Title: Quantum Raychaudhuri equation

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Abstract: We compute quantum corrections to the Raychaudhuri equation, by replacing classical geodesics with quantal (Bohmian) trajectories, and show that they prevent focusing of geodesics, and the formation of conjugate points. We discuss implications for the Hawking-Penrose singularity theorems, and for curvature singularities. Reference: arXiv: 1311.6539

Quantum Raychaudhuri Equation

Saurya Das
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CANADA

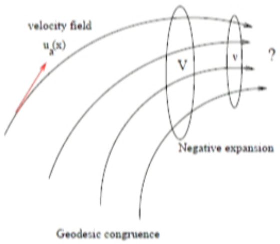
Perimeter Institute, 20 February 2014

Plan

- Raychaudhuri equation - review and importance
- Quantum Raychaudhuri equation
- Implications for singularity theorems
- Implications for curvature singularities
- Applications: Einstein equation of state, cosmology

S. Das, arXiv:1311.6539

Raychaudhuri equation



Expansion $\theta = \frac{d \ln V}{d\lambda}$ (< 0 in above figure) (λ = affine parameter)

$$\frac{du_{a;b}}{d\lambda} = u_{a;b;c}\ u^c = \left[u_{a;c;b} + R_{cba}{}^d u_a\right] u^c \qquad (1)$$

$$= \underbrace{(u_{a;c}u^c)_{;b}}_{=0 \text{ (geodesic equation)}} - u^c{}_{;b} u_{a;c} + R_{cba}{}^d u^c u_d \qquad (2)$$

$$= -u^c{}_{;b} u_{a;c} + R_{cbad}\ u^c u^d\ . \qquad (3)$$

$h_{ab} = g_{ab} - u_a u_b$ (induced 3-metric)

$u_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$ = Trace + Traceless symmetric + anti-symmetric

expansion shear twist

$$h^{ab}\,\frac{du_{a;b}}{d\lambda} = Tr\left(\frac{du_{a;b}}{d\lambda}\right) = Tr\left(-u^c{}_{;b}u_{a;c} + R_{cbad}\;u^c u^d\right)$$

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \underbrace{\omega_{ab}\omega^{ab}}_{=0} - \underbrace{R_{cd}u^c u^d}_{>0} < 0$$

hypersurface orthog. *strong energy cond.*

If $\theta_0 = \theta(0) < 0$ (initially converging)

Focus/caustic for $\lambda \leq \frac{3}{|\theta_0|}$ *(finite proper time!)*

Raychaudhuri equation (1955) (also, Landau & Lifshitz)

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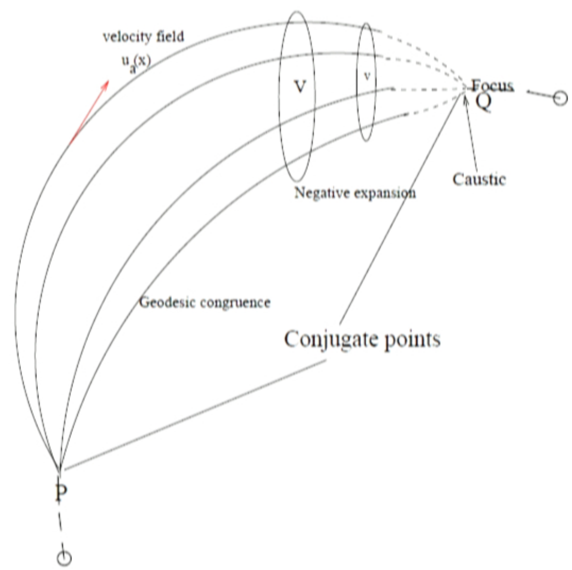
velocity field
u_a(x)
V
v
Focus
Q
Caustic
Negative expansion
Geodesic congruence
Conjugate points
P

Singularity theorems

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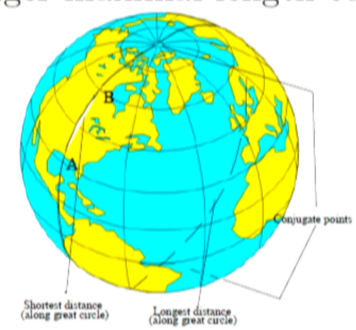

- Maximal geodesics predicted by global arguments, on the other hand.
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Note

