

Title: Geometrical dependence of information in 2d critical systems

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Abstract: In both classical and quantum critical systems, universal contributions to the mutual information and Renyi entropy depend on geometry. I will first explain how in 2d classical critical systems on a rectangle, the mutual information depends on the central charge in a fashion making its numerical extraction easy, as in 1d quantum systems. I then describe analogous results for 2d quantum critical systems. Specifically, in special 2d quantum systems such as quantum dimer/Lifshitz models, the leading geometry-dependent term in the Renyi entropies can be computed exactly. In more common 2d quantum systems, numerical computations of a corner term hint toward the existence of a universal quantity providing a measure of the number of degrees of freedom analogous to the central charge.

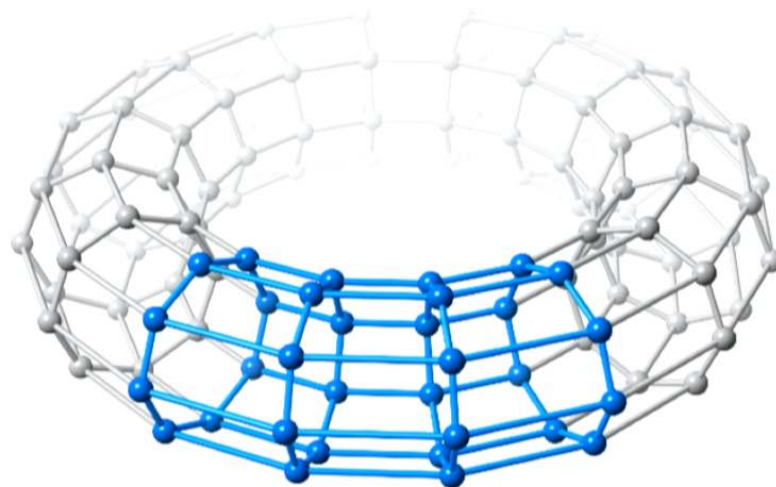
I'll talk about mutual information/entanglement in critical 2d classical/quantum models.

Because of the long-range correlations in a critical system, universal subleading terms depend on the geometry.

Choosing the geometry appropriately makes their calculation quite amenable to numerical computations.

Exact computations are possible in some important cases.

A useful geometry comes from cutting a **torus into two cylinders**:



One can **vary the size** of the regions being entangled **without changing the length of the boundary** between them!

This allows critical properties to be probed accurately numerically.

Many collaborators on four papers:

On the numerical side:

Roger Melko with Stephen Inglis, Hyejin Ju, and Ann Kallin

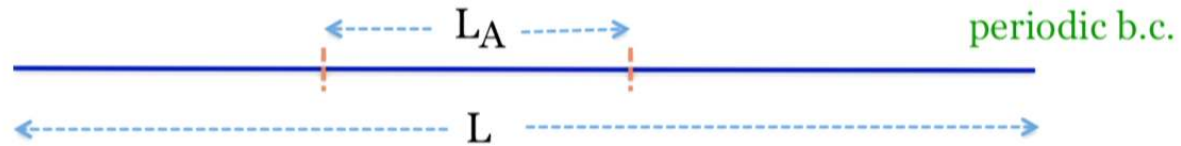
On the theory side:

Jean-Marie Stephan

On the important-contributions side:

Matt Hastings, Rajiv Singh, and Miles Stoudenmire

A paradigmatic result in 1+1d critical systems:



reduced density matrix

central charge of conformal field theory

$$\mathcal{S}_n = \frac{1}{n-1} \ln \text{tr} \rho_A^n = \frac{c}{6} \left(1 + \frac{1}{n} \right) \ln \left[\frac{L}{\pi} \sin \frac{\pi L_A}{L} \right] + \dots,$$

Renyi index

Holzhey, Larson and Wilczek; Vidal et al; Calabrese and Cardy

This result is both **practically** and **conceptually** important.

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Zamolodchikov's c-theorem:

Its definition can be extended off criticality to give a quantity that **decreases in RG flows**.

The entanglement entropy is usually the **easiest way to numerically extract c** from a lattice model.

No Fermi velocity, no fitting bulk terms.

This suggests that in more general situations, information may provide other **easily computable universal** quantities providing a **measure of the number of degrees of freedom**, generalizing the **central charge**.

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2d classical critical points



Since there's such a beautiful formula for 1d quantum critical points, shouldn't there be a similar formula for 2d classical critical points?

A classical analog of entanglement entropy is the **Renyi mutual information**.

