Title: Geometrical dependence of information in 2d critical systems

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Abstract: In both classical and quantum critical systems, universal contributions to the mutual information and Renyi entropy depend on geometry. I will first explain how in 2d classical critical systems on a rectangle, the mutual information depends on the central charge in a fashion making its numerical extraction easy, as in 1d quantum systems. I then describe analogous results for 2d quantum critical systems. Specifically, in special 2d quantum systems such as quantum dimer/Lifshitz models, the leading geometry-dependent term in the Renyi entropies can be computed exactly. In more common 2d quantum systems, numerical computations of a corner term hint toward the existence of a universal quantity providing a measure of the number of degrees of freedom analogous to the central charge.

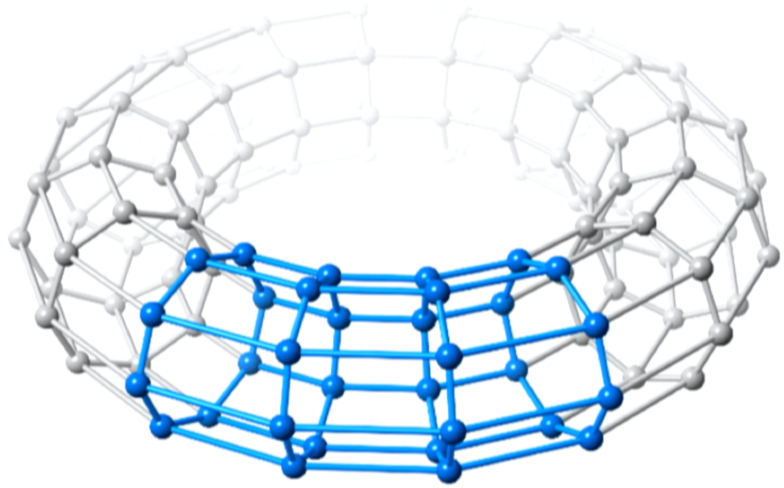
I'll talk about mutual information/entanglement in critical 2d classical/quantum models.

Because of the long-range correlations in a critical system, universal subleading terms depend on the geometry.

Choosing the geometry appropriately makes their calculation quite amenable to numerical computations.

Exact computations are possible in some important cases.

A useful geometry comes from cutting a torus into two cylinders:



One can vary the size of the regions being entangled without changing the length of the boundary between them!

This allows critical properties to be probed accurately numerically.

Many collaborators on four papers:

On the numerical side:

Roger Melko with Stephen Inglis, Hyejin Ju, and Ann Kallin

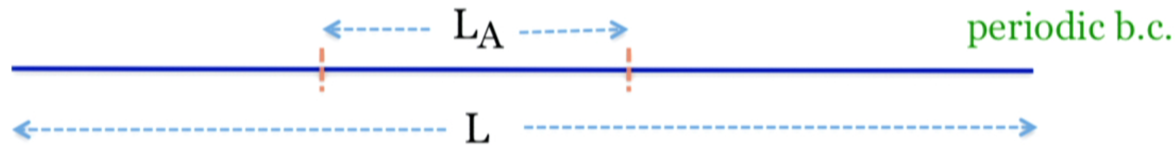
On the theory side:

Jean-Marie Stephan

On the important-contributions side:

Matt Hastings, Rajiv Singh, and Miles Stoudenmire

A paradigmatic result in 1+1d critical systems:



reduced density matrix

central charge of conformal field theory

$$\mathcal{S}_n = \frac{1}{n-1} \ln \operatorname{tr} \rho_A^n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln\left[\frac{L}{\pi} \sin \frac{\pi L_A}{L}\right] + \ldots,.$$

Renyi index

Holzhey, Larson and Wilczek; Vidal et al; Calabrese and Cardy

This result is both practically and conceptually important.

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It provides a measure of the number of degrees of freedom.

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Zamolodchikov's c-theorem:

Its definition can be extended off criticality to give a quantity that decreases in RG flows.

The entanglement entropy is usually the easiest way to numerically extract c from a lattice model.

No Fermi velocity, no fitting bulk terms.

This suggests that in more general situations, information may provide other easily computable universal quantities providing a measure of the number of degrees of freedom, generalizing the central charge.

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2d classical critical points

Since there's such a beautiful formula for 1d quantum critical points, shouldn't there be a similar formula for 2d classical critical points?

A classical analog of entanglement entropy is the Renyi mutual information.