- *Generality:* for all reasonable spacetimes (gravity universal & attractive)
- Fluid picture: velocity field $u^a(x)$
- Also, from the geodesic equation: $\frac{D^2\eta^a}{d\lambda^2} = -R^a{}_{bfc}u^b u^c \eta^f$

  η^a connecting neighboring geodesics $\to 0$ for finite λ
- But, classical $\to$ will get modified
- Expectation values, Ehrenfest type theorem, ...
- Find a 'quantum velocity field'

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Non-relativistic limit

$$u^a(x) \to v^a(\vec{x},t) \quad {\scriptstyle (a\,=\,1,2,3)}\ , \quad u^0 = 1\ , \ \lambda \to t$$

$$R_{cd}u^c u^d \to \nabla^2 V = 4\pi G\rho \geq 0,$$

$$\tfrac{d\vec{v}}{dt} = -\vec{\nabla}V(\vec{x},t)$$

$$\tfrac{d\theta}{dt} = -\tfrac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - \nabla^2 V < 0 \ \leftarrow \text{(focusing in finite time)}$$

Quantum fluid picture

$$\psi(\vec{x},t) = \mathcal{R}e^{iS}$$

(Normalizable, single-valued, $\mathcal{R}$, $S=$ Real.
E.g. complete set of H-atom bound states and scattering wavefunctions, $e^2/4\pi\epsilon_0 \to GMm$)

$$\vec{v}\,(\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\,\mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\,\vec{\nabla}S(\vec{x},t) \leftarrow$$ quantum velocity field!

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Real and imaginary parts $\rightarrow$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$ (Probability conservation)

$$m\frac{d\vec{v}}{dt} = -m\vec{\nabla}V + \underbrace{\frac{\hbar^2}{2m}\vec{\nabla}\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}_{V_Q}$$ (Newton's law + quantum potential V_Q!)

- Initially, particles distributed as $\rho(0) = |\psi(0)|^2$ ('quantum equilibrium')
- Prob. conservation $\Rightarrow$ they remain distributed as $\rho(t) = |\psi(t)|^2$
- Each particle follows individual trajectories, subjected to $V + V_Q$
  quantal/Bohmian trajectories
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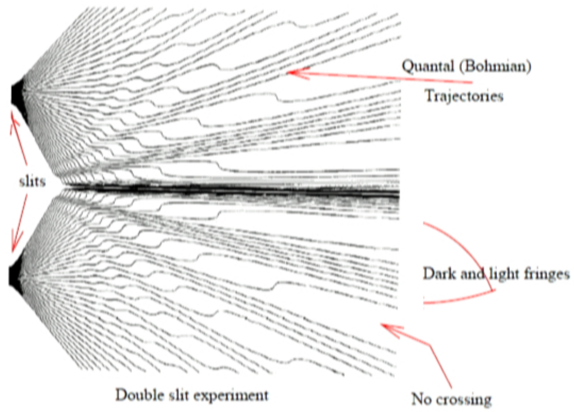

(C. Phillipides, C. Dewdney, B. J. Hiley, *Quantum Interference and the Quantum Potential*, il nuovo cimento B, **52**, no.1 (1979) 15-28)

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- Only dynamical input → Schrödinger equation
- Classical limit: trajectories → with $\hbar \to 0$ (interference disappears)
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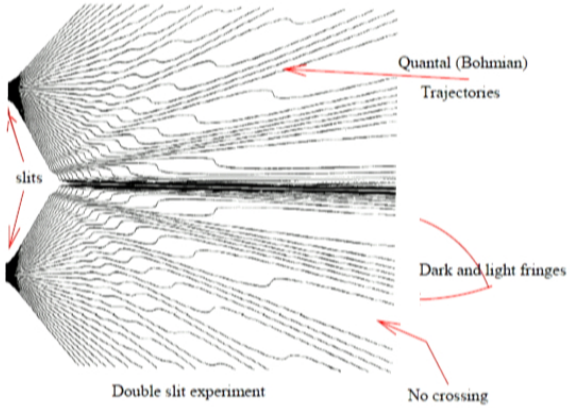
Valid picture!



