Title: On null singularities in general relativity

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Abstract: This talk will discuss the formation, structure and interaction of null singularities for
 the Einstein equations, as well as what this all means for the singular boundary
 of generic space times within black hole regions.


Null singularities in general relativity
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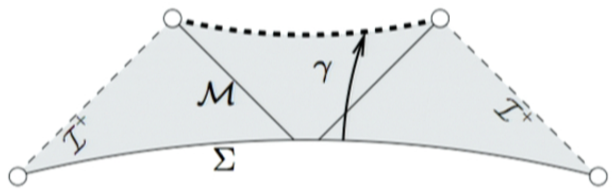
Outline

1. Schwarzschild, Reissner–Nordström/Kerr and the strong cosmic censorship conjecture
2. The blue-shift effect in linear theory
3. A fully non-linear toy-model under spherical symmetry
4. The real thing: the vacuum Einstein equations without symmetry

Schwarzschild



The Schwarzschild spacetime is geodesically incomplete–there are observers–like poor γ–who live only for finite proper time. **All** such observers are **torn apart** by infinite tidal forces. The spacetime is *inextendible* as a Lorentzian manifold with C^0 metric.

Is this prediction stable to arbitrary perturbation of initial data?

Strong cosmic censorship

Conjecture (Strong cosmic censorship, PENROSE 1972). *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal Cauchy development is future* <u>*inextendible*</u> *as a suitably regular Lorentzian manifold.*

One should think of this conjecture as a statement of *global uniqueness*, or, in more colloquial language:

"The future is uniquely determined by the present".

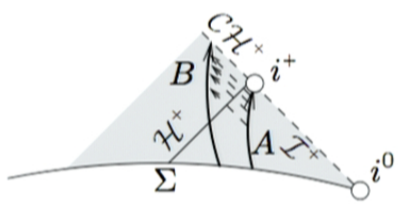
The **in**extendibility requirement of the conjecture is *true* then in Schwarzschild, but *false* in Reissner–Nordström and Kerr for $Q \neq 0$, $a \neq 0$ respectively.

Thus, within the class of explicit stationary solutions, it is *extendibility* that is generic, not *inextendibility*, which only holds with $a = Q = 0$!

Why would one ever conjecture then that strong cosmic censorship holds?

Blue-shift instability (PENROSE, 1968)

A possible mechanism for instability is the celebrated blue-shift effect, first pointed out by PENROSE:



PENROSE argued that this would cause linear perturbations to blow-up in some way on a Reissner–Nordström background. Subsequent numerical study by SIMPSON–PENROSE on Maxwell fields (1972).

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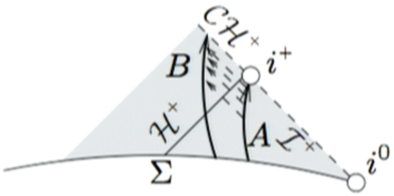

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The blue-shift effect in linear theory

The simplest mathematical realisation of the PENROSE heuristic account of the blue shift instability can be given as a corollary of a general recent result on the Gaussian beam approximation on Lorentzian manifolds, due to SBIERSKI. This gives:

Theorem 1 (SBIERSKI, 2012). *In subextremal Reissner–Nordström or Kerr, let Σ be a two-ended asymptotically flat Cauchy surface and choose a spacelike hypersurface $\widetilde{\Sigma}$ transverse to $\mathcal{CH}^+$, let $E_\Sigma[\psi]$, $E_{\widetilde{\Sigma}}[\psi]$ denote the energy measured with respect to the normal of Σ, $\widetilde{\Sigma}$, respectively.*

Then

$$\sup_{\psi \in C^\infty : E_\Sigma[\psi]=1} E_{\widetilde{\Sigma}}[\psi] = \infty.$$

