Title: Towards a 1/c Expansion in 2d Conformal Field Theory

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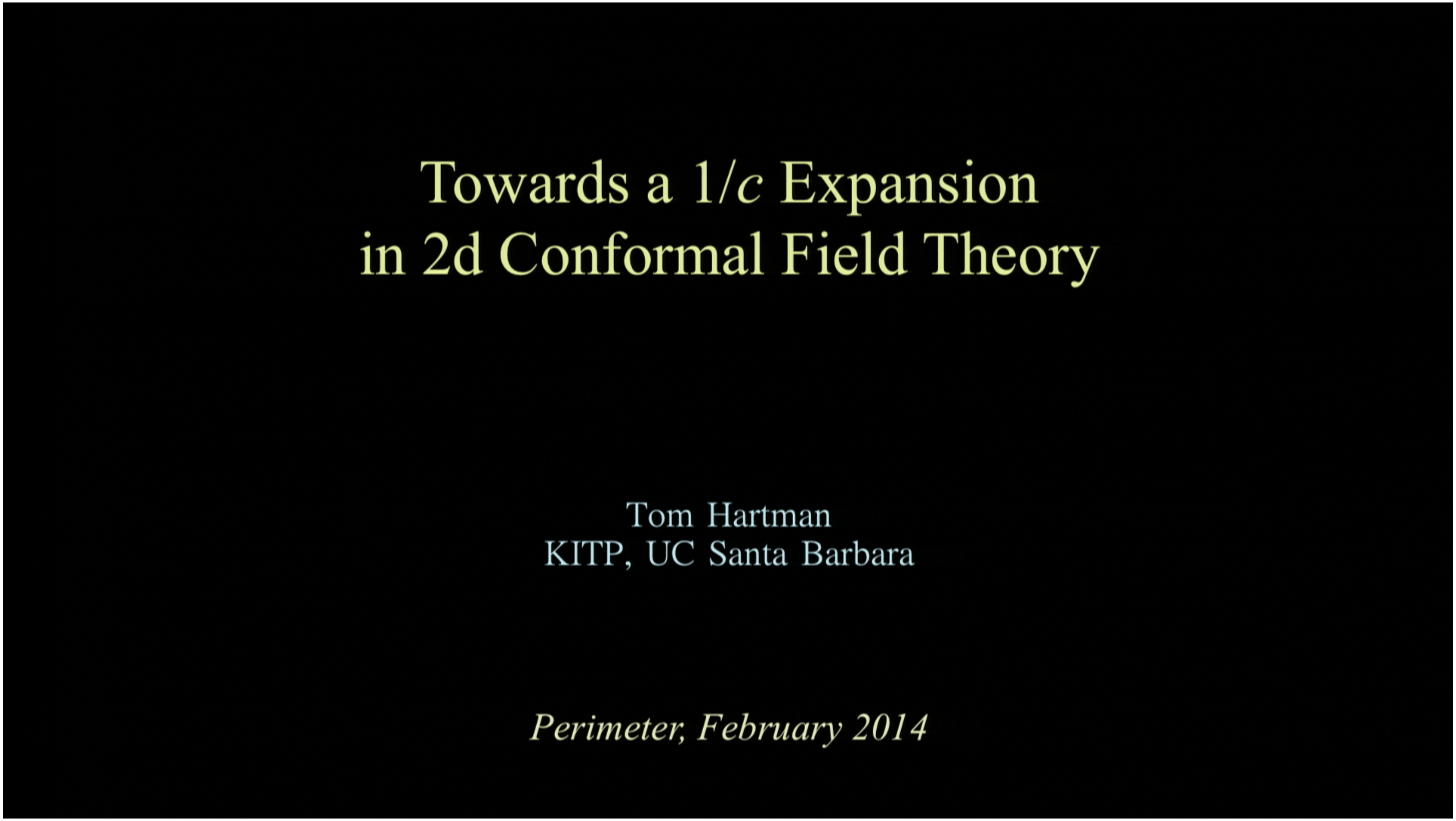
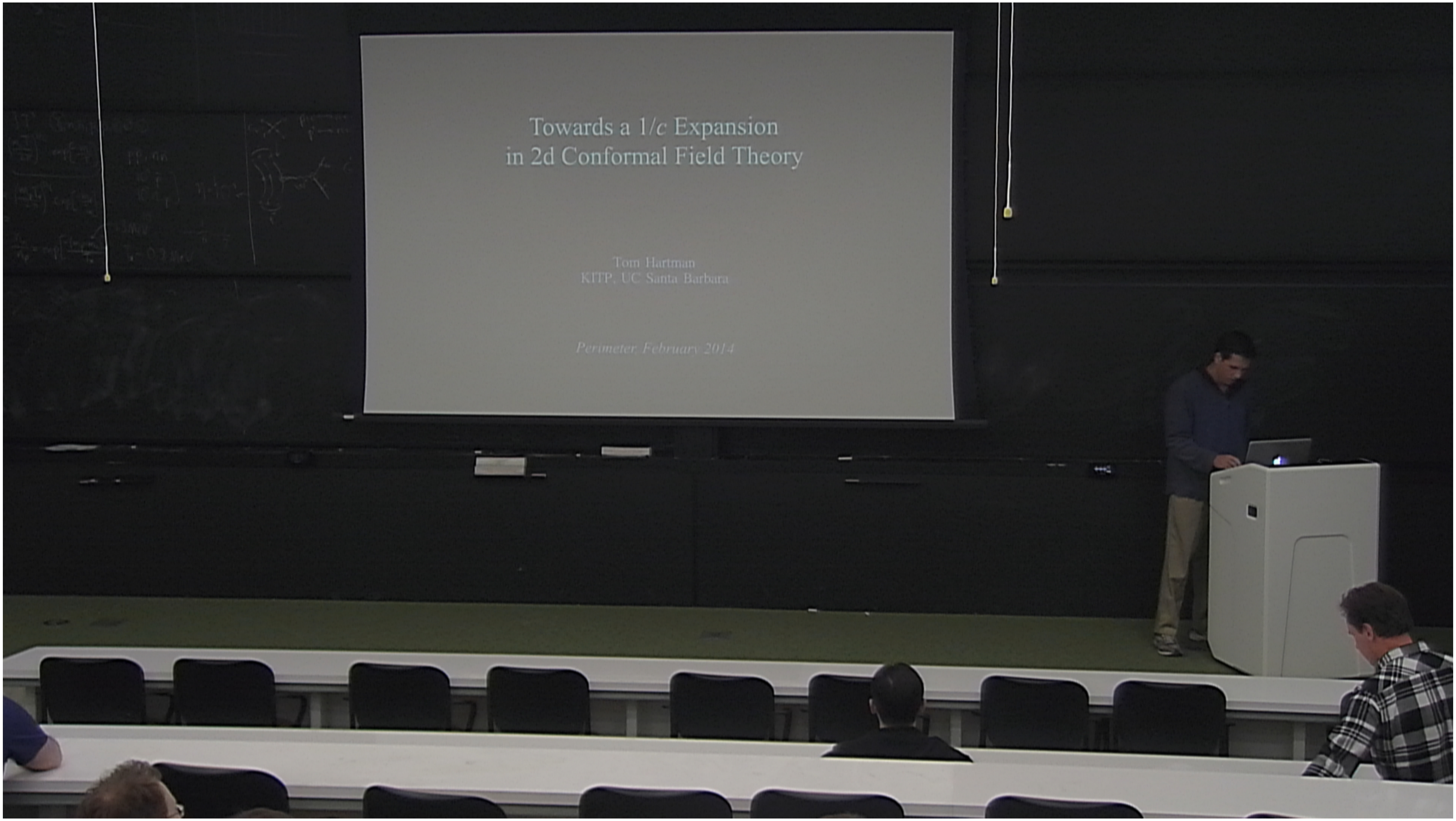
Abstract: I will describe progress in deriving 3d gravity directly from 2d conformal field theory at large central charge 'c'. In a large class of CFTs, using general arguments like modular invariance, crossing symmetry, and the OPE expansion, the spectrum, the entanglement entropy, and certain partition functions can be computed to leading order in a 1/c expansion. The results agree with universal features of 3d gravity required by black hole thermodynamics and the Ryu-Takayanagi formula; furthermore, the relevant 3d geometries appear automatically from CFT calculations in this regime.



Towards a $1/c$ Expansion in 2d Conformal Field Theory

Tom Hartman
KITP, UC Santa Barbara

Perimeter, February 2014



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2d CFTs have Virasoro symmetry, with a parameter

$$\text{central charge} = c \sim \#\ \text{d.o.f.}$$

Certain CFTs with large c are dual to gravity in AdS$_3$, and

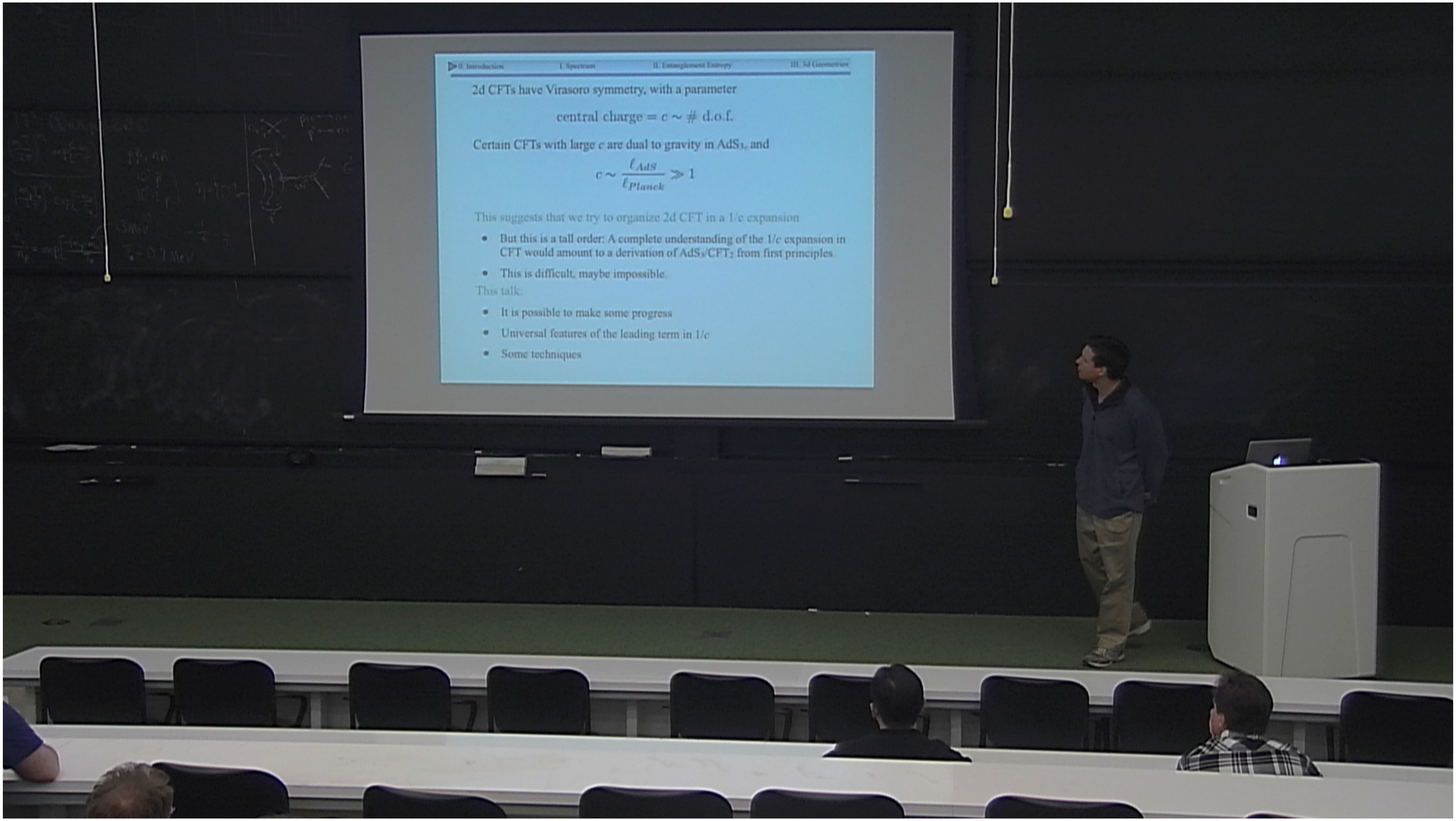
$$c \sim \frac{\ell_{AdS}}{\ell_{Planck}} \gg 1$$

This suggests that we try to organize 2d CFT in a 1/c expansion

- But this is a tall order: A complete understanding of the $1/c$ expansion in CFT would amount to a derivation of AdS$_3$/CFT$_2$ from first principles.
- This is difficult, maybe impossible.

