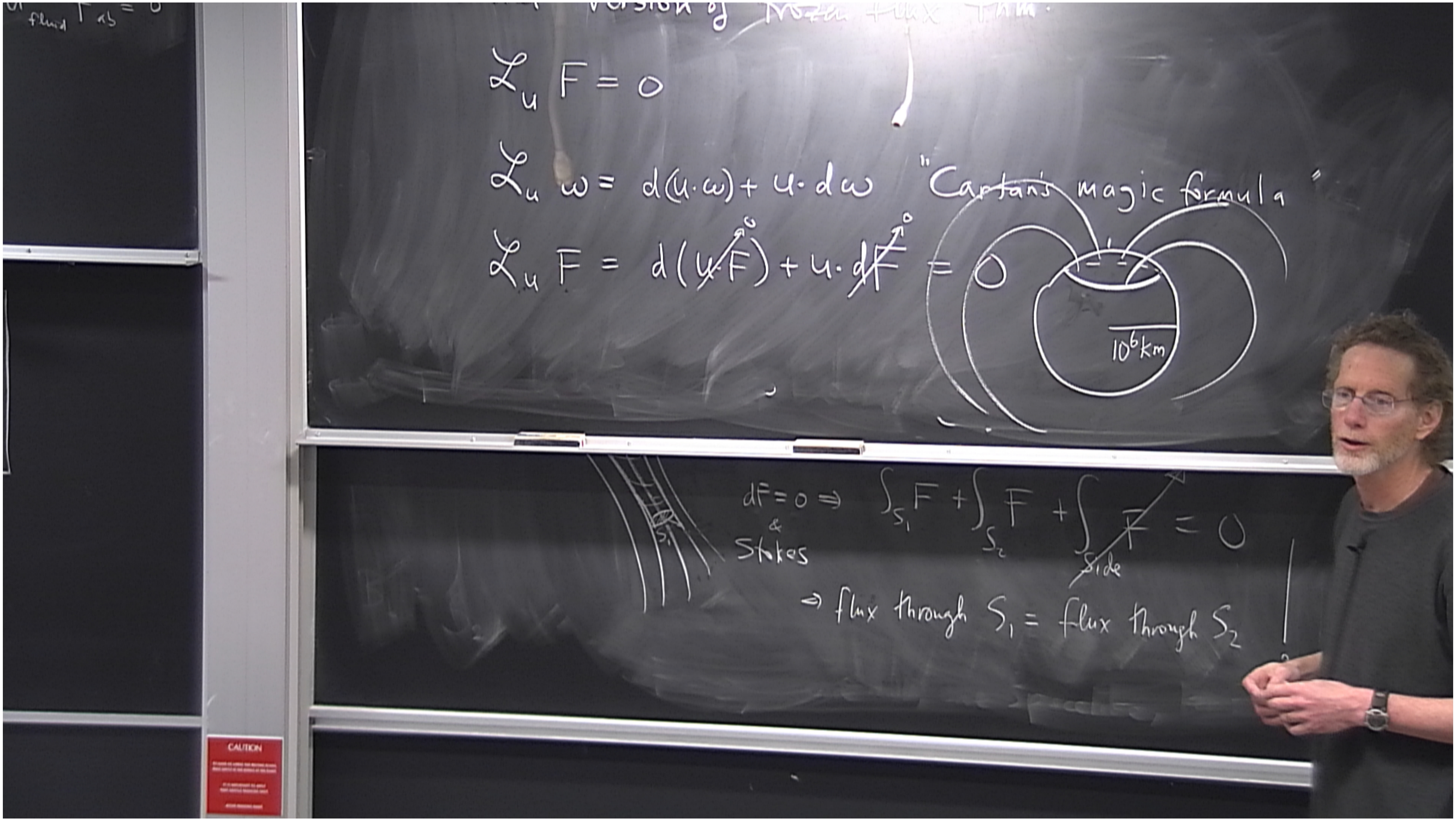
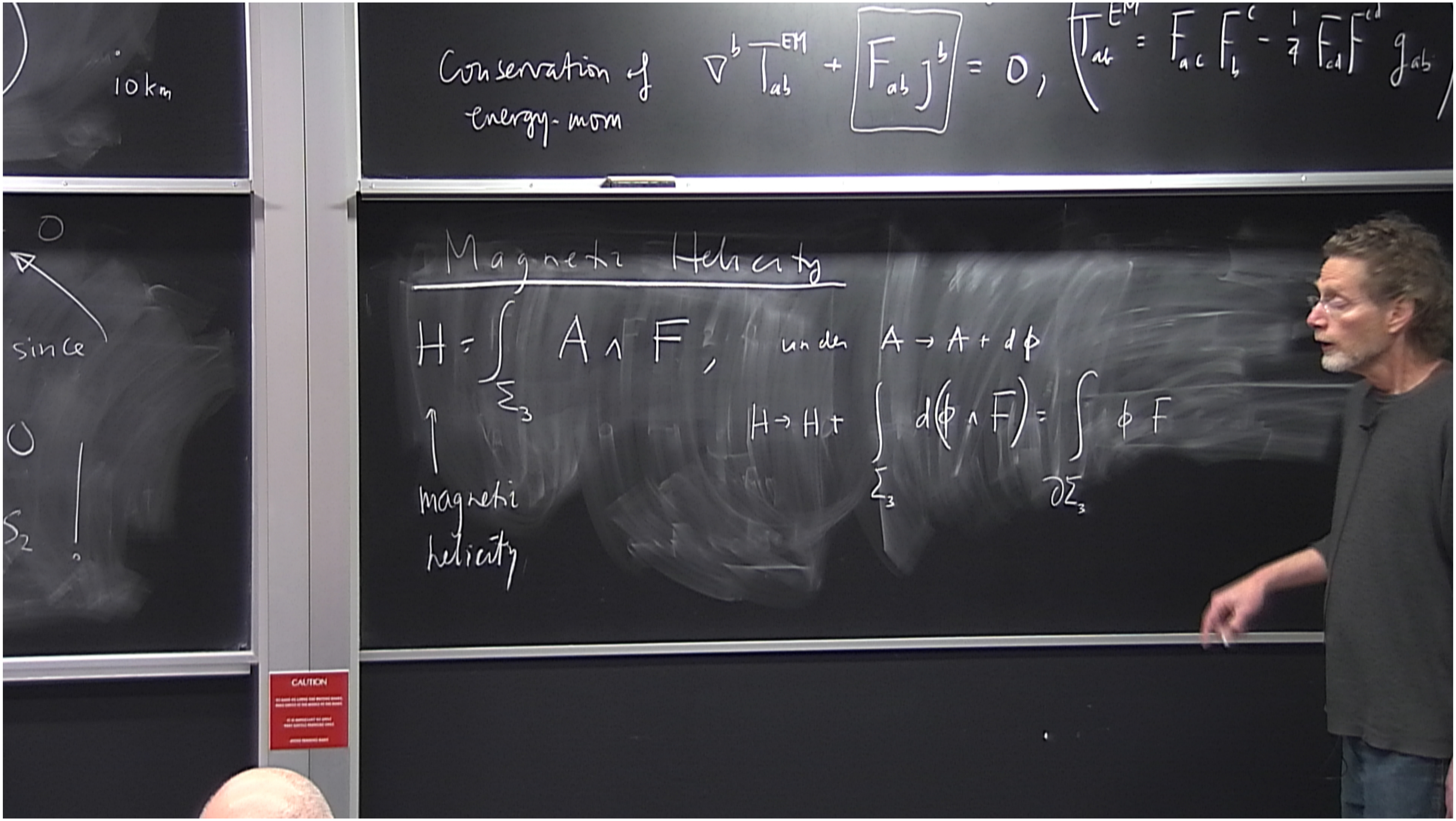