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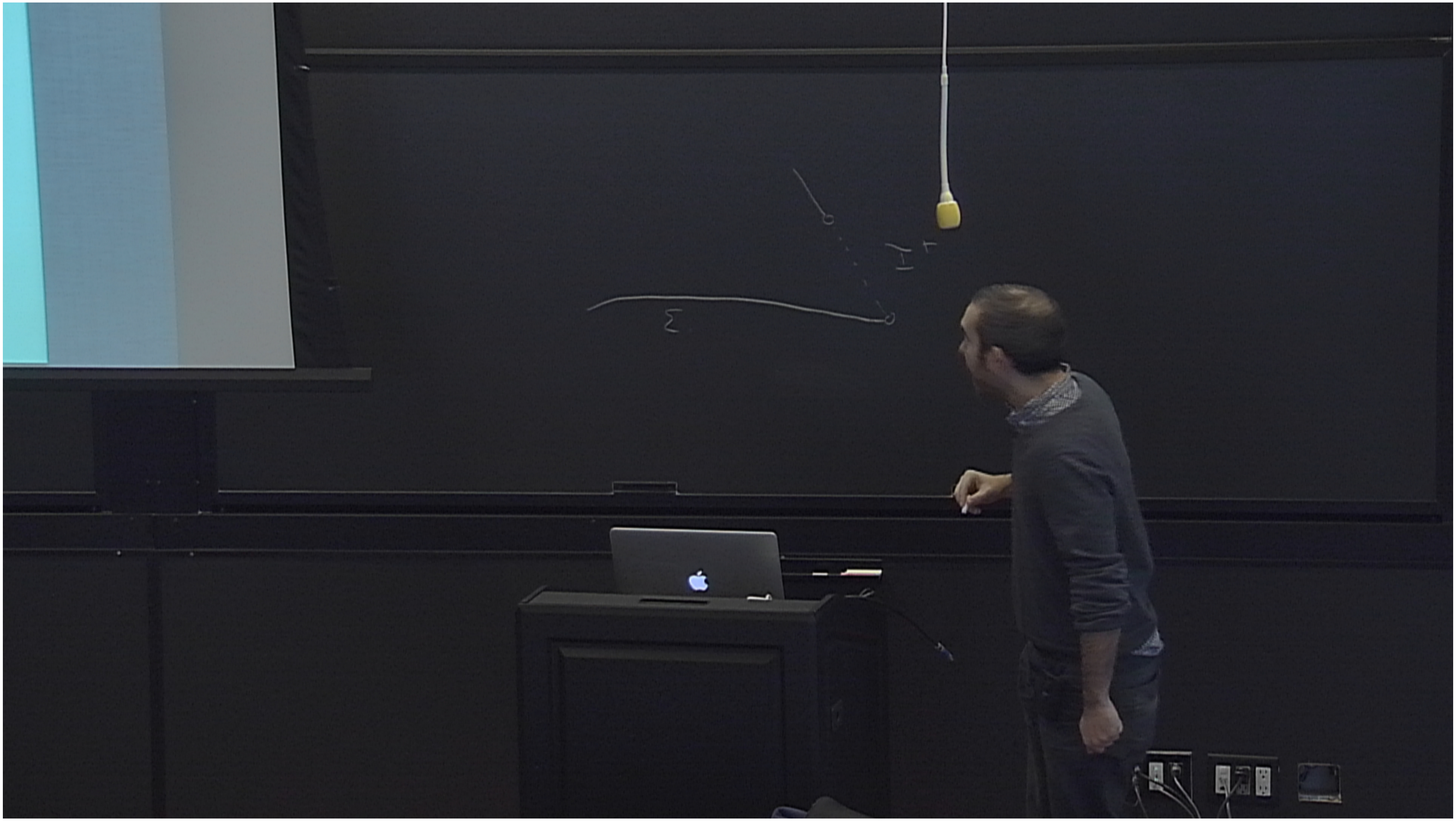

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On the other hand, the radiation emitted to the black hole from initially *localised data* should in fact decay and a priori this decay could compete with the blue-shift effect. We have, however:

Theorem 2 (M.D. 2003). *In subextremal Reissner–Nordström, for sufficiently regular solutions of* $\Box\psi = 0$ *of intially compact support, then if the spherical mean* ψ_0 *satisfies*

$$|\partial_v \psi_0| \geq c v^{-4} \tag{1}$$

along the event horizon $\mathcal{H}^+$*, for some constant* $c > 0$ *and all sufficiently large* v*, then* $E_{\widetilde{\Sigma}}[\psi] = \infty$.

The lower bound (1) is conjecturally true for generic initial data of compact support, cf. BICAK, GUNDLACH–PRICE–PULLIN, ...

The blow-up given by the above theorem, if it indeed occurs is, however, in a sense weak!

In particular, the L^∞ norm of the solution remains bounded.

Theorem 3 (A. Franzen, 2013). *In subextremal Reissner–Nordström or Kerr with* $M > Q \neq 0$ *or* $M > a \neq 0$*, respectively, let* ψ *be a sufficiently regular solution of the wave equation. Then*

$$|\psi| \leq C$$

globally in the black hole interior up to and including $\mathcal{CH}^+$.

The above result generalised a previous result (M.D. 2003) concerning spherically symmetric solutions in the Reissner–Nordström case.

The first input into the proof is an upper bound for the decay rate of a scalar field along the event horizon $\mathcal{H}^+$ of a general Kerr metric which follows from work of M.D.–RODNIANSKI on the wave equation on exterior Kerr:

$$\int_v^\infty |\partial_v \psi|^2 \leq v^{-1-\delta}$$

(A similar estimate holds in the much easier Reissner–Nordström case.)

If one "naively" extrapolates the linear behaviour of $\Box\psi = 0$ to the non-linear $\mathrm{Ric}(g) = 0$, where we think of ψ representing the metric itself in perturbation theory, whereas derivatives of ψ representing the Christoffel symbols, this suggests that the metric may extend continuously to the Cauchy horizon whereas the Christoffel symbols blow up, failing to be square integrable.

On the other hand, if one believes the original intuition, then the non-linearities of the Einstein equations should induce blow-up earlier.

Which of the two scenario holds?