Partition function: $$Z(\beta)=\sum_i e^{-\beta E_i}$$

Boltzmann weight: $$p_i=\frac{1}{Z(\beta)}e^{-\beta E_i}$$

Take two subregions A and B, and let i_A and i_B be configurations within each.

$$p_{i_A}=\sum_{i_B} p_{i_A,i_B} \qquad S_n(A)=\frac{1}{1-n}\ln\sum_{i_A} p_{i_A}^n$$

The RMI is then

$$I_n(A,B)=S_n(A)+S_n(B)-S_n(A\cup B)$$

Wilms, Troyer and Verstraete; Iaconis, Inglis, Kallin and Melko

The RMI is defined so that bulk terms cancel. In 2d

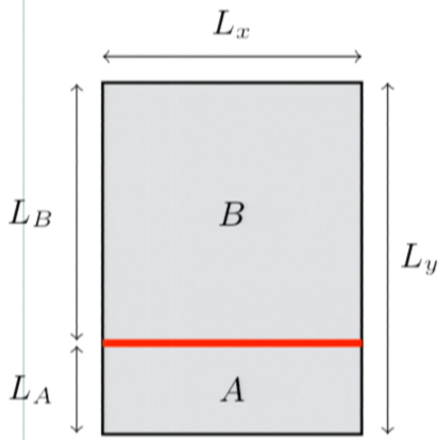
$$I_n = a_n L + G_n + o(1)$$

length of boundary separating A and B

The Geometric Mutual Information

By using CFT, the GMI can be computed in many 2d situations.

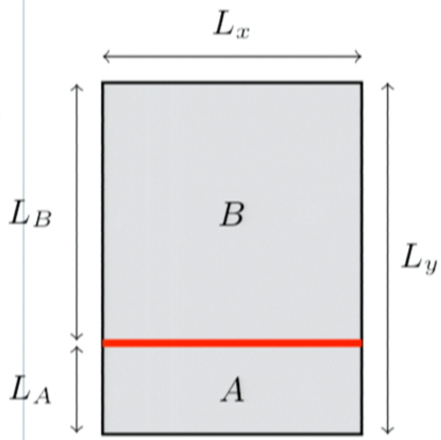
By varying the region size while keeping L fixed, G can be accurately simulated.



Rewriting using “replicas”:

$$I_n(A,B) = \frac{1}{1-n}\log\left(\frac{Z[A,n,\beta]Z[B,n,\beta]}{Z(\beta)^n Z(n\beta)}\right)$$

$$Z[A,n,\beta] = \sum_{i_A}\sum_{i_{B_1},\ldots,i_{B_n}} e^{-\beta\sum_{k=1}^{n} E_{i_A,i_{B_k}}}$$



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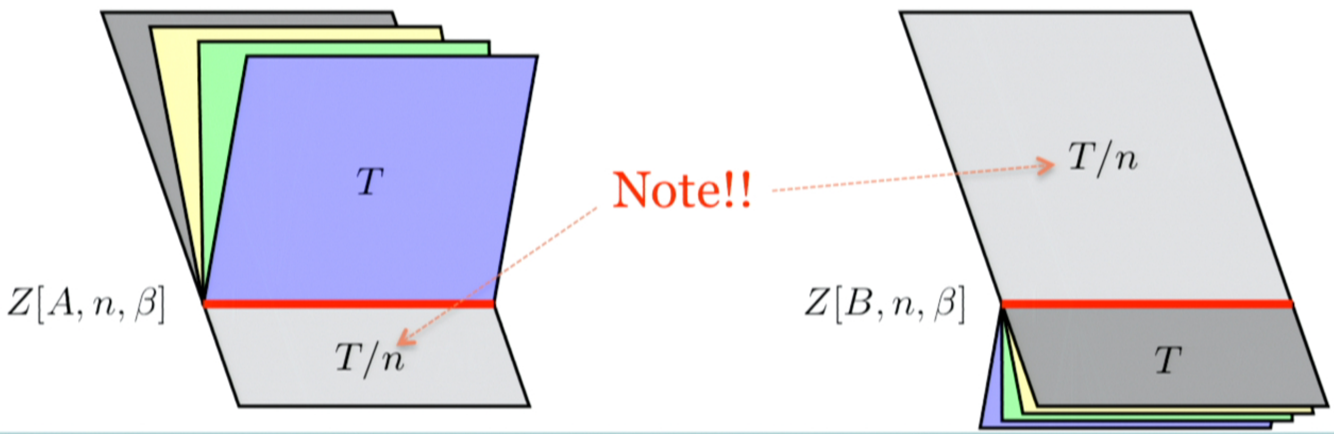
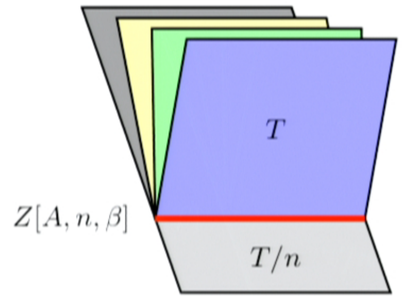