This talk:

- It is possible to make some progress
- Universal features of the leading term in $1/c$
- Some techniques



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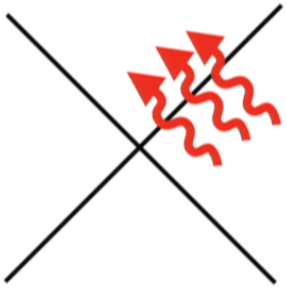
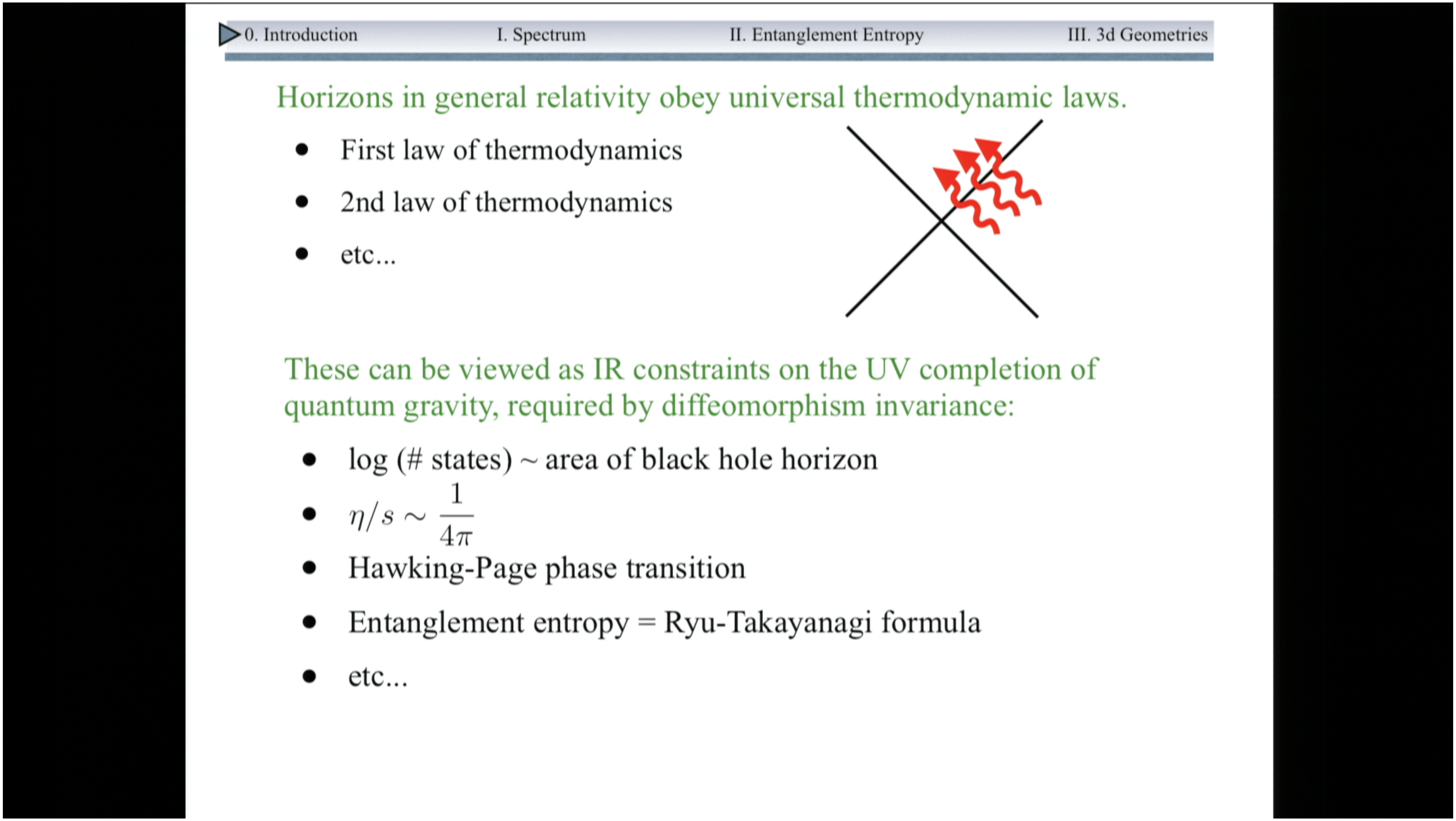
Horizons in general relativity obey universal thermodynamic laws.



- First law of thermodynamics
- 2nd law of thermodynamics
- etc...



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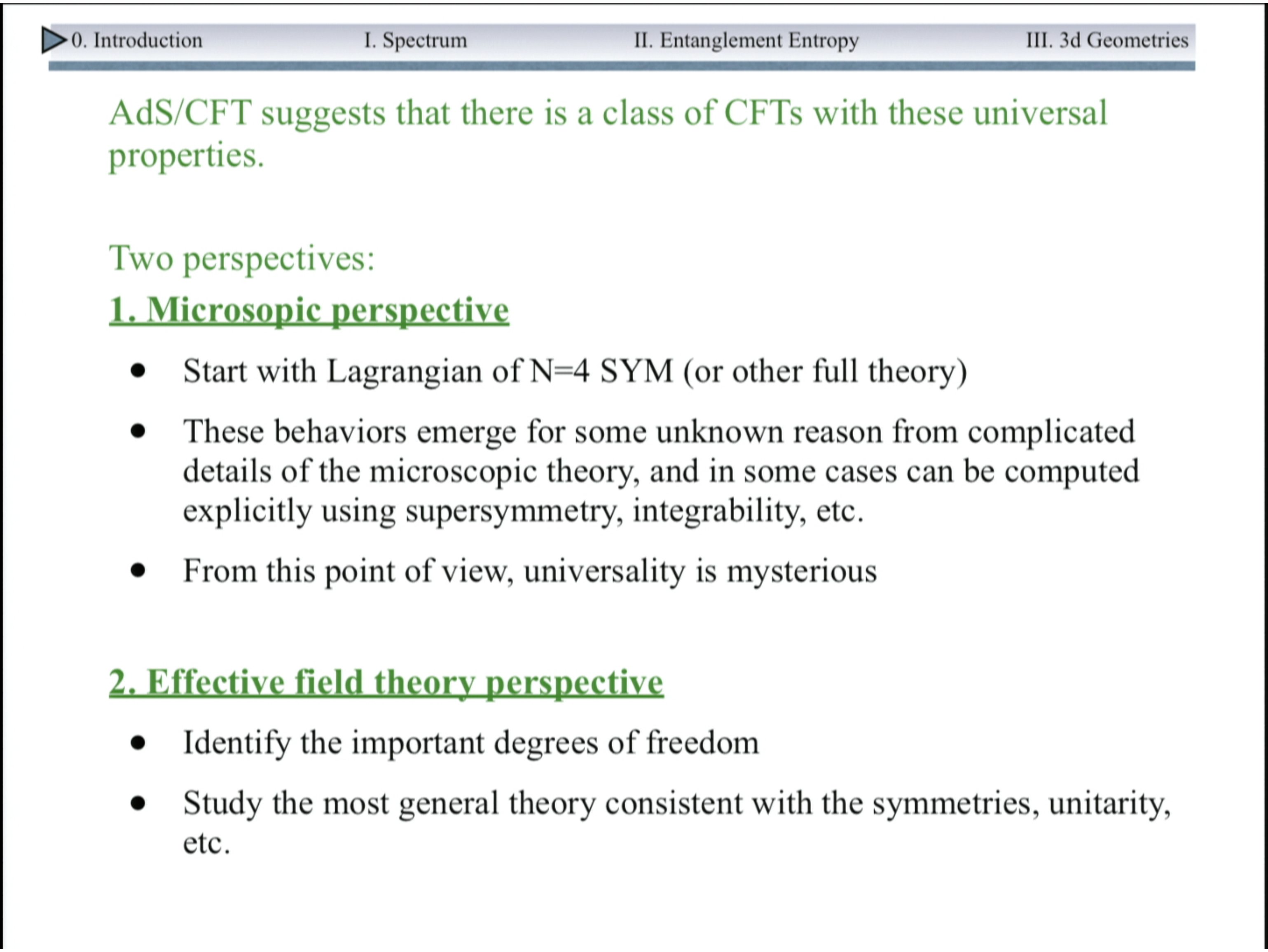
- etc...

These can be viewed as IR constraints on the UV completion of quantum gravity, required by diffeomorphism invariance:

- log (# states) ~ area of black hole horizon
- $\eta/s \sim \dfrac{1}{4\pi}$
- Hawking-Page phase transition
- Entanglement entropy = Ryu-Takayanagi formula
- etc...



AdS/CFT suggests that there is a class of CFTs with these universal properties.

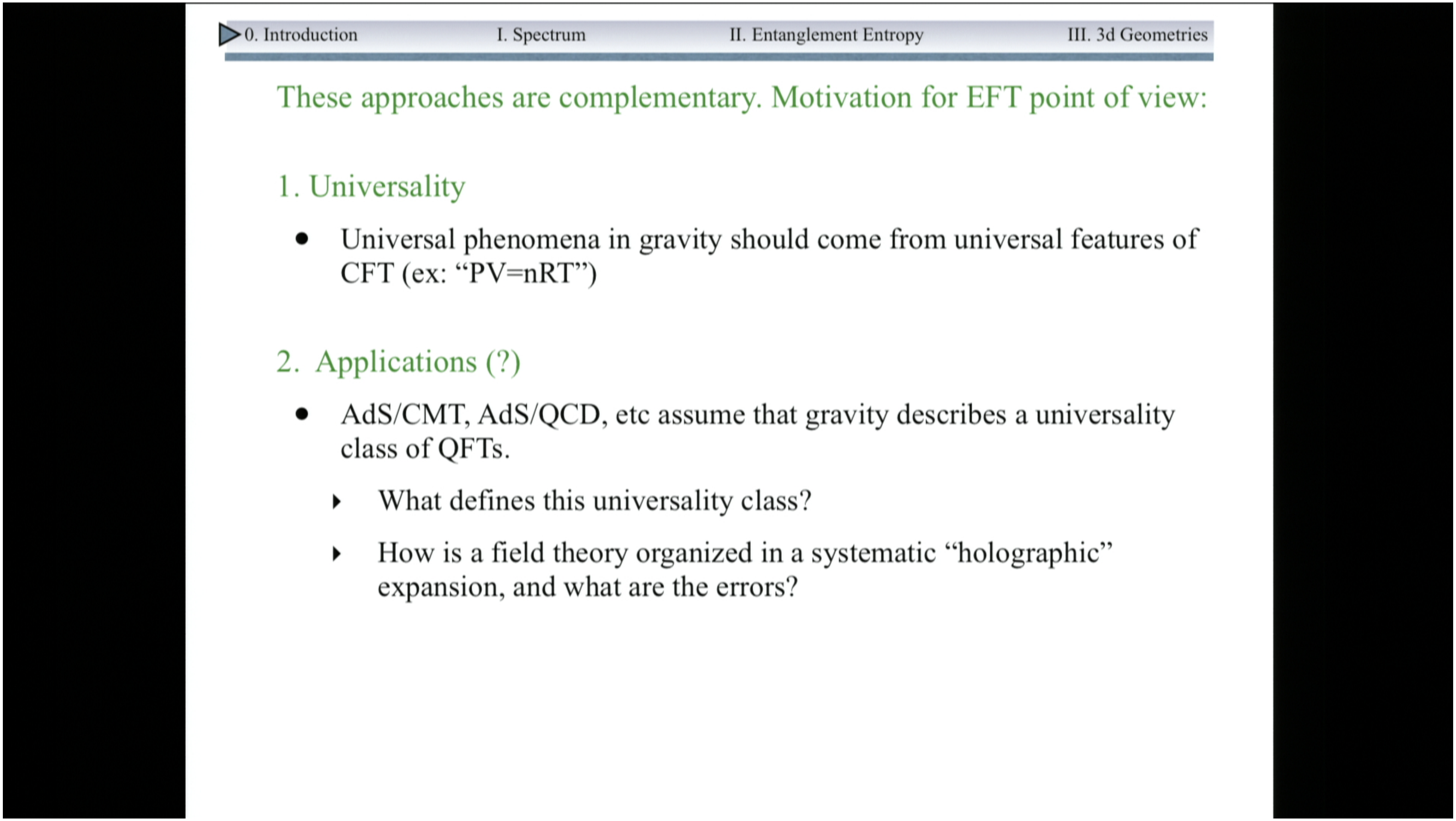
Two perspectives:

1. Microsopic perspective

- Start with Lagrangian of N=4 SYM (or other full theory)
- These behaviors emerge for some unknown reason from complicated details of the microscopic theory, and in some cases can be computed explicitly using supersymmetry, integrability, etc.
- From this point of view, universality is mysterious

2. Effective field theory perspective

- Identify the important degrees of freedom
- Study the most general theory consistent with the symmetries, unitarity, etc.