Fully non-linear toy-models under spherical symmetry

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The Einstein–Maxwell–(real) scalar field model under spherical symmetry

The simplest toy model which allows for the study of this problem in spherical symmetry with a true wave-like degree of freedom is that of a self-gravitating *real-valued* scalar field in the presence of a self-gravitating electromagnetic field.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi (T^{\phi}_{\mu\nu} + T^{F}_{\mu\nu})$$

$$T^{\phi}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi$$

$$T^{F}_{\mu\nu} = \frac{1}{4\pi} (g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta})$$

$$\Box_g \psi = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad dF = 0$$

$$(\mathcal{M}, g, \phi), \qquad g = -2\Omega^2 du dv + r^2 d\sigma_{\mathbb{S}^2}$$

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{1}{r}\partial_v r \partial_u r + \frac{1}{4}\Omega^2 r^{-3} Q^2,$$

$$\partial_u \partial_v \log \Omega^2 = -4\pi \partial_u \phi \partial_v \phi + \frac{\Omega^2}{4r^2} + \frac{1}{r^2}\partial_v r \partial_u r - \frac{\Omega^2 Q^2}{2r^4},$$

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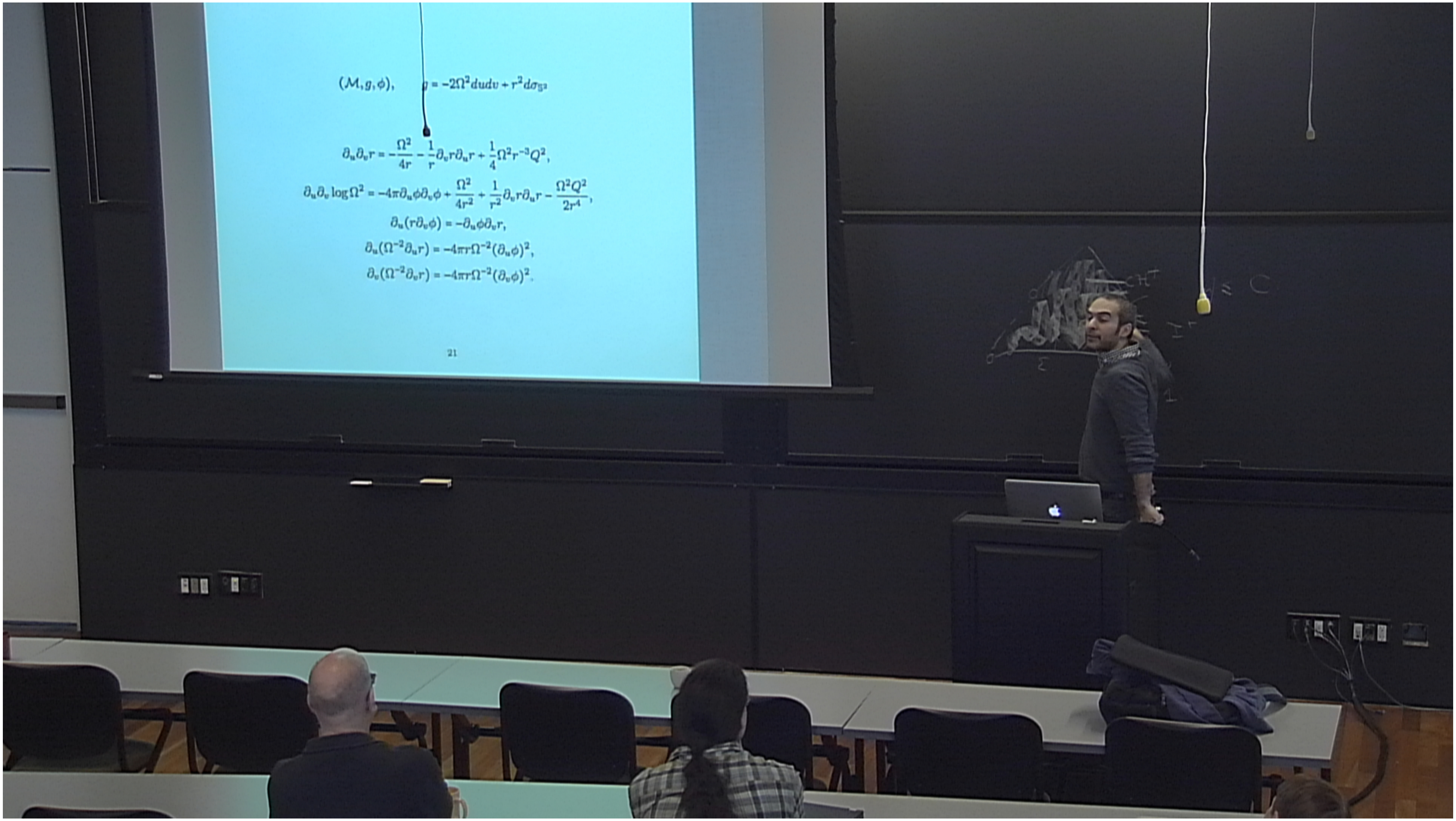
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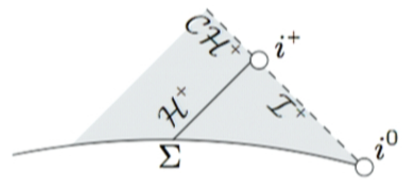
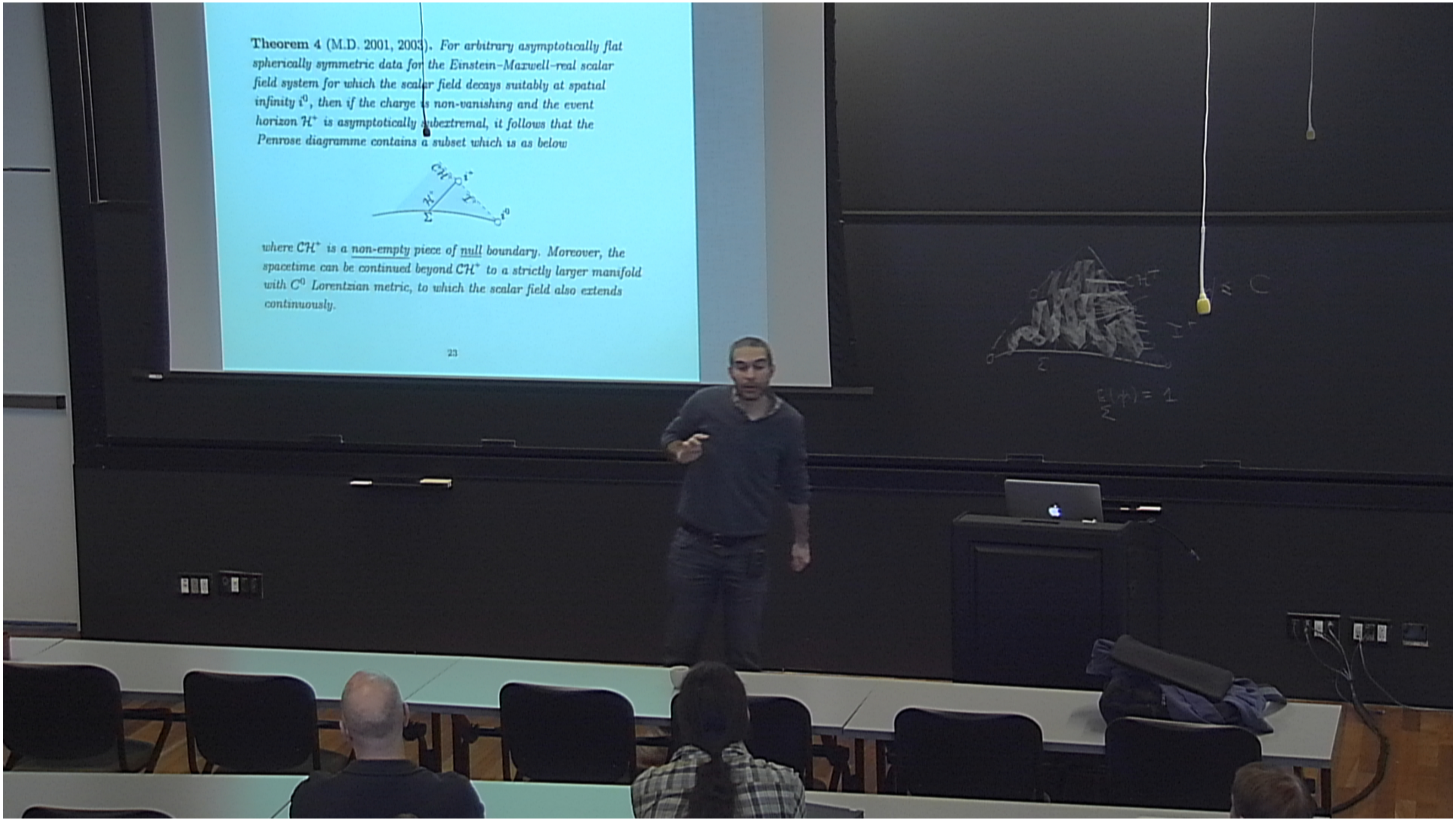
Theorem 4 (M.D. 2001, 2003). *For arbitrary asymptotically flat spherically symmetric data for the Einstein–Maxwell–real scalar field system for which the scalar field decays suitably at spatial infinity* i^0*, then if the charge is non-vanishing and the event horizon* $\mathcal{H}^+$ *is asymptotically subextremal, it follows that the Penrose diagramme contains a subset which is as below*



