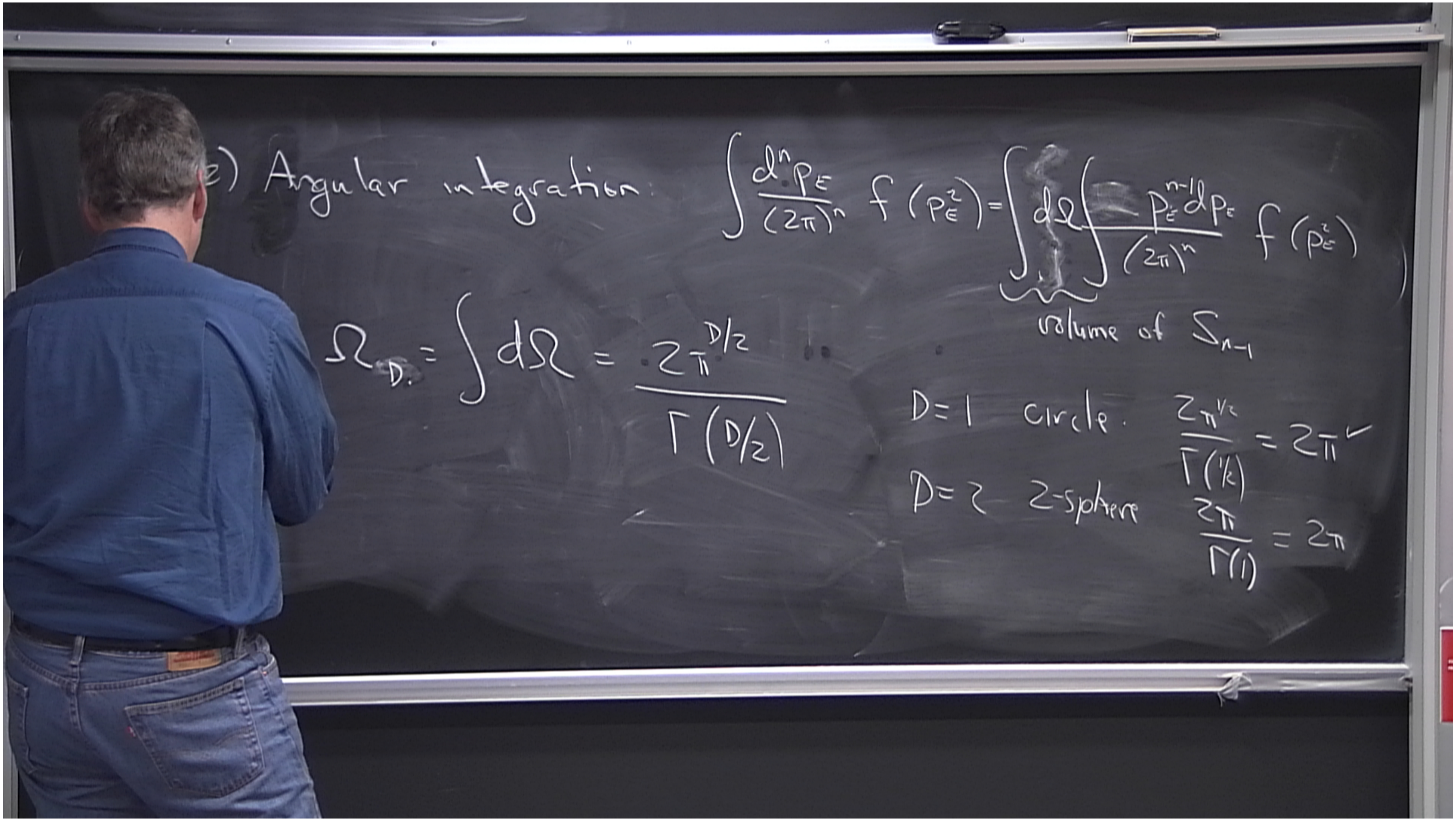
Title: Introduction to Effective Field Theories - Lecture 9

Date: Feb 07, 2014 09:00 AM

URL: http://pirsa.org/14020088

Abstract:


2) Angular integration: $\int \frac{d^n p_E}{(2\pi)^n} f(p_E^2) = \underbrace{\int d\Omega}_{\text{volume of } S_{n-1}} \int \frac{p_E^{n-1} dp_E}{(2\pi)^n} f(p_E^2)$
$\Omega_D = \int d\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$
$D=1$ circle. $\frac{2\pi^{1/2}}{\Gamma(1/2)} = 2\pi$ ✓
$D=2$ 2-sphere $\frac{2\pi}{\Gamma(1)} = 2\pi$

2) Angular integration: $\displaystyle\int \frac{d^n p_E}{(2\pi)^n} f(p_E^2) = \underbrace{\int d\Omega}_{\text{volume of } S_{n-1}} \int \frac{p_E^{n-1} dp_E}{(2\pi)^n} f(p_E^2)$

$$\underline{\text{volume of } S_{D-1}} = \Omega_D = \int d\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

$D=2$ circle $\dfrac{2\pi}{\Gamma(1)} = 2\pi$

$D=3$ 2-sphere $\dfrac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi$

$$\left[\int_{-\infty}^{\infty} e^{-x_1^2/2} dx_1\right]^n = \int d^n x \, e^{-x^2/2} = \Omega \int_0^\infty r^{n-1} dr \, e^{-r^2/2}$$

$$I = \frac{\Omega_{n-1}}{2(2\pi)^n} \int dp_E^2 \, \frac{(p_E^2)^{\frac{(n-2)}{2}+A}}{(p_E^2+m^2)^B}$$

$$p_E^2 = m^2 u$$

$$= \frac{\Omega_{n-1}}{2(2\pi)^n} \, m^{n+2A-2B} \int_0^\infty du \, \frac{u^{A+\frac{n}{2}-1}}{(1+u)^B}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int \frac{u^{x-1} \, du}{(u+1)^y} = \frac{\Gamma(x)\Gamma(y-x)}{\Gamma(y)}$$

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$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int_0^\infty \frac{u^{x-1}\,du}{(u+1)^y} = \frac{\Gamma(x)\Gamma(y-x)}{\Gamma(y)}$$