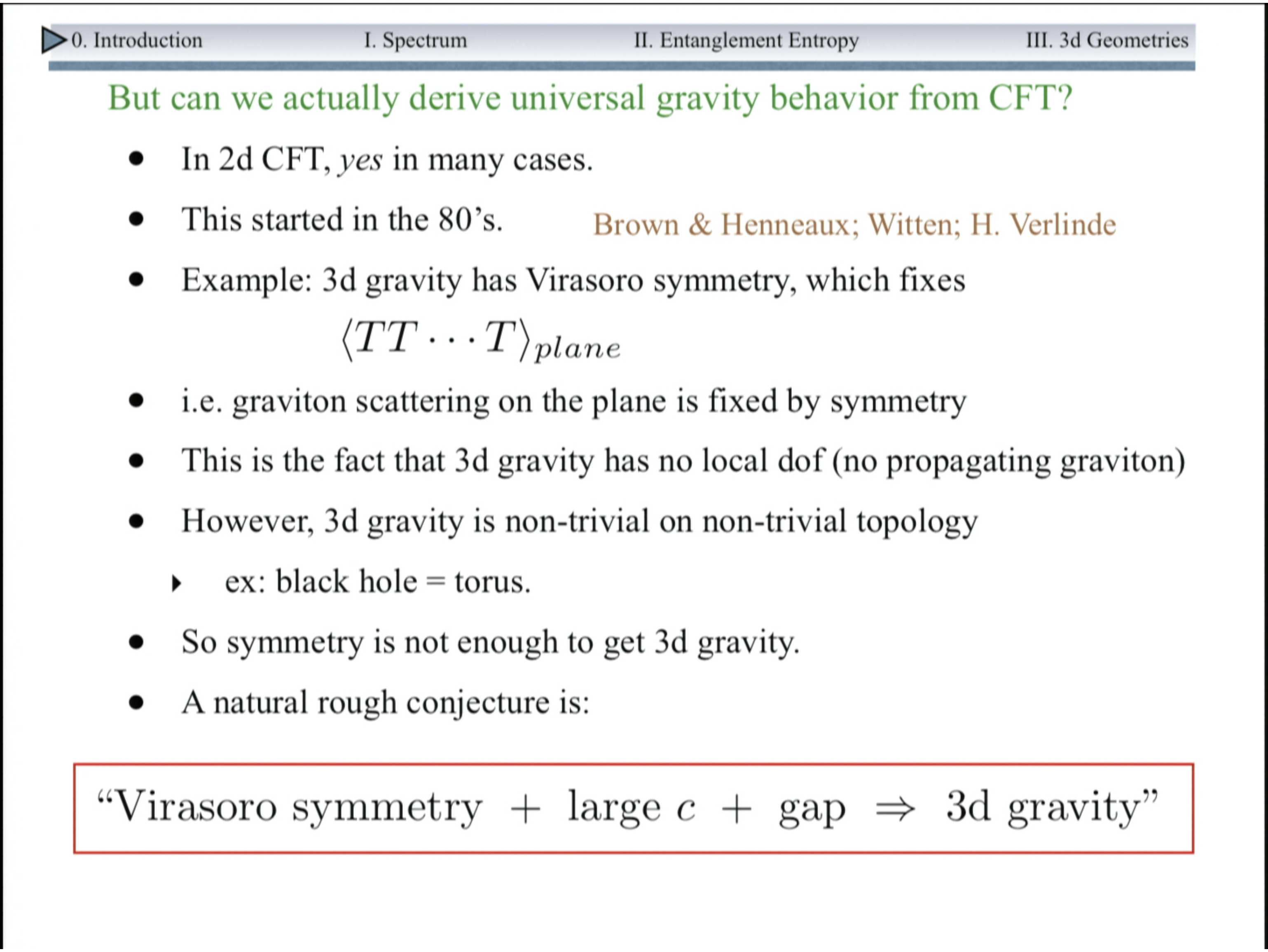
These approaches are complementary. Motivation for EFT point of view:

1. Universality

- Universal phenomena in gravity should come from universal features of CFT (ex: “PV=nRT”)

2. Applications (?)

- AdS/CMT, AdS/QCD, etc assume that gravity describes a universality class of QFTs.
    - What defines this universality class?
    - How is a field theory organized in a systematic “holographic” expansion, and what are the errors?



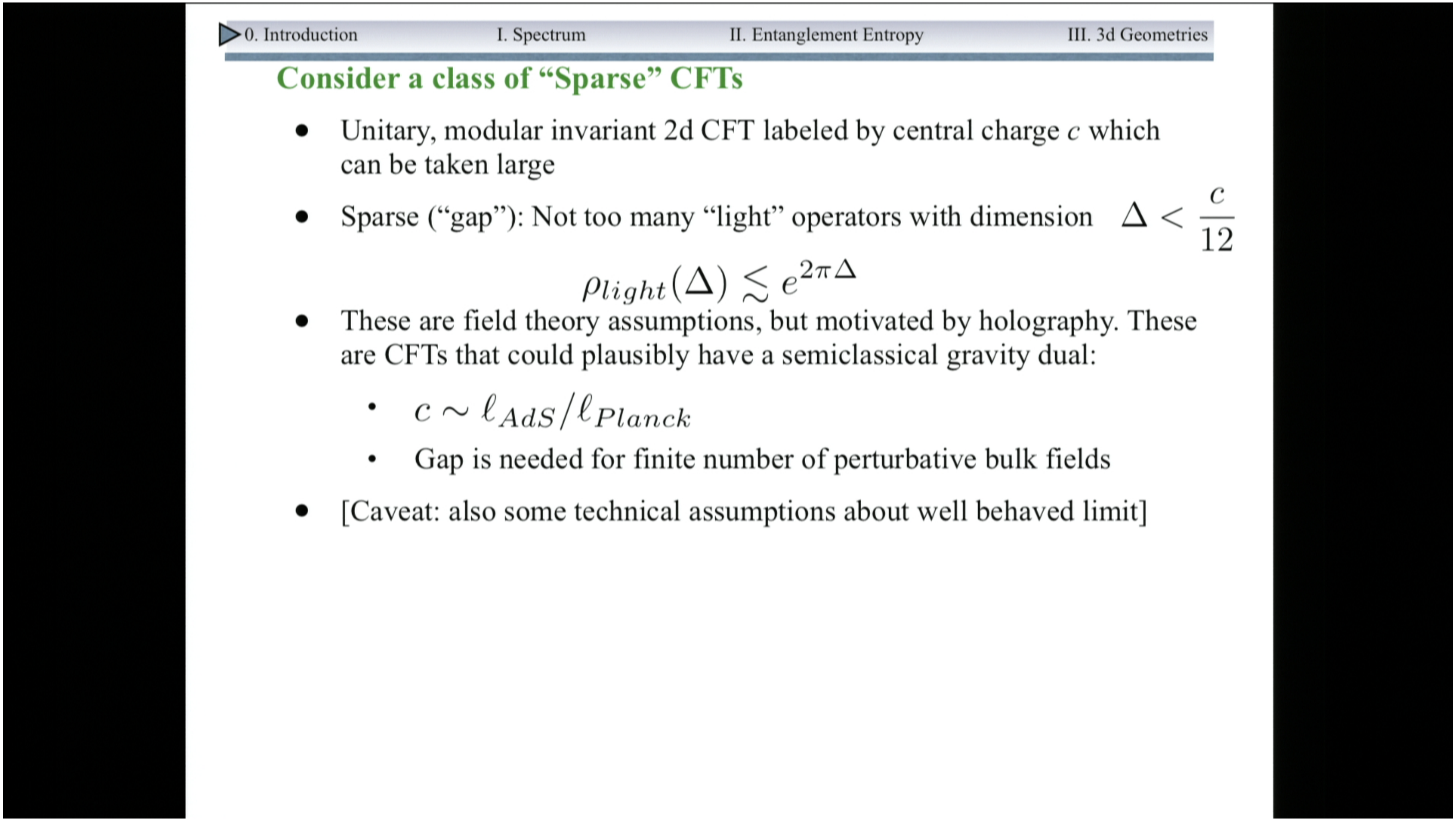
But can we actually derive universal gravity behavior from CFT?

- In 2d CFT, *yes* in many cases.
- This started in the 80's. Brown & Henneaux; Witten; H. Verlinde
- Example: 3d gravity has Virasoro symmetry, which fixes

$$\langle TT \cdots T \rangle_{plane}$$

- i.e. graviton scattering on the plane is fixed by symmetry
- This is the fact that 3d gravity has no local dof (no propagating graviton)
- However, 3d gravity is non-trivial on non-trivial topology
  - ex: black hole = torus.
- So symmetry is not enough to get 3d gravity.
- A natural rough conjecture is:

$$\boxed{\text{“Virasoro symmetry} \ + \ \text{large } c \ + \ \text{gap} \ \Rightarrow \ \text{3d gravity”}}$$

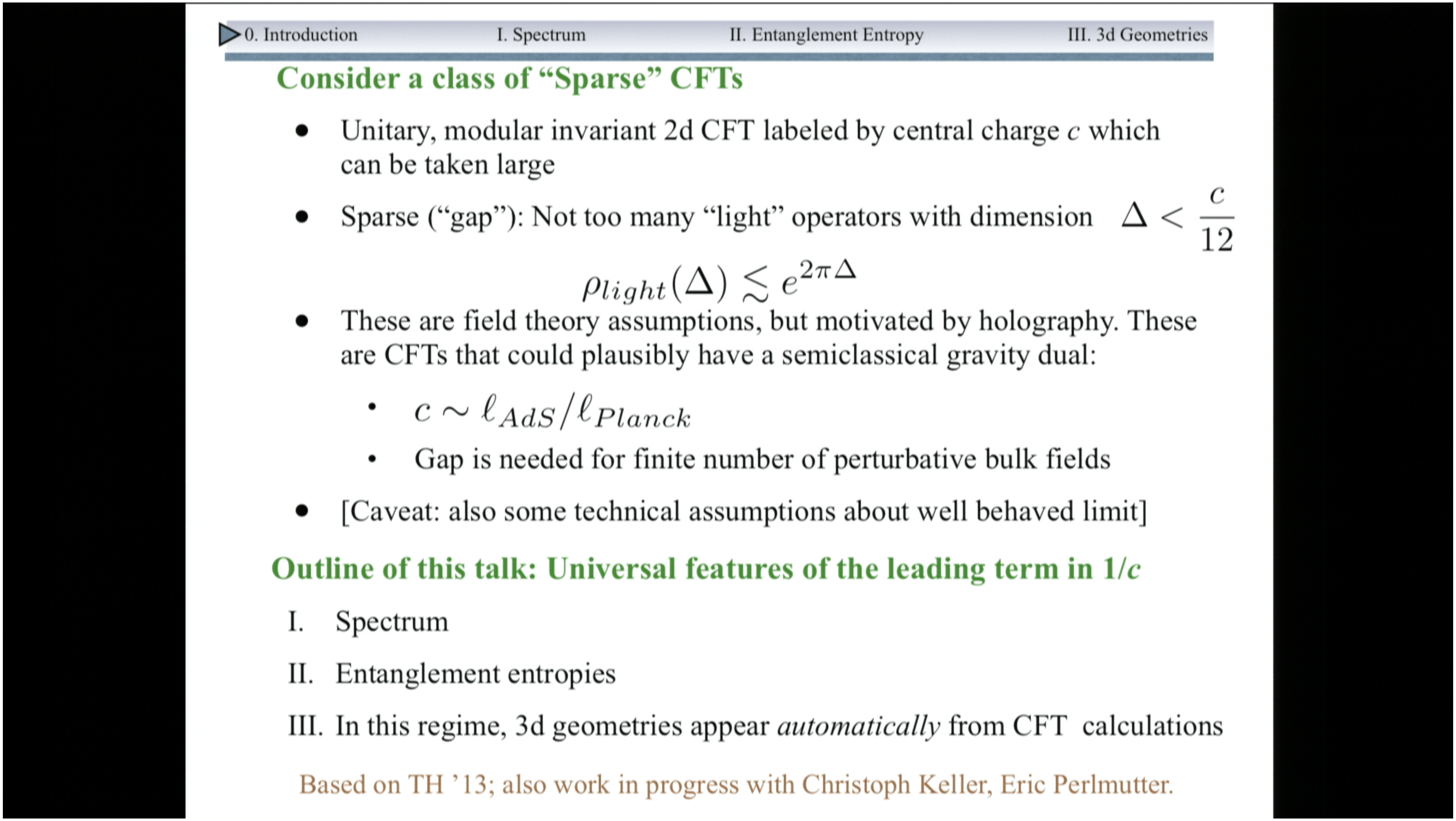


Consider a class of "Sparse" CFTs

- Unitary, modular invariant 2d CFT labeled by central charge c which can be taken large
- Sparse ("gap"): Not too many "light" operators with dimension $\Delta < \frac{c}{12}$

