Title: Aharonov-Bohm effect in symmetry protected states

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Abstract: Symmetry protected topological (SPT) states are generalizations of topological band insulators to interacting systems. They possess a gapped bulk spectrum together with symmetry protected edge states, with no topological order. There has been recently an intense effort to classify SPT states both in terms of group cohomology as well as from the point of view of effective field theories. An interesting related question is to understand the structute of lattice models that realize SPT physics. In this talk, I shall present a class of lattice models describing the egde of non-chiral two-dimensional bosonic SPT states protected by Z_N symmetry. A crucial aspect of the construction relies on finding the correct non-trivial Z_N symmetry realizations on the edge consistent with all the possible classes of SPT states. Then I shall discuss the Aharonov-Bohm effect on the many-body SPT state by studying this many-body effect on the aforementioned gapless edge states. The effect of a Z_N gauge flux on the egde states is formulated in terms of twisted boundary conditions of the lattice models. The low energy spectral shifts due to the gauge flux are shown to depend on each of the SPT classes in a predictable way. I shall, in the course of this talk, present numerical results of exact diagonalization of our lattice Hamiltonians that support this analysis. This work is done in collaboration with Juven Wang and
appears in arXiv:1310.8291.

Bosonic Symmetry-Protected Edge States

Luiz H. Santos

Perimeter Institute For Theoretical Physics

Emergence in Complex Systems, PI, 10-Feb-2014

This talk

- SPT states are a new class of (mostly theoretically studied) interacting quantum states of matter with a finite gap to all bulk excitations and non-trivial edge states protected by symmetry.
- I will make an attempt to describe bosonic SPT states in 2D protected by $\mathbb{Z}_N$ symmetry from the perspective of their 1D edge.
- Analysis of the edge phyics will be based on an effective field theory (review) and, most importantly, on lattice models (new), which I will show how to construct in the course of this talk.
- I will show how the interaction of these edge states with a gauge flux (many-body Aharonov-Bohm effect) can capture the signature of the non-trivial SPT order.

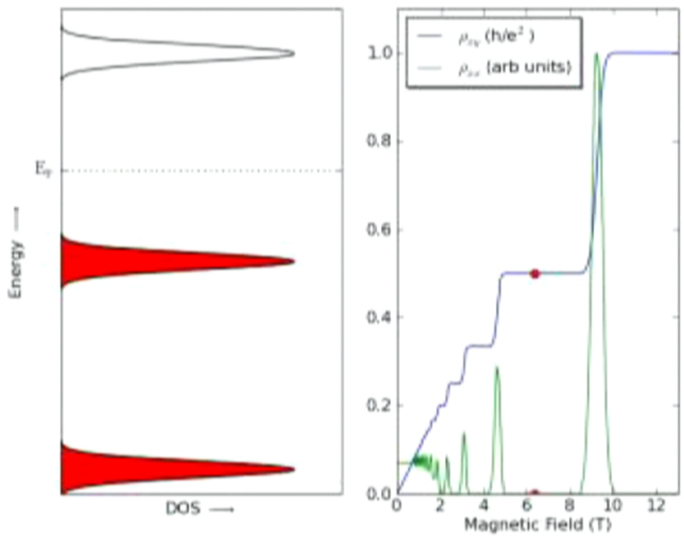
"A Symmetry-Protected Many-Body Aharonov-Bohm Effect ".
arXiv: 1310.8291

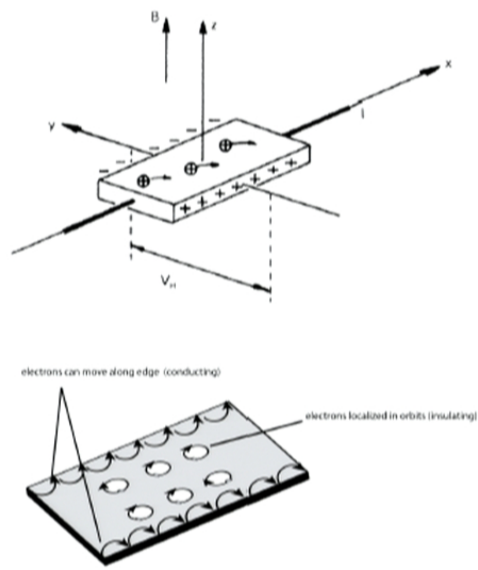
With Juven Wang
(MIT and Perimeter)

Thank discussions with: X. Chen, L. Cincio, D. Gaiotto, T. Senthil, G. Vidal, A. Vishwanath and X.-G. Wen.

Integer Quantum Hall Effect

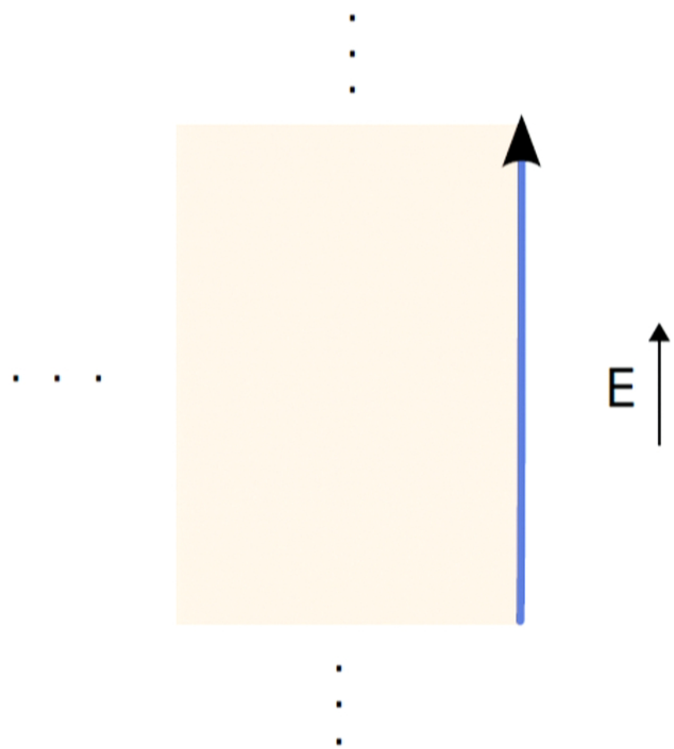
- Hall Conductance: $\sigma_{Hall} = n(e^2/h)$
- Chiral Edge states