Upshot:

$$\int \frac{d^n p}{(2\pi)^n} \frac{(p^2)^A}{(p^2+m^2)^B} = \frac{i}{(2\pi)^n}\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]\frac{1}{2}\, m^{n+2(A-B)}\, \frac{\Gamma\left(A+\frac{n}{2}\right)\Gamma\left(B-A-\frac{n}{2}\right)}{\Gamma(\;B\;)}$$

Upshot: $$\int \frac{d^n p}{(2\pi)^n} \frac{(p^2)^A}{(p^2+m^2)^B} = \frac{i}{(2\pi)^n}\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]\frac{1}{2}\, m^{n+2(A-B)}\, \frac{\Gamma\left(A+\frac{n}{2}\right)\Gamma\left(\overset{\to 0}{B-A-\frac{n}{2}}\right)}{\Gamma(B)}$$

log divergence: $n+2(A-B)=0$ $\quad B-A-\frac{n}{2}=0$

$\Gamma(1+n)=n!$ $\quad \Gamma(1)=0!=1$

$$\Gamma(z)=\frac{\Gamma(z+1)}{z}$$

$(z+1)\Gamma(z+1)=\Gamma(z)$ $\;\rightarrow\; \Gamma(1+z)=\dfrac{\Gamma(z+2)}{z+1}$

quadratic div: $n+2(A-B)=2$

so $B-A-n/2=-1$

Upshot: $$\int \frac{d^n p}{(2\pi)^n} \frac{(p^2)^A}{(p^2+m^2)^B} = \frac{i}{(2\pi)^n}\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]\frac{1}{2}\, m^{n+2(A-B)}\, \frac{\Gamma\left(A+\frac{n}{2}\right)\Gamma\left(\overbrace{B-A-\frac{n}{2}}^{\to 0}\right)}{\Gamma(B)}$$

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log divergence: $n+2(A-B)=0$ $\quad B-A-\frac{n}{2}=0$

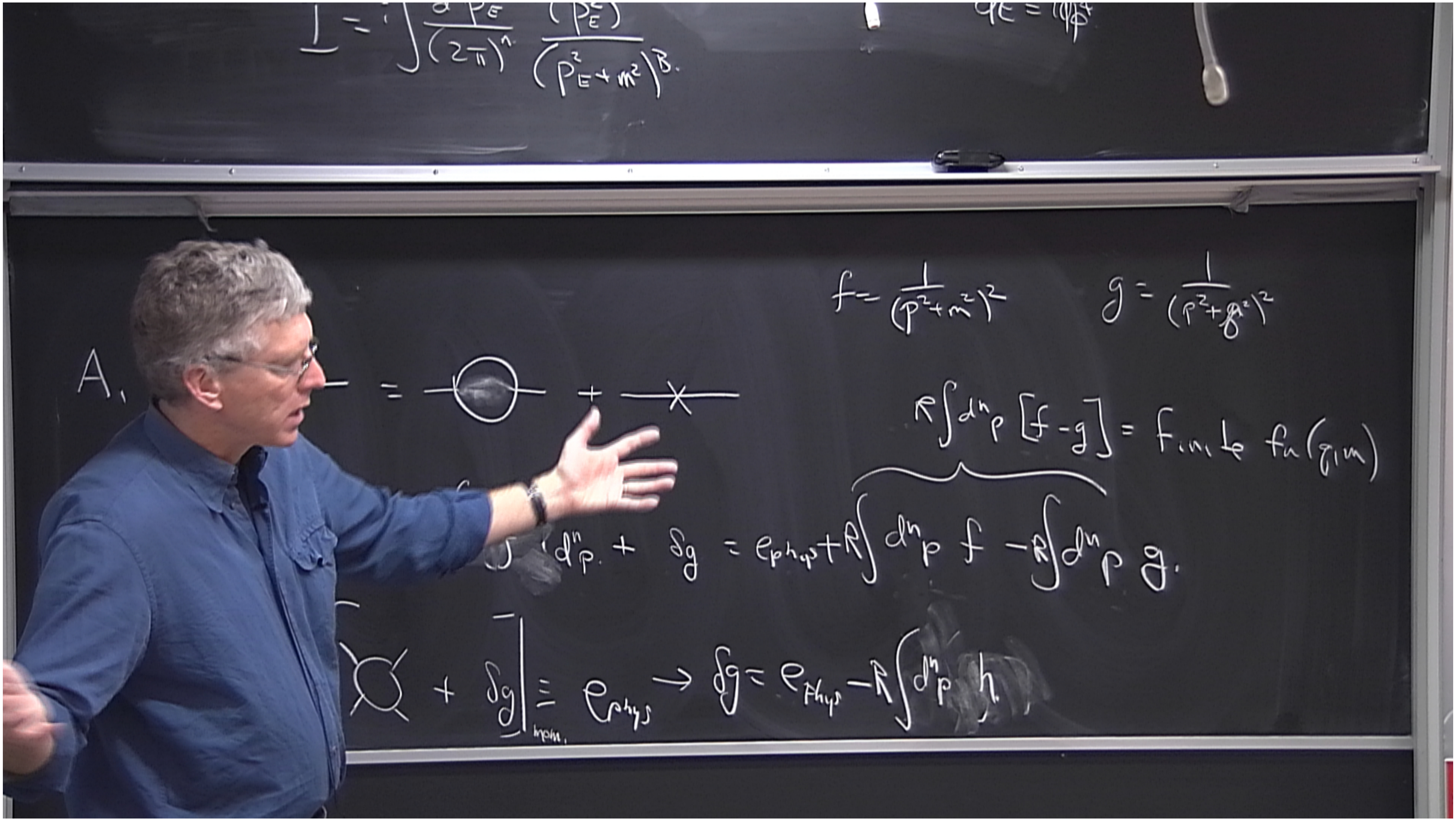
$$\Gamma(1+n)=n! \qquad \Gamma(1)=0!=1$$