$$\rho_{light}(\Delta) \lesssim e^{2\pi\Delta}$$

- These are field theory assumptions, but motivated by holography. These are CFTs that could plausibly have a semiclassical gravity dual:
  - $c \sim \ell_{AdS}/\ell_{Planck}$
  - Gap is needed for finite number of perturbative bulk fields
- [Caveat: also some technical assumptions about well behaved limit]



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Outline of this talk: Universal features of the leading term in 1/*c*

I. Spectrum

II. Entanglement entropies

III. In this regime, 3d geometries appear *automatically* from CFT calculations

Based on TH ’13; also work in progress with Christoph Keller, Eric Perlmutter.

Part I: The Spectrum of CFT at Large c

General comments

- Diff invariance in quantum gravity strongly constrains the high-E spectrum (UV/IR):

$$\log(\#\ \text{states}) = S_{BH} \approx \frac{1}{4G_N} \times \text{horizon area}$$

- Gravity question: To what extent does the low-E spectrum of quantum gravity determine/constrain the high-E spectrum?

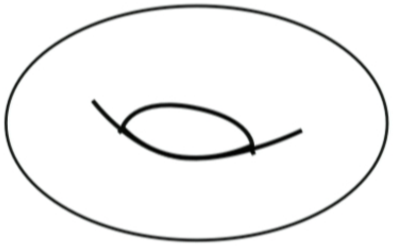
- Aim is to answer the corresponding question in CFT. The result is that the leading term in a 1/c expansion is completely fixed.

Thermal (torus) partition function:

$$Z(\beta) = \mathrm{Tr}\ e^{-\beta H} = \sum e^{-\beta(\Delta - \frac{c}{12})}$$

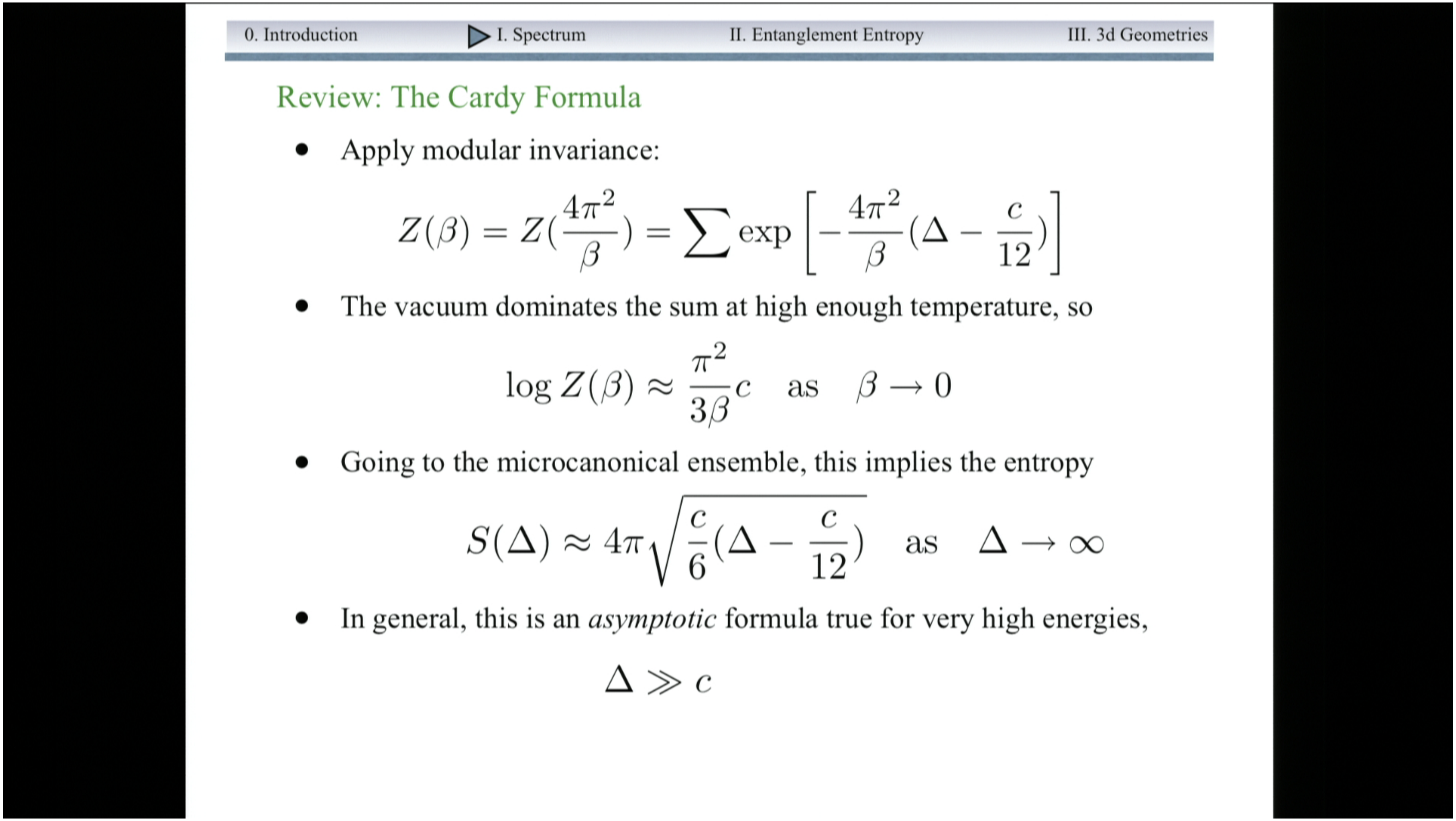
Modular invariance relates low-E to high-E spectrum:

$$Z(\beta) = Z(\frac{4\pi^2}{\beta})$$



Two types of constraints from modular invariance are well known:

- Cardy formula: universal spectrum as $\Delta \to \infty$
- Modular bootstrap: bounds on the dimension of the lightest primaries



Review: The Cardy Formula

- Apply modular invariance:

$$Z(\beta) = Z(\frac{4\pi^2}{\beta}) = \sum \exp\left[-\frac{4\pi^2}{\beta}(\Delta - \frac{c}{12})\right]$$

- The vacuum dominates the sum at high enough temperature, so

$$\log Z(\beta) \approx \frac{\pi^2}{3\beta}c \quad \text{as} \quad \beta \to 0$$

- Going to the microcanonical ensemble, this implies the entropy

$$S(\Delta) \approx 4\pi\sqrt{\frac{c}{6}(\Delta - \frac{c}{12})} \quad \text{as} \quad \Delta \to \infty$$

- In general, this is an *asymptotic* formula true for very high energies,

$$\Delta \gg c$$

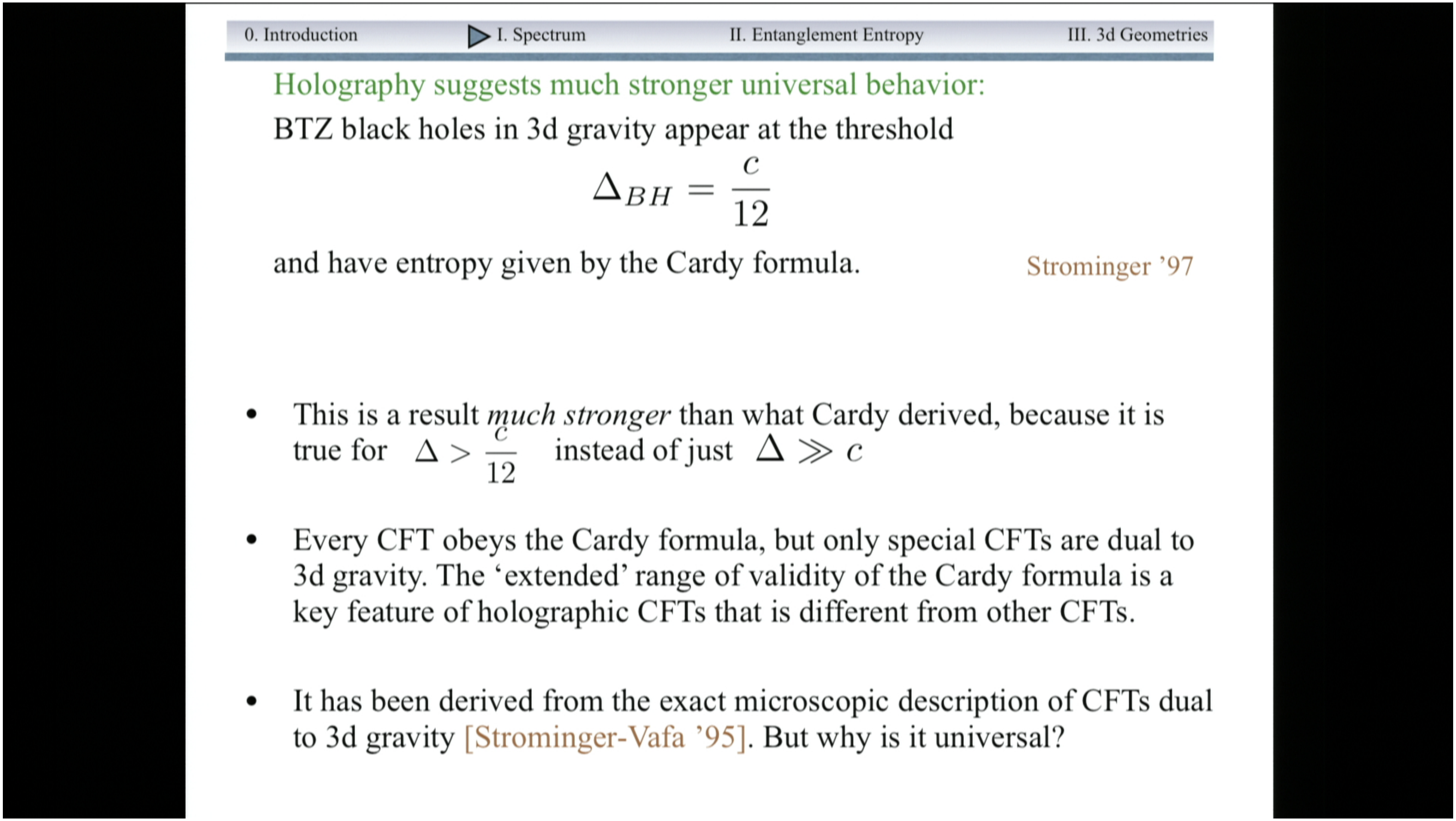
Holography suggests much stronger universal behavior:

BTZ black holes in 3d gravity appear at the threshold

$$\Delta_{BH} = \frac{c}{12}$$

and have entropy given by the Cardy formula. Strominger '97

- This is a result *much stronger* than what Cardy derived, because it is true for $\Delta > \frac{c}{12}$ instead of just $\Delta \gg c$
- Every CFT obeys the Cardy formula, but only special CFTs are dual to 3d gravity. The 'extended' range of validity of the Cardy formula is a key feature of holographic CFTs that is different from other CFTs.
- It has been derived from the exact microscopic description of CFTs dual to 3d gravity [Strominger-Vafa '95]. But why is it universal?