Figs. from Wikipedia and 1998 Nobel Prize Announcement

Anomaly in the QHE



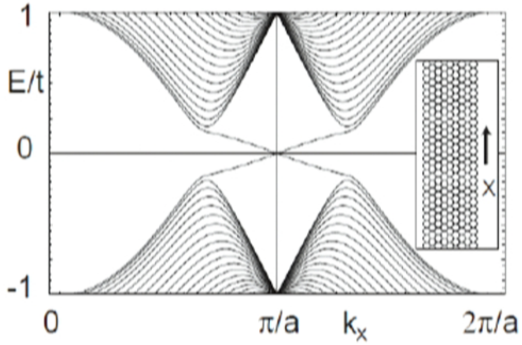
$$\partial_\mu J^\mu_{\text{bound.}} = \sigma_{\text{Hall}} E$$

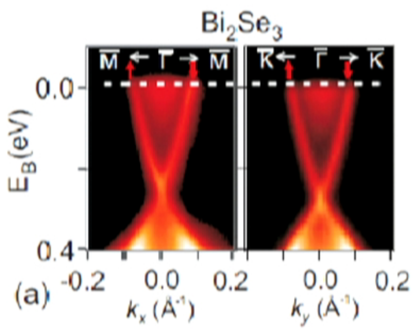
- Boundary theory breaks $U(1)$ symmetry!
- Physically, current flows into the bulk

Luiz H. Santos | Bosonic Symmetry-Protected Edge States

Non-Interacting $\mathbb{Z}_2$ Topological Insulator

kane and Mele 05





kane and Hasan RMP 10

Kramers doublet

$$\mathcal{T}\,\psi_L\mathcal{T}^{-1} = \psi_R$$
$$\mathcal{T}\,\psi_R\mathcal{T}^{-1} = -\psi_L$$
$$\mathcal{T}^2 = -1.$$

Gap opening perturbation $(\psi_L^\dagger\psi_R + h.c)$ breaks $\mathcal{T}$ symmetry.

Non-Interacting Fermionic SPT states.

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Schnyder *et. al.* (08) and Kitaev (09)

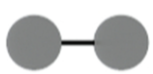
- Classification based on single particle Bloch states
- Interactions regarded as small perturbative effect.
- Short range entangled states (no topological order)

Bosonic SPT States

- Interacting Bosonic SPT states.
- Ex.: $S = 1$ Antiferromagnetic Chain. [Haldane (83), Affleck-Kennedy-Lieb-Tasaki (87)]

$$H_{AF} = J \sum_i \boldsymbol{\sigma}_i^R \cdot \boldsymbol{\sigma}_{i+1}^L, \quad J > 0.$$

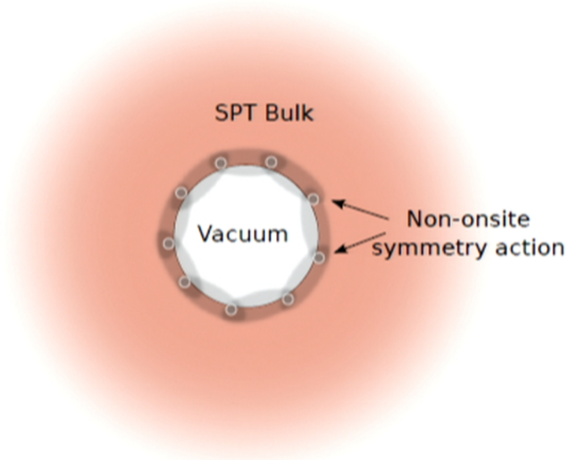
 Projection on S = 1 space

 VBS (Valence Bond Solid)

Boundary d.o.f ($S = 1/2$) are Krammers pairs, so symmetry is realized projectively. This is the hallmark of 1d SPT states.

Bosonic SPT States

- Projective representations of a group G are classified by the 2nd cohomology group $\mathcal{H}^2(G, U(1))$. Chen et al (11), Turner et al (11), Schuch et al. (11)
- In $(d+1)$ space-time dimensions, this idea has been extended leading to a classification of bosonic SPT states terms of the $(d+1)$-th cohomology group $\mathcal{H}^{(d+1)}(G, U(1))$. Chen-Gu-Liu-Wen (12)



$j = 1, ..., M$ (No. of bdry d.o.f.)

$$U_{\mathrm{bdry}}(g) \neq \prod_{j=1}^{M} U_j(g), \quad g \in \mathbb{Z}_N$$

N classes in $2d$:

$$\mathcal{H}^3(\mathbb{Z}_N, U(1)) = \mathbb{Z}_N$$

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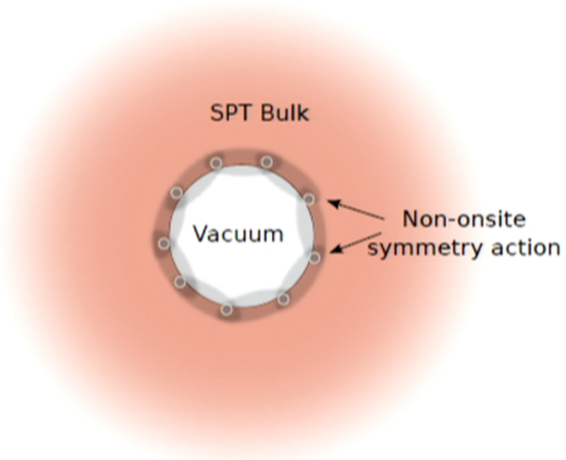

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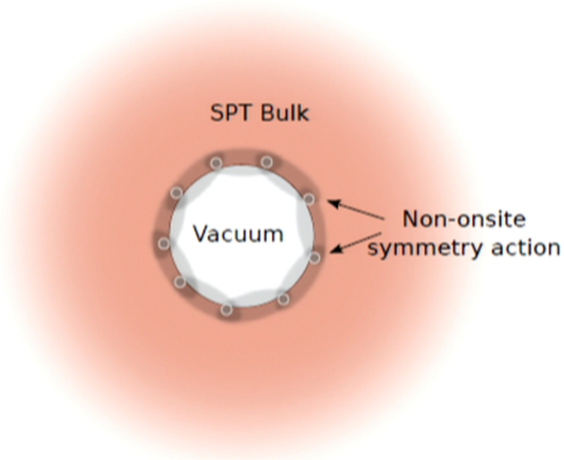