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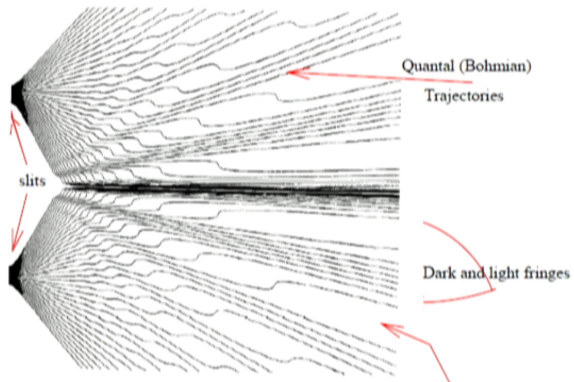

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Quantum Raychaudhuri equation

$V \to V + V_Q/m$ in Raychaudhuri equation

$$\frac{d\theta}{dt} = -\frac{1}{3}\,\theta^2 - \sigma_{ab}\sigma^{ab} - \nabla^2 V + \underbrace{\frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}_{\text{quantum correction}}$$

quantum correction

attractive or repulsive?

focusing or defocusing?

$$\psi = \psi_0 e^{-r^2/L^2} \;\to\; \frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right) \sim +\frac{1}{L^4} \;\leftarrow \textit{Repulsive at short distances}$$

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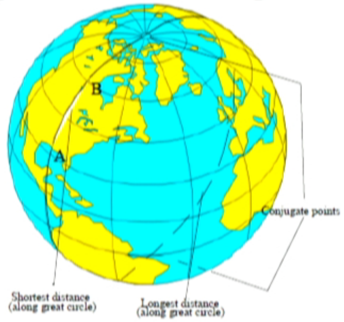

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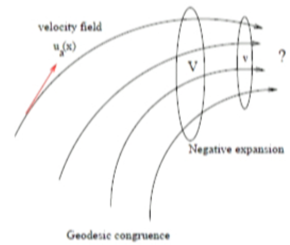
Raychaudhuri equation



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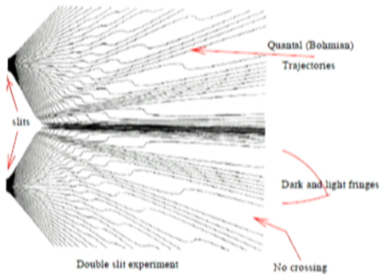
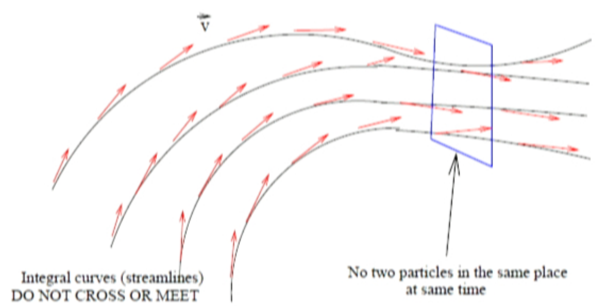


No-crossing of quantal (Bohmian) trajectories

$$\vec{v}\ (\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\ \mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\ \vec{\nabla}S(\vec{x},t)\ = \text{single valued}$$



Trajectories do not end in a caustic/focus. They go on forever.

Relativistic generalization

$$\left[\partial^2 + \frac{m^2c^2}{\hbar^2} - \epsilon_1 R - \epsilon_2 \frac{i}{2} f_{cd}\sigma^{cd}\right]\Phi = 0$$

$(\Phi = \mathcal{R}e^{iS}$, Normalizable, single valued)

$$k_a = \partial_a S \ , \quad u_a = c\frac{dx_a}{d\lambda} = \frac{\hbar k_a}{m} \ , \quad \vec{v} = \frac{d\vec{x}}{dt} = -c^2 \frac{\vec{\nabla} S}{\partial^0 S}$$

Imaginary part: $\partial^a \left(\mathcal{R}^2 \partial_a S\right) = \frac{\epsilon_2}{2} \ f_{cd}\sigma^{cd}\mathcal{R}^2$

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$$u^b{}_{;a}u^a = -\frac{\epsilon_1 \hbar^2}{m^2} \ R^{;b} + \frac{\hbar^2}{m^2}\left(\frac{\partial^2 \mathcal{R}}{\mathcal{R}}\right)^{;b}$$

(i.e. geodesic equation + $V_Q = \frac{\hbar^2}{m^2}\frac{\partial^2 \mathcal{R}}{\mathcal{R}}$)