$$\Gamma\left(\epsilon-\tfrac{1}{2}\right)=\frac{\Gamma\left(\epsilon+\frac{1}{2}\right)}{\epsilon-\frac{1}{2}} \qquad \Gamma(z)=\frac{\Gamma(z+1)}{z}$$

$$(z+1)\Gamma(z+1)=\Gamma(z) \rightarrow \Gamma(1+z)=\frac{\Gamma(z+2)}{z+1}$$

quadratic div: $n+2(A-B)=2$

so $B-A-n/2=-1$

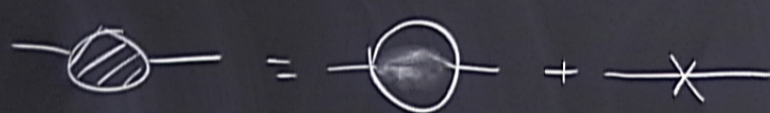


f = 1/(p²+m²)²
g = 1/(p²+μ²)²
A,
R∫dⁿp [f − g] = finite fn(μ, m)
dⁿp. + δg = e_phys + R∫dⁿp f − R∫dⁿp g.
+ δg = e_phys → δg = e_phys − R∫dⁿp h.

$$I = i\int \frac{d^n p_E}{(2\pi)^n} \frac{(p_E^2)}{(p_E^2+m^2)^\beta}$$

$$f = \frac{1}{(p^2+m^2)^2} \qquad g = \frac{1}{(p^2+\mu^2)^2}$$

A_1



$$R\int d^n p\, [f - g] = \text{finite fn}(\mu, m)$$

$$= R\int f\, d^n p + \delta g = e_{phys} + \overbrace{R\int d^n p\, f - R\int d^n p\, g}$$

A_2

$$\Big[\ \ + \delta g\Big]_{\text{norm.}} = e_{phys} \rightarrow \delta g = e_{phys} - R\int d^n p\, h$$

Power-counting: $\mathcal{L} = f^4 \sum_{i = \text{all interactions}} \mathcal{O}\left(\frac{\phi}{v}, \frac{\partial\phi}{Mv}\right) = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\mathcal{L}_0 = \frac{f^4}{v^2}\left[\frac{\partial_\mu\phi\,\partial^\mu\phi}{M^2} + \frac{m^2\phi^2}{M^2}\right]$$

E external lines

$A_E(q)$

$q \sim$ external energies

Power-counting: $\mathcal{L} = f^4 \sum_{i = \text{all interactions}} \mathcal{O}\left(\frac{\phi}{v}, \frac{\partial\phi}{Mv}\right) = \mathcal{L}_0 + \mathcal{L}_{int}$

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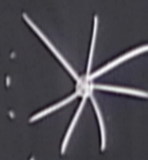
$\downarrow$ E external lines

$$A_E(q) = \sum_{D=0}^{D} A_{DE}\, q^D$$

$\uparrow$ $q \sim$ external energies

/ external energies

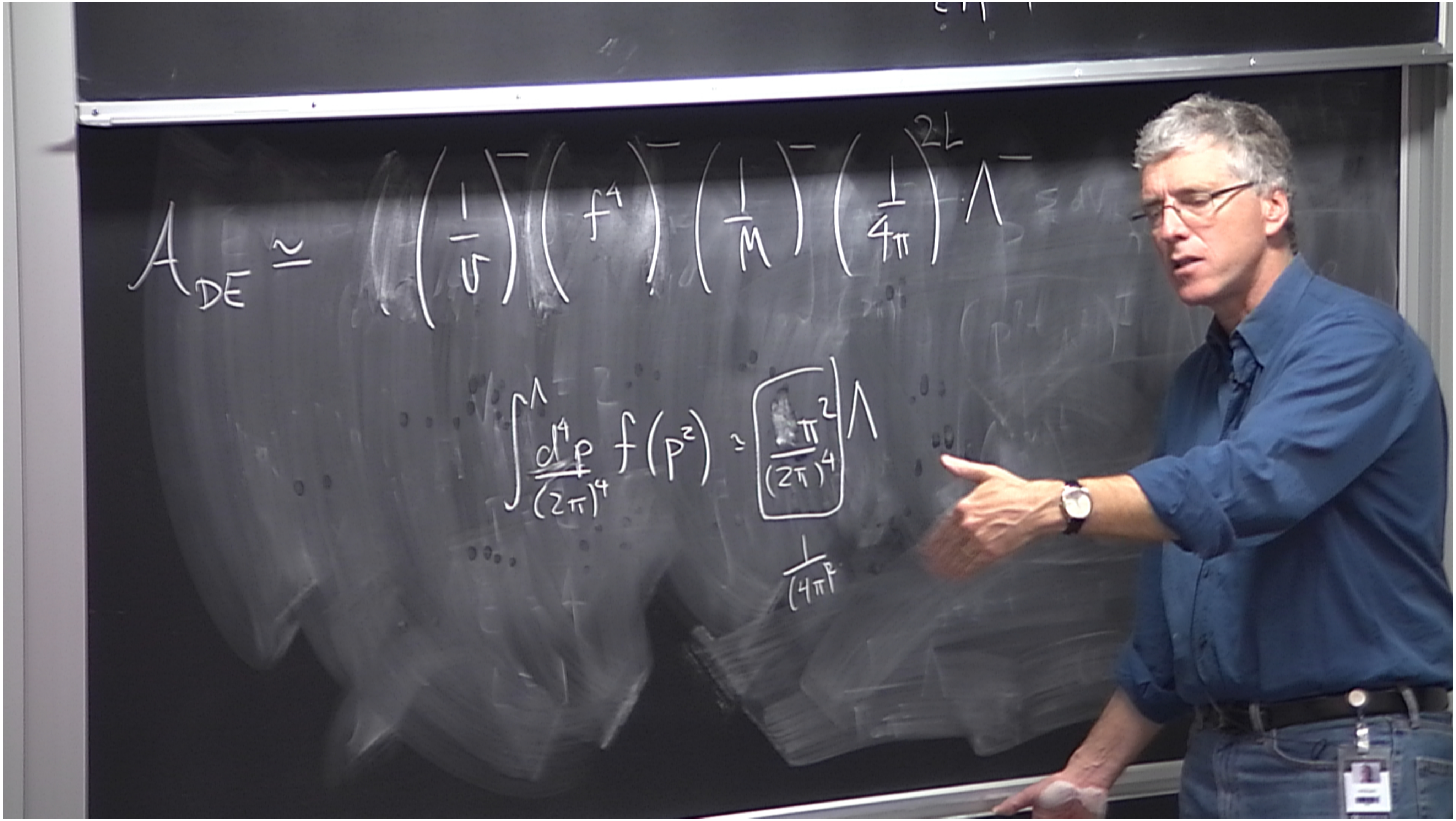
Consider a graph with

V_{dk} vertices with d derivatives and k fields : 