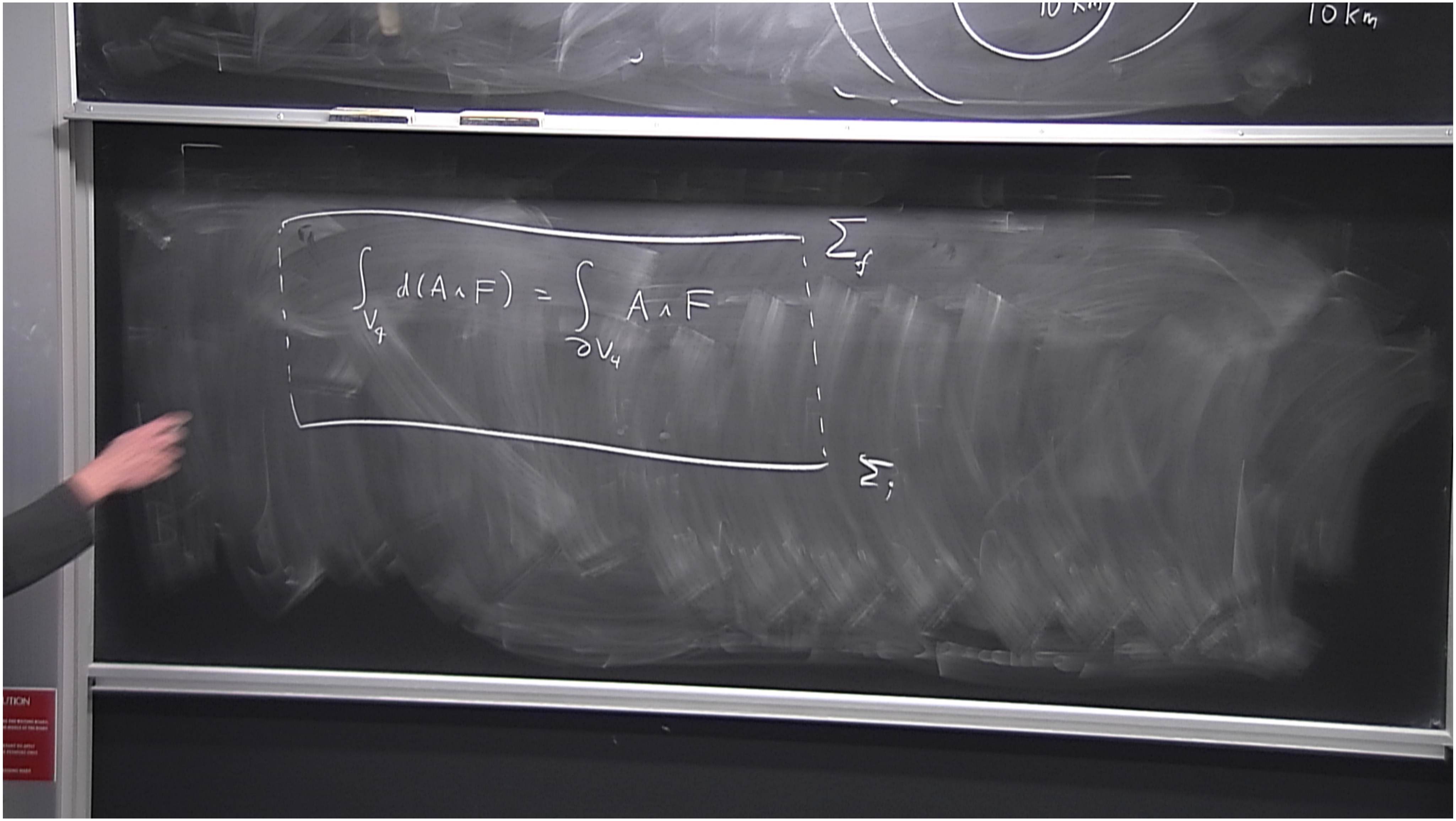
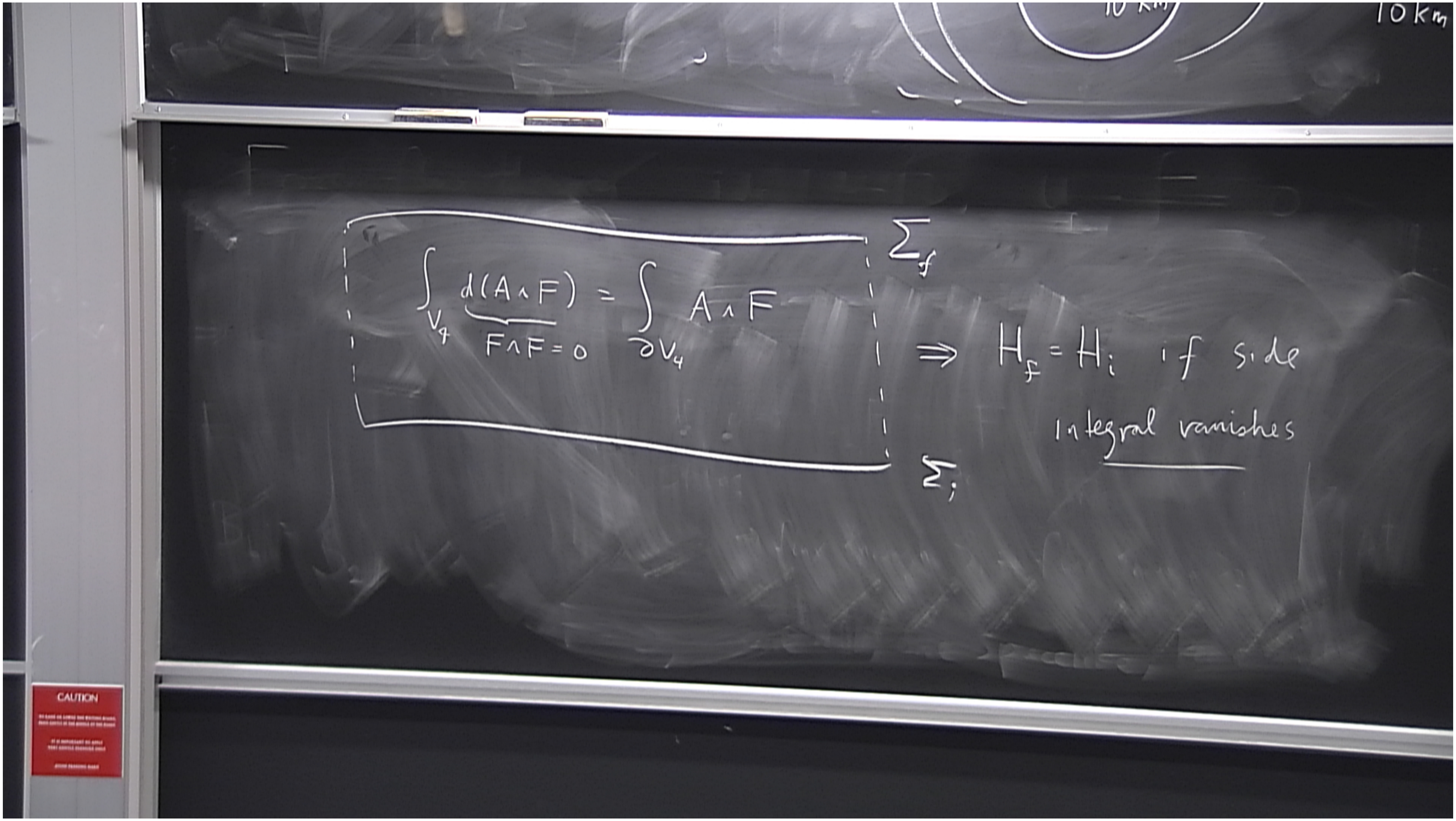
CAUTION