Partition function: $Z(\beta) = \sum_i e^{-\beta E_i}$

Boltzmann weight: $p_i = \frac{1}{Z(\beta)} e^{-\beta E_i}$

Take two subregions A and B , and let i_A and i_B be configurations within each.

$$p_{i_A} = \sum_{i_B} p_{i_A, i_B} \quad S_n(A) = \frac{1}{1-n} \ln \sum_{i_A} p_{i_A}^n$$

The **RMI** is then

$$I_n(A, B) = S_n(A) + S_n(B) - S_n(A \cup B)$$

Wilms, Troyer and Verstraete; Iaconis, Inglis, Kallin and Melko

The RMI is defined so that bulk terms cancel. In 2d

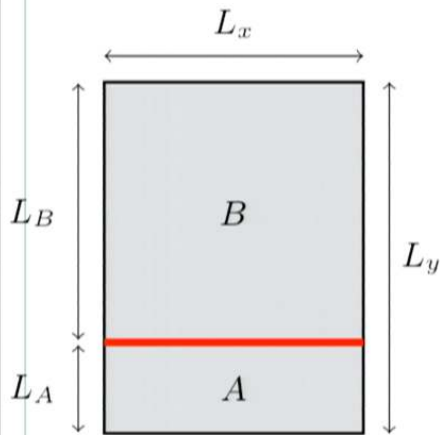
$$I_n = a_n L + G_n + o(1)$$

length of boundary separating A and B

The **Geometric Mutual Information**

By using CFT, **the GMI can be computed** in many 2d situations.

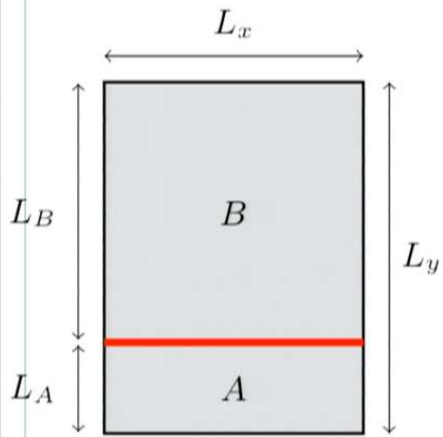
By varying the region size while keeping L fixed, **G can be accurately simulated.**



Rewriting using “replicas”:

$$I_n(A, B) = \frac{1}{1-n} \log \left(\frac{Z[A, n, \beta] Z[B, n, \beta]}{Z(\beta)^n Z(n\beta)} \right)$$

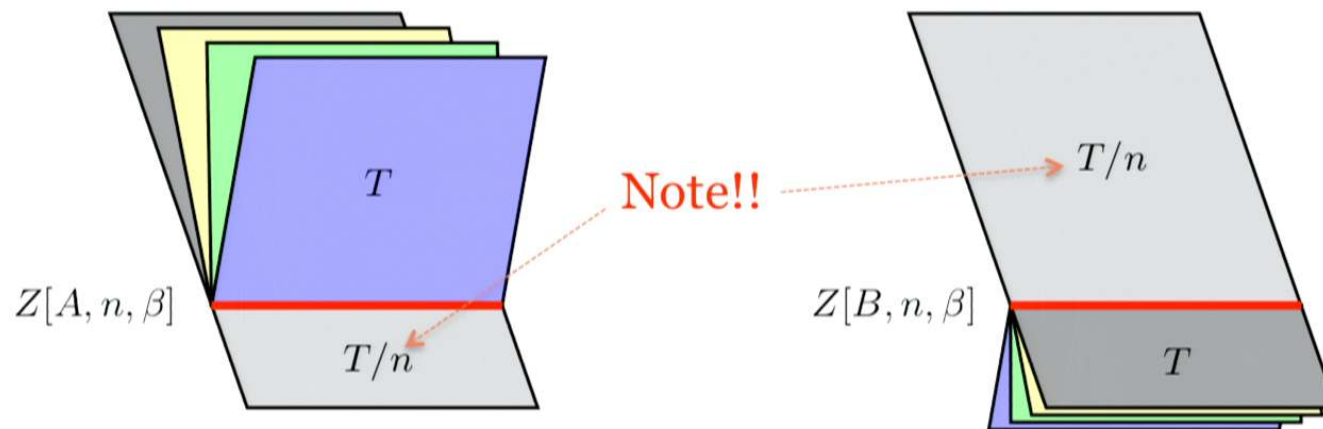
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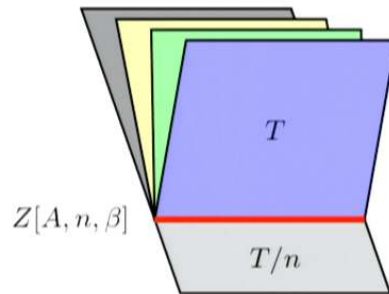
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The RMI exhibits critical behavior at both $T=T_c$ and $T=nT_c$