where $\mathcal{CH}^+$ *is a* *non-empty* *piece of* *null* *boundary. Moreover, the spacetime can be continued beyond* $\mathcal{CH}^+$ *to a strictly larger manifold with* C^0 *Lorentzian metric, to which the scalar field also extends continuously.*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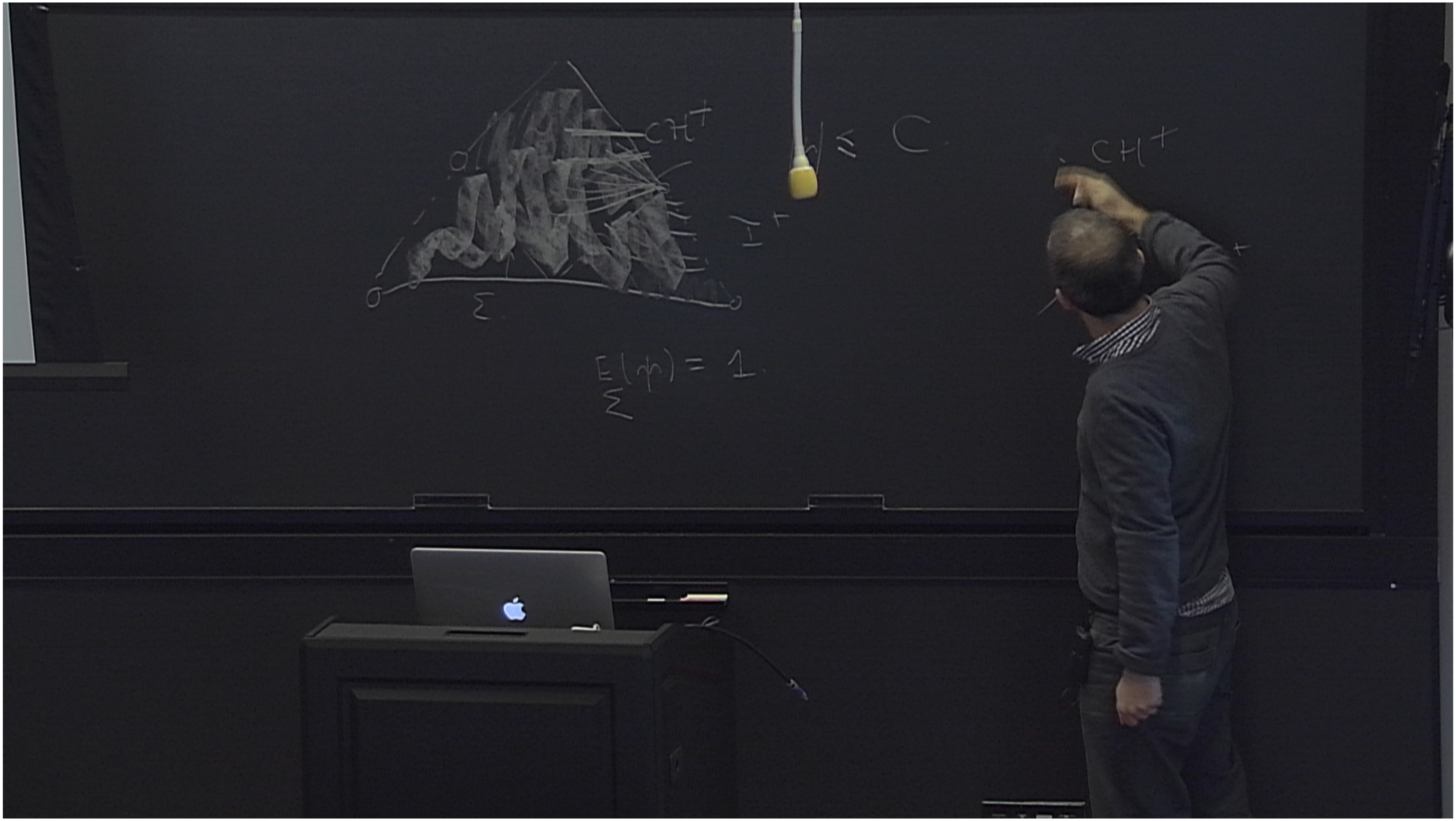
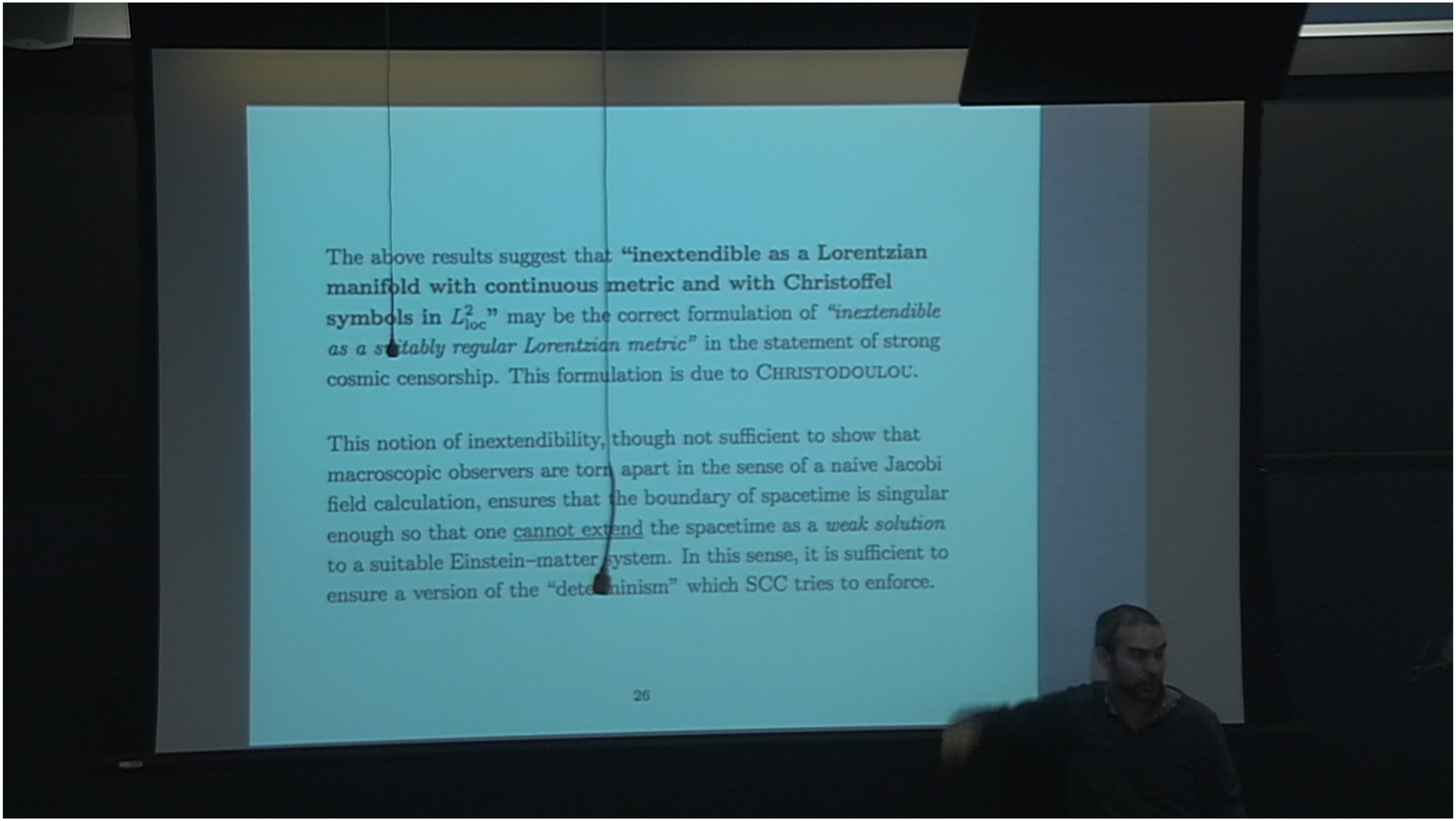

