Title: The Sherrington-Kirkpatrick model and its diluted version

Date: Feb 20, 2014 02:30 PM

URL: http://pirsa.org/14020151

Abstract: I will talk about two types of random processes -- the classical Sherrington-Kirkpatrick (SK) model of spin glasses and its diluted version. One of the main motivations in these models is to find a formula for the maximum of the process, or the free energy, in the limit when the size of the system is getting large. The answer depends on understanding the structure of the Gibbs measure in a certain sense, and this structure is expected to be described by the so called Parisi solution in the SK model and MÃ©zard-Parisi solution in the diluted SK model. I will explain what these are and mention some results in this direction.

Sherrington-Kirkpatrick model and its diluted version

Dmitry Panchenko


and its diluted version
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The Sherrington-Kirkpatrick model

Splitting a group of people into two:

$\{1, \ldots, N\}$	– a group of N people
$\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, +1\}^N$	– labels of 2 groups
(g_{ij})	– interactions between i & j

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Model typical behavior: (g_{ij}) - i.i.d. standard Gaussian.

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Giorgio Parisi 1980:

$$\lim_{N\to\infty} \frac{1}{N^{3/2}} \mathbb{E} \max_{\sigma} \sum_{i<j} g_{ij}\sigma_i\sigma_j = 0.76\ldots.$$

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Example. $N = 10,000$:

2462 enemies (optimal) vs. 2500 enemies (random)

The Sherrington-Kirkpatrick model

Hamiltonian:

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \le i,j \le N} g_{ij}\, \sigma_i \sigma_j$$

$\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, +1\}^N$, (g_{ij}) - i.i.d. $N(0,1)$.

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$$R_{1,2} = \frac{\sigma^1 \cdot \sigma^2}{N} = \frac{1}{N} \sum_{1 \le i \le N} \sigma_i^1 \sigma_i^2.$$

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Invariance under orthogonal transformations!

Diluted version of the SK model

Each person interacts with finitely many others:

$$H_N(\sigma) = \sum_{k \le \pi(\lambda N)} g_k \, \sigma_{i_k} \sigma_{j_k}$$

$\pi(\lambda N)$ is Poisson(λN), λ – **connectivity parameter**,
$(i_k, j_k)_{k \ge 1}$ – i.i.d. uniform on $\{1, \ldots, N\}$.

Smooth approximation of maximum

The free energy:

$$F_N(\beta) = \frac{1}{N\beta}\,\mathbb{E}\log\sum_{\sigma}\exp\beta H_N(\sigma),$$

where $\beta = 1/T > 0$ – **inverse temperature parameter**.

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Notice that:

$$\frac{1}{N}\,\mathbb{E}\max H_N(\sigma) \leq F_N(\beta) \leq \frac{1}{N}\,\mathbb{E}\max H_N(\sigma) + \frac{\log 2}{\beta}.$$

The free energy

SK model:

$$\lim_{N\to\infty} F_N(\beta) = \text{ Parisi formula (1980)}$$

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Diluted SK model:

$$\lim_{N\to\infty} F_N(\beta) = \text{ Mézard-Parisi formula (2001)}$$

Open problem.

The Gibbs measure

Part of the story not included in this talk:

Structure of the Gibbs measure

$\Downarrow$

Formula for the free energy

The Gibbs measure:

$$G_N(\sigma) = \frac{\exp \beta H_N(\sigma)}{Z_N}, \text{ where } Z_N = \sum_{\sigma} \exp \beta H_N(\sigma).$$

Main question: How does G_N look like asymptotically as $N \to \infty$?

Asymptotic Gibbs measures

Asymptotic Gibbs measures

- Sample i.i.d. **replicas** $(\sigma^\ell)_{\ell\geq 1}$ from G_N and consider

$$\left.\begin{array}{c} \sigma^1 = \left(\sigma_1^1, \ldots, \sigma_N^1, \ldots\right) \\ \sigma^2 = \left(\sigma_1^2, \ldots, \sigma_N^2, \ldots\right) \\ \vdots \\ \sigma^\ell = \left(\sigma_1^\ell, \ldots, \sigma_N^\ell, \ldots\right) \\ \vdots \end{array}\right\} \in \left\{-1, +1\right\}^{\mathbb{N}\times\mathbb{N}}$$

Asymptotic Gibbs measures

Symmetries:

$$\left(\sigma_{\rho(i)}^{\pi(\ell)}\right)_{i,\ell\geq 1} \stackrel{d}{=} \left(\sigma_i^{\ell}\right)_{i,\ell\geq 1}$$

for all permutations π (replica symmetry), ρ (symmetry between sites).