At $T=T_c$: the “fan” is critical, the “original” system at low temp

At $T=nT_c$: the original system is critical, the fan at high temp



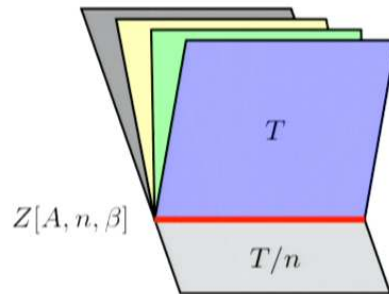
Philosophical digression:

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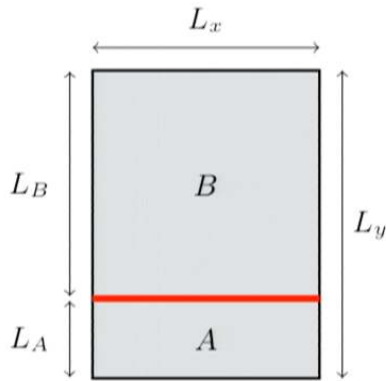


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The computation of the partition functions on a rectangle is standard in 2d CFT. Kleban and Vassileva

At $T=nT_c$ with free boundary conditions on the outside:



$$G_n = \frac{c}{2} \left(\frac{n}{n-1} \right) \ln \left(\frac{f(L_A / L_x) f(L_B / L_x)}{\sqrt{L_x} f(L_y / L_x)} \right)$$

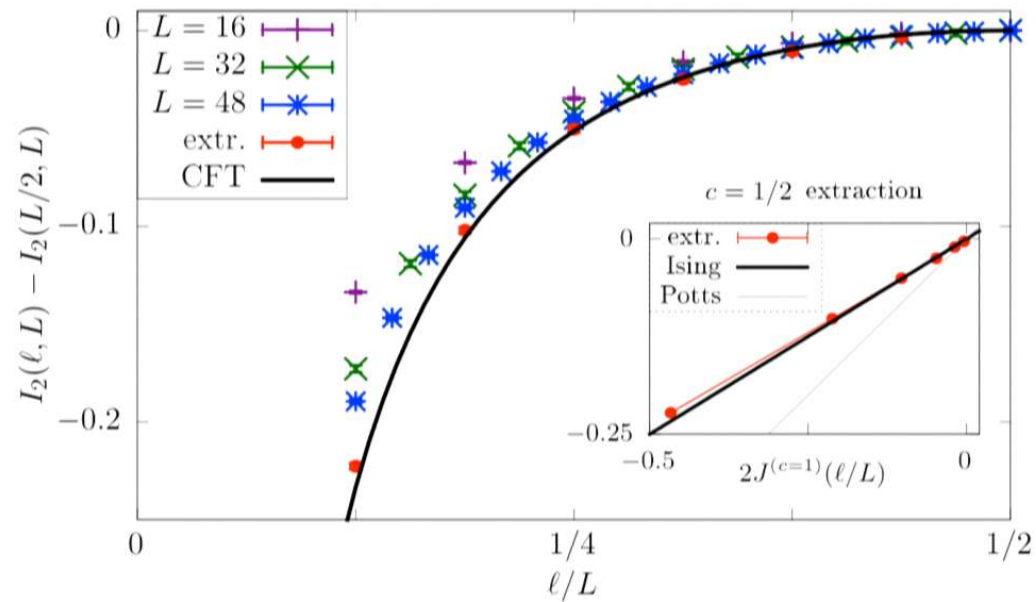
$$f(u) = e^{-\pi u/12} \prod_{k=1}^{\infty} (1 - e^{-2\pi k u}).$$

The central charge is just a coefficient in the GMI!