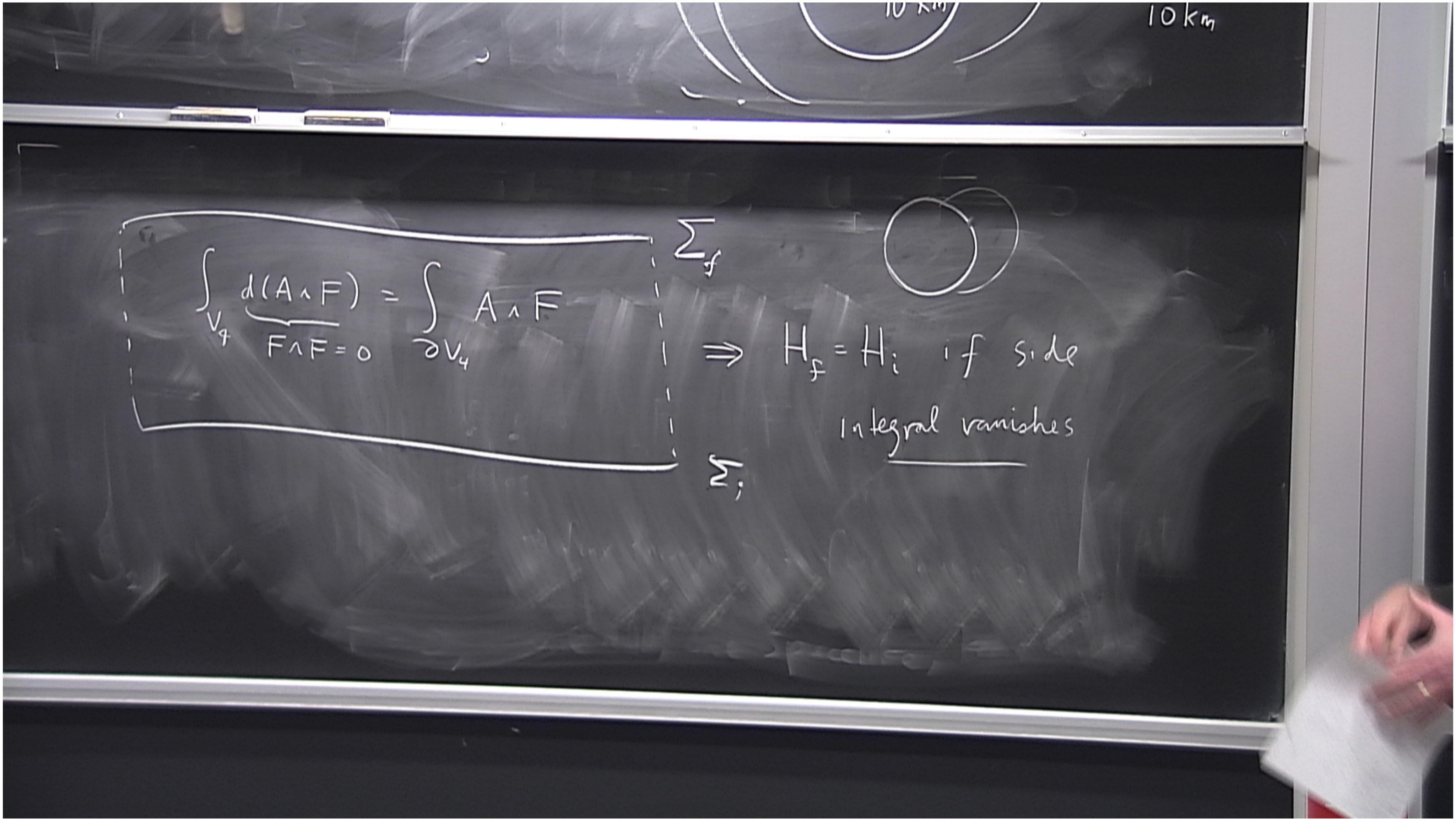

$\mathcal{L}_u F = 0$
$\mathcal{L}_u \omega = d(u\cdot\omega) + u\cdot d\omega$ "Cartan's magic formula"
$\mathcal{L}_u F = d(u\cdot F) + u\cdot dF = 0$
10^6 km
$dF = 0 \Rightarrow \int_{S_1} F + \int_{S_2} F + \int_{side} F = 0$
& Stokes
S_1
$\Rightarrow$ flux through S_1 = flux through S_2
CAUTION

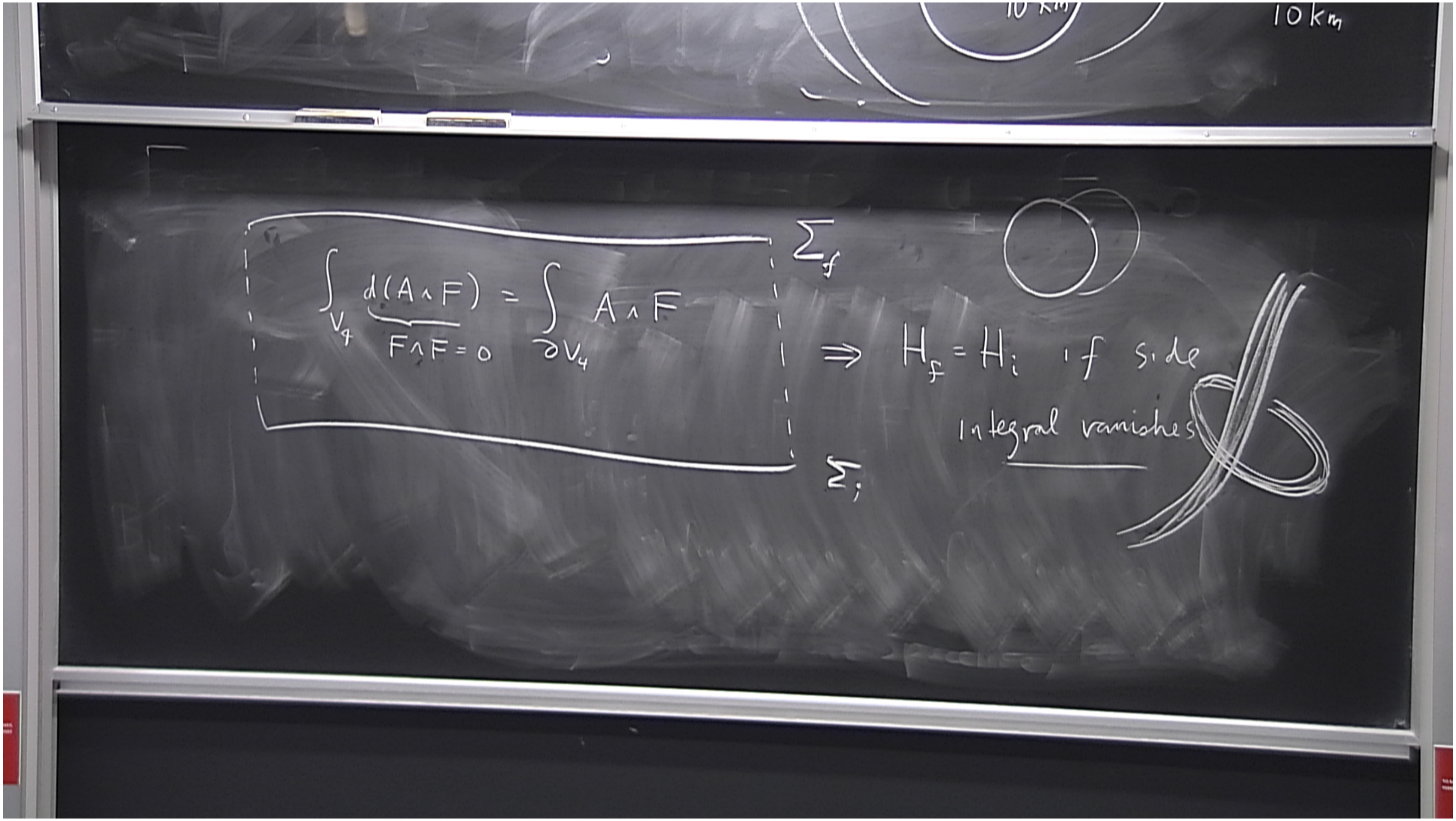


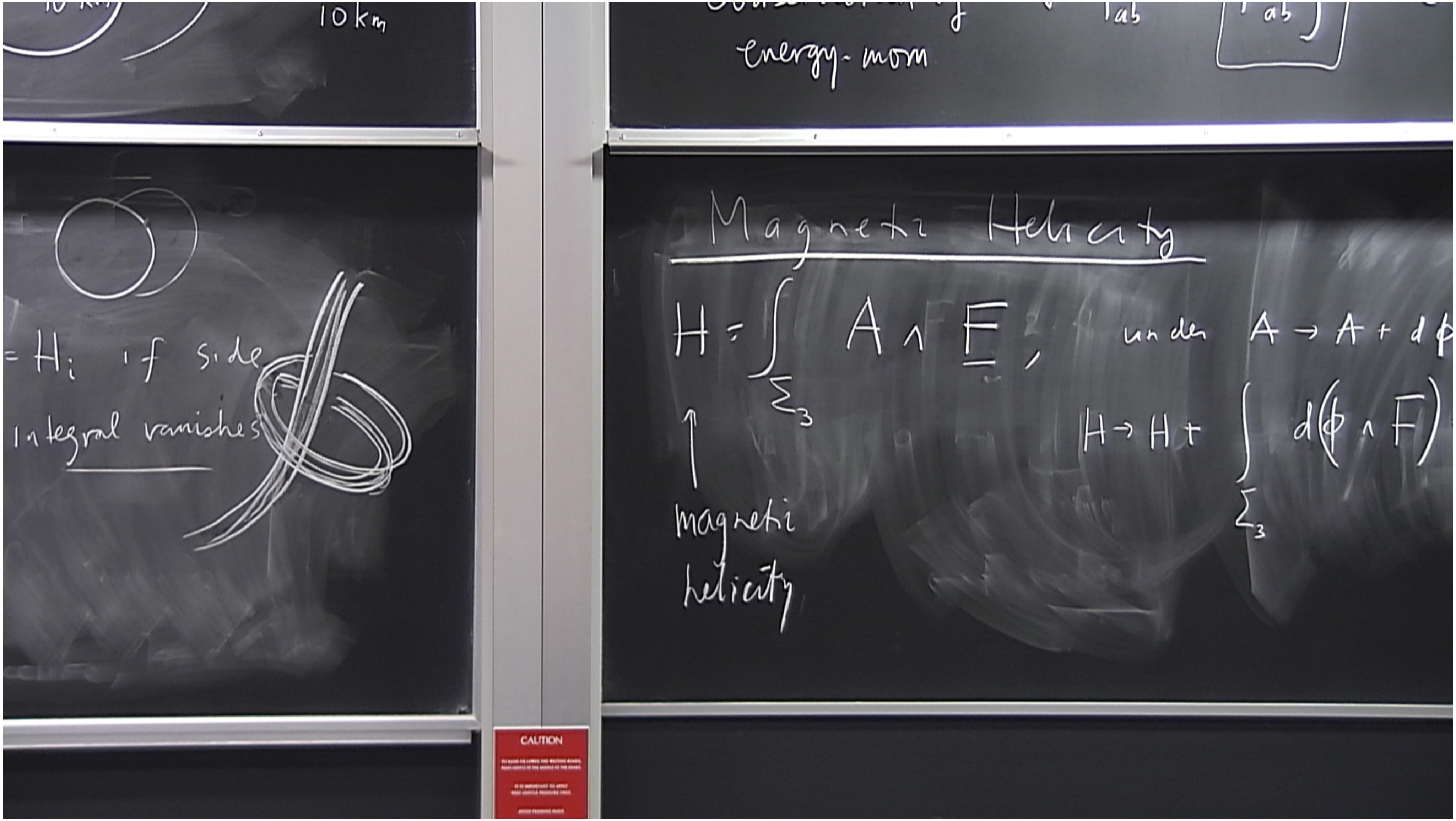
10km
Conservation of energy-mom
$\nabla^b T^{EM}_{ab} + F_{ab} J^b = 0,$
$T^{EM}_{ab} = F_{ac} F_b{}^c - \frac{1}{4} F_{cd} F^{cd} g_{ab}$
since
Magnetic Helicity
$H = \int_{\Sigma_3} A \wedge F;$
under $A \to A + d\phi$
$H \to H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial\Sigma_3} \phi F$
magnetic helicity
CAUTION