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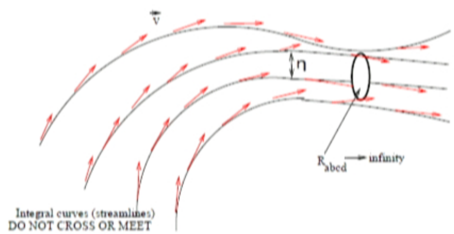
Curvature singularities

$$\frac{D^2\eta^a}{d\lambda^2} = -R^a{}_{bfc}u^b u^c \eta^f - \frac{\hbar^2}{m^2}\left[\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)^{;a}\right]_{;c}\eta^c$$

$\uparrow$ *quantum*

But $\vec{\eta} \neq 0$ anymore



Therefore, $R_{abcd}R^{abcd} \to \infty$ regions are not accessible!

Relativistic generalization

$$\left[\partial^2 + \frac{m^2c^2}{\hbar^2} - \epsilon_1 R - \epsilon_2 \frac{i}{2} f_{cd}\sigma^{cd}\right]\Phi = 0$$

$(\Phi = \mathcal{R}e^{iS}$, Normalizable, single valued)

$$k_a = \partial_a S \ , \quad u_a = c\frac{dx_a}{d\lambda} = \frac{\hbar k_a}{m} \ , \quad \vec{v} = \frac{d\vec{x}}{dt} = -c^2 \frac{\vec{\nabla} S}{\partial^0 S}$$

Imaginary part: $\partial^a \left(\mathcal{R}^2 \partial_a S\right) = \frac{\epsilon_2}{2} \ f_{cd}\sigma^{cd}\mathcal{R}^2$

Real part: $k^2 = \frac{(mc)^2}{\hbar^2} - \epsilon_1 R + \frac{\partial^2 \mathcal{R}}{\mathcal{R}}$

$$u^b{}_{;a}u^a = -\frac{\epsilon_1 \hbar^2}{m^2} \ R^{;b} + \frac{\hbar^2}{m^2}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)^{;b}$$

(i.e. geodesic equation + $V_Q = \frac{\hbar^2}{m^2}\frac{\partial^2\mathcal{R}}{\mathcal{R}}$)

Quantum Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\ \theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d$$

$$-\frac{\epsilon_1\hbar^2}{m^2} h^{ab} R_{;a;b} - \frac{\hbar^2}{m^2} h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$$

↑ *quantum corrections* ↑

Null geodesics: $\frac{d\theta}{d\lambda} = -\frac{1}{3}\ \theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d - \epsilon_1\hbar^2 h^{ab}R_{;a;b} - \hbar^2 h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$

$[\theta = Tr(k_{a;b})]$

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Application

Einstein Equation of State

(S. Braunstein [York] and SD, in progress)

Reference: T. Jacobson, Phys. Rev. Lett. **75** 1260 (1995).

$$\delta Q = \int T_{ab}\chi^a d\Sigma^b = -\kappa \int \lambda T_{ab} k^a k^b d\lambda d\mathcal{A}$$

$$\chi^a = -\kappa\lambda k^a \ , d\Sigma^a = k^a d\lambda d\mathcal{A}$$

$$\delta A = \int \theta d\lambda d\mathcal{A} \quad (d\mathcal{A} = \text{horizon area element})$$

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\ \theta^2 - R_{ab}k^a k^b + \Gamma \leftarrow \ \text{correction}$$

$$\theta = -\lambda R_{ab}k^a k^b + \lambda\Gamma \ (\text{small } \lambda)$$

$$\delta\mathcal{A} = -\int \lambda R_{ab}k^a k^b dl d\mathcal{A} + \int \lambda\Gamma d\lambda d\mathcal{A}$$

$$S = S(\mathcal{A}),\ S' = dS/d\mathcal{A} \equiv S_1(\kappa),\ T = \hbar\kappa/2\pi$$

$$\delta Q = TdS = \frac{\hbar\kappa}{2\pi}\ S'(\mathcal{A})\ \delta\mathcal{A}\ ,$$

$$T_{ab}k^a k^b = \frac{\hbar S_1(\kappa)}{2\pi}\left[R_{ab}k^a k^b - \Gamma\right]$$

$$\frac{2\pi}{\hbar S_1(\kappa)} T_{ab} = R_{ab} + f g_{ab} - \Gamma n_a n_b \quad (n^a n_a = 0 \text{ and } k^a n_a = -1)$$

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$$T_{ab}^{\ ;a} = \left(R_{ab} - \frac{1}{2} g_{ab} R\right)^{;a} = 0 \ \Rightarrow$$

$$f g_{ab} + \frac{1}{2} g_{ab} R = \Lambda g_{ab} + \frac{2\pi}{\hbar\kappa S_1(\kappa)} \Gamma n_a n_b$$

Tracing $f = -\frac{R}{2} + \Lambda$.