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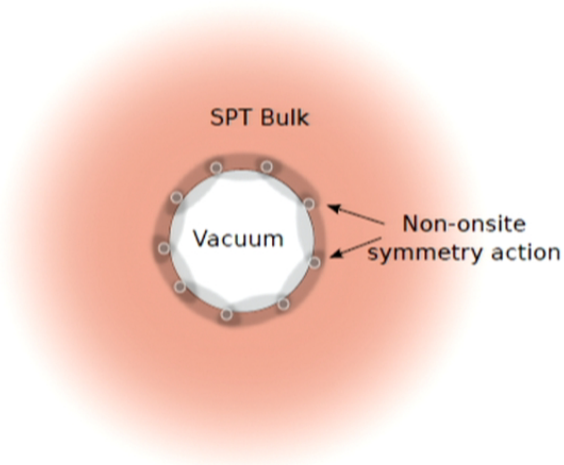

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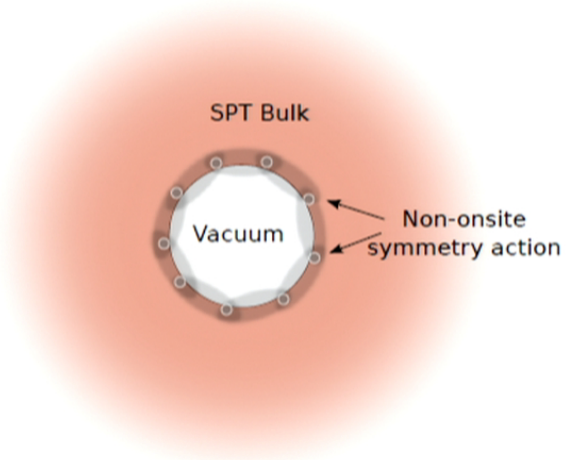

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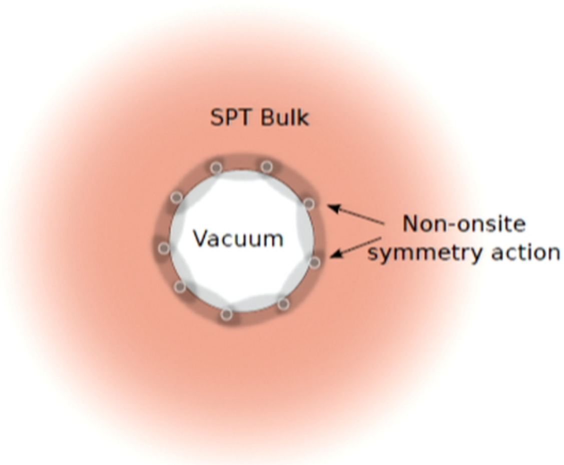

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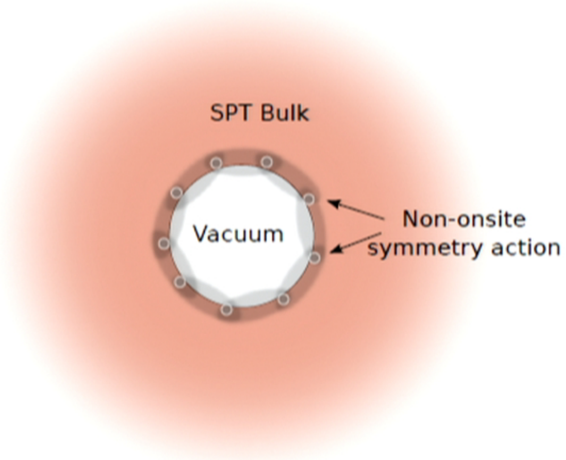

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2D Bosonic SPT states: Effective field theory

- In 2D we can conveniently study gapped states in terms of an effective Chern-Simons field theory Lu and Vishwanath (12)

$$\mathcal{L}_{2d} = \frac{1}{4\pi}\,\varepsilon^{\mu\nu\lambda}\,a_\mu^{\mathsf{T}}\cdot K\cdot\partial_\nu\,a_\lambda\,,$$

where K is the “K matrix” that couples the N-tuplet of gauge fields.

- No topological order:

$$\text{GS degeneracy} = |\det(K)| = 1.$$

Non-degenerate ground state.

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Edge Instability

$$\mathcal{L}_{1d} = \frac{1}{4\pi}\left(K_{ij}\partial_t\Phi_i\partial_x\Phi_j + V_{ij}\partial_x\Phi_i\partial_x\Phi_j\right)$$

- Without symmetry restrictions, perturbations

$$U_i \sim \cos\left(\Phi_i\right)$$

destabilize this picture and generically lead to a gapped edge.

$\Longrightarrow$

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Trivial $\mathbb{Z}_N$ state

$$[\Phi_1(x), \partial_y \Phi_2(y)] = i 2\pi \delta(x-y)$$

$$S_N = \prod_j e^{i \frac{2\pi}{N} n_j} = e^{i \frac{2\pi}{N} \int_0^L dx\, \rho(x)} = e^{i \frac{1}{N} \int_0^L dx\, \partial_x \Phi_2(x)}$$

$$S_N : \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \to \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + \frac{2\pi}{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So, $\cos(\Phi_2)$ is a perturbation that does not break the symmetry and leads to gapped and symmetric state. In this case, the edge is regarded as trivial.

$\mathbb{Z}_N$ symmetry

- How do we construct such a class of 1d system describing the boundary of a 2d bosonic SPT ?
- For a “trivial ”edge the transformation

$$\Phi_1 \to \Phi_1 + \frac{2\pi}{N}, \quad \Phi_2 \to \Phi_2$$

is implemented by

$$S = e^{i\frac{1}{N}\int_0^L dx\,\partial_x \Phi_2(x)}.$$

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$$\Phi_1 \to \Phi_1 + \frac{2\pi}{N}, \quad \Phi_2 \to \Phi_2 + p\,\frac{2\pi}{N}, \; p = 0, 1, ..., N-1$$

is implemented by

$$S = e^{i\,\frac{1}{N}\int_0^L dx\,\partial_x\Phi_2(x) + i\,p\,\frac{1}{N}\int_0^L dx\,\partial_x\Phi_1(x)} \sim e^{i\,\frac{2\pi}{N}(n+pm)}$$

- The operator $e^{i\,p\,\frac{1}{N}\int_0^L dx\,\partial_x\Phi_1(x)}$ shall then be associated to a non-onsite symmetry realization.