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New approach, directly at large *c*

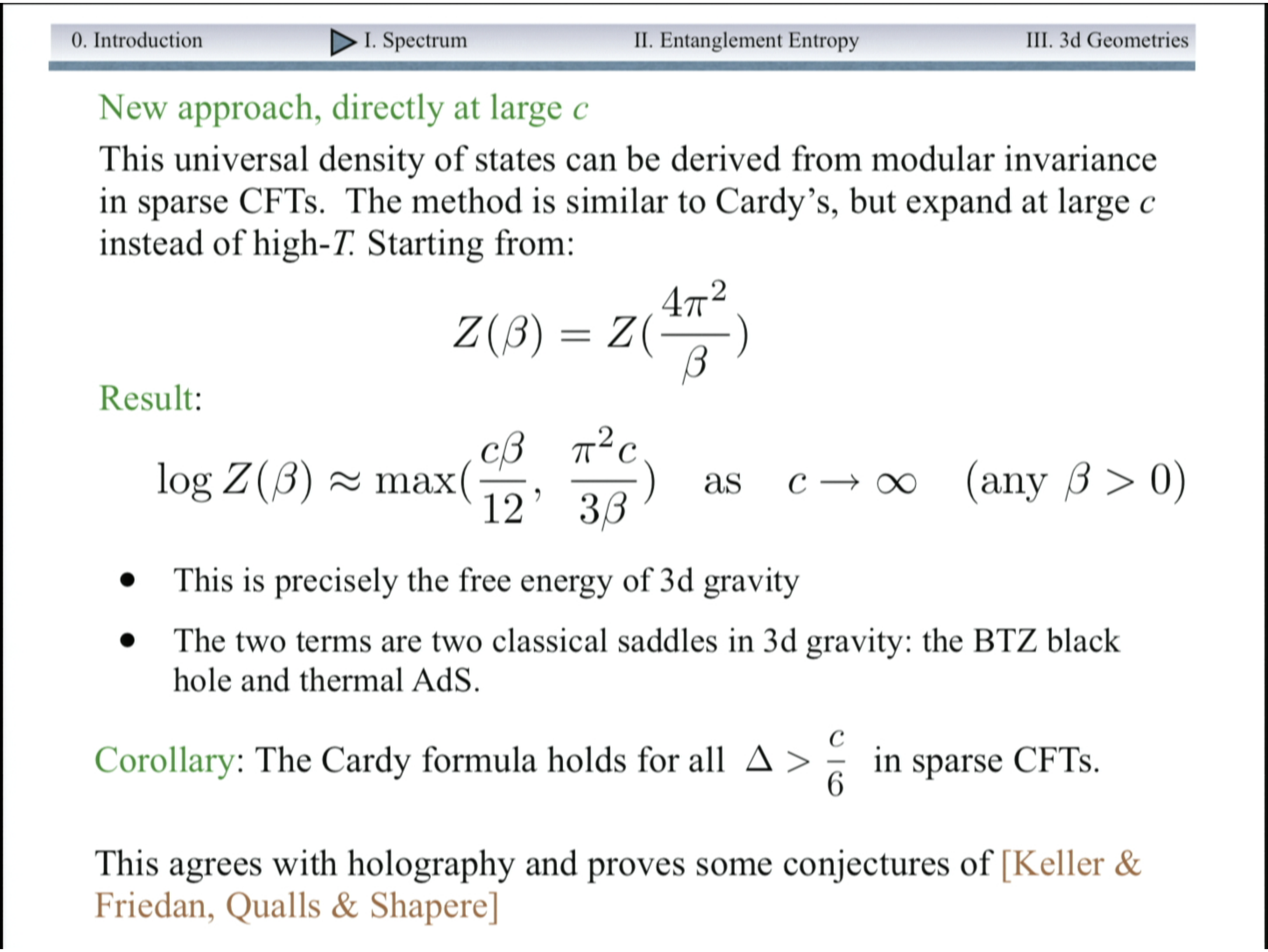
This universal density of states can be derived from modular invariance in sparse CFTs. The method is similar to Cardy's, but expand at large *c* instead of high-*T*. Starting from:

$$Z(\beta) = Z(\frac{4\pi^2}{\beta})$$

Result:

$$\log Z(\beta) \approx \max(\frac{c\beta}{12}, \frac{\pi^2 c}{3\beta}) \quad \text{as} \quad c \to \infty \quad (\text{any } \beta > 0)$$

- This is precisely the free energy of 3d gravity
- The two terms are two classical saddles in 3d gravity: the BTZ black hole and thermal AdS.



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Corollary: The Cardy formula holds for all $\Delta > \frac{c}{6}$ in sparse CFTs.

This agrees with holography and proves some conjectures of [Keller & Friedan, Qualls & Shapere]

This is one piece of a bigger picture

- Universal black hole thermodynamics = Universal behavior of the torus partition function in sparse CFTs

Higher genus partition functions also play an important role:

- Not just of mathematical interest! Higher genus partition functions encode information about groundstate entanglement (on the plane)
- This will also show us more explicitly how to interpret the 3d geometries in 2d CFT; 3d geometries will appear automatically during the CFT calculation.

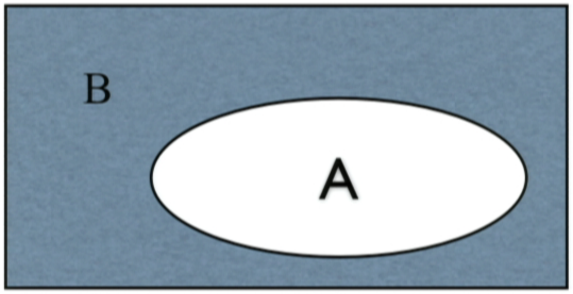
Part II: Entanglement Entropy at Large c

Entanglement entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho_A = \mathrm{tr}_B\ \rho$$

$$S_A = -\mathrm{tr}\ \rho_A \log \rho_A$$



Example: thermal entropy

A=system, B=bath, state=equilibrium

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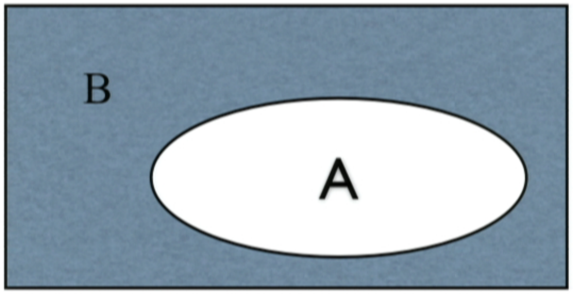


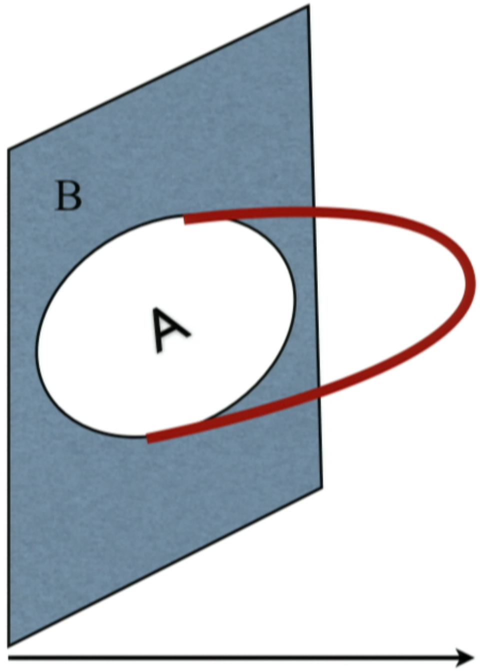
Example: thermal entropy

A=system, B=bath, state=equilibrium

In 1+1 dimensions:

Space is a line, so **A** consists of one or more intervals:

In holographic theories, the entanglement entropy is computed by a simple geometric formula:

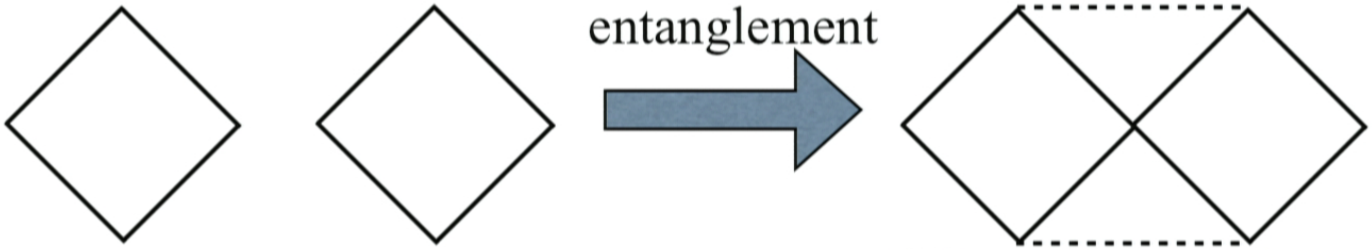


$$S_A = \frac{\text{Area(minimal surface)}}{4G_N}$$

This generalizes the Bekenstein-Hawking entropy to other types of surfaces, including Rindler horizons.

Conjecture:
  Ryu & Takayanagi ’06
Derivation:
  Casini, Huerta, Myers ’12
  Lewkowycz & Maldacena ’13

There is a rough idea that emergent geometry comes from entanglement:



ex: Maldacena '01; Ryu & Takayanagi; van Raamsdonk '10; Bianchi & Myers '12; Maldacena & Susskind '13

With this motivation in mind, we will compute entanglement entropies in the $1/c$ expansion.

These partition functions can be computed analytically in CFT in (at least) 4 situations:

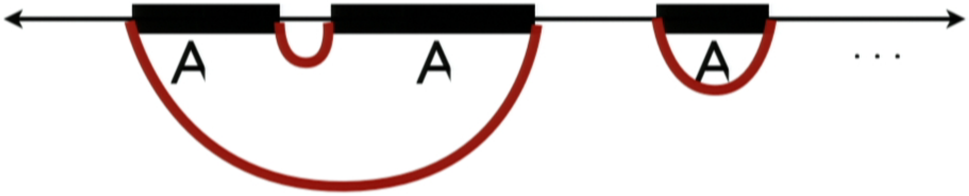
- "A" is a connected region (single interval)

Holzhey, Larsen, Wilczek '94
Calabrese & Cardy '04

$$S_\ell = \frac{c}{3} \log \left(\frac{\ell}{\epsilon_{UV}} \right)$$

universal!