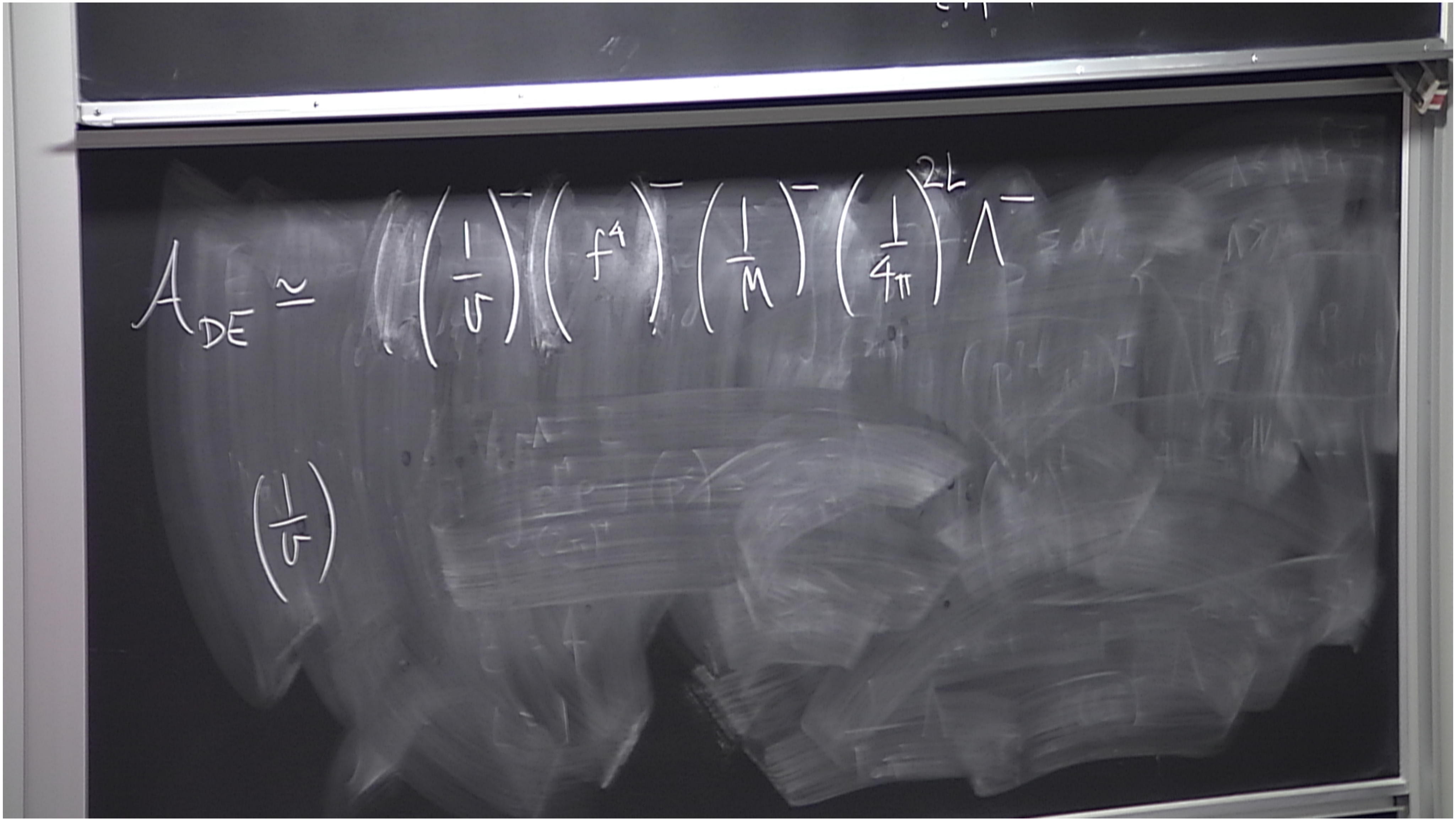
k lines

I internal lines : $\frac{v^2 M^2}{f^4}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2}$ —— $\sim \frac{f^4 p^d}{M^d f^k}\delta(\Sigma p)(2\pi)^4$

total # of integrations $= I - \sum_{kd} V_{dk} + 1 := L$

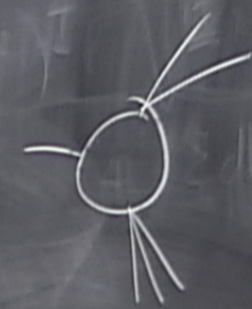


A_DE ≃ (1/v)(f^4)(1/M)(1/4π)^2L Λ
∫^Λ d^4p/(2π)^4 f(p^2) = π^2/(2π)^4 Λ
1/(4π)^2


$A_{DE} \simeq \left(\frac{1}{v}\right)^{-} \left(f^4\right)^{-} \left(\frac{1}{M}\right)^{-} \left(\frac{1}{4\pi}\right)^{2L} \Lambda^{-}$
$\left(\frac{1}{v}\right)$

$$A_{DE} \simeq \left(\frac{1}{v}\right)^{E}\left(f^{4}\right)^{-}\left(\frac{1}{M}\right)^{-}\left(\frac{1}{4\pi}\right)^{2L}\Lambda^{-}$$

$$\left(\frac{1}{v}\right)^{-2I+\sum_{kd} k V_{dk}} = \left(\frac{1}{v}\right)^{E}$$