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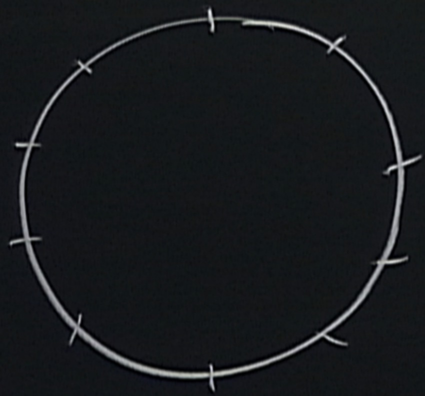
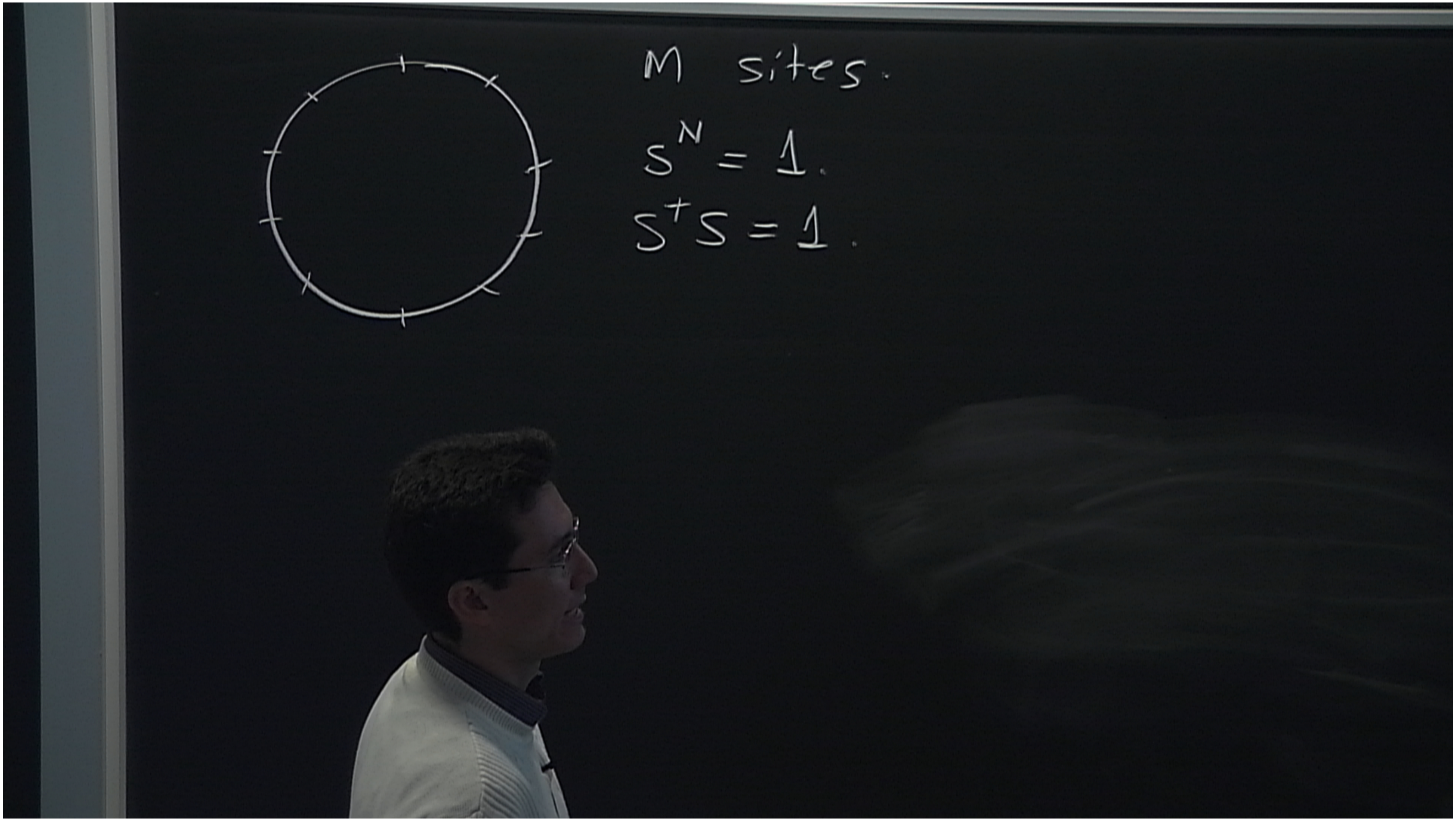


M sites.

$S^N = \mathbb{1}$.

$S^\dagger S = \mathbb{1}$.


M sites.
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$\mathbb{Z}_N$ symmetry - Lattice

- $\mathbb{Z}_N$ representation for each site $j = 1, ..., M$ on the 1d edge

$$\tau^N = 1, \quad \sigma^N = 1, \quad \tau^\dagger \sigma \tau = \omega \sigma, \quad \omega = e^{i\frac{2\pi}{N}}$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega^{N-1} \end{pmatrix} \sim e^{i\Phi_1}$$

$$\tau = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$\sigma \to \omega \sigma,$$

$$\Phi_1 \to \Phi_1 + \frac{2\pi}{N}$$

M sites.

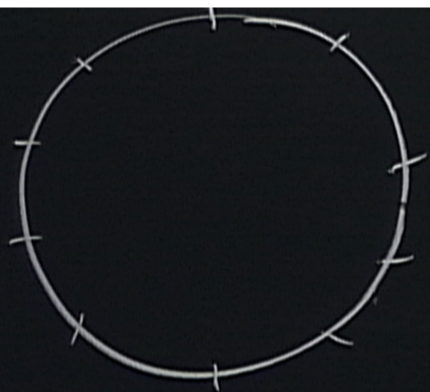
$S^N = 1$.

