Title: Gauging symmetry of 2D topological phases

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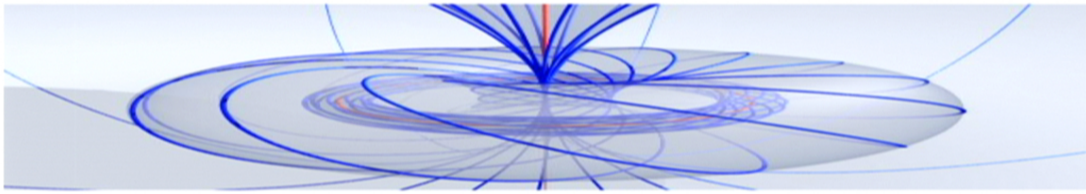
Abstract: I will discuss the mathematical framework for gauging a local unitary finite group symmetry of a 2D topological phase of matter.

On Gauging Symmetry of 2D Topological Phases of Matter



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Joint work with M. Barkeshli, P. Bonderson, and M. Cheng

2D Topological Phases of Matter (Bosonic)

- **Anyon systems in 2D are algebraically modeled well by finite tensor categories C---unitary modular categories (UMCs)**
- **Examples:**

Toric code: {1,e,m,ε}

Abelian Z_3 CS theory K=$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ or $SU(3)_1$ or $U(1)_3$: {1,a,$\bar{a}$}

$SU(2)_k$ for the statistics sectors of Read-Rezayi states: {j/2,j=0,…,k}

k=1, semion, k=2 anti-Pfaffian (Ising sister) for ν=$\frac{5}{2}$, k=4 for ν=$\frac{8}{3}$

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3fermion theory (Toric code sister)

Anyons: {1,e,m,ε} , e,m,ε all fermions

Abelian Chern-Simons Theory with

$$K=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

Or

quantum group $SO(8)_1$

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Possible Symmetries

Particle-hole symmetry of Abelian Z_3 CS:

Z_2, a$\leftrightarrow$ $\bar{a}$

Electric-magnetic duality of T.C.:

Z_2, e$\leftrightarrow$m

Permutation symmetry of 3-fermions:

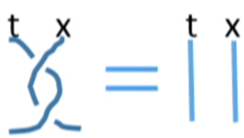
S_3, e$\leftrightarrow$m$\leftrightarrow$ ε

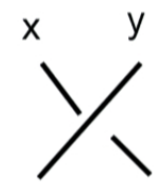
Unitary Modular Category

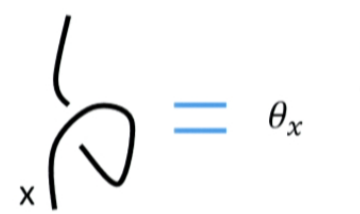
- A finite set L---anyon types, $x \in L$
- Fusion rules: $x \otimes y = \oplus N_{xy}^{z}\, z$
- Braiding $c_{x,y}$: $x \otimes y \longrightarrow y \otimes x$
- A topological twist θ_x: $x \longrightarrow x$

A theory is modular if the only transparent particle is trivial, ie anyons can be detected by full braidings.

A particle t is transparent if for all x,







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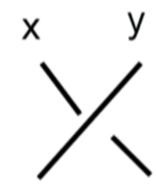

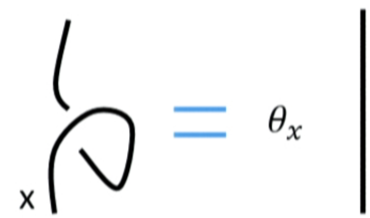

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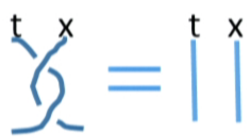

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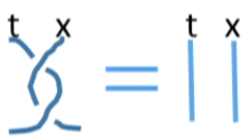