Stephan, Inglis, Fendley and Melko

Because the aspect ratio can be varied without changing L , it is easy to numerically **extract the universal subleading GMI from the RMI**.
Using **the transfer-matrix ratio trick**,

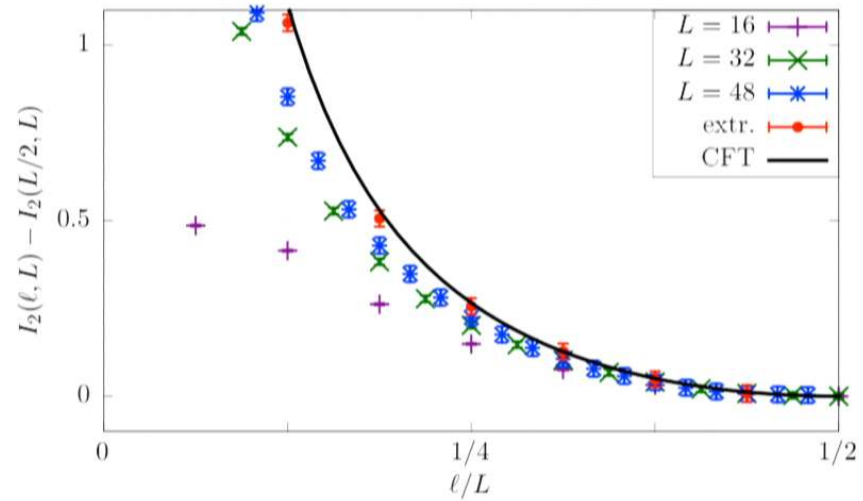
Ising at $T=nT_c$



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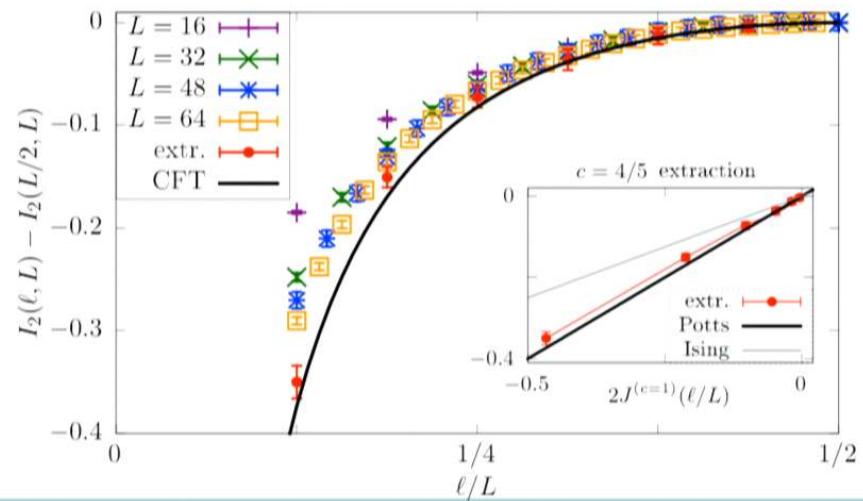
Get excited-state dimensions:

Ising at $T=T_c$



This works for strongly interacting models too.

3-state Potts at $T=nT_c$



Entanglement in 2d quantum systems



I'll start with special type of system, **conformal quantum critical points**, that have much in common with 2d classical systems.

Here we have derived some exact results.

Then I'll move on to more familiar systems.

A conformal quantum critical point in 2+1d has ground-state wave function built from the Boltzmann weights of a 2d classical system.

They are ground states of frustration-free/Rokhsar-Kivelson Hamiltonians.

Examples include the square-lattice quantum dimer model, the RVB state, and the quantum Lifshitz field theory.

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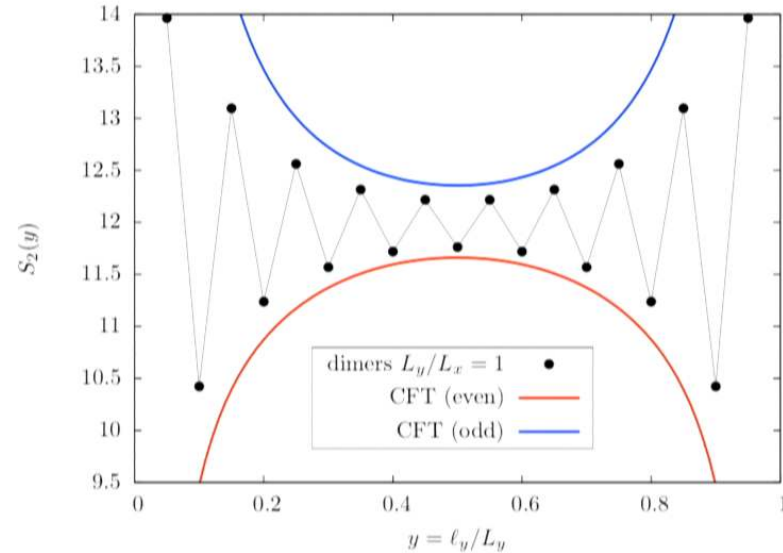
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The computation of the geometry-dependent subleading term in the Renyi entropy at a CQCP is very similar to that of the GMI.