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Aldous-Hoover representation: There exists $\sigma : [0,1]^4 \to \{-1,+1\}$ such that

$$\left(\sigma_i^{\ell}\right)_{i,\ell\geq 1} \stackrel{d}{=} \Big(\sigma\big(w, u_\ell, v_i, x_{i\ell}\big)\Big)_{i,\ell\geq 1}$$

where $w, (u_\ell), (v_i)$ and $(x_{i\ell})$ are i.i.d. uniform on $[0,1]$.

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Role of $x_{i\ell}$: flip a $\{-1,+1\}$-valued coin with the mean

$$\bar{\sigma}(w, u_\ell, v_i) = \int_0^1 \sigma\big(w, u_\ell, v_i, x\big)\, dx.$$

Geometric interpretation

A configuration $\sigma \in \{-1, +1\}^N$ is replaced by a **function**

$$\bar{\sigma}(w, u, \cdot) \in \{\|\bar{\sigma}\|_\infty \leq 1\} \cap L^2([0,1], dv).$$

$\left(\bar{\sigma}(w, u_\ell, \cdot)\right)_{\ell \geq 1}$ – i.i.d. replicas from the random measure

$$G = du \circ \left(u \to \bar{\sigma}(w, u, \cdot)\right)^{-1}.$$

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G – **asymptotic Gibbs measure**. Why $L^2([0,1],dv)$?

$$R_{\ell,\ell'} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^\ell\sigma_i^{\ell'} \xrightarrow{d} \int_0^1 \bar{\sigma}(w,u_\ell,v)\bar{\sigma}(w,u_{\ell'},v)\,dv.$$

Ultrametric Parisi solution

- The Gibbs measure lives on a sphere: $G\big(\|\sigma\| = \text{const}\big) = 1$.
- **Ultrametricity:** Sample $\sigma^1, \sigma^2, \sigma^3$ from G,

$$\|\sigma^2 - \sigma^3\| \leq \max\Big(\|\sigma^1 - \sigma^2\|, \|\sigma^1 - \sigma^3\|\Big).$$

$\forall r \geq 0$, equivalence relation on the support of G:

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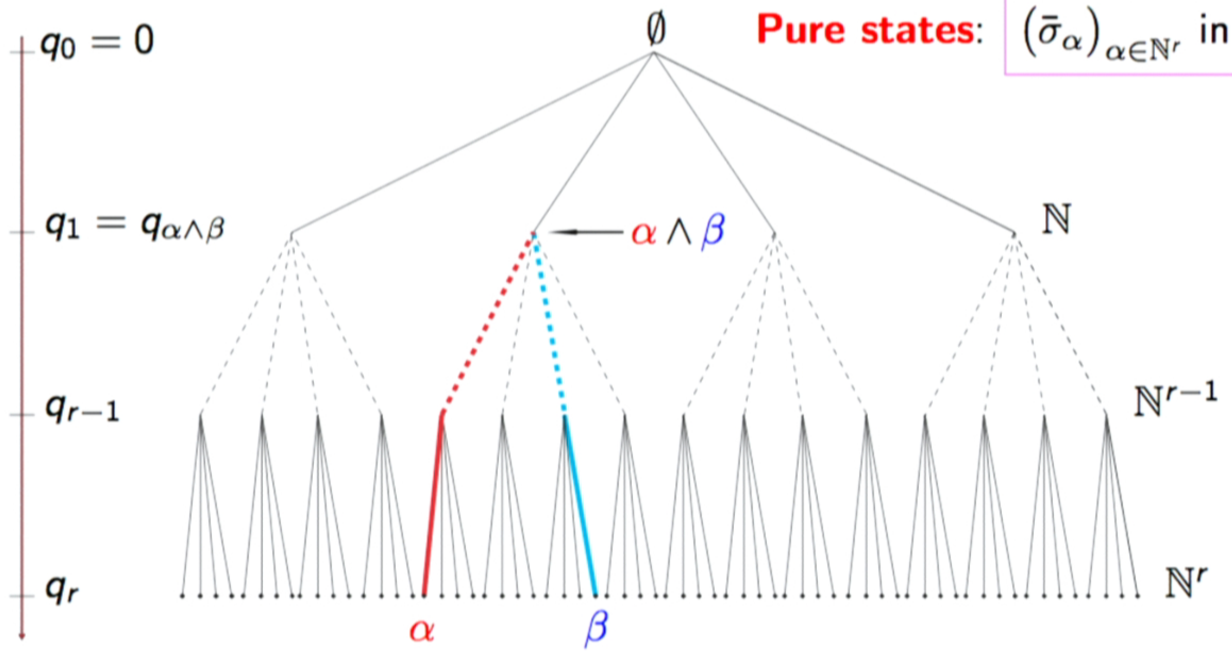
Ultrametricity = clustering!