- Multiple intervals (in general depend on full details of CFT)
  - free field theory
  - In a small-interval expansion
  - In a limit of large central charge, expect a universal answer from Ryu-Takayanagi:



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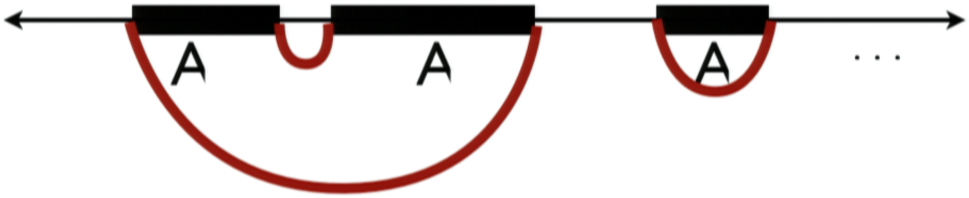

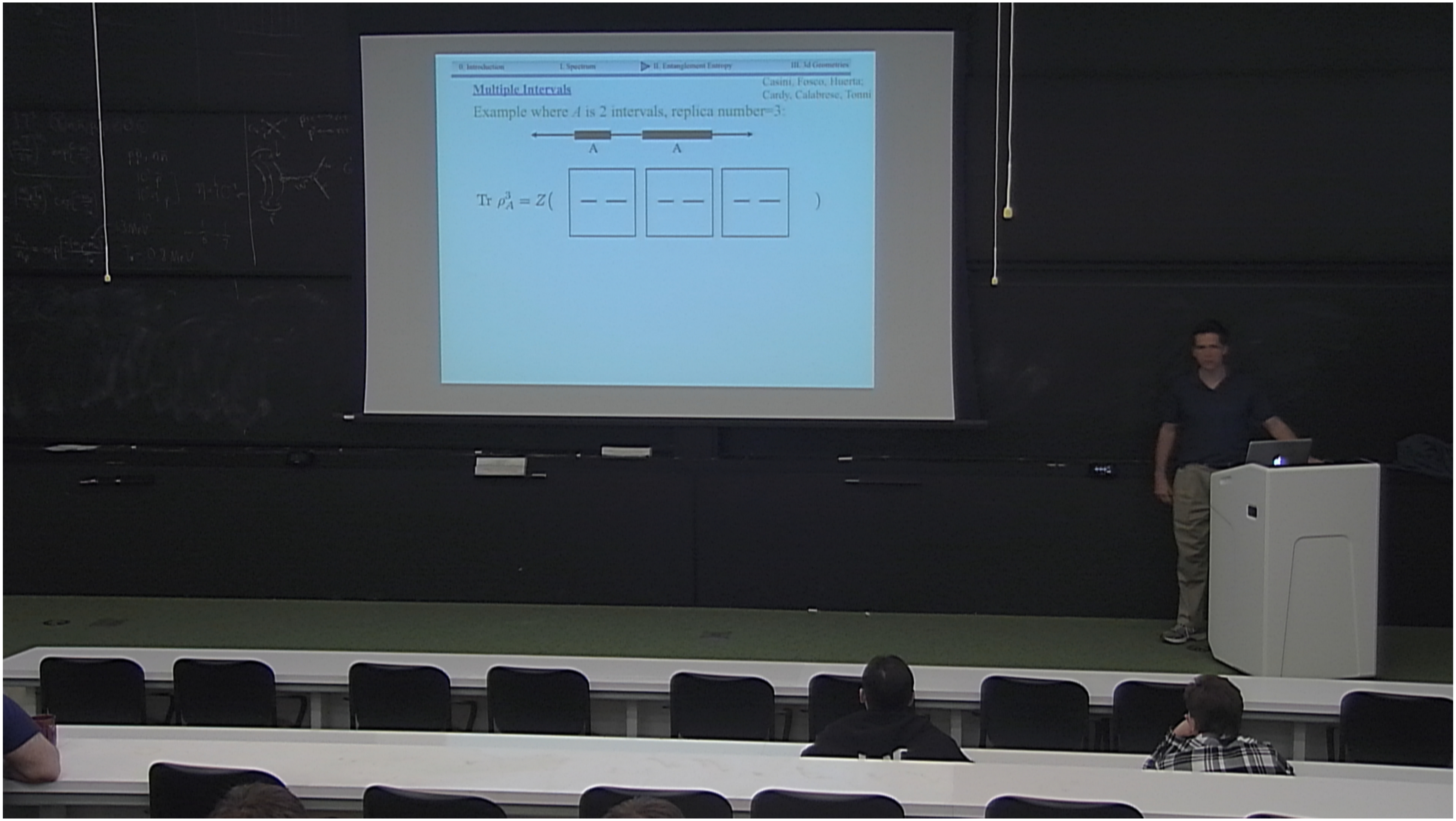

0. Introduction I. Spectrum ▷ II. Entanglement Entropy III. 3d Geometries

Multiple Intervals

Casini, Fosco, Huerta;
Cardy, Calabrese, Tonni

Example where A is 2 intervals, replica number=3:

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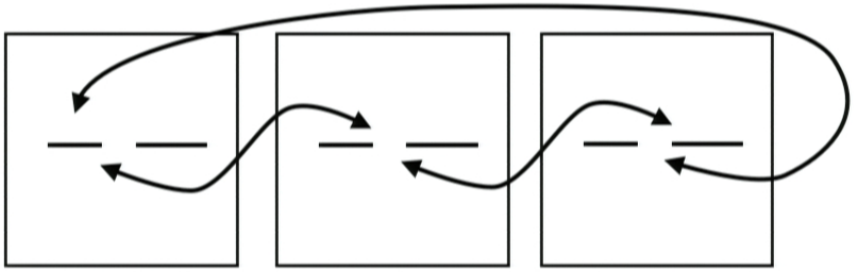
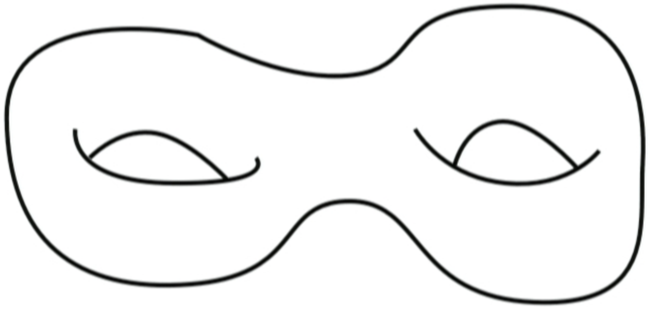


$$)$$

This is a Riemann surface with nontrivial topology.

This example (2 slits, 3 replicas) has genus 2:

$$= Z($$



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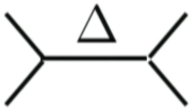
2pt functions are fixed by conformal invariance (single interval).

4pt functions are not fixed, but are constrained to have the form

$$\langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle = \sum_{\Delta}$$



$$= \sum_{\Delta} c_{\Delta}^2 \mathcal{F}(\Delta, H_n, z) \mathcal{F}(\bar{\Delta}, H_n, \bar{z})$$

Virasoro Conformal Blocks

OPE coefficient

$$H_n = \frac{c}{24}(n - 1/n) = \text{ dimension of twist operator}$$

First applied in this context by
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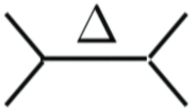

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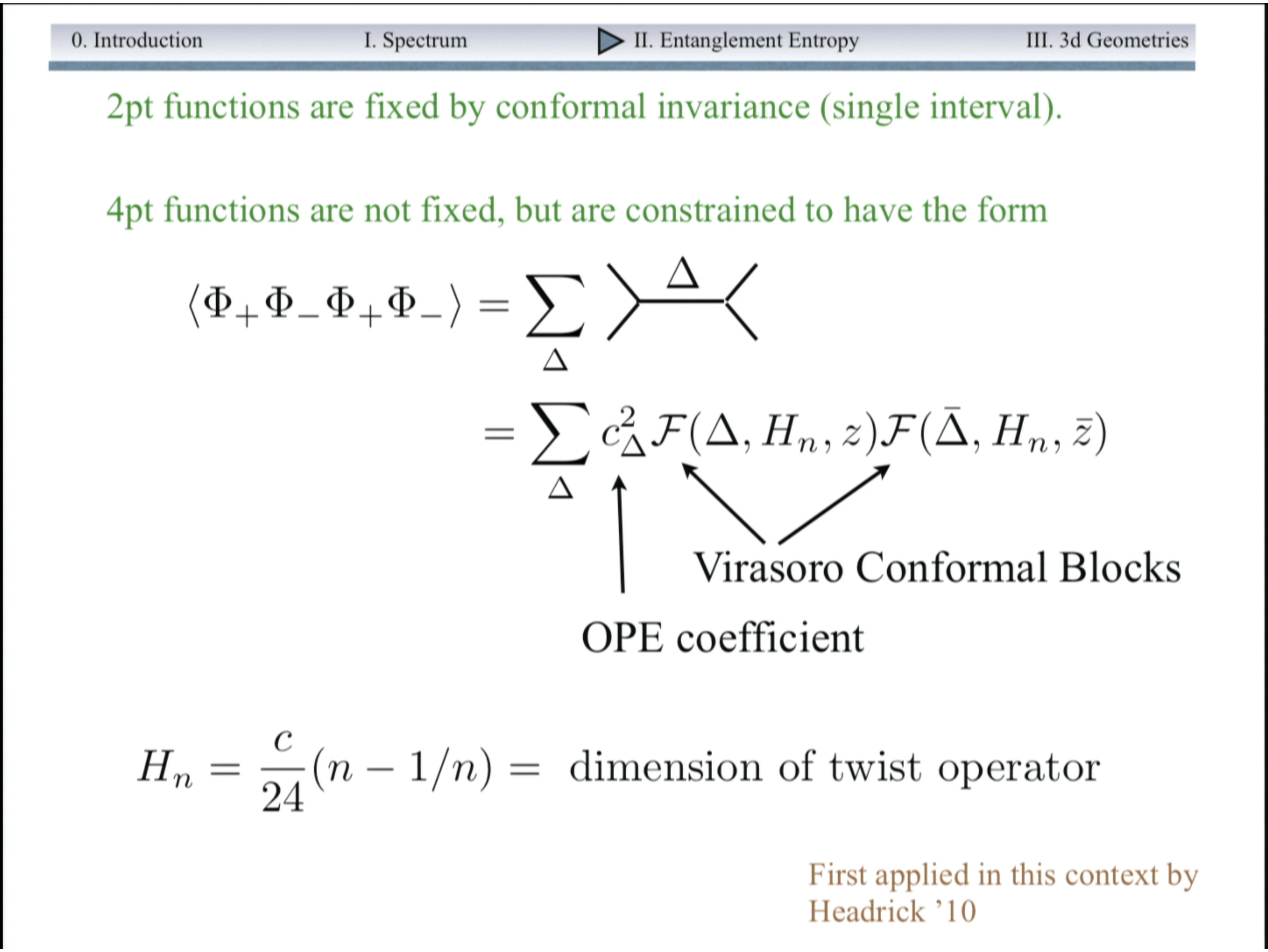

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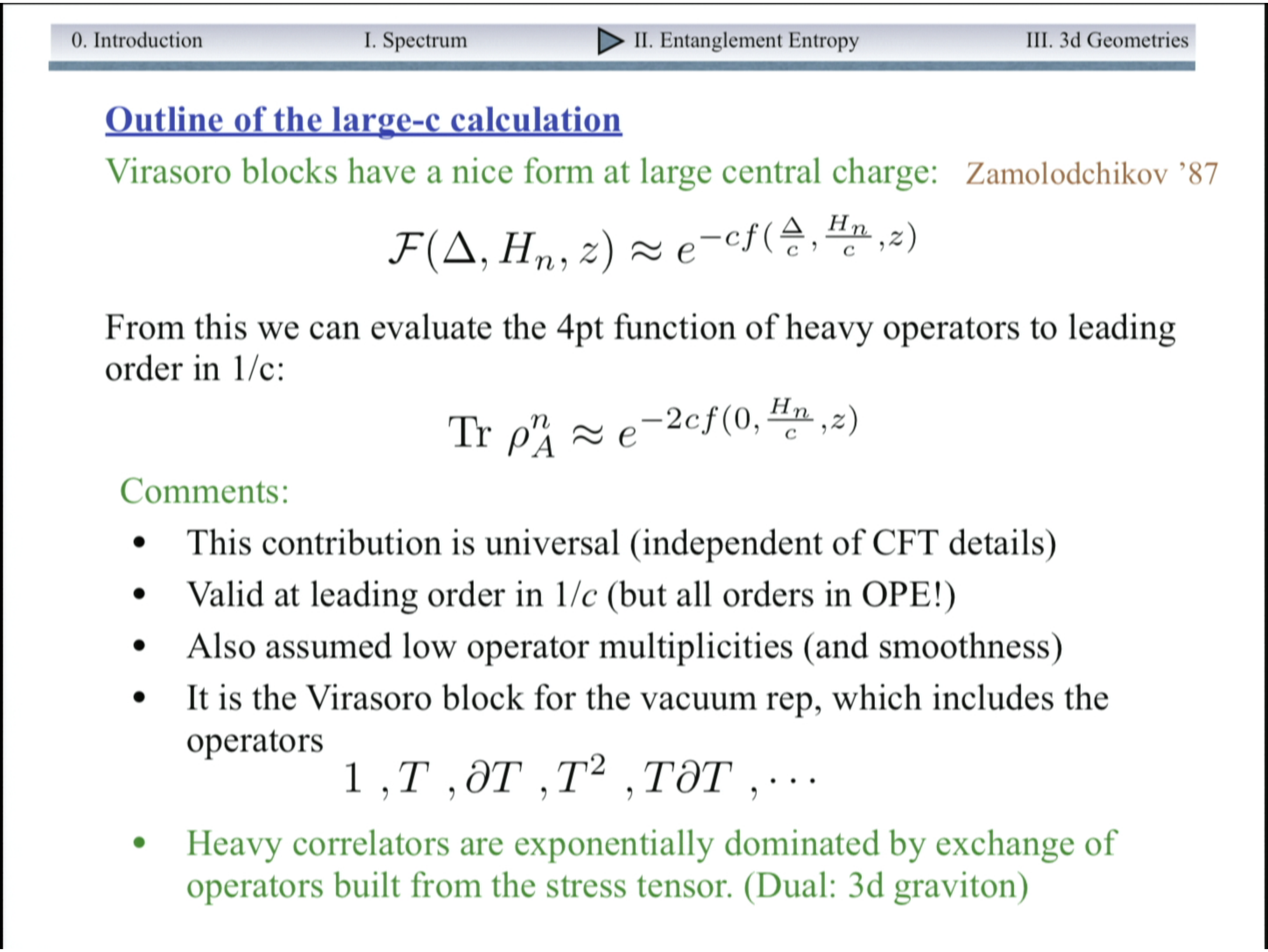
Outline of the large-c calculation