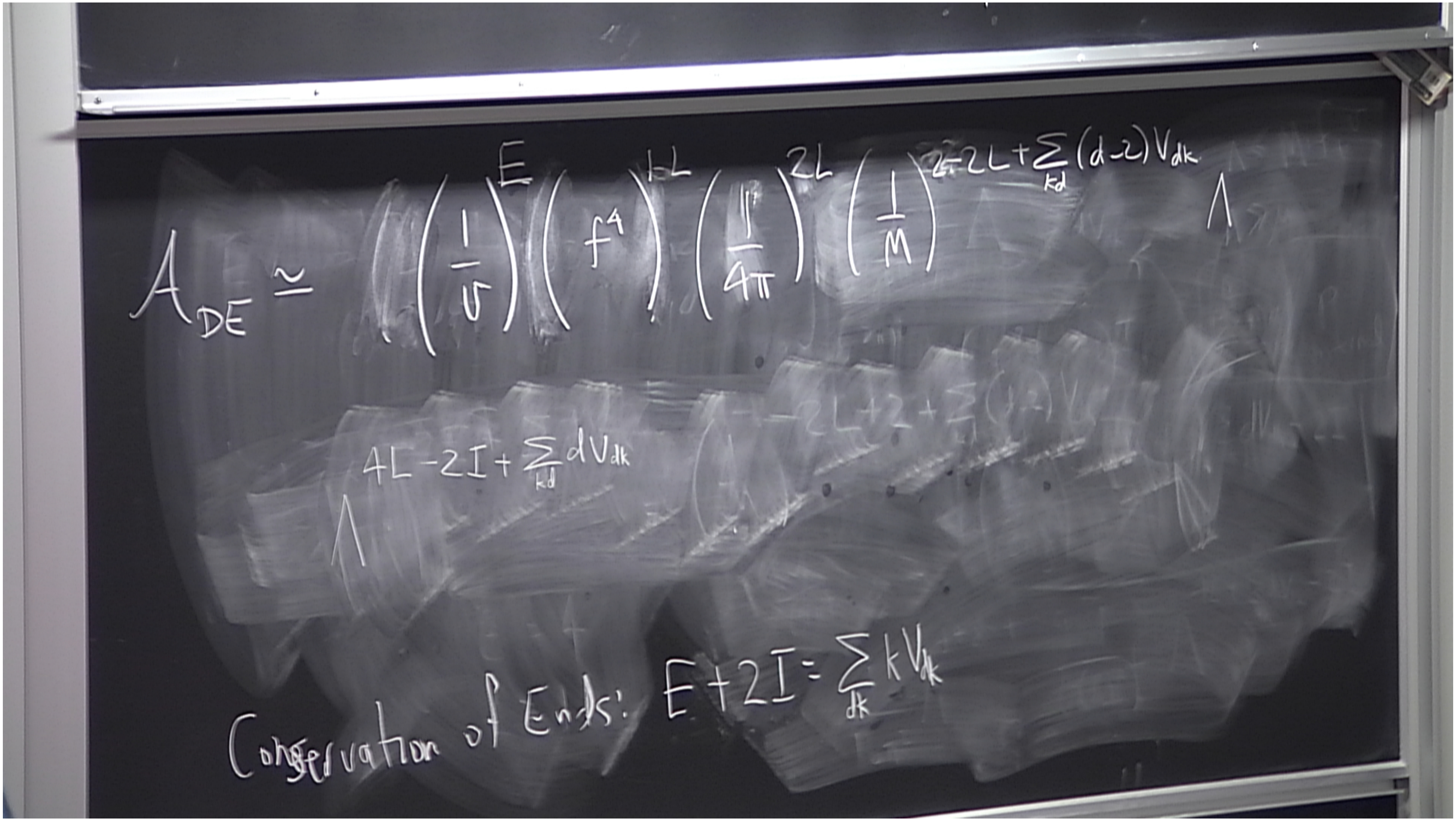


Conservation of Ends: $E+2I=\sum_{dk} k V_{dk}$

$$A_{DE} \simeq \left(\frac{1}{v}\right)^{E} \left(f^4\right)^{1-L} \left(\frac{1}{4\pi}\right)^{2L} \left(\frac{1}{M}\right)^{2E-2L+\sum_{kd}(d-2)V_{dk}} \Lambda$$