CH+
Σ
E(ψ) = 1


The above results suggest that "inextendible as a Lorentzian manifold with continuous metric and with Christoffel symbols in L^2_{loc}" may be the correct formulation of "inextendible as a suitably regular Lorentzian metric" in the statement of strong cosmic censorship. This formulation is due to CHRISTODOULOU.
This notion of inextendibility, though not sufficient to show that macroscopic observers are torn apart in the sense of a naive Jacobi field calculation, ensures that the boundary of spacetime is singular enough so that one cannot extend the spacetime as a weak solution to a suitable Einstein–matter system. In this sense, it is sufficient to ensure a version of the "determinism" which SCC tries to enforce.
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The vacuum Einstein equations without symmetry

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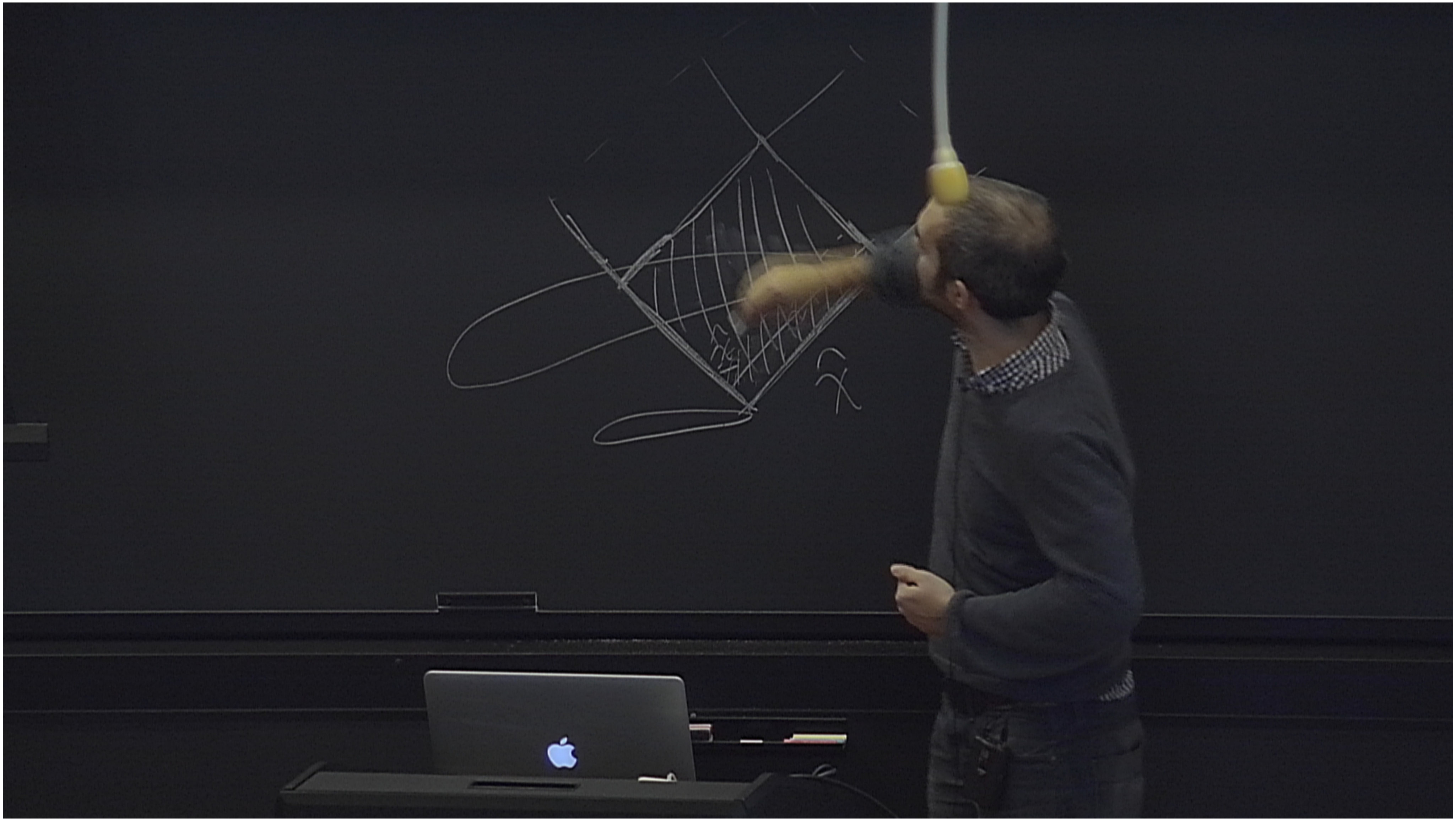
The first question one might ask is, can one construct weak null singularities for the vacuum and are they “stable” to perturbation?