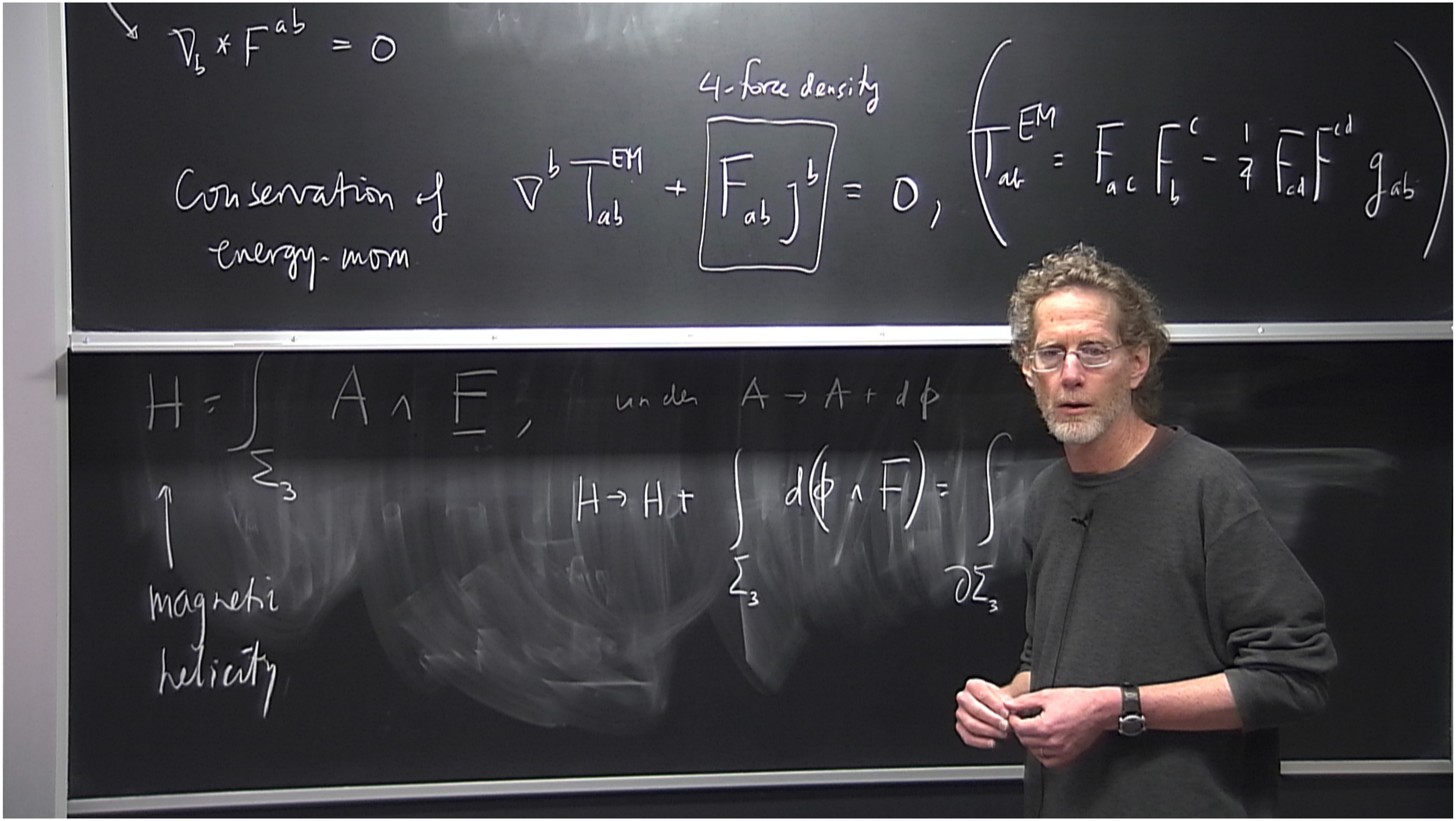

10 km
10 km
$\int_{V_4} d(A \wedge F) = \int_{\partial V_4} A \wedge F$
Σ_f
Σ_i

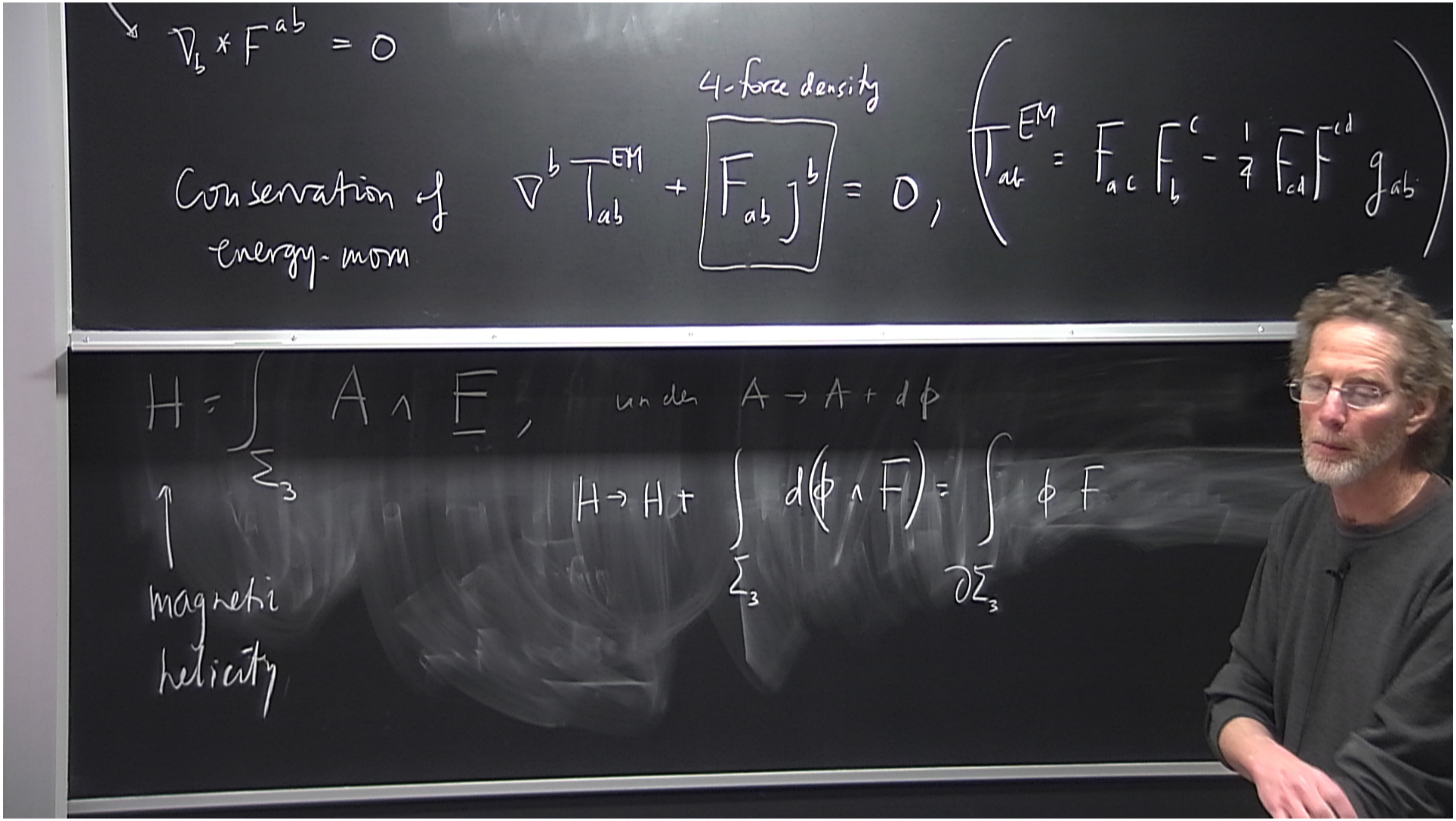

10 km
10 km
$\int_{V_4} d(A \wedge F) = \int_{\partial V_4} A \wedge F$
$F \wedge F = 0$
Σ_f
Σ_i
$\Rightarrow H_f = H_i$ if side integral vanishes
CAUTION