$S^\dagger S = 1$.

$$S = \prod_{j=1}^{M} \tau_j \prod_{j=1}^{M} U_{j,j+1}(\sigma_j^\dagger \sigma_{j+1}).$$

$$U_{j,j+1} = e^{\frac{i}{N} Q(\sigma_j^\dagger \sigma_{j+1})},$$

$$Q = q_0 + q_1 \sigma_j^\dagger \sigma_{j+1} + \dots + q_{N-1}(\sigma_j^\dagger \sigma_{j+1}).$$



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$$U_{j,j+1} = e^{\frac{i}{N} Q(\sigma_j^\dagger \sigma_{j+1})}$$

$$S^N = \prod_j U^N_{j,j+1} = 1.$$

$$U^N_{j,j+1} = (\sigma_j^\dagger \sigma_{j+1})^p, \quad p = 0, \dots, N-1$$

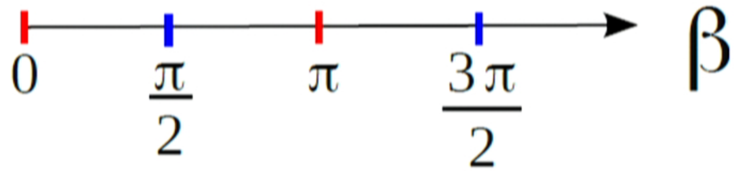
$$\prod_j (\sigma_j^\dagger \sigma_{j+1})^p = 1.$$

$$Q = q_0 + q_1 \sigma_j^\dagger \sigma_{j+1} + \dots + q_{N}$$

$\mathbb{Z}_2$ symmetry - Lattice

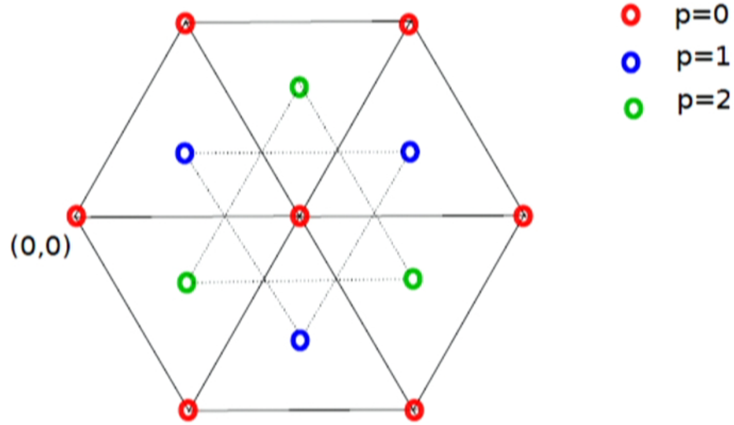
$$S = \prod_{j=1}^{M} \sigma_j^x \prod_{j=1}^{M} U(\sigma_j^z \sigma_{j+1}^z), \quad U(\sigma_j^z \sigma_{j+1}^z) = e^{\frac{i}{2}(\alpha + \beta \sigma_j^z \sigma_{j+1}^z)}$$

$$U^2 = e^{i\alpha}\left[\cos(\beta) + i \sin(\beta)\, \sigma_j^z \sigma_{j+1}^z\right] = (\sigma_j^z \sigma_{j+1}^z)^p$$



$$S = \prod_{j=1}^{M} \sigma_j^x \prod_{j=1}^{M} e^{i p \frac{\pi}{4}(1 - \sigma_j^z \sigma_{j+1}^z)}, \quad p = 0, 1$$

$\mathbb{Z}_3$ symmetry - Lattice



$$S_3^{(p)} = \prod_{j=1}^{M} \tau_j \prod_{j=1}^{M} e^{\frac{i}{3} Q_3^{(p)}(\sigma_j^\dagger \sigma_{j+1})},$$

$$Q_3^{(p)}(\sigma_j^\dagger \sigma_{j+1}) = q_0^{(p)} + q_1^{(p)}(\sigma_j^\dagger \sigma_{j+1}) + \bar{q}_1^{(p)}(\sigma_j^\dagger \sigma_{j+1})^2$$

$$q_0^{(p)} = -p\frac{2\pi}{3}, \quad q_1^{(p)} = p\frac{\pi}{3}(1 + i/\sqrt{3}), \quad p = 0, 1, 2.$$

$\mathbb{Z}_2$ symmetry - Lattice

3 conditions determine $U(\sigma_j^z, \sigma_{j+1}^z)$:

- $\prod_{j=1}^{M} \sigma_j^x \prod_{j=1}^{M} U(\sigma_j^z, \sigma_{j+1}^z) = \prod_{j=1}^{M} U(\sigma_j^z, \sigma_{j+1}^z) \prod_{j=1}^{M} \sigma_j^x$

$$\Rightarrow U(\sigma_j^z, \sigma_{j+1}^z) = U(\sigma_j^z\, \sigma_{j+1}^z)$$

- $S^\dagger \cdot S = 1 \Rightarrow U = e^{\frac{i}{2}(\alpha + \beta\, \sigma_j^z\, \sigma_{j+1}^z)}, \quad \alpha, \beta \in \mathbb{R}$

- $S^2 = \prod_{j=1}^{M} U^2(\sigma_j^z\, \sigma_{j+1}^z) = 1$

$$\Rightarrow U^2(\sigma_j^z\, \sigma_{j+1}^z) = (\sigma_j^z\, \sigma_{j+1}^z)^p, \quad p = 0, 1 \bmod 2$$

Lattice models

We seek lattice Hamiltonians such that

- $[H_N^{(p)}, T] = 0, \;\; [H_N^{(p)}, S_N^{(p)}] = 0$

(where T is the lattice translation operator).

$$H_N^{(p)} = \sum_{j=1}^{M} h_{N,j}^{(p)}$$

$$h_{N,j}^{(p)} = \tau_j + \left(S_N^{(p)}\right) \tau_j \left(S_N^{(p)}\right) + \ldots + \left(S_N^{(p)}\right)^{-(N-1)} \tau_j \left(S_N^{(p)}\right)^{(N-1)} + h.c.$$

This Hamiltonian yields a gapped symmetric ground state in the trivial SPT phase.