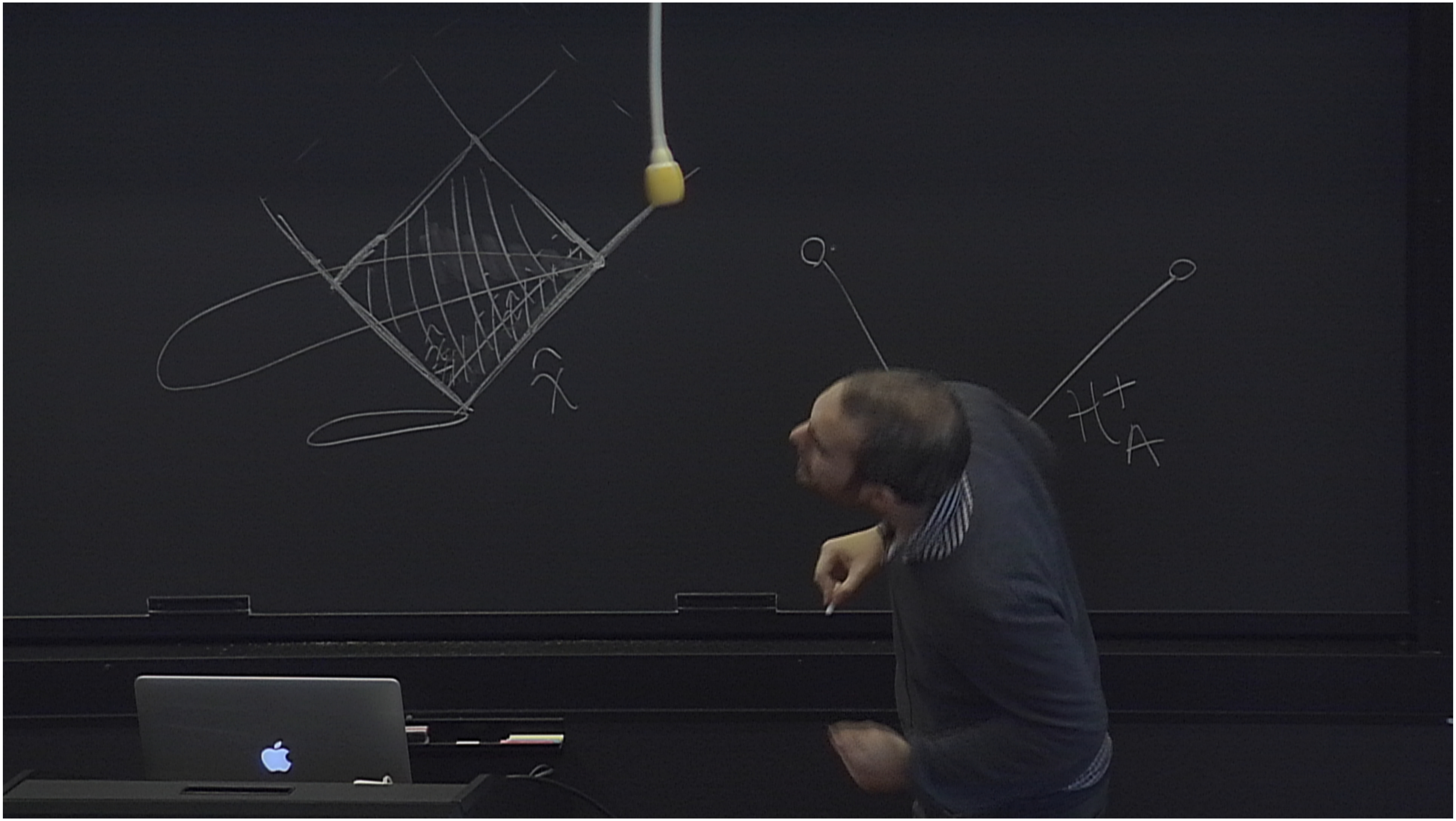
This has recently been resolved in a remarkable new result of LUK