10 km
$\int_{V_4} d(A \wedge F) = \int_{\partial V_4} A \wedge F$
$F \wedge F = 0$
Σ_f
Σ_i
$\Rightarrow H_f = H_i$ if side integral vanishes

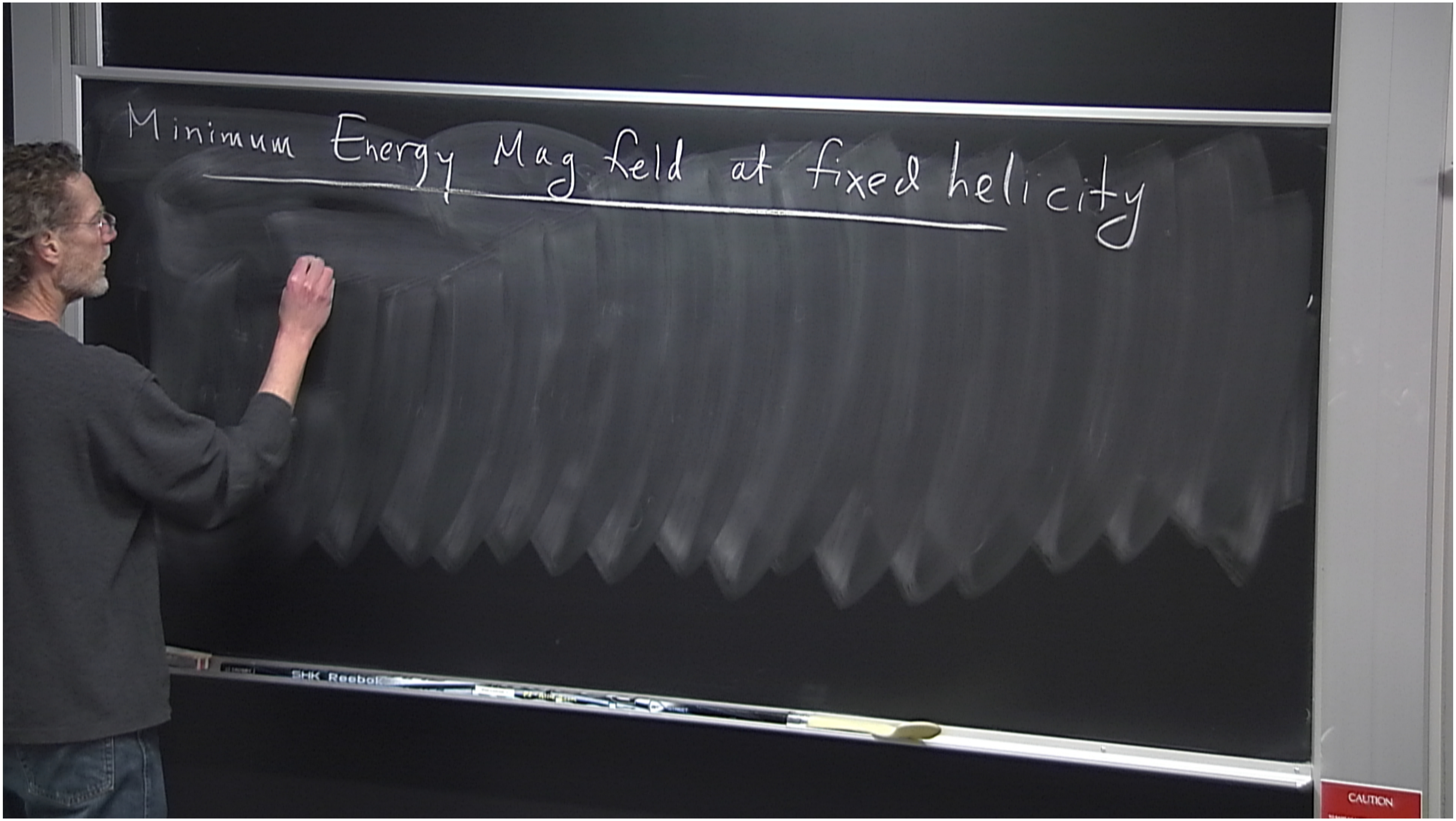

10 km
Σf
∫V4 d(A∧F) = ∫∂V4 A∧F
F∧F = 0
Σi
⇒ Hf = Hi if side
integral vanishes

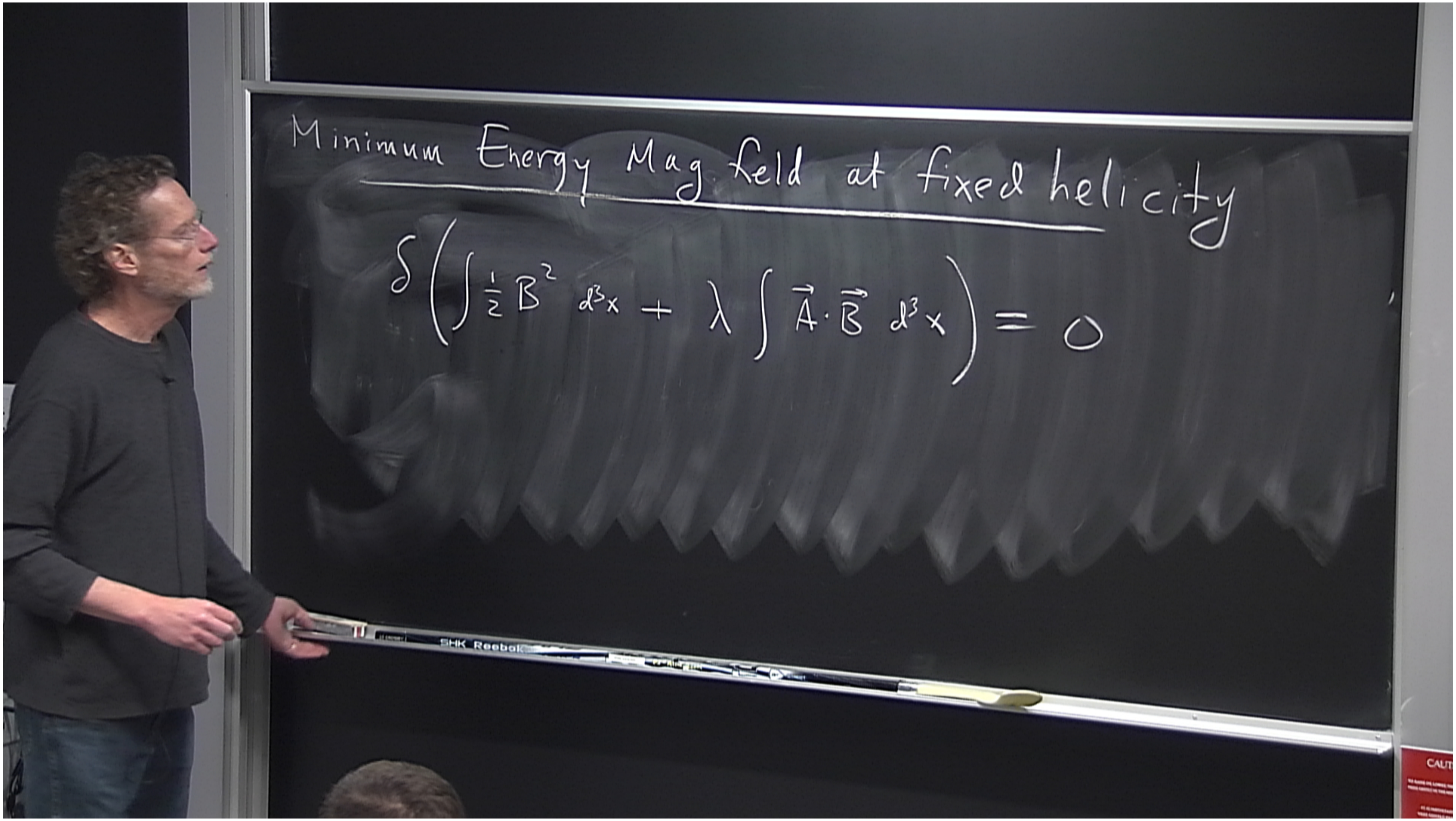

10km
energy-mom
= H_i if side
integral vanishes
Magneti Helicity
H = ∫_Σ3 A ∧ F ; under A → A + dφ
H → H + ∫_Σ3 d(φ ∧ F)
magneti helicity
CAUTION