For a 20 x 20 torus split into 2 cylinders, finite-size effects are large. They do go away!



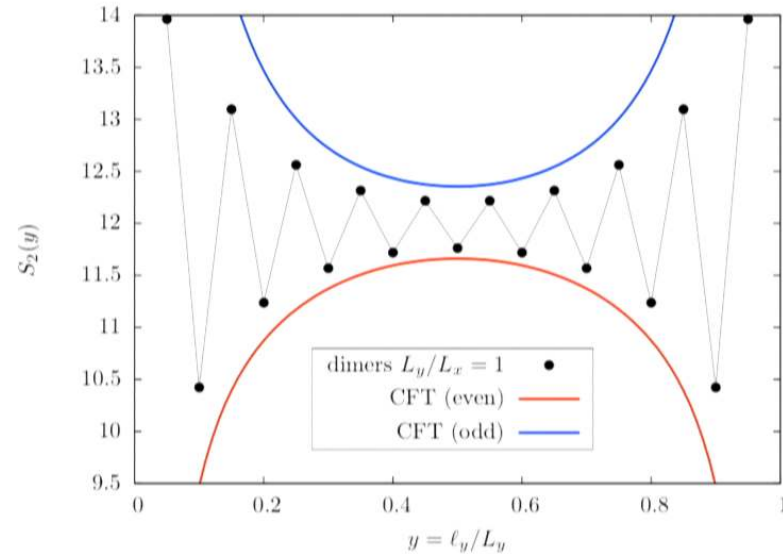
$$s_n^{(\text{even})}(y, \tau) = \frac{n}{1-n} \ln \left(\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \times \frac{\theta_3(2y\tau)\theta_3(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)} \right)$$

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This is one of the [few exact entanglement calculations](#) possible for 2+1d quantum critical points (such finite-size effects are not yet known even for free fermions).

It is not known how to do such exact computations even in all CQCPs, much less Lorentz-invariant theories. Nevertheless, the previous curve [fits the numerical data very well](#) for the 2d quantum transverse-field Ising model, fitting only the overall coefficient.

Inglis and Melko

This correspondence remains very mysterious. It works well for pi-flux fermions, but is not quite exact (off by about 1%).

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Corners



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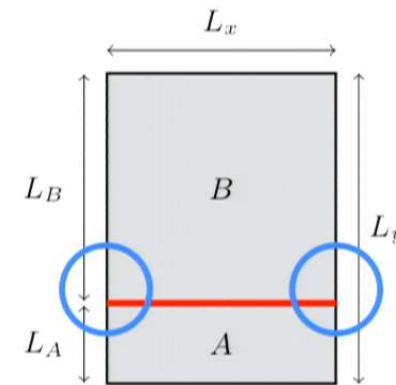


A famous result of Cardy and Peschel gives the contribution of a **corner** to the free energy of a 2d CFT in a region with linear size L :

$$\propto c \ln L$$

This shows up in the GMI:

$$G_n = \frac{c}{2} \left(\frac{n}{n-1} \right) \ln \left(\frac{f(L_A / L_x) f(L_B / L_x)}{\sqrt{L_x} f(L_y / L_x)} \right)$$



Coefficients of logs are typically universal.

So does similar behavior occur in higher dimensions?

The **same** $c \ln L$ occurs as a subleading term in the entanglement entropy at a CQCP in 2+1d, e.g.

$$S = \alpha L - \frac{c}{9} \ln(L) + \dots$$

for region A a rectangle surrounded by region B . This is “hearing the shape of a quantum drum”! **Fradkin and Moore**

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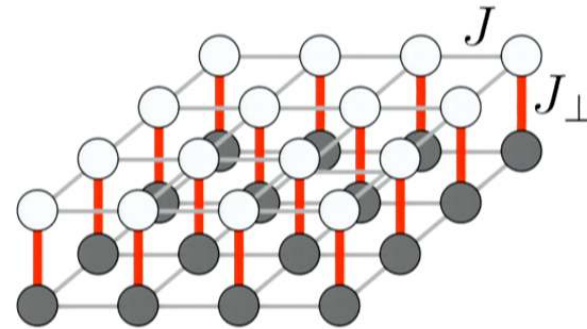
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Using the Numerical Linked-Cluster Expansion of **Rigol, Bryant and Singh**, we explored corner contributions to the Renyi entanglement entropy in a much more conventional 2+1d model.