$\mathbb{Z}_2$ Lattice models

$$S = \prod_{j=1}^{M} \sigma_j^x \prod_{j=1}^{M} e^{i p \frac{\pi}{4} (1 - \sigma_j^z \sigma_{j+1}^z)}, \quad p = 0, 1$$

- Trivial $p = 0$ class:

$$H_2^{(0)} = \sum_{j=1}^{M} \sigma_j^x, \quad \Rightarrow \text{Gapped symmetric ground state}$$

- Non-trivial $p = 1$ SPT class

$$H_2^{(1)} = \sum_{j=1}^{M} \left(\sigma_j^x - \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z \right).$$

Levin-Gu (12); Chen-Wen(12)

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Levin-Gu (12); Chen-Wen(12)

Lattice models

We seek lattice Hamiltonians such that

- $[H_N^{(p)}, T] = 0, \; [H_N^{(p)}, S_N^{(p)}] = 0$

(where T is the lattice translation operator).

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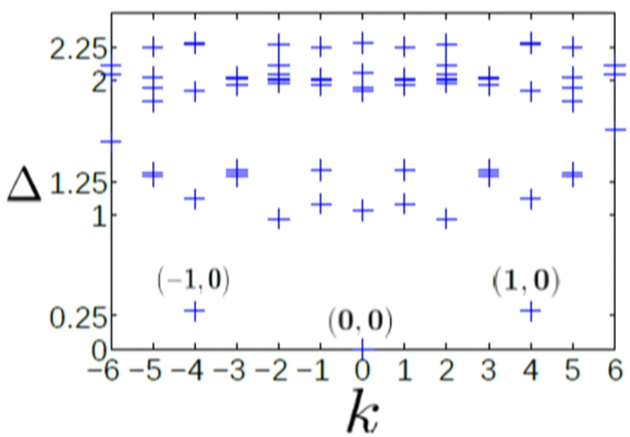
$$h_{N,j}^{(p)} = \tau_j + \left(S_N^{(p)}\right) \tau_j \left(S_N^{(p)}\right) + \ldots + \left(S_N^{(p)}\right)^{-(N-1)} \tau_j \left(S_N^{(p)}\right)^{(N-1)} + h.c.$$

This Hamiltonian yields a gapped symmetric ground state in the trivial SPT phase.

$\mathbb{Z}_3$ Lattice models

$$H_3^{(p)} = \sum_{j=1}^{M} \left\{ \tau_j \left[\frac{5}{3} + \frac{\omega + \bar{\omega}}{3} \left(\sigma_{j-1}^{\dagger} \sigma_j + \sigma_{j-1} \sigma_j^{\dagger} \right) + \left(\frac{(1+\omega)}{3} \sigma_j^{\dagger} \sigma_{j+1} + \frac{2\bar{\omega}}{3} \sigma_{j-1}^{\dagger} \sigma_{j+1} + \frac{2\omega}{3} \sigma_{j-1}^{\dagger} \sigma_j^{\dagger} \sigma_{j+1}^{\dagger} + h.c. \right) \right] + h.c. \right\}.$$



- $\Delta(n,m) = A n^2 + B m^2$
  $n, m \in \mathbb{Z}$
- $S \,|\, n, m\rangle = e^{i \frac{2\pi}{3}(n+pm)} \,|\, n, m\rangle$

Same spectrum for $p = 1, 2$ but states transform differently !

Luiz H. Santos | Bosonic Symmetry-Protected Edge States

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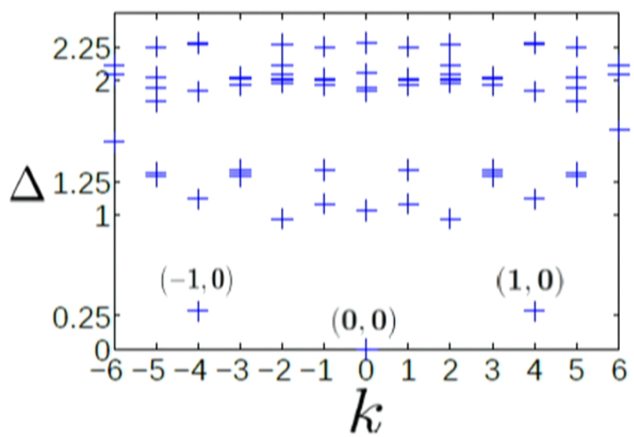

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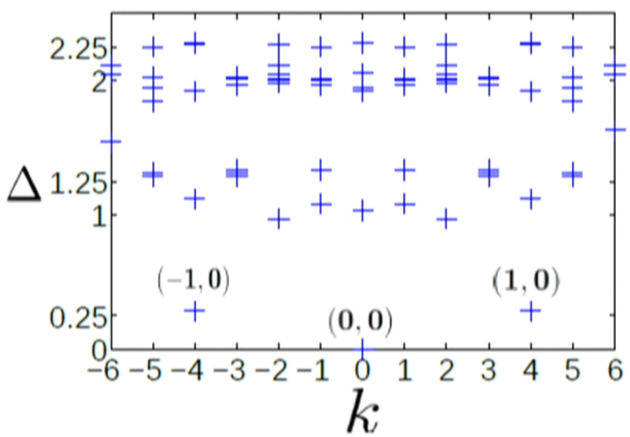

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Jordan-Wigner Transformation

$$\sigma_j^x = i\,\alpha_j\,\beta_j$$

$$\sigma_j^y = \Big(\prod_{n<j} i\,\alpha_n\,\beta_n\Big)\,\alpha_j$$

$$\sigma_j^z = \Big(\prod_{n<j} i\,\alpha_n\,\beta_n\Big)\,(-\beta_j)$$

$$\{\alpha_j\,,\alpha_\ell\} = 2\,\delta_{j,\ell}, \quad \{\beta_j\,,\beta_\ell\} = 2\,\delta_{j,\ell},$$
$$\{\alpha_j\,,\beta_\ell\} = 0.$$

$$H_{\mathbb{Z}_2} = \sum_{j=1}^{M} \left(\sigma_j^x - \sigma_{j-1}^z\sigma_j^x\sigma_{j+1}^z\right) = H_{\text{odd}} + H_{\text{even}}$$

$$H_{\text{odd/even}} = \sum_{j\,\in\,\text{odd/even}} \Big(i\,\alpha_j\,\beta_j - i\,\alpha_{j-1}\,\beta_j\Big) \Longrightarrow \text{Gapless spectrum}$$

$$S = \Big(\prod_j i\,\alpha_{2j-1}\,\beta_{2j-1}\Big)\Big(\prod_j i\,\alpha_{2j}\,\beta_{2j}\Big)\prod_j e^{i\,\frac{\pi}{4}(1 - i\,\alpha_{2j-1}\,\beta_{2j} - i\,\alpha_{2j}\,\beta_{2j+1})}$$