The RMI exhibits critical behavior at both $T = T_c$ and $T = nT_c$

At $T = T_c$: the "fan" is critical, the "original" system at low temp

At $T = nT_c$: the original system is critical, the fan at high temp



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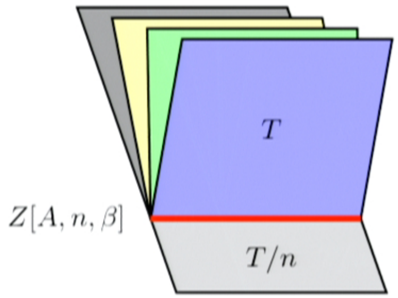
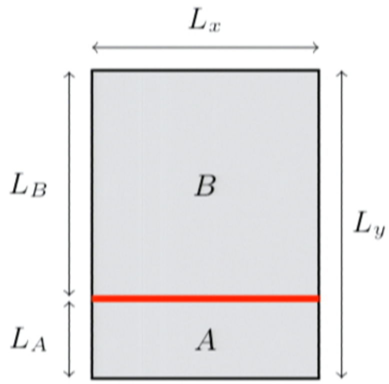


Philosphical digression:

If one could compute the RMI for all n at a given T, then one could reconstruct the physics at all temperatures.

The computation of the partition functions on a rectangle is standard in 2d CFT. Kleban and Vassileva

At $T=nT_c$ with free boundary conditions on the outside:



$$G_n = \frac{c}{2}\left(\frac{n}{n-1}\right)\ln\left(\frac{f(L_A/L_x)f(L_B/L_x)}{\sqrt{L_x}\,f(L_y/L_x)}\right)$$

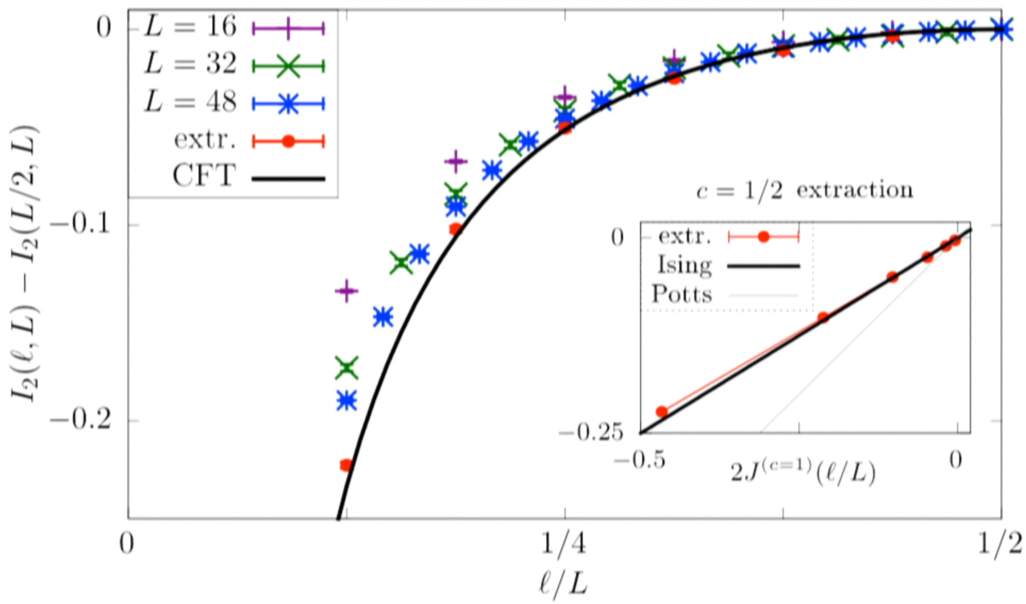
$$f(u) = e^{-\pi u/12}\prod_{k=1}^{\infty}\left(1-e^{-2\pi k u}\right).$$

The central charge is just a coefficient in the GMI!