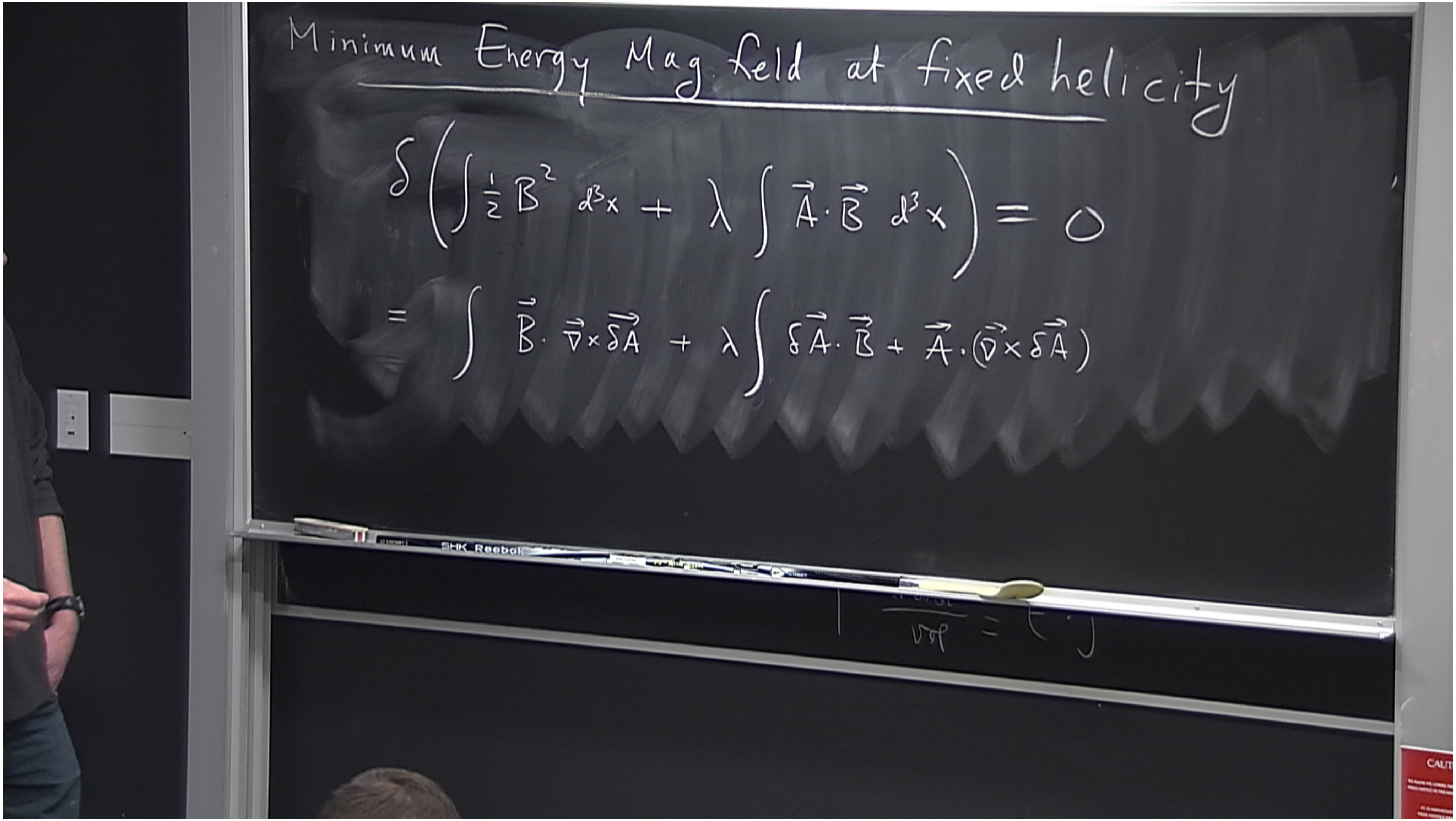

$\nabla_b * F^{ab} = 0$
4-force density
Conservation of energy-mom
$\nabla^b T_{ab}^{EM} + F_{ab} j^b = 0, \quad \left(T_{ab}^{EM} = F_{ac} F_b{}^c - \frac{1}{4} F_{cd} F^{cd} g_{ab} \right)$
$H = \int_{\Sigma_3} A \wedge E$, under $A \to A + d\phi$
$H \to H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial \Sigma_3}$
magneti helicity



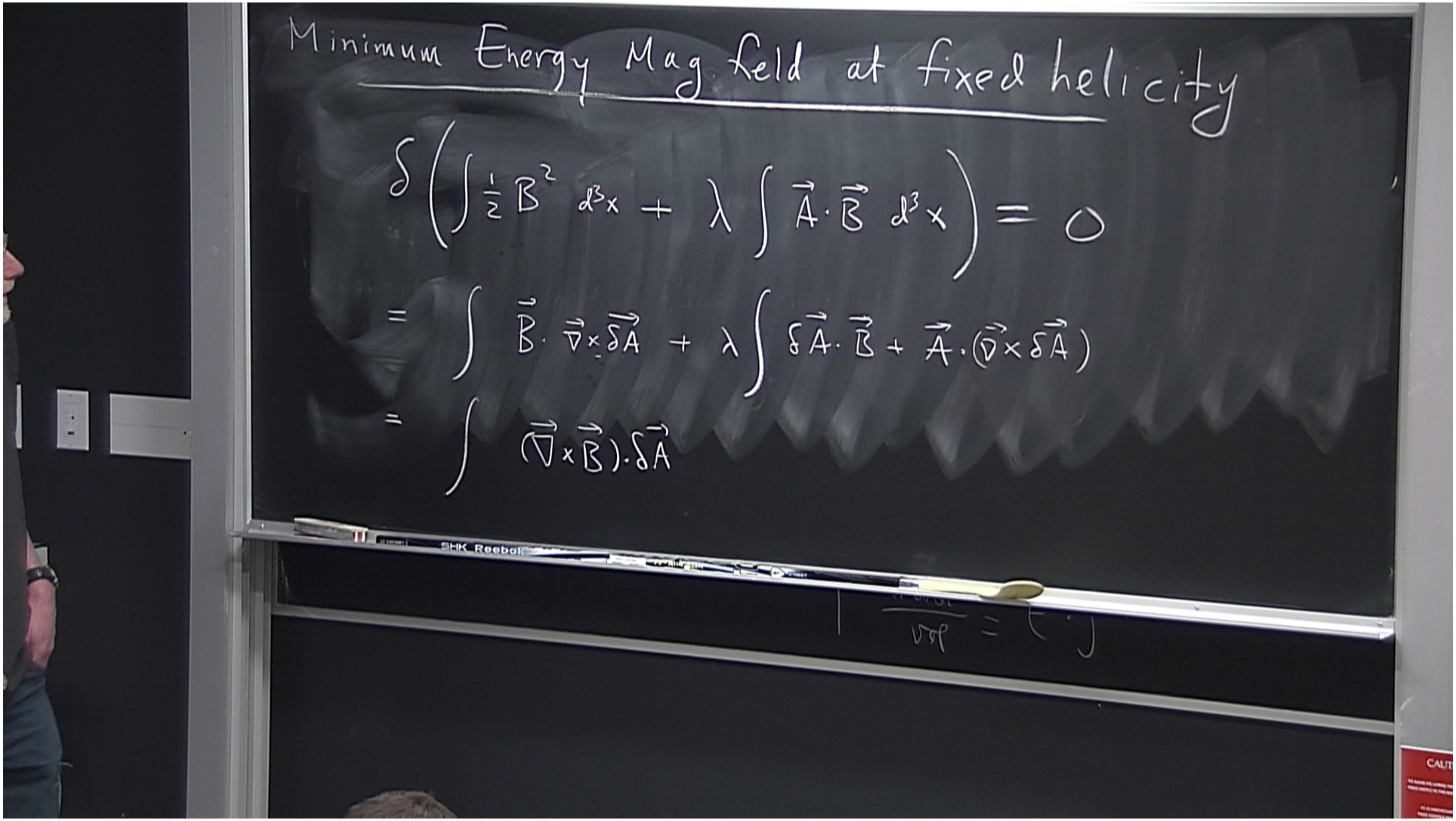
$\nabla_b * F^{ab} = 0$
4-force density
Conservation of energy-mom
$\nabla^b T_{ab}^{EM} + F_{ab} j^b = 0$,
$\left(T_{ab}^{EM} = F_{ac} F_b{}^c - \frac{1}{4} F_{cd} F^{cd} g_{ab} \right)$
$H = \int_{\Sigma_3} A \wedge E$,
under $A \to A + d\phi$
$H \to H + \int_{\Sigma_3} d(\phi \wedge F) = \int_{\partial \Sigma_3} \phi F$
magnetic helicity