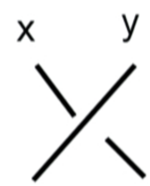

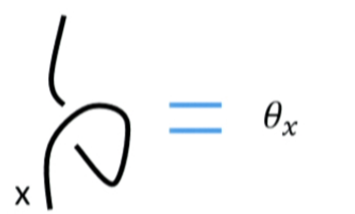

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Classification of Unitary Modular Categories

rank=1,2,3,4 (with Rowell and Stong), finite when rank fixed (with Bruillard, Ng, Rowell)

	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 $\mathbb{Z}_3$	NA 8 Ising	NA 2 (SO(3),5) BU	
A 5 Toric Code	A 4 $\mathbb{Z}_4$	NA 4 Fib x Semion BU	NA 2 (SO(3),7) BU	NA 3 DFib BU

The ith-row is the classification of all rank=i unitary modular tensor categories. Middle symbol: fusion rule. Upper left corner: A=abelian theory, NA=non-abelian. Upper right corner number=the number of distinct theories. Lower left corner BU=there is a universal braiding anyon.

Topological Symmetry

- **The full symmetry of a finite set X of n elements is the permutation group S_n.**

A finite group G is a symmetry of X if there is a map from G to S_n.

- **The symmetries of a finite tensor category C form a categorical group---a finite 2-group.**

Given a unitary modular category C, its full symmetry 2-group is parametrized by a triple (Π_1, Π_2, φ),

where Π_1 is a finite group, Π_2 is a commutative finite group with a Π_1 action, and $\varphi \in H^3(\Pi_1, \Pi_2)$.

A finite group G is a symmetry of a UMC C if there is a 2-group map from the 2-group $\underline{G}$ to the 2-group (Π_1, Π_2, φ). This is equivalent to a group map from G to Π_1 such that the pull back of φ to H^3(G,Π_2) vanishes---note Π_2 acts on G $\subset \Pi_1$. Then possible symmetries are in 1-1 correspondence with H^2(G,Π_2) .

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Symmetries of Abelian Theories

- An **abelian theory** is given by **a pair (A,q),** where A is a commutative finite group and q(x) is the topological twist of anyon type x $\in$A.

- The finite group Π_1 is the orthogonal group of (A,q):

  O(A,q)={s$\in$ Aut(A): s(q(x))=q(x) for all x$\in$A}

  and $\Pi_2 \cong$A
- There is a 1-1 correspondence between **symmetry** of (A,q) with **pairs (ρ,μ),** where ρ: G⟶O(A,q) is a group map such that $\rho^*(\varphi)$=0 $\in$ H^3(G,A) and $\mu \in H^2$(G,A).

Symmetry Data

- **Particle-hole symmetry of Z_3 CS:**

Π_1=Z_2, a$\leftrightarrow$ $\bar{a}$, Π_2=Z_3, φ =0 $\in H^3(Z_2, Z_3)$, $H^2(Z_2, Z_3)$=0.

- **Electric-magnetic duality of T.C.:**

Π_1=Z_2, e$\leftrightarrow$m, Π_2=$Z_2 \oplus Z_2$, φ =0$\in H^3(Z_2,Z_2 \oplus Z_2)$, $H^2(Z_2,Z_2 \oplus Z_2)=Z_2$.

- **Permutation symmetry of 3-fermions:**

Π_1=S_3, e$\leftrightarrow$m$\leftrightarrow$ε,Π_2=$Z_2 \oplus Z_2$,φ =0 $\in H^3(S_3,Z_2 \oplus Z_2)$, $H^2(S_3,Z_2 \oplus Z_2)=Z_2 \oplus Z_2$(?).

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Gauging Topological Symmetry

Given a UMC C, then gauging a symmetry (G, ρ, μ) of C:

Step I:

Add flux sectors to obtain a modular G-crossed category C^{ext}=$\bigoplus_g C_g$, where C_e=C.