Stephan, Inglis, Fendley and Melko

Because the aspect ratio can be varied without changing L, it is easy to numerically extract the universal subleading GMI from the RMI. Using the transfer-matrix ratio trick,

Ising at T=nT_c



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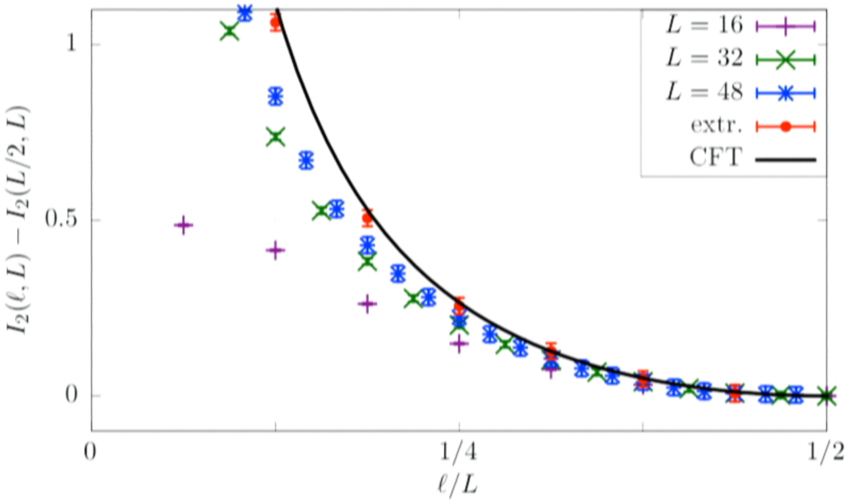
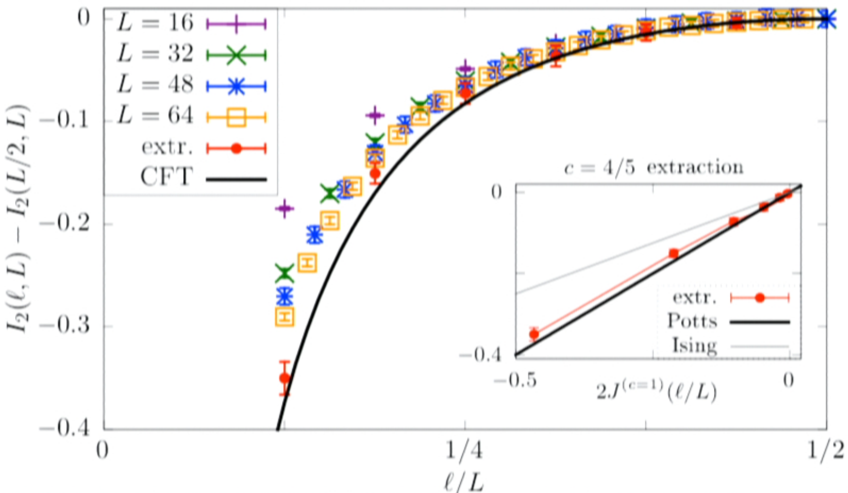
Get excited-state dimensions:



Ising at $T=T_c$

This works for strongly interacting models too.



3-state Potts at $T=nT_c$

Entanglement in 2d quantum systems

I'll start with special type of system, conformal quantum critical points, that have much in common with 2d classical systems.

Here we have derived some exact results.

Then I'll move on to more familiar systems.

A conformal quantum critical point in 2+1d has ground-state wave function built from the Boltzmann weights of a 2d classical system.

They are ground states of frustration-free/Rokhsar-Kivelson Hamiltonians.

Examples include the square-lattice quantum dimer model, the RVB state, and the quantum Lifshitz field theory.