$$\Lambda^{4L}$$

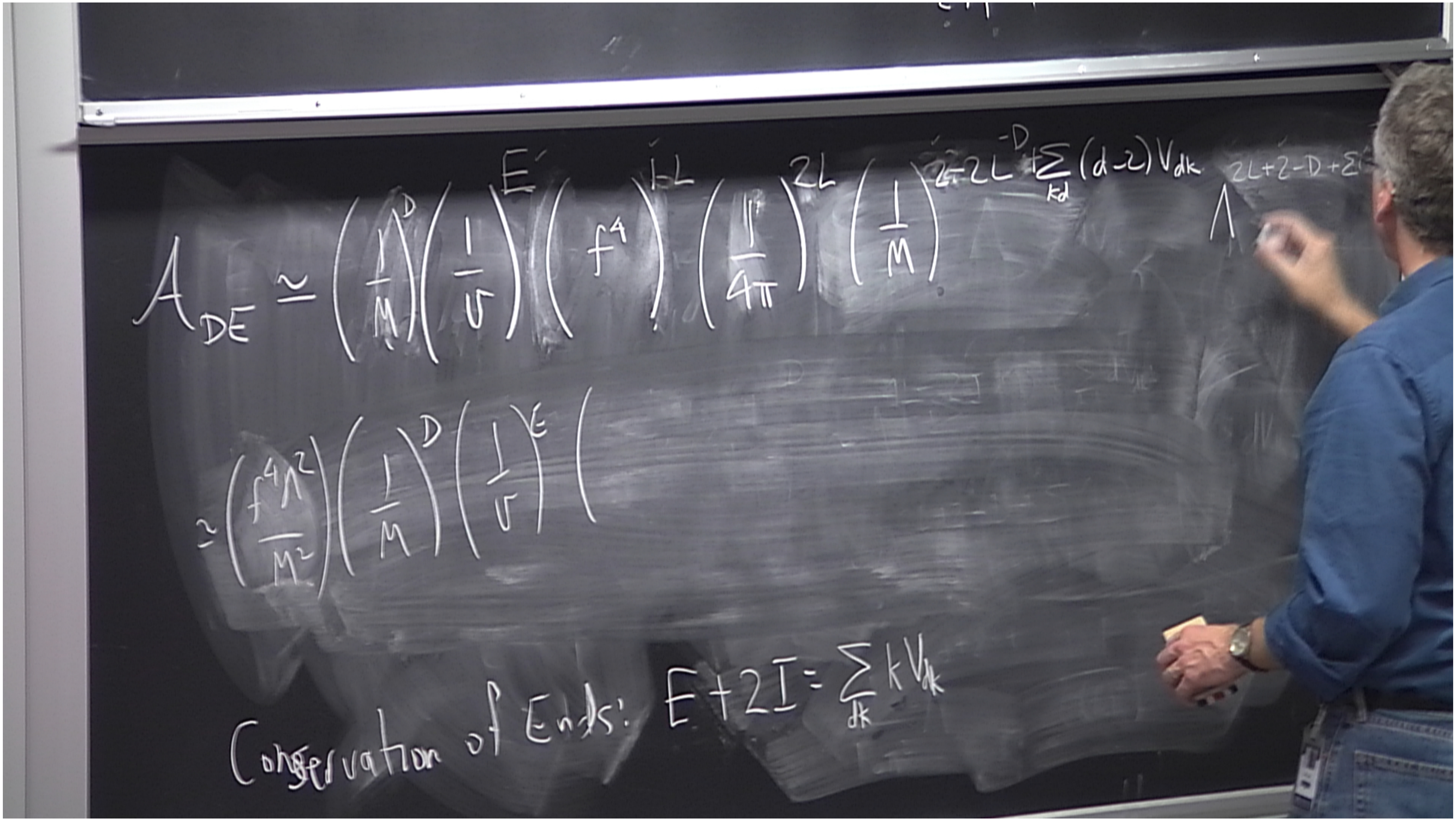
Conservation of Ends: $E + 2I = \sum_{dk} k V_{dk}$

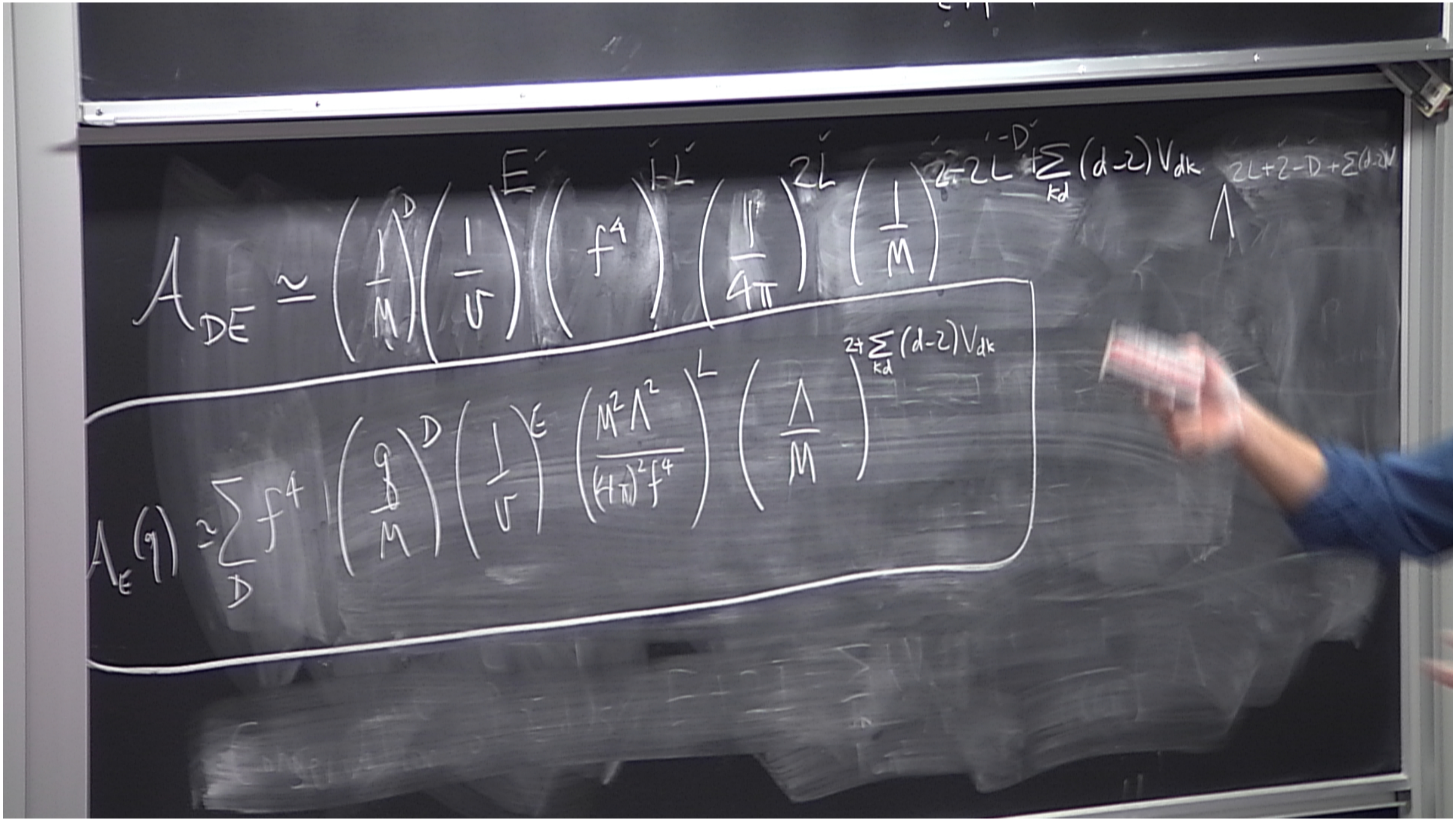

A_DE ≃ (1/v)^E (f^4) (1/4π)^2L (1/M)^-2L+Σ_kd (d-2)V_dk Λ
4L - 2I + Σ_kd d V_dk
Λ
Conservation of Ends: E + 2I = Σ_dk k V_dk