Ultrametric Parisi solution: r-RSB case

Pure states: $(\bar{\sigma}_\alpha)_{\alpha \in \mathbb{N}^r}$ in $L^2[0,1]$



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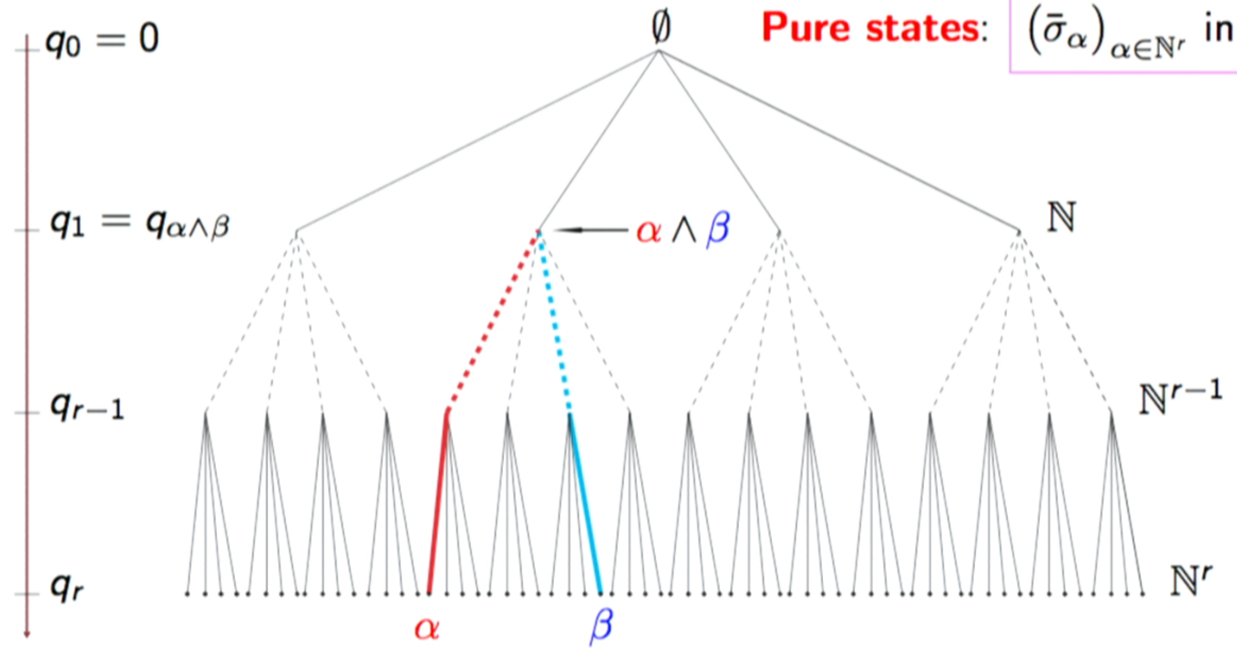

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Mézard-Parisi solution in diluted models