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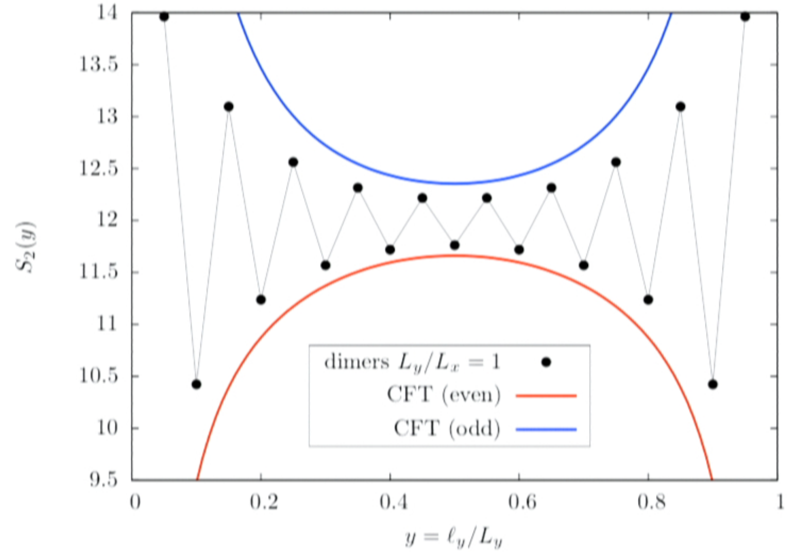
For a 20 x 20 torus split into 2 cylinders, finite-size effects are large. They do go away!



$$s_n^{(\mathrm{even})}(y,\tau) = \frac{n}{1-n}\ln\left(\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \times \frac{\theta_3(2y\tau)\theta_3(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)}\right)$$

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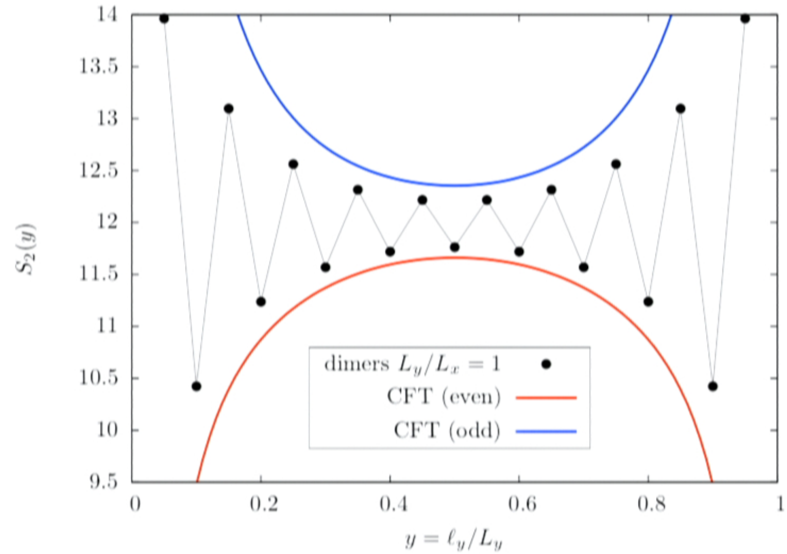

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This is one of the few exact entanglement calculations possible for 2+1d quantum critical points (such finite-size effects are not yet known even for free fermions.

It is not known how to do such exact computations even in all CQCPs, much less Lorentz-invariant theories. Nevertheless, the previous curve fits the numerical data very well for the 2d quantum transverse-field Ising model, fitting only the overall coefficient. Inglis and Melko

This correspondence remains very mysterious. It works well for pi-flux fermions, but is not quite exact (off by about 1%). Stephan, unpublished

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Corners

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This shows up in the GMI:

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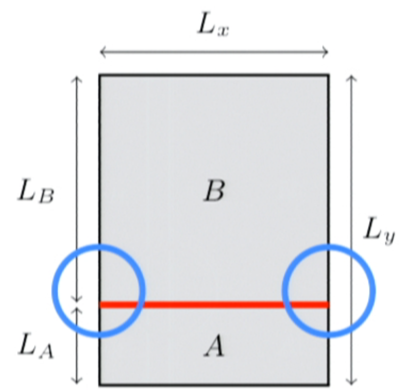

Coefficients of logs are typically universal.

So does similar behavior occur in higher dimensions?

The same $c \ln L$ occurs as a subleading term in the entanglement entropy at a CQCP in 2+1d, e.g.

$$S = \alpha L - \frac{c}{9}\ln(L) + \ldots$$

for region A a rectangle surrounded by region B. This is "hearing the shape of a quantum drum"!

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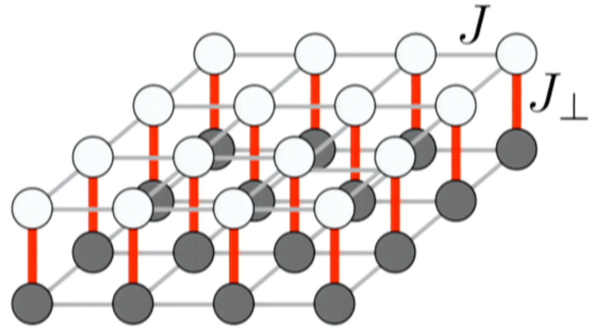
Using the Numerical Linked-Cluster Expansion of Rigol, Bryant and Singh, we explored corner contributions to the Renyi entanglement entropy in a much more conventional 2+1d model.