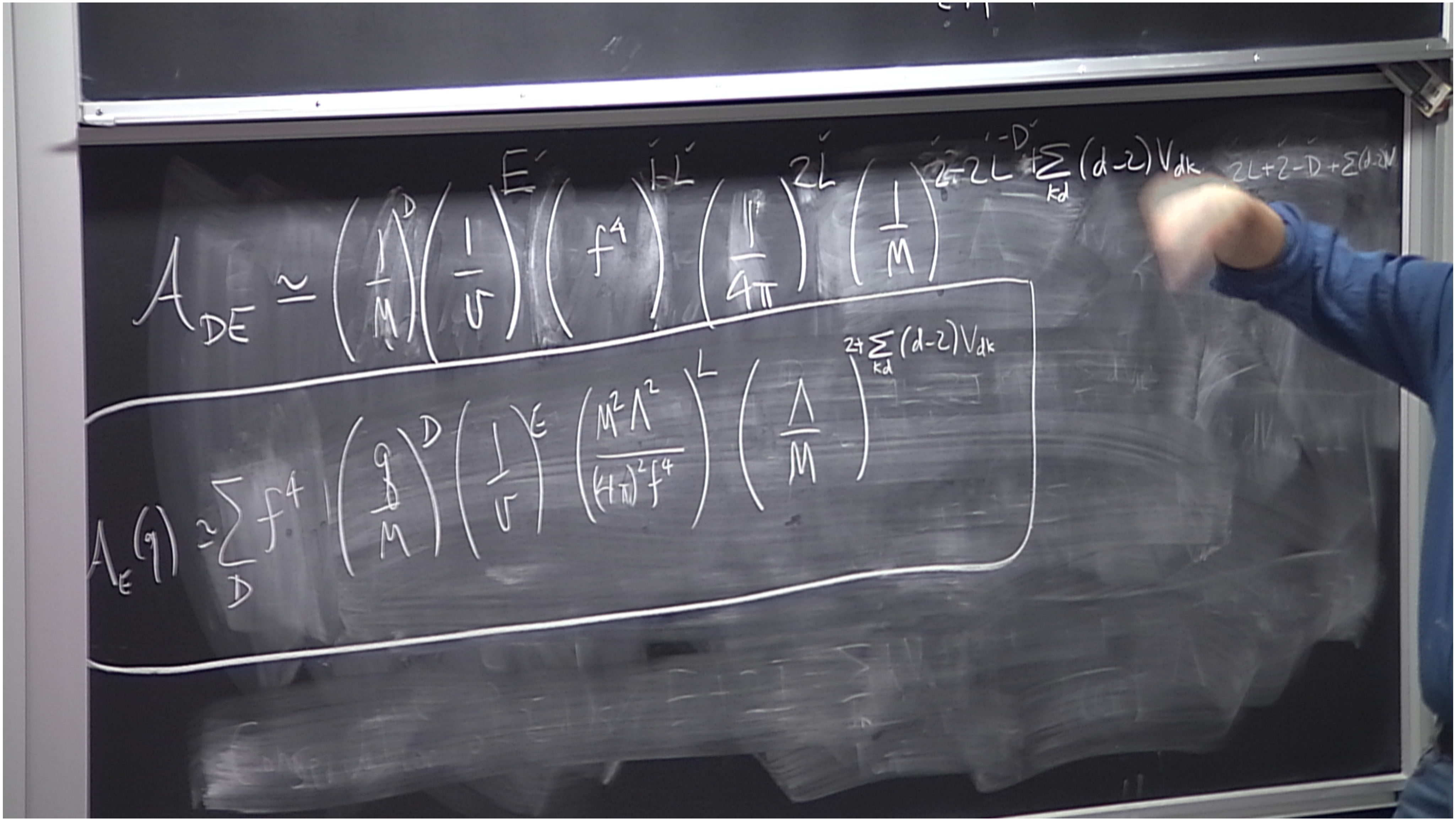
$$A_{DE} \simeq \left(\frac{1}{v}\right)^{E} \left(f^4\right)^{1-L} \left(\frac{1}{4\pi}\right)^{2L} \left(\frac{1}{M}\right)^{2-2L+\sum_{kd}(d-2)V_{dk}} \quad \Lambda$$

$$\int (d^4p)^L \left(\frac{1}{p^2}\right)^I \prod_{kd} p^{dV_{kd}}$$

Conservation of Ends: $E + 2I = \sum_{dk} k V_{dk}$


$A_{DE} \simeq \left(\frac{1}{M}\right)^{D} \left(\frac{1}{v}\right)^{E} \left(f^4\right)^{1-L} \left(\frac{1}{4\pi}\right)^{2L} \left(\frac{1}{M}\right)^{2-2L-D+\sum_{kd}(d-2)V_{dk}} \Lambda^{2L+2-D+\Sigma}$
$\simeq \left(\frac{f^4\Lambda^2}{M^2}\right) \left(\frac{1}{M}\right)^{D} \left(\frac{1}{v}\right)^{E} ($
Conservation of Ends: $E + 2I = \sum_{dk} k V_{dk}$