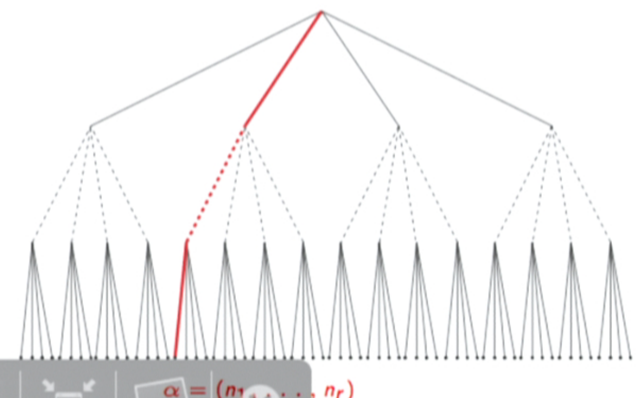
1. Overlap structure is the same as in the Parisi solution.

Pure state spin magnetizations $\boxed{\bar{\sigma}_\alpha(w, v_i)}$ **???**

2. Weights (p_α) are independent of $\big(\bar{\sigma}_\alpha(w, v_i)\big)$.
3. If $\alpha = (n_1, \ldots, n_r) \in \mathbb{N}^r$,

$$\bar{\sigma}_\alpha(w, v_i) \overset{d}{=} \mathcal{T}\big(v^i_\emptyset, v^i_{n_1}, \ldots, v^i_{n_1 \ldots n_r}\big)$$

where all $v^{\text{index}}_{\text{index}}$ are i.i.d. $U[0, 1]$.



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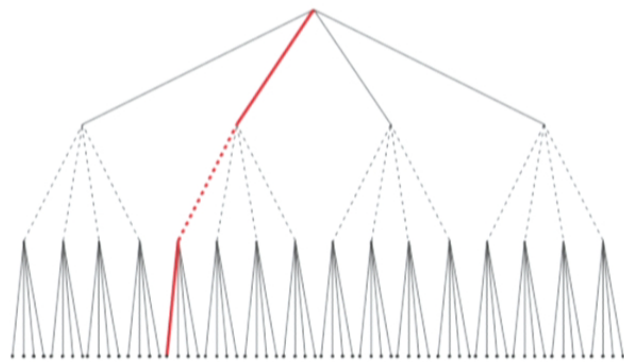
$\mathcal{T}$ – **functional order parameter**



Toward the Mézard-Parisi solution

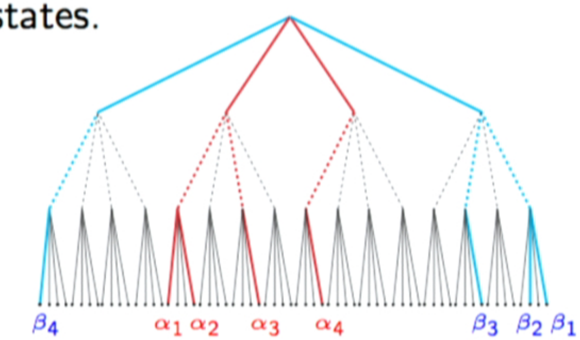
1. Parisi solution for the overlaps holds in all these models.
2. Weights (p_α) are independent of $\left(\bar{\sigma}_\alpha(w, v_i)\right)$.
3. **Hierarchical exchangeability**:

$$\left(\bar{\sigma}_{\pi(\alpha)}(w, v_i)\right)_{\alpha \in \mathbb{N}^r, i \in \mathbb{N}} \stackrel{d}{=} \left(\bar{\sigma}_\alpha(w, v_i)\right)_{\alpha \in \mathbb{N}^r, i \in \mathbb{N}}$$

for any bijection $\pi : \mathbb{N}^r \to \mathbb{N}^r$ such that

$$\pi(\alpha) \wedge \pi(\beta) = \alpha \wedge \beta \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^r,$$

i.e. π preserves distances between pure states.



Toward the Mézard-Parisi solution

[Austin-P'13] **Hierarchical Aldous-Hoover representation**:

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The Mézard-Parisi solution predicts complete symmetry:

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The Ghirlanda-Guerra identities (1998)

Sample $\sigma^1, \ldots, \sigma^n, \sigma^{n+1}$ from $G = G_w$ and recall the notation

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