Kallin, Stoudenmire, Fendley, Singh and Melko

Heisenberg bilayer:



$$H = J\sum_{\langle i,j\rangle}\left(\mathbf{S}_{1i}\cdot\mathbf{S}_{1j}+\mathbf{S}_{2i}\cdot\mathbf{S}_{2j}\right)+J_{\perp}\sum_{i}\mathbf{S}_{1i}\cdot\mathbf{S}_{2i}$$

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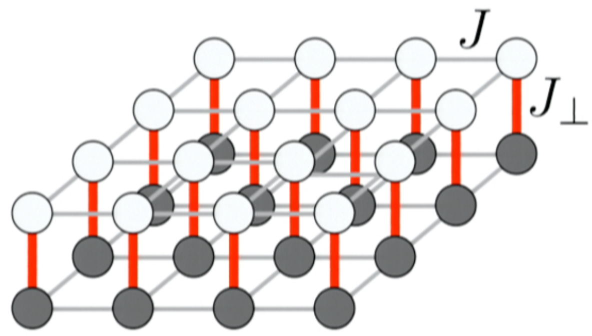

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The bilayer (as opposed to the square lattice) Heisenberg model has a quantum critical point at $J_{\perp} / J = 2.522$.

It is in the same universality class as the 3d classical Heisenberg model.

Wang, Beach and Sandvik

With $\vec{\phi}$ a suitably coarse-grained staggered magnetization, the critical region is described by Landau-Ginzburg action

$$\int d^2xdt\left(\frac{\partial\vec{\phi}}{\partial t}\cdot\frac{\partial\vec{\phi}}{\partial t}-\nabla\vec{\phi}\cdot\nabla\vec{\phi}-\mu^2\vec{\phi}\cdot\vec{\phi}-g(\vec{\phi}\cdot\vec{\phi})^2\right)$$