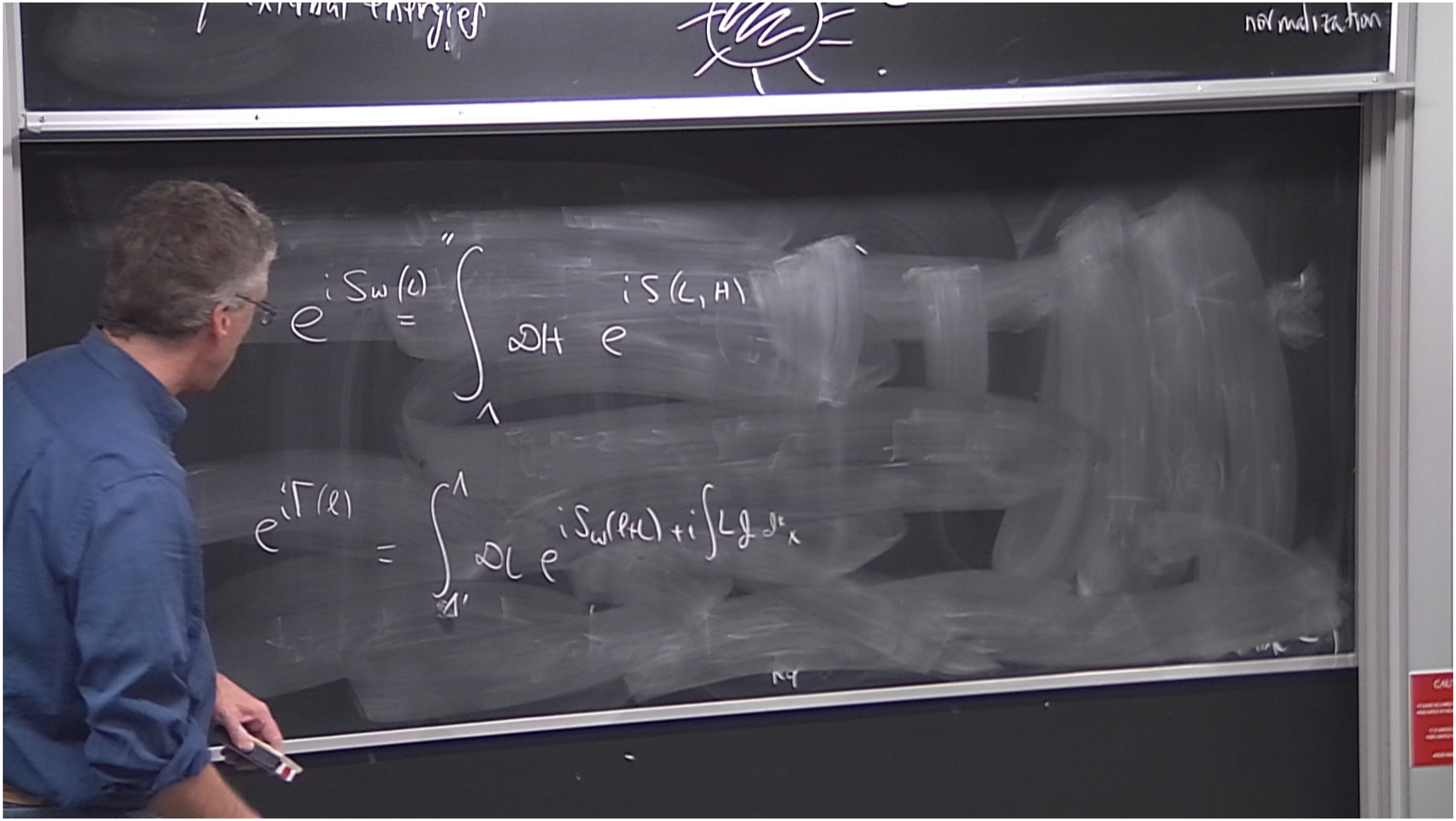

$\mathcal{A}_{DE} \simeq \left(\frac{1}{M}\right)^{D}\left(\frac{1}{v}\right)^{E}\left(f^4\right)^{1-L}\left(\frac{1}{4\pi}\right)^{2L}\left(\frac{1}{M}\right)^{2-2L-D+\sum_{kd}(d-2)V_{dk}} \Lambda^{2L+2-D+\sum(d-2)V}$
$\mathcal{A}_E(q) \simeq \sum_D f^4 \left(\frac{q}{M}\right)^{D}\left(\frac{1}{v}\right)^{E}\left(\frac{M^2\Lambda^2}{(4\pi)^2 f^4}\right)^{L}\left(\frac{\Lambda}{M}\right)^{2+\sum_{kd}(d-2)V_{dk}}$



$\mathcal{A}_{DE} \simeq \left(\frac{1}{M}\right)^{D} \left(\frac{1}{v}\right)^{E} \left(f^4\right)^{1-L} \left(\frac{1}{4\pi}\right)^{2L} \left(\frac{1}{M}\right)^{2-2L+\sum_{kd}(d-2)V_{dk}}$
$2L+2-D+\Sigma(d-2)V$
$\mathcal{A}_E(q) \simeq \sum_D f^4 \left(\frac{q}{M}\right)^D \left(\frac{1}{v}\right)^E \left(\frac{M^2\Lambda^2}{(4\pi)^2 f^4}\right)^L \left(\frac{\Lambda}{M}\right)^{2+\sum_{kd}(d-2)V_{dk}}$

$$V(\phi) = f^4 \sum_n c_n \left(\frac{\phi}{v}\right)^n$$

$$\text{eg } n=2 \longleftrightarrow c_2 \frac{f^4}{v^2} \phi^2$$


normalization
$e^{iS_W(L)} = \int^{\Lambda} \mathcal{D}H \, e^{iS(L,H)}$
$e^{i\Gamma(\ell)} = \int_{\Lambda'}^{\Lambda} \mathcal{D}L \, e^{iS_W(\ell+L) + i\int L J \, d^4x}$

... ternal energies

normalization

$$e^{iS_W(L,\Lambda)} \overset{\text{``}}{=} \int^{\Lambda} \mathcal{D}H \; e^{iS(L,H)}$$

$$e^{iS_W(\ell,\Lambda')} = \int_{\Lambda'}^{\Lambda} \mathcal{D}L \; e^{iS_W(\ell+L) + i\int L \mathcal{J}\, d^4x}$$

M

Λ

Λ'

normalization

$$e^{iS_W(L,\Lambda)} = \int^{\Lambda} \mathcal{D}H \; e^{iS(L,H)}$$

$$e^{iS_W(\ell,\Lambda')} = \int_{\Lambda'}^{\Lambda} \mathcal{D}L \; e^{iS_W(\ell+L) + i\int L J \, d^4x}$$

M

Λ

Λ'

$\frac{1}{M^4}(\partial\xi\,\partial\xi)^2$

$\frac{1}{\Lambda^4}(\partial\xi\,\partial\xi)^2$