Virasoro blocks have a nice form at large central charge: Zamolodchikov '87

$$\mathcal{F}(\Delta, H_n, z) \approx e^{-cf(\frac{\Delta}{c}, \frac{H_n}{c}, z)}$$

From this we can evaluate the 4pt function of heavy operators to leading order in 1/c:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-2cf(0, \frac{H_n}{c}, z)}$$



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Comments:

- This contribution is universal (independent of CFT details)
- Valid at leading order in $1/c$ (but all orders in OPE!)
- Also assumed low operator multiplicities (and smoothness)
- It is the Virasoro block for the vacuum rep, which includes the operators

$$1\ , T\ , \partial T\ , T^2\ , T\partial T\ , \cdots$$

- Heavy correlators are exponentially dominated by exchange of operators built from the stress tensor. (Dual: 3d graviton)

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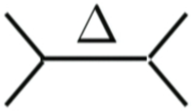

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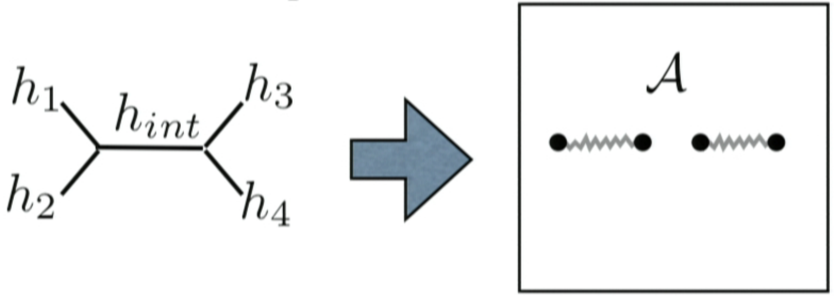
How to compute *f*: Zamolodchikov '87

(Derived using Liouville CFT, but applies in general.)

1. Find a flat *SL(2,C)* connection on the Riemann surface, with holonomies related to the operator dimensions:



$$\mathrm{Tr}\ P\exp\left(\oint \mathcal{A}\right) = -2\cos\pi\sqrt{1-\frac{24h}{c}}$$

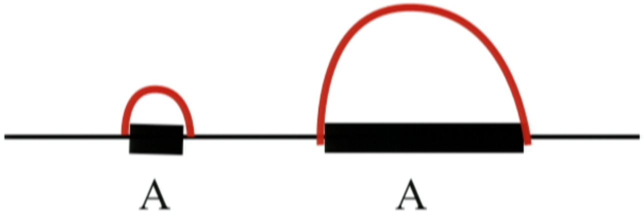
2. Write $\mathcal{A}_w = \begin{pmatrix} 0 & 1 \\ -T(w) & 0 \end{pmatrix}$; this *T(w)* is interpreted as a stress tensor

3. Compute *f* from a Ward identity, $\partial_z f = \text{res } T(w) \text{ at } w \sim z$

For the vacuum block, this amounts to imposing **trivial monodromy**. In general this can be done numerically; in the limit n-->1, it is easy to solve analytically. Now using

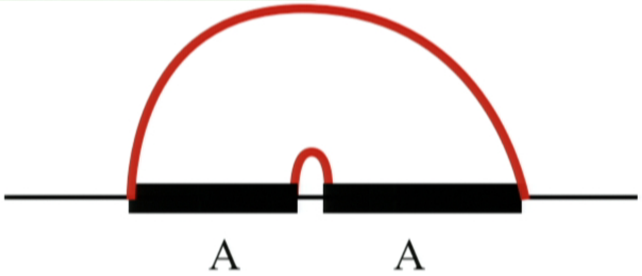
$$S_A = -\partial_n \mathrm{Tr}\ \rho_A^n|_{n=1}$$

s-channel OPE:



$$S_A = \frac{c}{3}\log(L_1) + \frac{c}{3}\log(L_2)$$

t-channel OPE:



$$S_A = \frac{c}{3}\log(L_3) + \frac{c}{3}\log(L_4)$$

Agrees with holographic Ryu-Takayanagi formula

(assuming no other non-perturbative contributions, ie non-geometric saddles)

What is an on-shell 3d geometry?

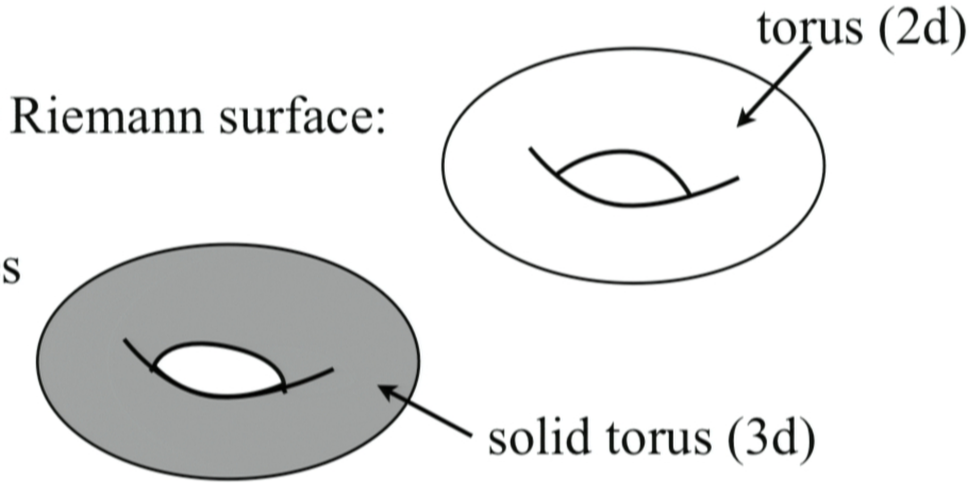
3d gravity has no propagating graviton, so all solutions of Einstein equation are *locally* AdS_3.

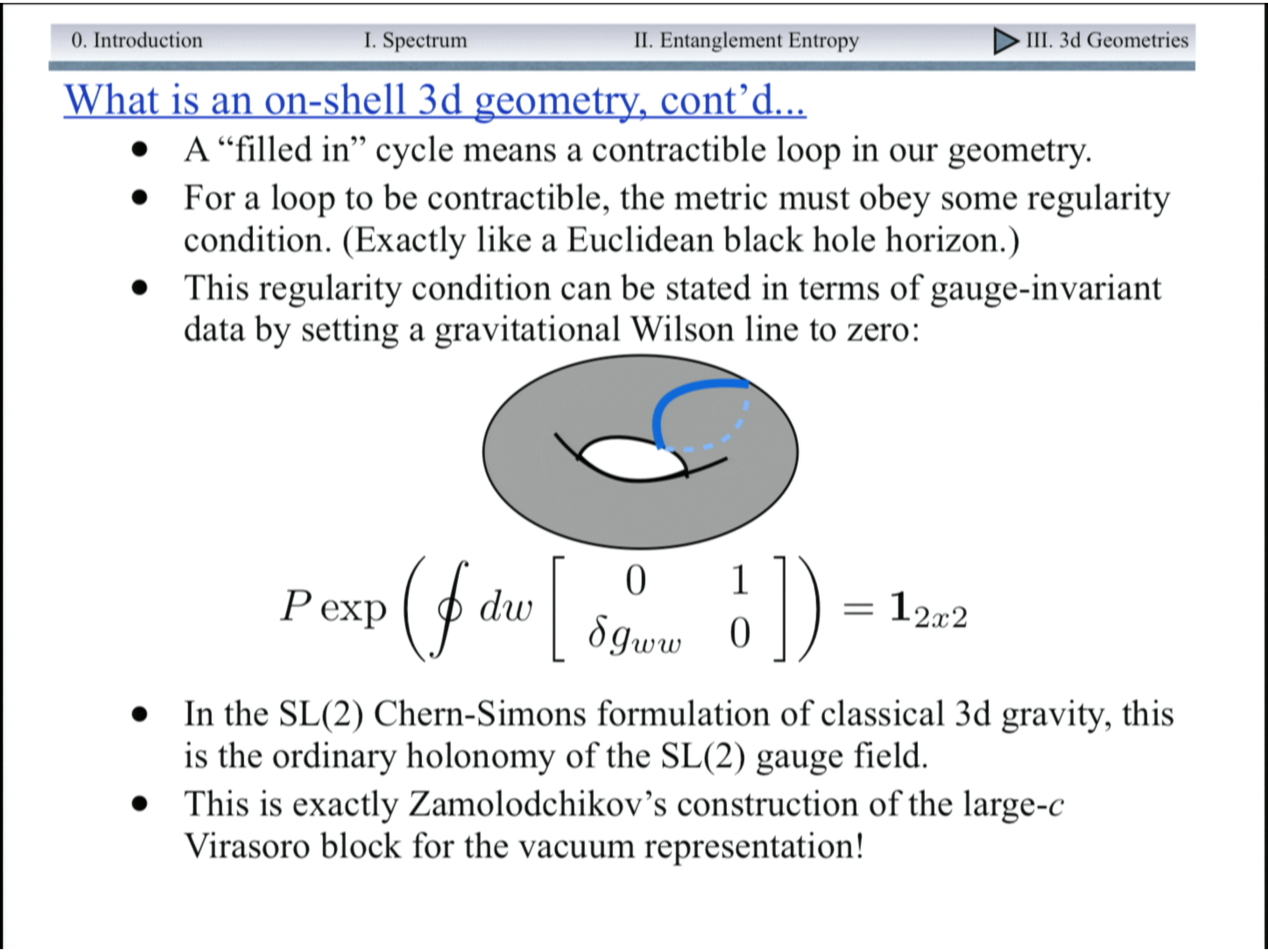
In Euclidean signature: hyperbolic 3-manifold.

To construct hyperbolic 3-manifolds:

1. Draw a genus-*g* Riemann surface:

2. "Fill in" *g* cycles





What is an on-shell 3d geometry, cont'd...

- A "filled in" cycle means a contractible loop in our geometry.
- For a loop to be contractible, the metric must obey some regularity condition. (Exactly like a Euclidean black hole horizon.)
- This regularity condition can be stated in terms of gauge-invariant data by setting a gravitational Wilson line to zero:

$$P\exp\left(\oint dw \begin{bmatrix} 0 & 1 \\ \delta g_{ww} & 0 \end{bmatrix}\right) = \mathbf{1}_{2x2}$$

- In the SL(2) Chern-Simons formulation of classical 3d gravity, this is the ordinary holonomy of the SL(2) gauge field.
- This is exactly Zamolodchikov's construction of the large-*c* Virasoro block for the vacuum representation!