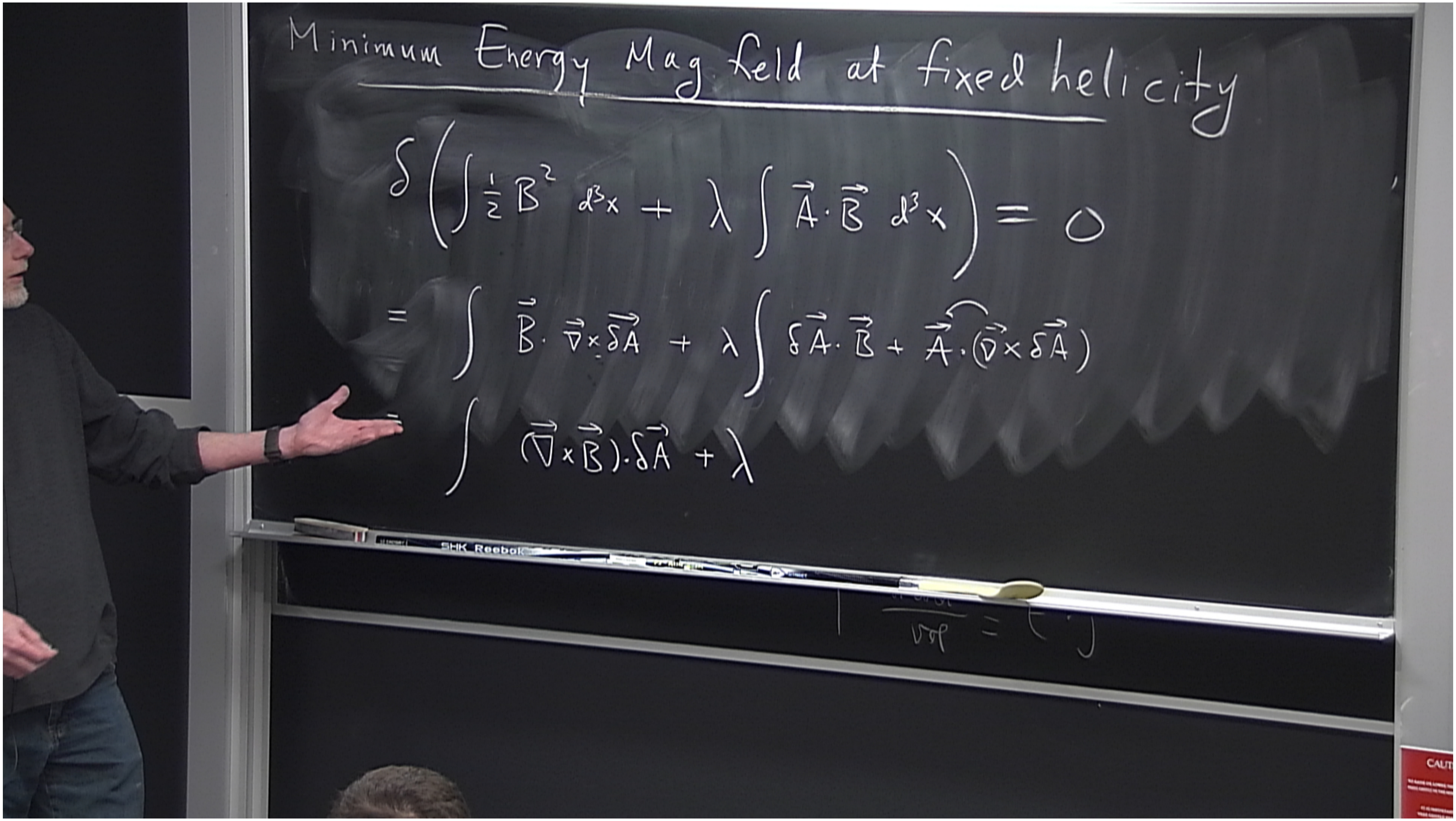

Minimum Energy Mag feld at fixed helicity
CAUTION

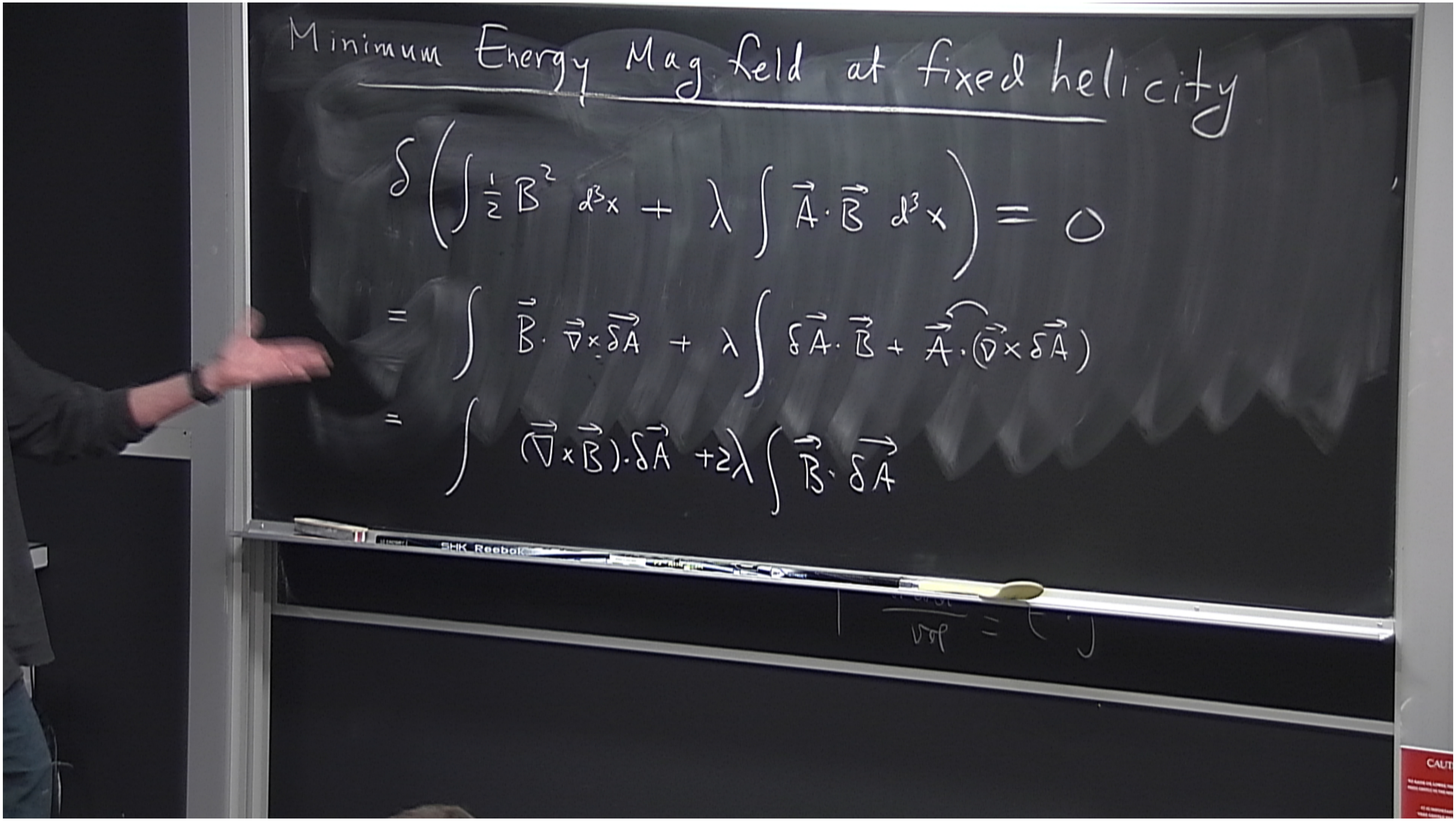

Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2} B^2 \, d^3x + \lambda \int \vec{A}\cdot\vec{B} \, d^3x\right) = 0$