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar S_1(\kappa)} T_{ab} + \frac{2\pi}{\hbar\kappa S_1(\kappa)} \Gamma n_a n_b \ .$$

(i) $2\pi/\hbar S_1(\kappa) = 8\pi G =$ constant

$$S \propto \mathcal{A}$$

(ii) $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} + \underbrace{\frac{8\pi G}{\kappa} \Gamma n_a n_b}_{=0}$

Since $\frac{8\pi G}{\kappa} n_a n_b h^{ab} \left(\frac{\partial^2 \mathcal{R}}{\mathcal{R}}\right) = 0$,

No corrections to Entropy law, or Einstein equations

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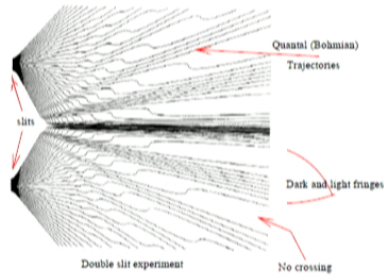
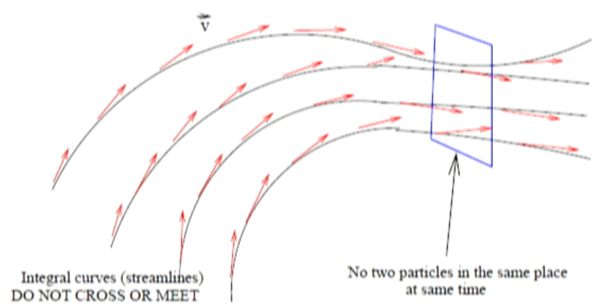


No-crossing of quantal (Bohmian) trajectories

$$\vec{v}\,(\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\,\mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\,\vec{\nabla}S(\vec{x},t) \;= \text{single valued}$$



Trajectories do not end in a caustic/focus. They go on forever.

$\boxed{\Lambda_Q}$

For wavefunctions considered earlier

$(\psi = \psi_0 e^{-r^2/L^2}\,,\ \psi = \psi_0 \tanh(r/L\sqrt{2})\ (g>0)\ ,\ \psi = \sqrt{2}\ \psi_0\ \mathrm{sech}(r/L)\ (g<0))$

$$\Lambda_Q = \frac{1}{L^2} = \frac{1}{(\text{Compton wavelength})^2} = \left(\frac{mc}{\hbar}\right)^2$$

$L = 1.4 \times 10^{26}\ m$ (size of observable universe)

$m \approx 10^{-68}\ kg = 10^{-32}\ eV\ ,\quad F = -\frac{Gm_1m_2}{r^2}e^{-r/L}$

$$\Lambda_Q = 10^{-52}m^{-2} = 10^{-123}(\text{in Planck units}) = H_0^2\ \text{(Observed, also coincidence problem)}$$

m = graviton (or photon) mass?

References with same/similar graviton/photon mass:
F. Zwicky, Publications of the Astronomical Society of the Pacific, Vol. 73, No. 434, p.314 (1961). C. M. Will, Phys. Rev. **D57** (1998) 2061. L. S. Finn, P. J. Sutton Phys. Rev. **D65** (2002) 044022. A. S. Goldhaber, M. M. Nieto, Rev. Mod. Phys. **82** (2010) 939. E. Berti, J, Gair, A. Sesana, Phys. Rev. **D84** (2011) 101501(R). J. R. Mureika, R. B. Mann, Mod. Phys. Lett. **A26** (2011) 171-181 [arXiv:1005.2214]. C. de Rham, G. Gabadadze, L. Heisenberg, D. Pirtskhalava, Phys. Rev. **D83** (2011) 103516 [arXiv:1010.1780]. S. Majid, arXiv:1401.0673.

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Summary and open problems

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  $\Rightarrow$ Classical $\rightarrow$ quantum Raychaudhuri equation
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- Also: corrections to geometrical optics (astrophysics?)
- Connection to path integrals
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- Numerically plotting quantal trajectories in curved spacetimes
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