$\mathbb{Z}_3$ Lattice models

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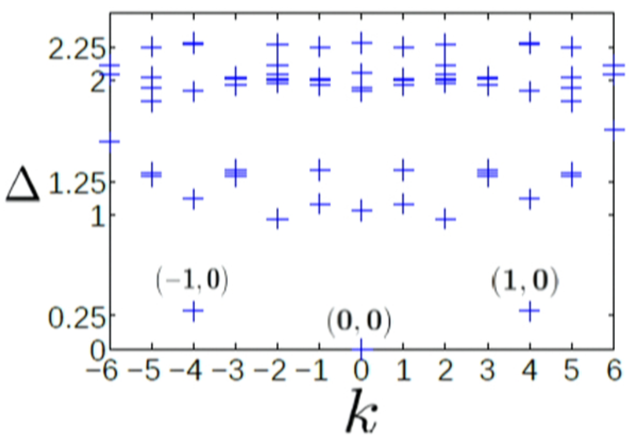

- $\Delta(n,m) = An^2 + Bm^2$
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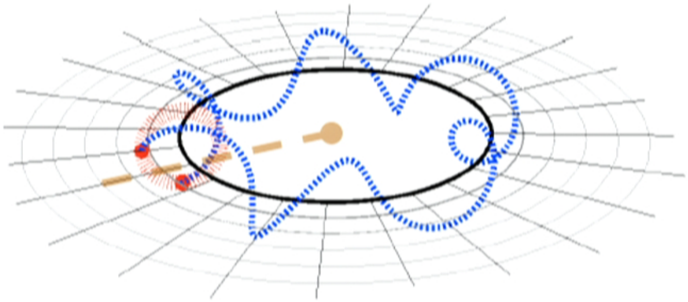
Aharonov-Bohm Effect: Many-Body



$$Z = \int \mathcal{D}\Phi_I \, e^{i \int d^2 x \, \mathcal{L}_{\text{edge}}[A]}$$

$$\mathcal{L}_{\text{edge}}[A] = \mathcal{L}_0$$

$$\phi_1(x+L) = \phi_1(x) + 2\pi\left(m + \frac{1}{N}\right)$$

$$\phi_2(x+L) = \phi_2(x) + 2\pi\left(n + \frac{p}{N}\right)$$

$$\tilde{\Delta}_N^{(p)}(n,m) = A\left(n + \frac{p}{N}\right)^2 + B\left(m + \frac{1}{N}\right)^2$$

$$\tilde{\mathcal{P}}_N^{(p)}(n,m) = \left(n + \frac{p}{N}\right)\left(m + \frac{1}{N}\right)$$

How to implement Twisted Boundary Conditions ?

- Consider the 1d quantum Ising model with p.b.c

$$H_{Ising} = -\sum_{j=1}^{M} \sigma_j^z \sigma_{j+1}^z + \sum_{j=1}^{M} \sigma_j^x, \quad (\sigma_{M+1}^z \equiv \sigma_1^z)$$

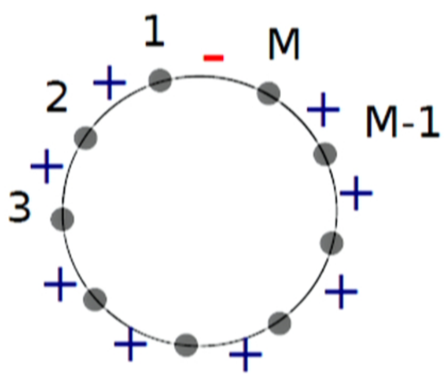
- $\mathbb{Z}_2$ symmetry: $S = \prod_{j=1}^{M} \sigma_j^x$
  $S^2 = 1, \quad S = e^{i\pi Q} \quad Q \in \{0, 1\}$ mod 2 (Z_2 charge)

- Translation symmetry: $T^\dagger \sigma_j^a T = \sigma_{j+1}^a$
  $T^M = 1, \quad T = e^{i \frac{2\pi}{M} k}, \quad k \in \mathbb{Z}$ mod M (lattice momentum)

- Twisted (anti-periodic) boundary conditions

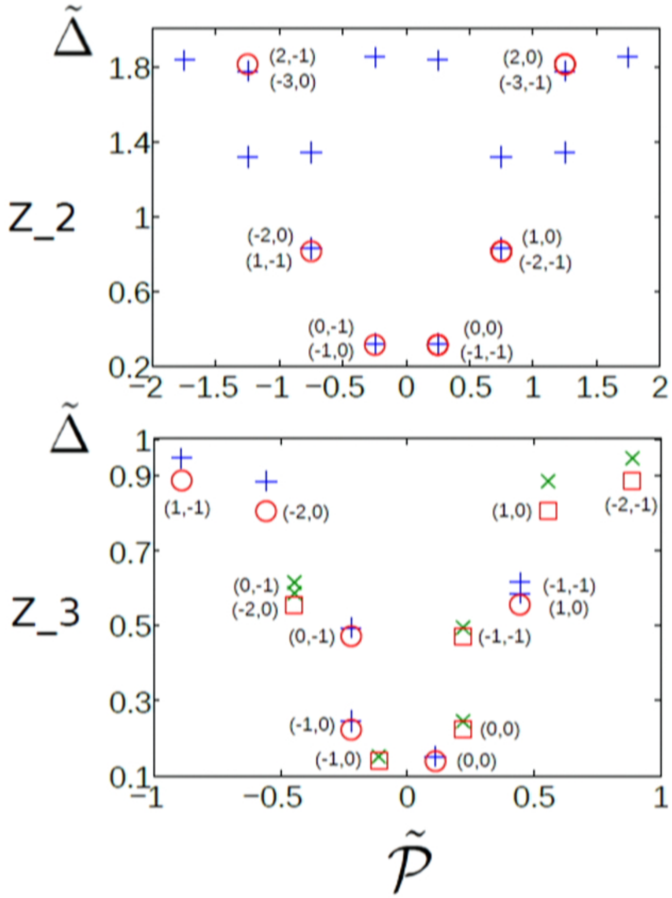
$$\tilde{H}_{Ising} = -\sum_{j=1}^{M} \sigma_j^z \sigma_{j+1}^z + \sum_{j=1}^{M} \sigma_j^x, \quad (\sigma_{M+1}^z \equiv -\sigma_1^z)$$