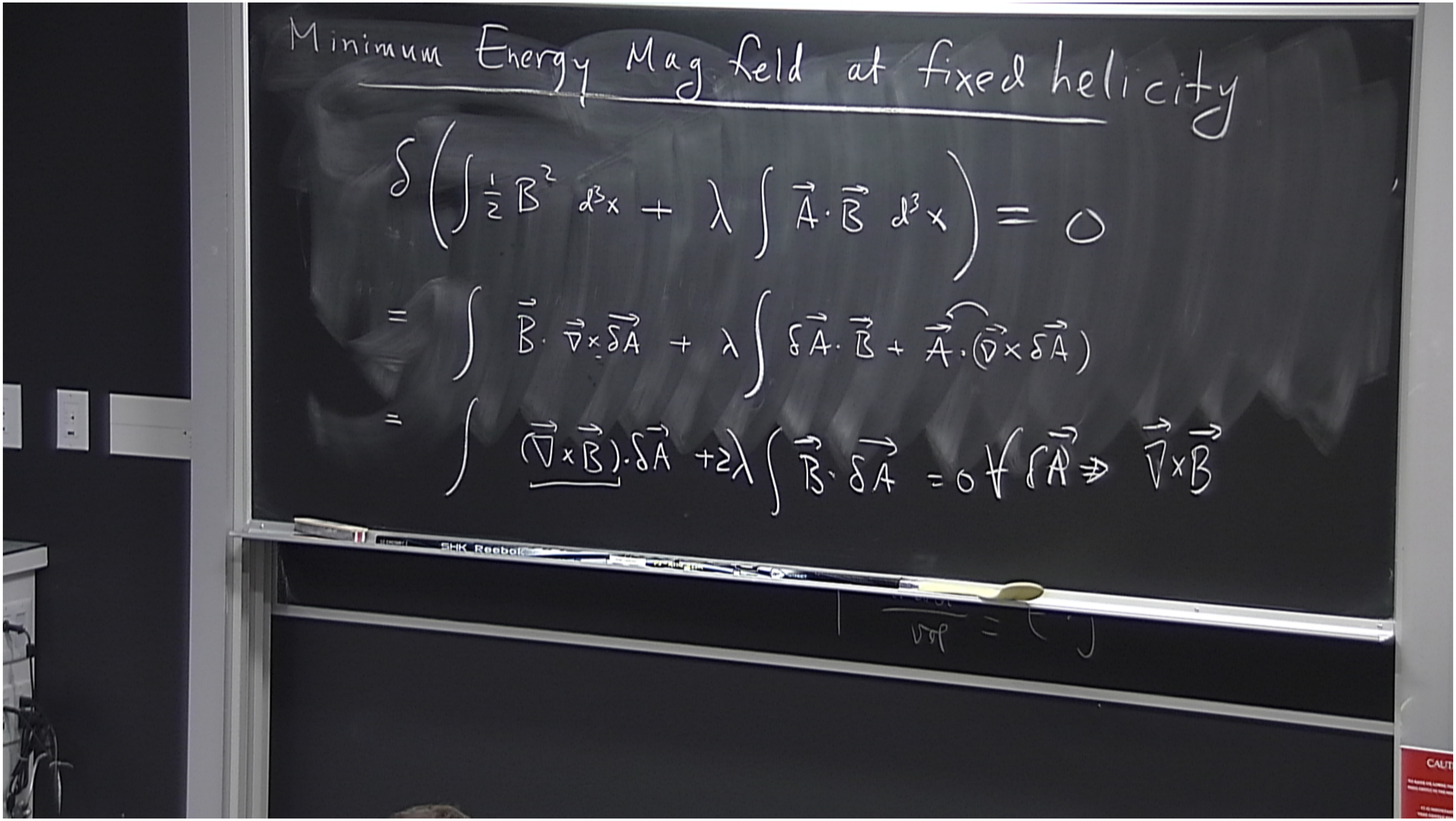


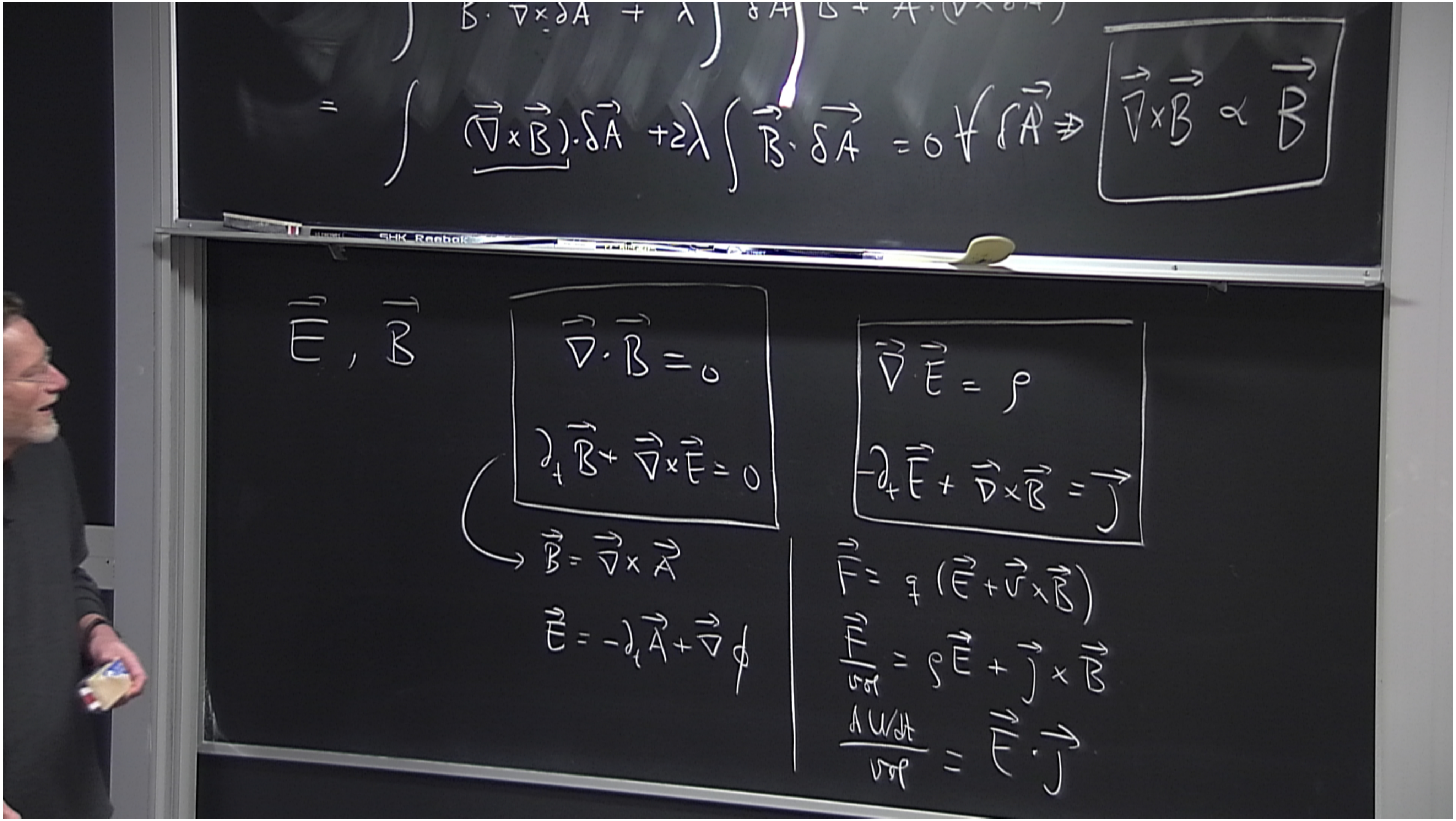
Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2} B^2 \, d^3x + \lambda \int \vec{A}\cdot\vec{B} \, d^3x\right) = 0$
$= \int \vec{B}\cdot\vec{\nabla}\times\delta\vec{A} + \lambda\int \delta\vec{A}\cdot\vec{B} + \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})$


Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2} B^2 \, d^3x + \lambda \int \vec{A}\cdot\vec{B} \, d^3x\right) = 0$
$= \int \vec{B}\cdot \vec{\nabla}\times\delta\vec{A} + \lambda \int \delta\vec{A}\cdot\vec{B} + \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})$
$= \int (\vec{\nabla}\times\vec{B})\cdot\delta\vec{A}$
SHK Reebok


Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2}B^2\, d^3x + \lambda \int \vec{A}\cdot\vec{B}\, d^3x\right) = 0$
$= \int \vec{B}\cdot\vec{\nabla}\times\delta\vec{A} + \lambda\int \delta\vec{A}\cdot\vec{B} + \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})$
$= \int (\vec{\nabla}\times\vec{B})\cdot\delta\vec{A} + \lambda$
SHK Reebok


Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2}B^2\, d^3x + \lambda \int \vec{A}\cdot\vec{B}\, d^3x\right) = 0$
$= \int \vec{B}\cdot\vec{\nabla}\times\delta\vec{A} + \lambda\int \delta\vec{A}\cdot\vec{B} + \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})$
$= \int (\vec{\nabla}\times\vec{B})\cdot\delta\vec{A} + 2\lambda\int \vec{B}\cdot\delta\vec{A}$


Minimum Energy Mag field at fixed helicity
$\delta\left(\int \frac{1}{2} B^2 \, d^3x + \lambda \int \vec{A}\cdot\vec{B} \, d^3x\right) = 0$
$= \int \vec{B}\cdot\vec{\nabla}\times\delta\vec{A} + \lambda\int \delta\vec{A}\cdot\vec{B} + \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})$
$= \int (\vec{\nabla}\times\vec{B})\cdot\delta\vec{A} + 2\lambda\int \vec{B}\cdot\delta\vec{A} = 0 \;\forall\, \delta\vec{A} \Rightarrow \vec{\nabla}\times\vec{B}$


= ∫ (∇×B)·δA + 2λ ∫ B·δA = 0 ∀ δA ⇒ ∇×B ∝ B
E, B
∇·B = 0
∂t B + ∇×E = 0
∇·E = ρ
−∂t E + ∇×B = J
B = ∇×A
E = −∂t A + ∇φ
F = q(E + v×B)
F/vol = ρE + J×B
dW/dt / vol = E·J