Kallin, Stoudenmire, Fendley, Singh and Melko

Heisenberg bilayer:

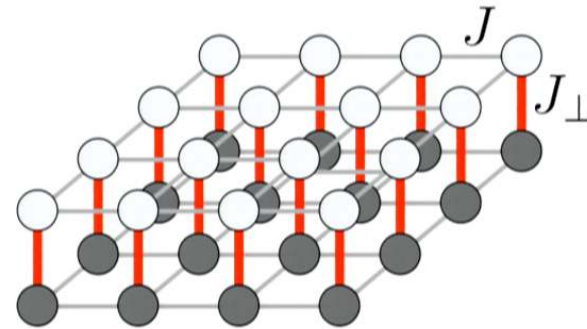


$$H = J \sum_{\langle i,j \rangle} (\mathbf{S}_{1i} \cdot \mathbf{S}_{1j} + \mathbf{S}_{2i} \cdot \mathbf{S}_{2j}) + J_{\perp} \sum_i \mathbf{S}_{1i} \cdot \mathbf{S}_{2i}$$

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The bilayer (as opposed to the square lattice) Heisenberg model has a quantum critical point at $J_{\perp} / J = 2.522$.

It is in the same universality class as the **3d classical Heisenberg model**.

Wang, Beach and Sandvik

With $\vec{\phi}$ a suitably coarse-grained staggered magnetization, the critical region is described by Landau-Ginzburg action

$$\int d^2x dt \left(\frac{\partial \vec{\phi}}{\partial t} \cdot \frac{\partial \vec{\phi}}{\partial t} - \nabla \vec{\phi} \cdot \nabla \vec{\phi} - \mu^2 \vec{\phi} \cdot \vec{\phi} - g(\vec{\phi} \cdot \vec{\phi})^2 \right)$$