Theorem 7 (LUK). *Let us be given characteristic data for the Einstein vacuum equations* $\mathrm{Ric}(g) = 0$ *defined on a bifurcate null hypersurface* $\mathcal{N}^{\mathrm{out}} \cup \mathcal{N}^{\mathrm{in}}$, *where* $\mathcal{N}^{\mathrm{out}}$ *is parameterised by affine parameter* $\underline{u} \in [0, \underline{u}^*)$, *and the data are regular on* $\mathcal{N}^{\mathrm{in}}$ *while singular on* $\mathcal{N}^{\mathrm{out}}$, *according to*

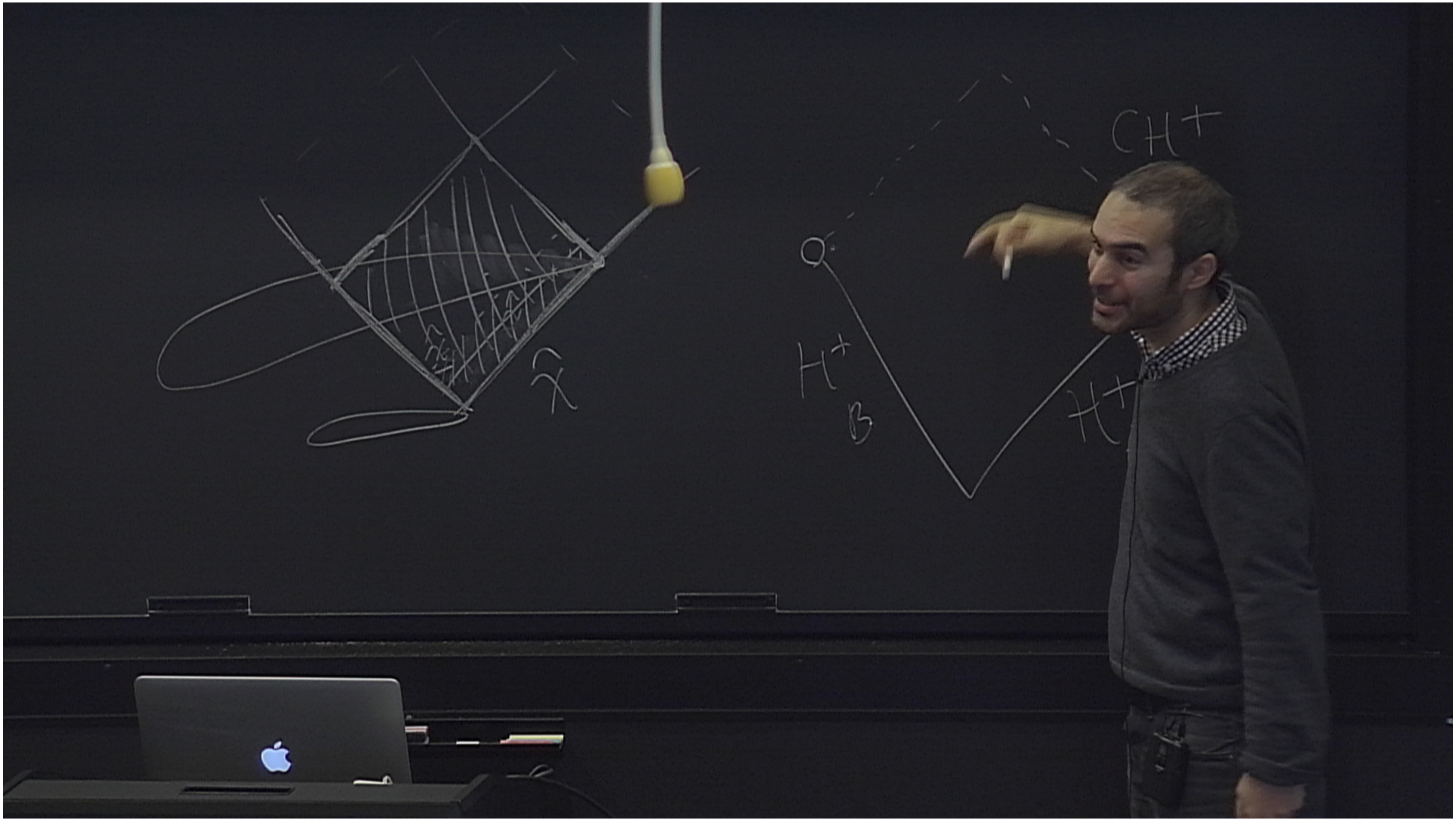
$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}, \tag{2}$$

for appropriate $p > 1$. *Then the solution exists in a region foliated by a double null foliation with level sets* u, $\bar{u}$ *covering the region* $0 \leq u < u^*$, $0 \leq \underline{u} < \underline{u}^*$ *for* $\underline{u}^*$ *as above and sufficiently small* u^*, *and the bound* (2) *propagates. The spacetime is continuously extendible beyond* $\underline{u} = \underline{u}^*$, *but the Christoffel symbols fail to be square integrable in this extension.*





$\hat{\chi}$
$\mathcal{H}^+_A$


CH+
H+
B
H+