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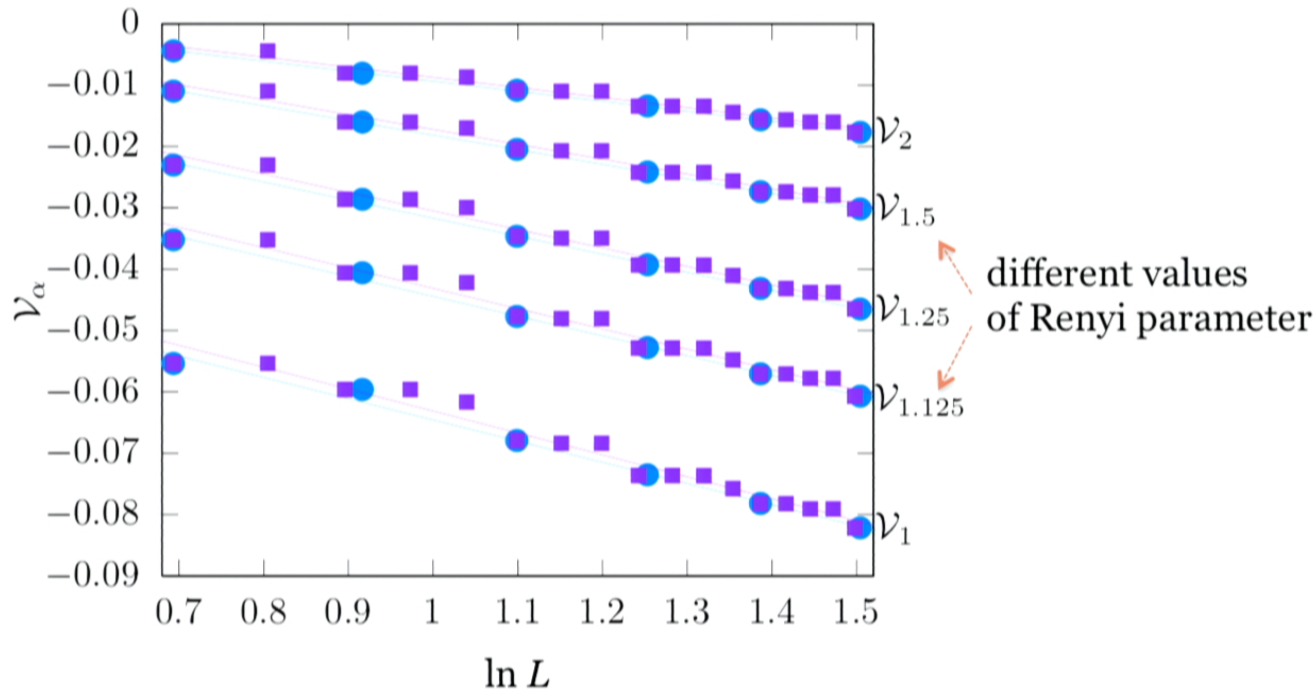
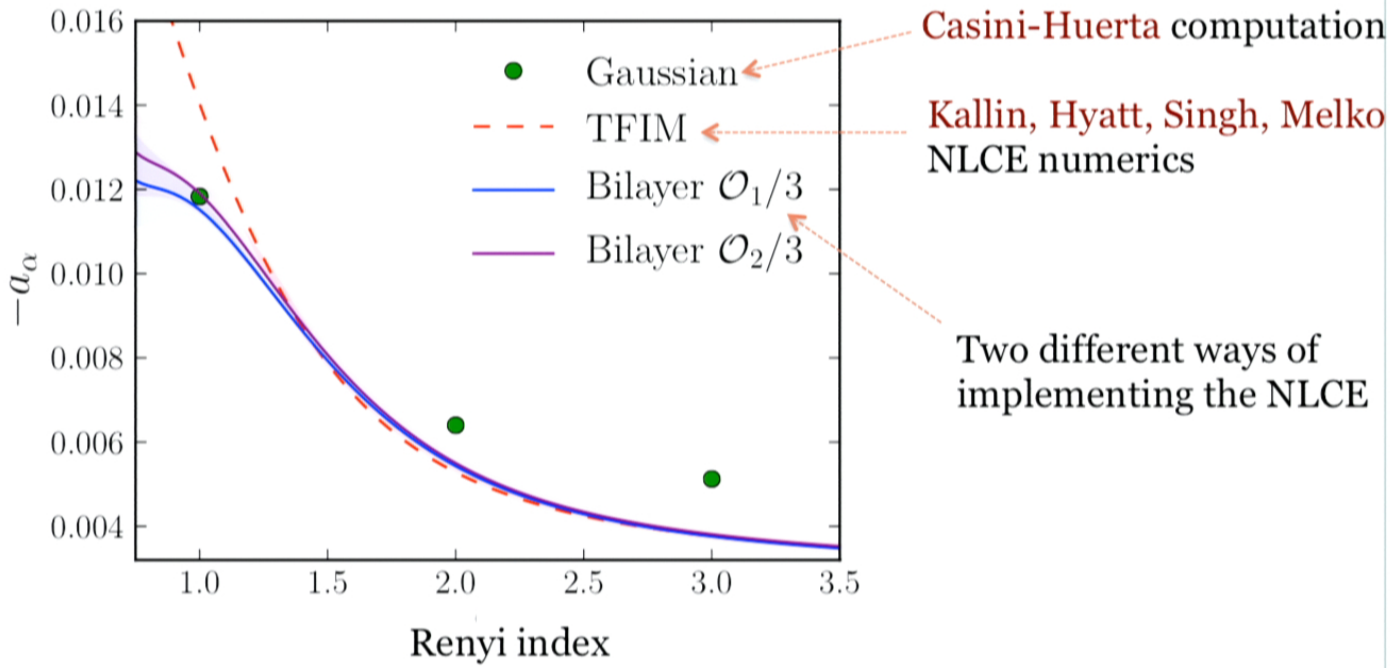
The corner contribution does scale with ln L as in the CQCPs:



Kallin, Stoudenmire, Fendley, Singh and Melko

Extracting the coefficient in front of the log:



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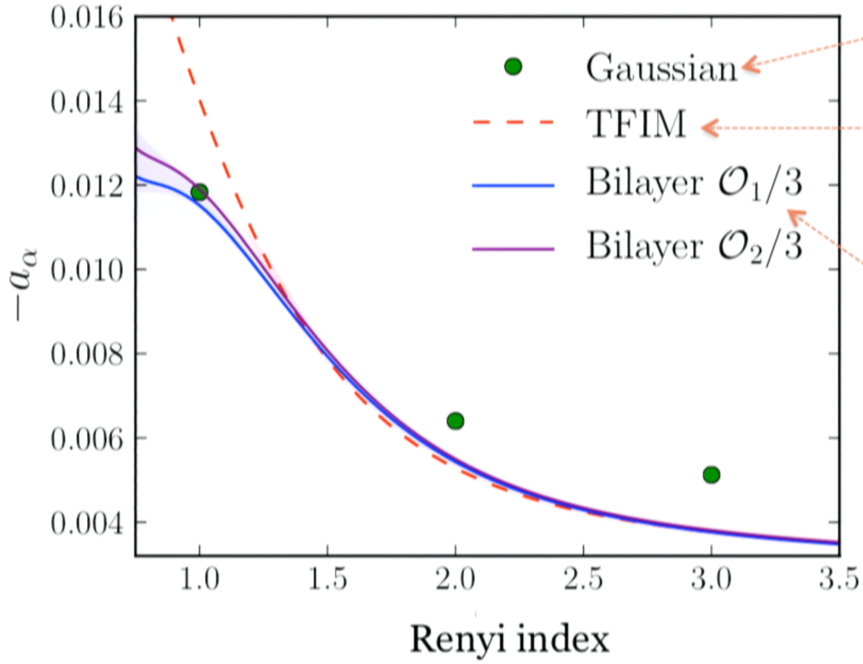

Casini-Huerta computation

Kallin, Hyatt, Singh, Melko NLCE numerics

Two different ways of implementing the NLCE

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This is a strong hint that the coefficient of a log provides a measure of the number of degrees of freedom, just like c.

The Landau-Ginzburg action for Ising is $O(N)\ \phi^4$ with $N=1$, for bilayer Heisenberg it is $N=3$.

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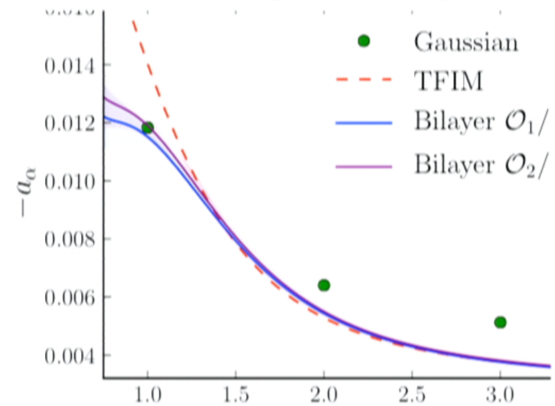
For interacting theories it won't be strictly additive, but these theories are "close" to free (the epsilon expansion works).



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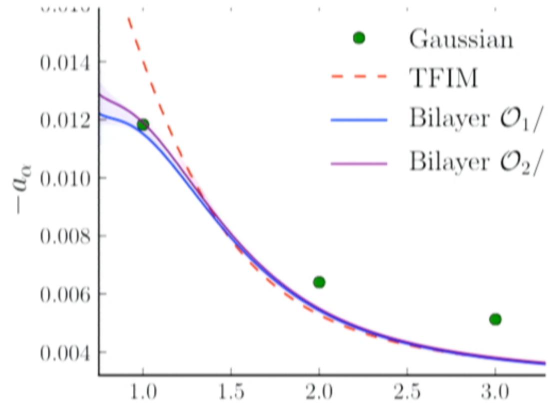

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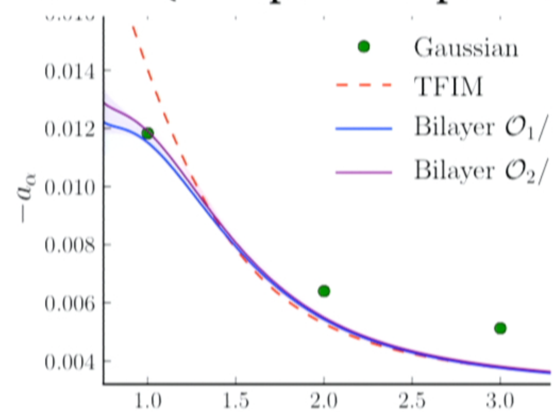

Some obvious further directions:

- Computing the GMI in the 2d XY model
  (low T phase not ordered)
- Computing GMI in higher d CFTs via Ryu-Takayanagi?
- Maybe corner term can be computed in large N?
- Nice to have corners for a model with gauge fields – maybe the J-Q model?
- Nice also to check a non-critical Goldtone phase, e.g. Heisenberg. NLCE though is more difficult