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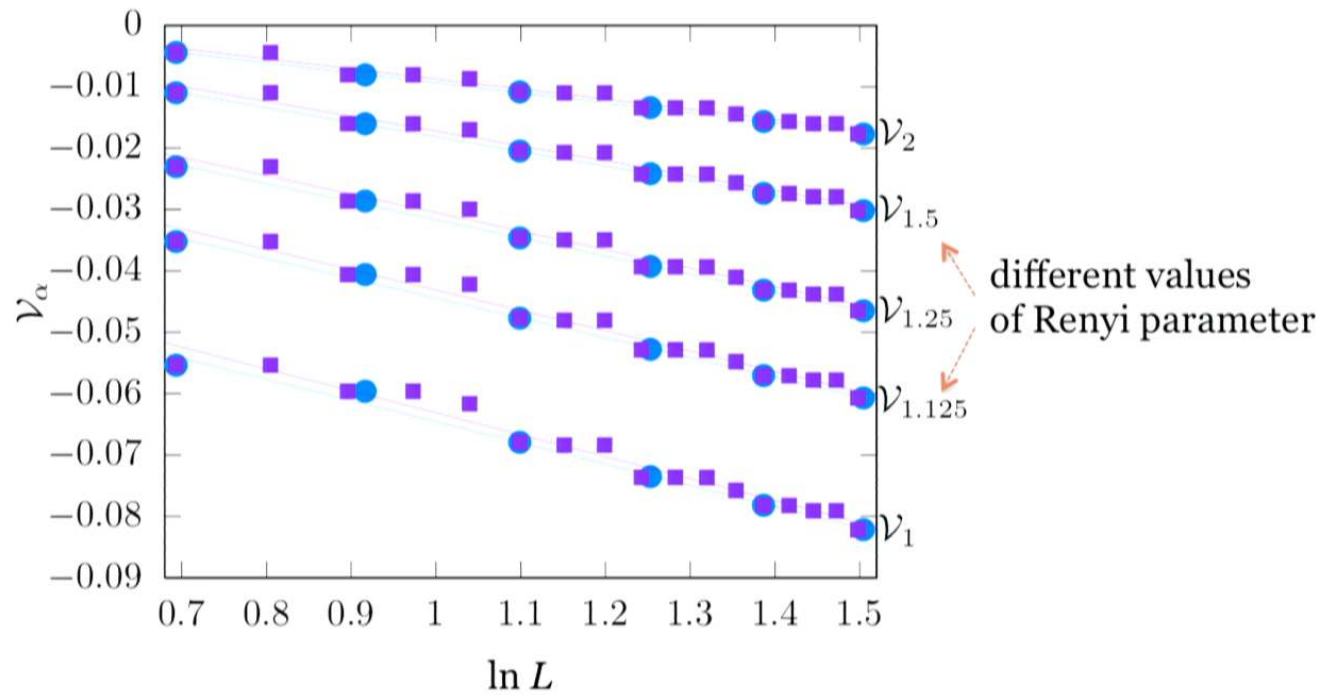
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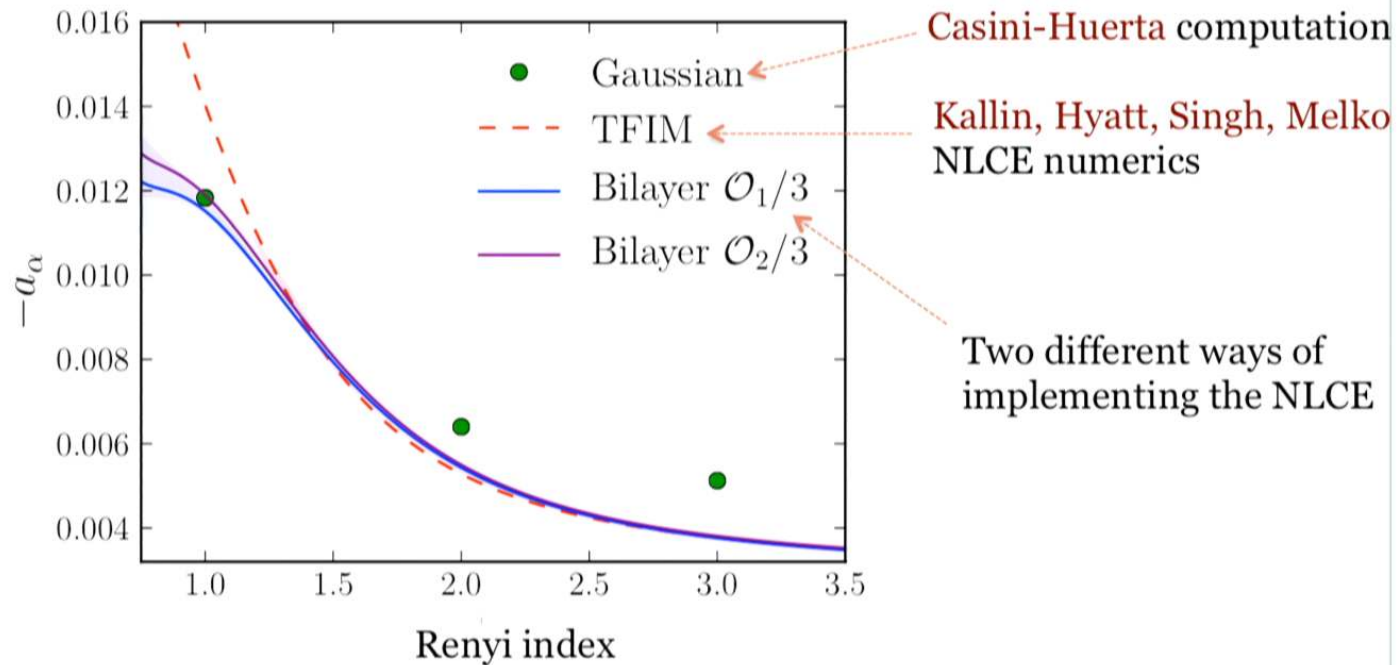
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The corner contribution does scale with $\ln L$ as in the CQCPs:



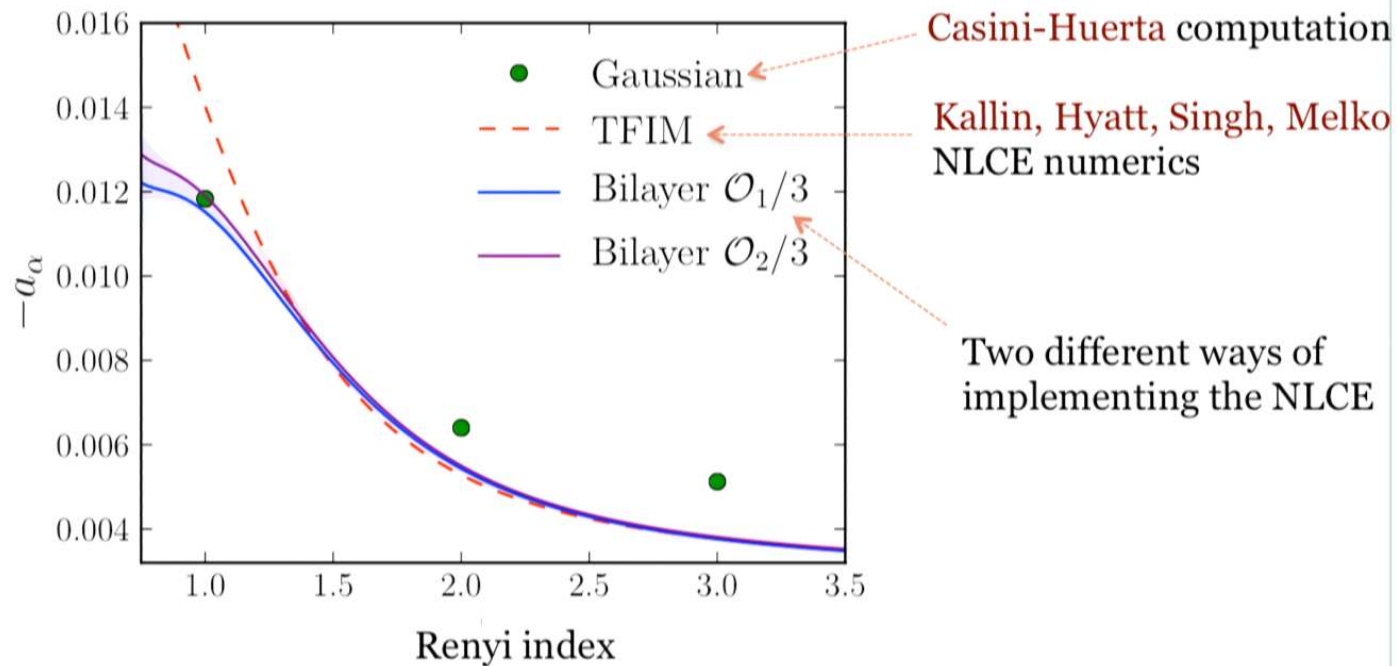
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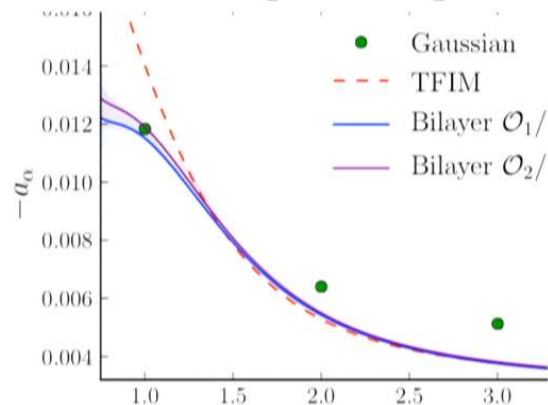
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The Landau-Ginzburg action for Ising is $O(N) \phi^4$ with $N=1$, for bilayer Heisenberg it is $N=3$.

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For interacting theories it won't be strictly additive, but these theories are "close" to free (the epsilon expansion works).

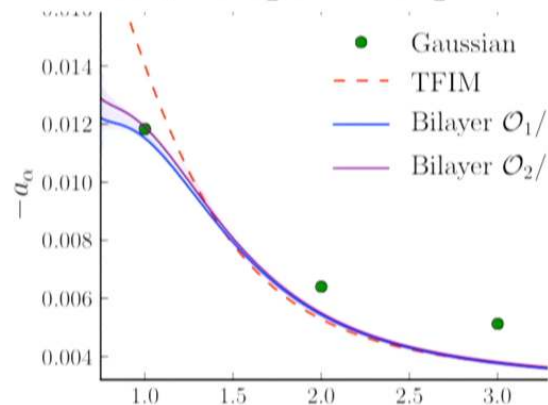


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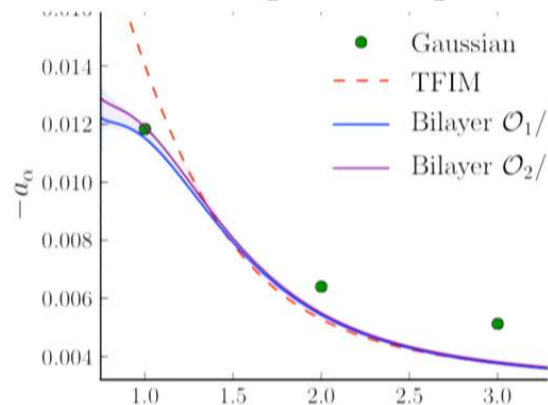


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Some obvious further directions:



- Computing the GMI in the 2d XY model (low T phase not ordered)
- Computing GMI in higher d CFTs via Ryu-Takayanagi?
- Maybe corner term can be computed in large N?
- Nice to have corners for a model with gauge fields – maybe the J-Q model?
- Nice also to check a non-critical Goldstone phase, e.g. Heisenberg. NLCE though is more difficult