There is an obstruction o(G,ρ,μ) $\in H^4$(G,U(1)). If it vanishes, then possible extensions are in 1-1 correspondence with classes of H^3(G,U(1)).

Step II:

Find the equivariant quotient C_{gd}=$(C^{ext})^G$ of C^{ext}---a unitary modular category, closely related to "orbifolding".

When C is trivial, then H^3(G,U(1)) classes correspond to SPTs, and the gauged theories are the corresponding quantum doubles.

Will illustrate by gauging the particle-hole symmetry of the bosonic Z_3.

Modular G-crossed Category

A G-crossed theory has two compatible structures:

- G-graded fusion: $C^{grade} = \bigoplus_g C_g$, $C_g \otimes C_h \subset C_{gh}$
- G-action: the action of g sends C_h to $C_{ghg^{-1}}$.

A G-crossed braiding

$$c_{x,y}: x \otimes y \longrightarrow y^g \otimes x, \text{ where } x \in C_g$$

For each commutative pair (g,h), there is a sector $V_{g,h}$ for the Verlinde algebra. Modular if the extended $\tilde{s}$–matrix is non-degenerate.

Gauging PH Symmetry of Bosonic $\boldsymbol{Z_3}$

Consider the particle-hole symmetry of Z_3---unique as $H^2(Z_2, Z_3)$=0.

Step 1:

Only one twist defect g in C_{-1}: g⊗g=1+a+$\bar{a}$. This theory is NOT braided---Tambara-Yamagami theory for Z_3. But it has a G-crossed braiding. There are two ways to have an defect as $H^3(Z_2$,U(1))=Z_2.

Step 2:

Taking the equivariant quotient results either $SU(2)_4$ or its cousin Jones-Kauffman theory at r=6---two metaplectic theories corresponding to the two classes in $H^3(Z_2$,U(1))=Z_2 as above.

Modular G-crossed Category

- The extended Verlinde algebra has 4 sectors: $V_{1,1}$, $V_{1,-1}$, $V_{-1,1}$, $V_{-1,-1}$, and $\tilde{s}$-, $\tilde{t}$-matrices form a rep. of SL(2,ℤ). Below the **s,t** are those of the Z_3 theory.
- The extended $\tilde{s}$-matrix $\tilde{s}=\begin{pmatrix} \boldsymbol{s} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\kappa \end{pmatrix}$
- The extended $\tilde{t}$ matrix $\tilde{t}=\begin{pmatrix} \boldsymbol{t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (-i\kappa)^{1/2} \\ 0 & 0 & (-i\kappa)^{1/2} & 0 \end{pmatrix}$

General Theory

- **Result of Step I:**

The number of defects in each sector C_g of the resulted modular G-crossed category C^{ext}=$\bigoplus_g C_g$ is the number of g fixed points, and $D^2{}_g$=D^2.

- **Result of Step II:**

The resulted unitary modular category C_{gd}=$(C^{ext})^G$ contains a sub-category Rep(G). The $D^2{}_{gd}$=$D^2\ |G|^2$, and topological central charges are equal.

- **Inverse process:**

When C_{gd}=$(C^{ext})^G$ is de-equivariantized w.r.t. to Rep(G), closely related to "condensing" Rep(G), we recover C as the untwisted sector of C^{ext} with the G-symmetry.

Challenge: Gauging $\boldsymbol{S_3}$ of 3fermion Theory

Many others:

particle-hole of Z_N, electric-magnetic of D(Z_N), C-F of D(S_3), Z_2 of two layers such as Ising×Ising, Fib×Fib,…

Gauging S_3 of 3fermion is hard. S_3=semi-product of Z_3 and Z_2, so

- Gauging done in two steps:
  first gauging the Z_3 and then the Z_2
- For one choice:

The result of gauging Z_3 is $SU(3)_3$ and the final result by further gauging Z_2 is some theory with 12 particle types of quantum dimensions: **1,1,2,3,3,4,4,4,3√2,3√2,3√2,3√2.**