Slogan:

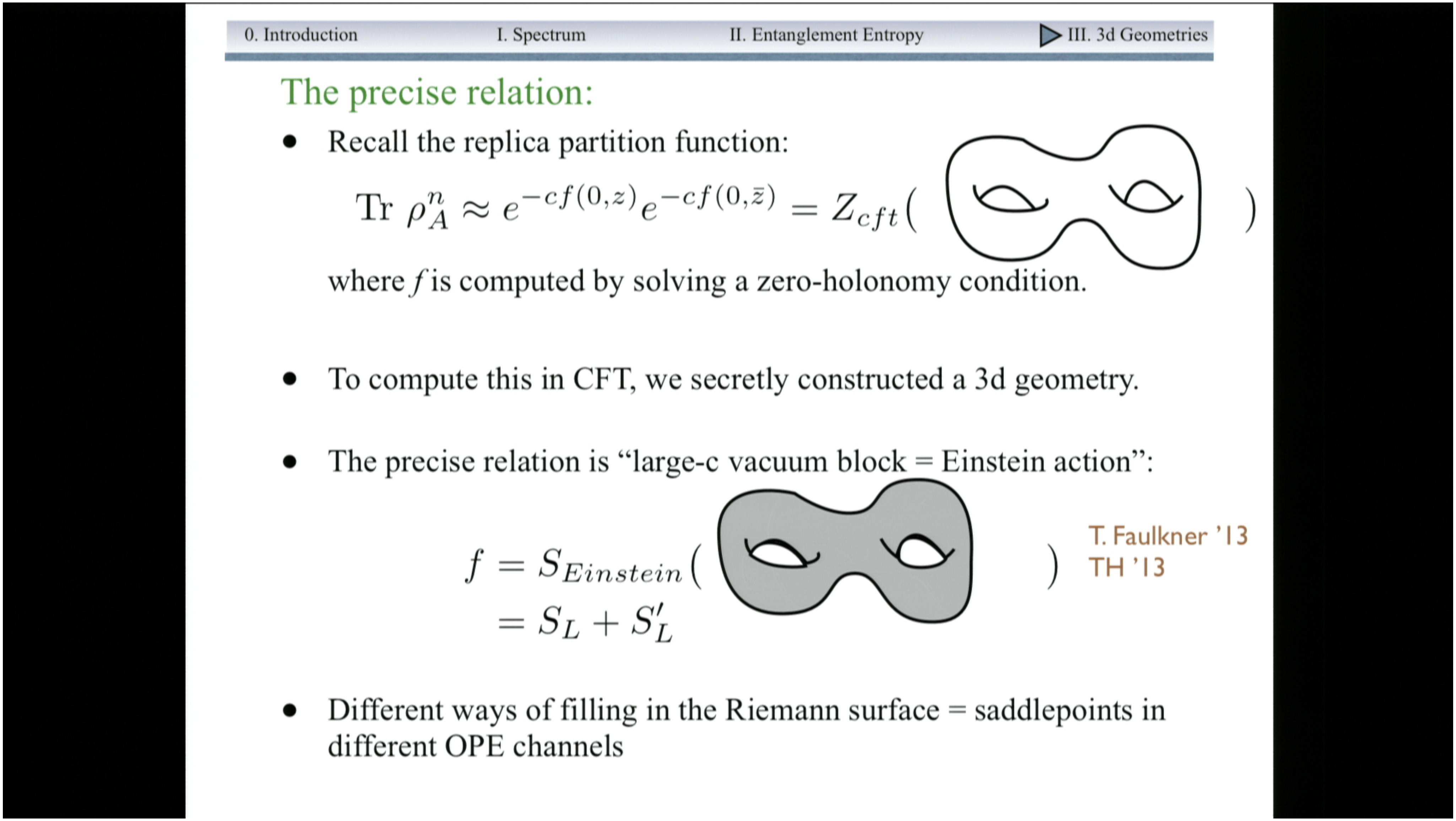
Virasoro vacuum block at large c

“=”

3d geometry

Witten 80’s
H.Verlinde 80’s
Coussaert, Henneaux
& van Driel 90’s
Krasnov 00’s
Takhtajan et al 00’s
Yin ’08

Disclaimer: This has a precise meaning for the replica manifolds we are considering here, but I do not know what exactly the statement should be on general Riemann surfaces.



The precise relation:

- Recall the replica partition function:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})} = Z_{cft}\Big(\qquad \Big)$$

where f is computed by solving a zero-holonomy condition.

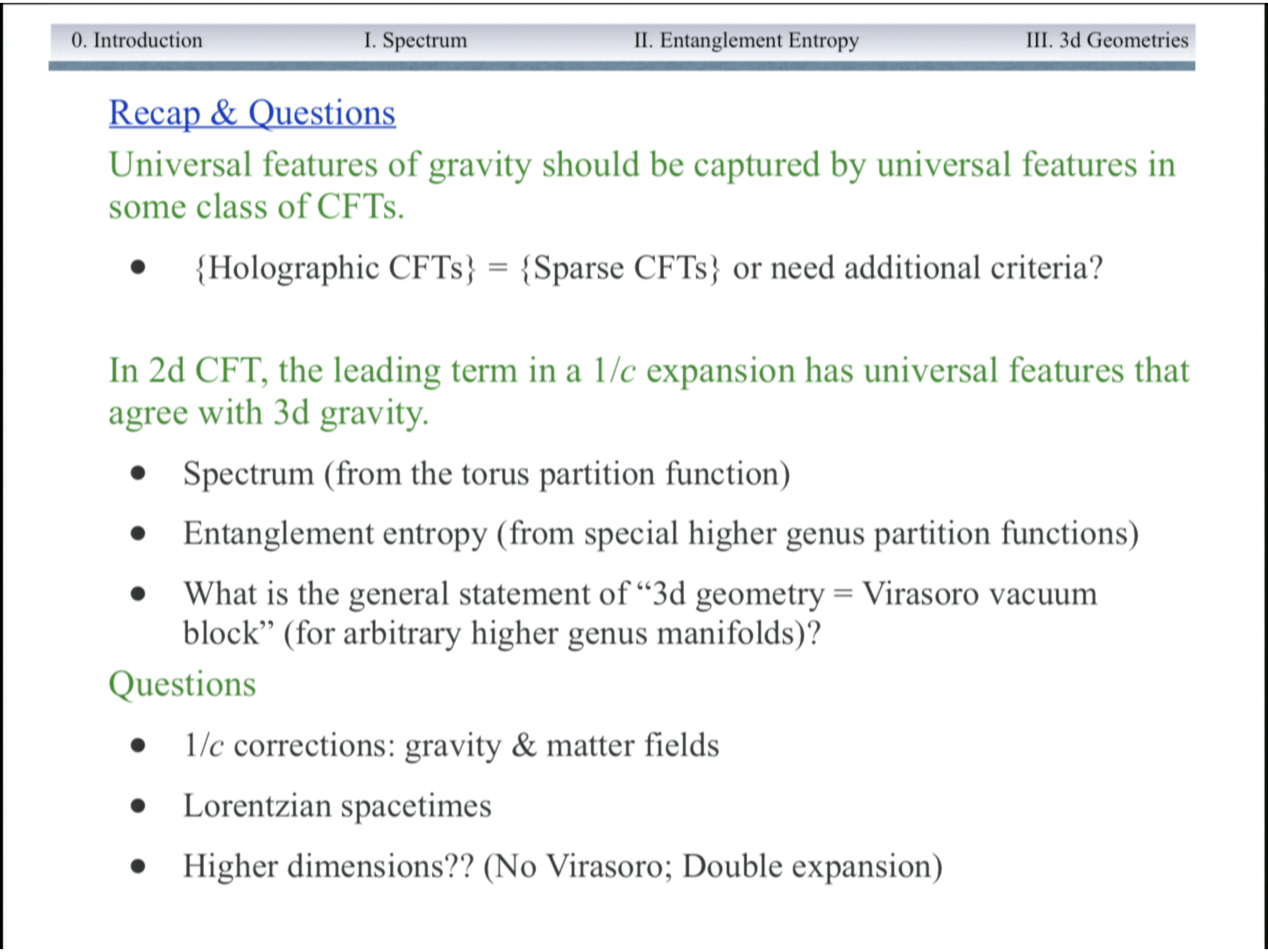
- To compute this in CFT, we secretly constructed a 3d geometry.

- The precise relation is "large-c vacuum block = Einstein action":

$$\begin{aligned} f &= S_{Einstein}\Big(\qquad \Big) \\ &= S_L + S_L' \end{aligned}$$

T. Faulkner '13
TH '13

- Different ways of filling in the Riemann surface = saddlepoints in different OPE channels



Recap & Questions

Universal features of gravity should be captured by universal features in some class of CFTs.

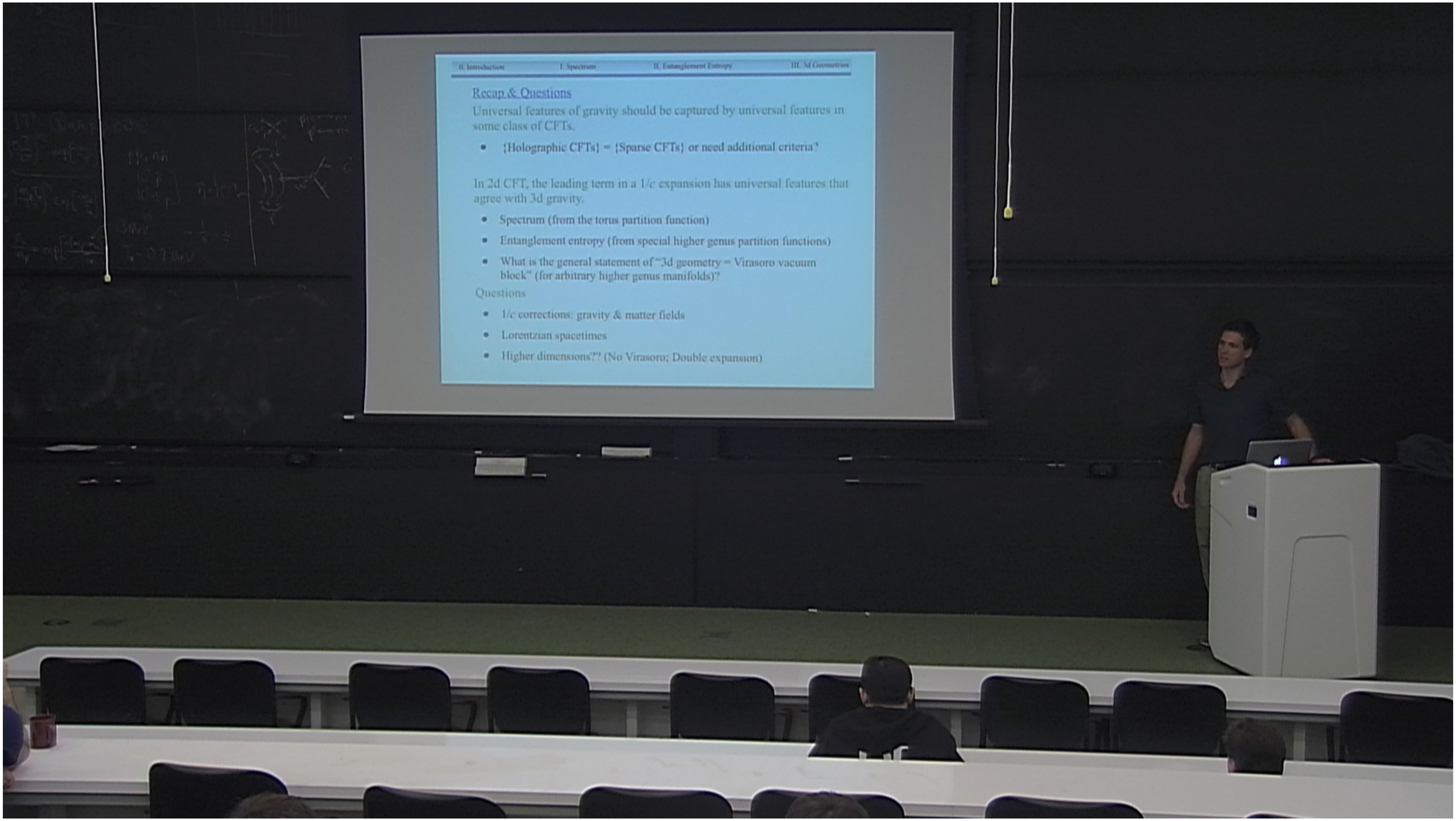
- {Holographic CFTs} = {Sparse CFTs} or need additional criteria?

In 2d CFT, the leading term in a $1/c$ expansion has universal features that agree with 3d gravity.

- Spectrum (from the torus partition function)
- Entanglement entropy (from special higher genus partition functions)
- What is the general statement of "3d geometry = Virasoro vacuum block" (for arbitrary higher genus manifolds)?

Questions

- $1/c$ corrections: gravity & matter fields
- Lorentzian spacetimes
- Higher dimensions?? (No Virasoro; Double expansion)



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