- Twisted boundary conditions $\sim \mathbb{Z}_2$ gauge flux $\sim$ Branch cut



- $\tilde{T} = T\,\sigma_1^x, \quad [\tilde{H}_{Ising}, \tilde{T}] = 0.$
- $\tilde{T}^M = \sigma_1^x \ldots \sigma_M^x = S$
- $\tilde{k} \in \mathbb{Z} + Q/2 \mod M.$

$\mathbb{Z}_2$ and $\mathbb{Z}_3$ SPT States - Twisted BC



$$\tilde{T}^{(p)} = T\, e^{\frac{i}{N} Q_N^{(p)}(\sigma_M^\dagger \sigma_1)} \tau_1,$$

$$[\tilde{H}_N^{(p)}, \tilde{T}^{(p)}] = 0.$$

Spectrum

$$\tilde{\Delta} = A\left(n + \frac{p}{N}\right)^2 + B\left(m + \frac{1}{N}\right)^2$$

$$\tilde{\mathcal{P}} = \left(n + \frac{p}{N}\right)\left(m + \frac{1}{N}\right)$$

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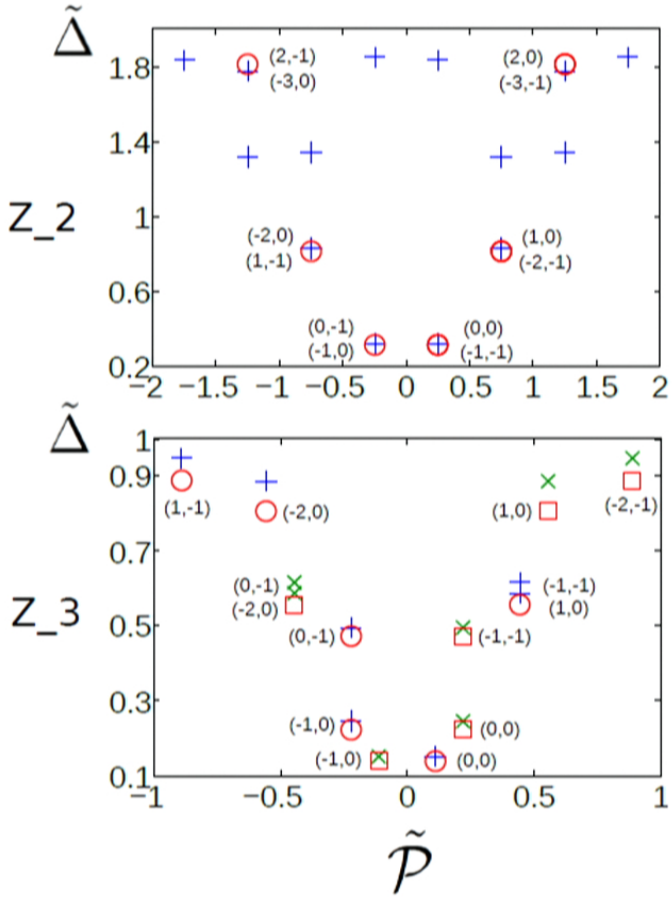

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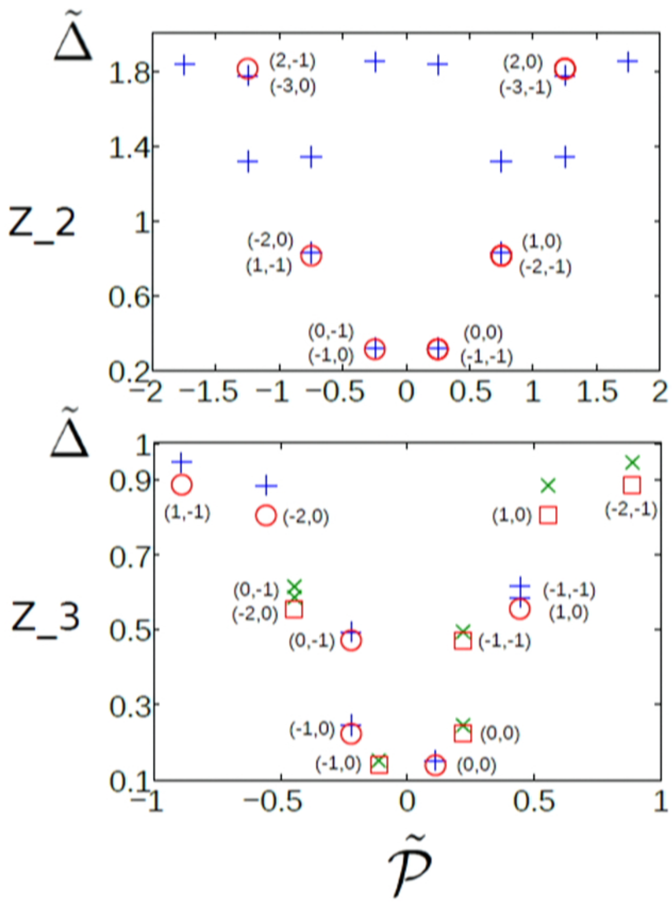

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Conclusions and Perspectives

- In this talk I described effective lattice models for the 1d edge of bosonic SPT states with $\mathbb{Z}_N$ symmetry.
- Method can be extend to the 1d edge of bosonic SPT state with symmetry $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots$ (work in progress)
- It is a useful framework for modeling other kinds of SPT boundary states (w/ different symmetries and in other dimensions).
- Is it possible to construct boundary states of fermionic SPT states in a similar way?