Title: 13/14 PSI - Quantum Information Review - Lecture 10

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Abstract:

- Distance measures
- Entropy

Before, we used a distance measure

$$\| |\psi\rangle - |\phi\rangle \| \qquad \max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \|$$

$$|\psi\rangle \qquad -|\psi\rangle \qquad \| |\psi\rangle - (-|\psi\rangle) \| = 2$$

distance measure

$$\max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \|$$

$$\| \, |\psi\rangle - (-|\psi\rangle) \| = 2$$

Fidelity

$F(|\psi\rangle, |\varphi\rangle) = |\langle\psi|\varphi\rangle|$ "overlap"

POVM $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$ on $|\varphi\rangle$

gives $|\psi\rangle\langle\psi|$ w.p. $F(|\psi\rangle, |\varphi\rangle)^2$

$$F(|\psi\rangle, \rho)^2 = tr(|\psi\rangle\langle\psi|\rho) = \langle\psi|\rho|\psi\rangle$$

in general: $F(\rho,\sigma) = \mathrm{tr}\,|\sqrt{\rho}\sqrt{\sigma}|$

$|A| = \sqrt{AA^\dagger}$ $\qquad = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$

<u>Uhlman's theorem</u>: $F(\rho,\sigma) = \max_{|\psi\rangle,|\varphi\rangle} |\langle\psi|\varphi\rangle|$

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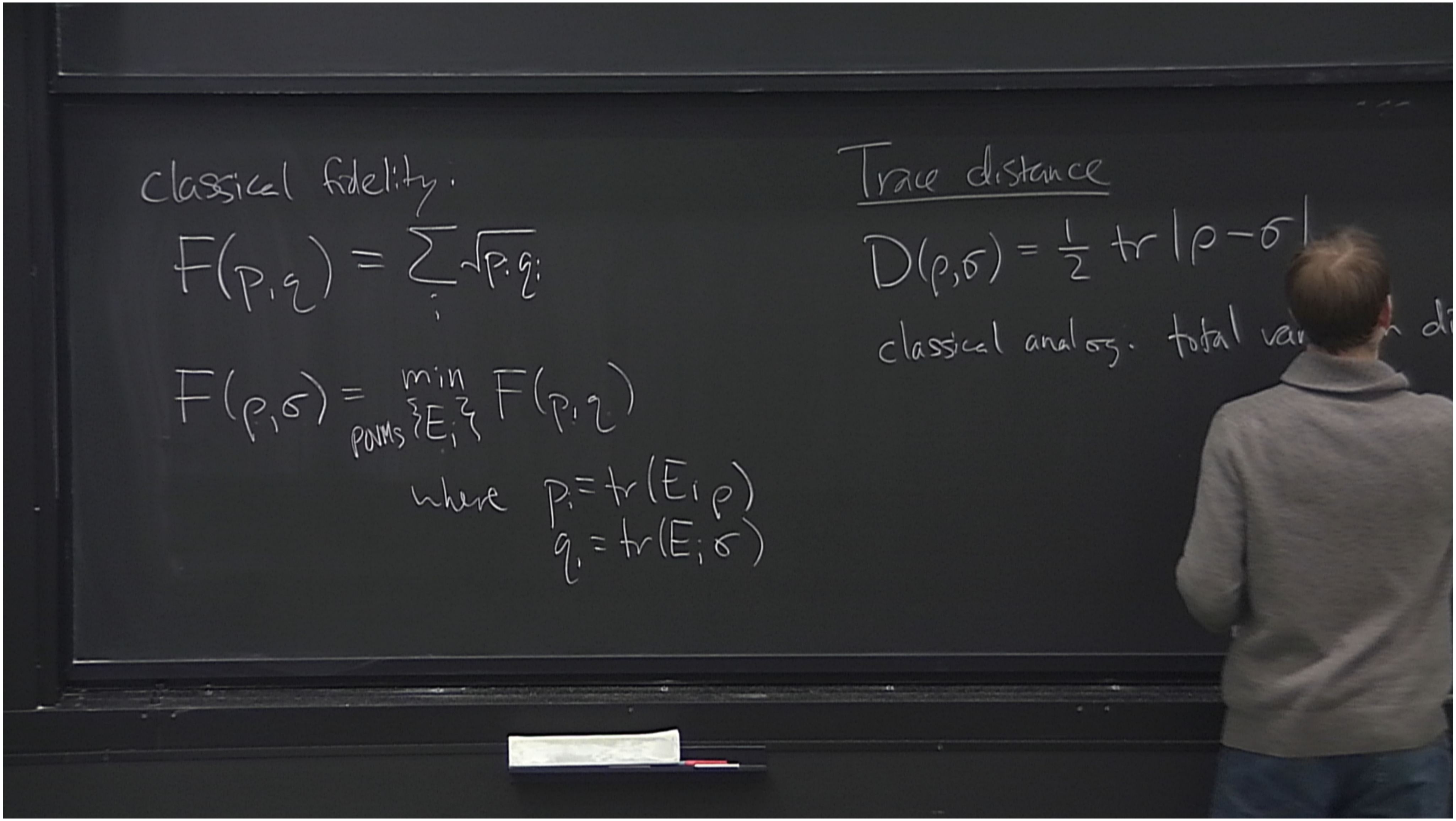
Properties:

- $0 \le F(\rho,\sigma) \le 1$
- $F(\rho,\sigma) = 0 \Longleftrightarrow \rho,$

Properties:

- $0 \leq F(\rho,\sigma) \leq 1$
- $F(\rho,\sigma) = 0 \iff \rho,\sigma$ have $\perp$ support
- $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho,\sigma)$ $\forall$ unitaries U
- $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho,\sigma)$ $\forall$ q. ops. $\mathcal{E}$


classical fidelity:
$F(p,q) = \sum_i \sqrt{p_i q_i}$
$F(\rho,\sigma) = \min_{\text{POVMs } \{E_i\}} F(p,q)$
where $p_i = tr(E_i \rho)$
$q_i = tr(E_i \sigma)$
Trace distance
$D(\rho,\sigma) = \frac{1}{2} tr|\rho-\sigma|$
classical analog: total va

Trace distance

$$D(\rho,\sigma) = \frac{1}{2} \operatorname{tr} |\rho - \sigma|$$

classical analog: total variation distance:

$$D(p,q) = \frac{1}{2} \sum_i |p_i - q_i|$$

max guessing probability: $\frac{1}{2}\left(1 + D(p,q)\right)$

2 distributions $(p, 1-p)$ $(q,$

$\ldots |\rho - \sigma|$

total variation distance:

$\ldots \sum_i |p_i - q_i|$

...bability: $\frac{1}{2}(1 + D(p,q))$

2 distributions $(p, 1-p)$ $(q, 1-q)$

$$D = \frac{1}{2}\left(|p-q| + |(1-p)-(1-q)|\right) = |p-q|$$

Pr (guessing correctly w/ best strategy) (for $p > q$)

$$= \frac{1}{2} p + \frac{1}{2}(1-q) = \frac{1}{2}(1 + p - q)$$

Theorem (Helstrom):

$$D(\rho,\sigma) = \max_{\text{POVMs } \{E_i\}} D(p,q)$$

where $p_i = tr(E_i \rho)$

$q_i = tr(E_i \sigma)$

Proof:

$$D(p,q) = \frac{1}{2} \sum_i |tr(E_i(\rho-\sigma))|$$

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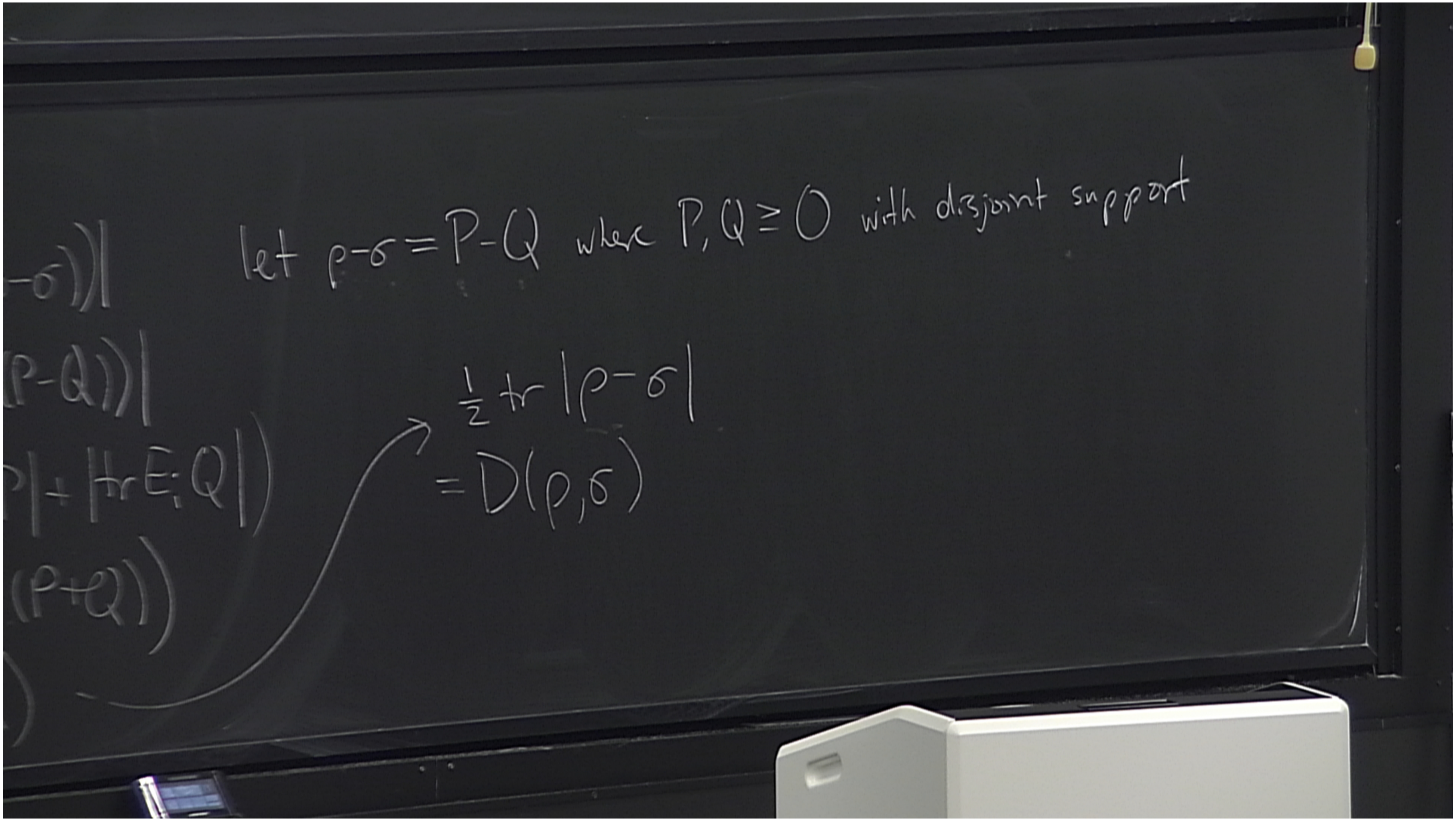
Proof:

let $\rho-\sigma = P-Q$ where $P, Q \geq 0$ wi

$$D(\rho,\sigma) = \frac{1}{2}\sum_i \left|\mathrm{tr}\left(E_i(\rho-\sigma)\right)\right|$$
$$= \frac{1}{2}\sum \left|\mathrm{tr}\left(E_i(P-Q)\right)\right|$$
$$\leq \frac{1}{2}\sum \left(\left|\mathrm{tr}\, E_i P\right| + \left|\mathrm{tr}\, E_i Q\right|\right)$$
$$= \frac{1}{2}\sum \mathrm{tr}\left(E_i(P+Q)\right)$$
$$= \frac{1}{2}\mathrm{tr}(P+Q)$$

$$\frac{1}{2}\mathrm{tr}\,|\rho-\sigma|$$


let $\rho-\sigma = P-Q$ where $P,Q \geq 0$ with disjoint support
$\frac{1}{2}\mathrm{tr}|\rho-\sigma|$
$= D(\rho,\sigma)$

This is achievable:

POVM $E_0 = \Pi_P$ (projection onto support of P)

$$E_1 = I - \Pi_P$$

$$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}(P+Q)$$

$$D(p,q) = \frac{1}{2}\left[\mathrm{tr}(\Pi_P \rho) + \mathrm{tr}((I-\Pi_P)\sigma\right]$$

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POVM $E_0 = \Pi_P$ (projection onto support of P)

$E_1 = I - \Pi_P$

$$D(\rho,\sigma) = \frac{1}{2}\, tr(P+Q)$$

$$D(p,q) = \frac{1}{2}\left[tr(\Pi_P \rho) + tr((I-\Pi_P)\sigma\right]$$

$$= \frac{1}{2}\left[tr(\Pi_P(P-Q)) + tr((I-\Pi_P)(P-Q))\right]$$

$$\frac{1}{2}\left(\mathrm{tr}\left(\Pi_P(\underbrace{\rho-\sigma}_{P-Q})\right) + \cancel{\mathrm{tr}(\sigma)}\right)$$

$$= \frac{1}{2}\left(1 + \mathrm{tr}\, P\right)$$

$\mathrm{tr}(\rho-\sigma) = 0 \qquad \underline{\mathrm{tr}(P) = \mathrm{tr}(Q)}$

$-Q)]$

<u>Properties</u>:

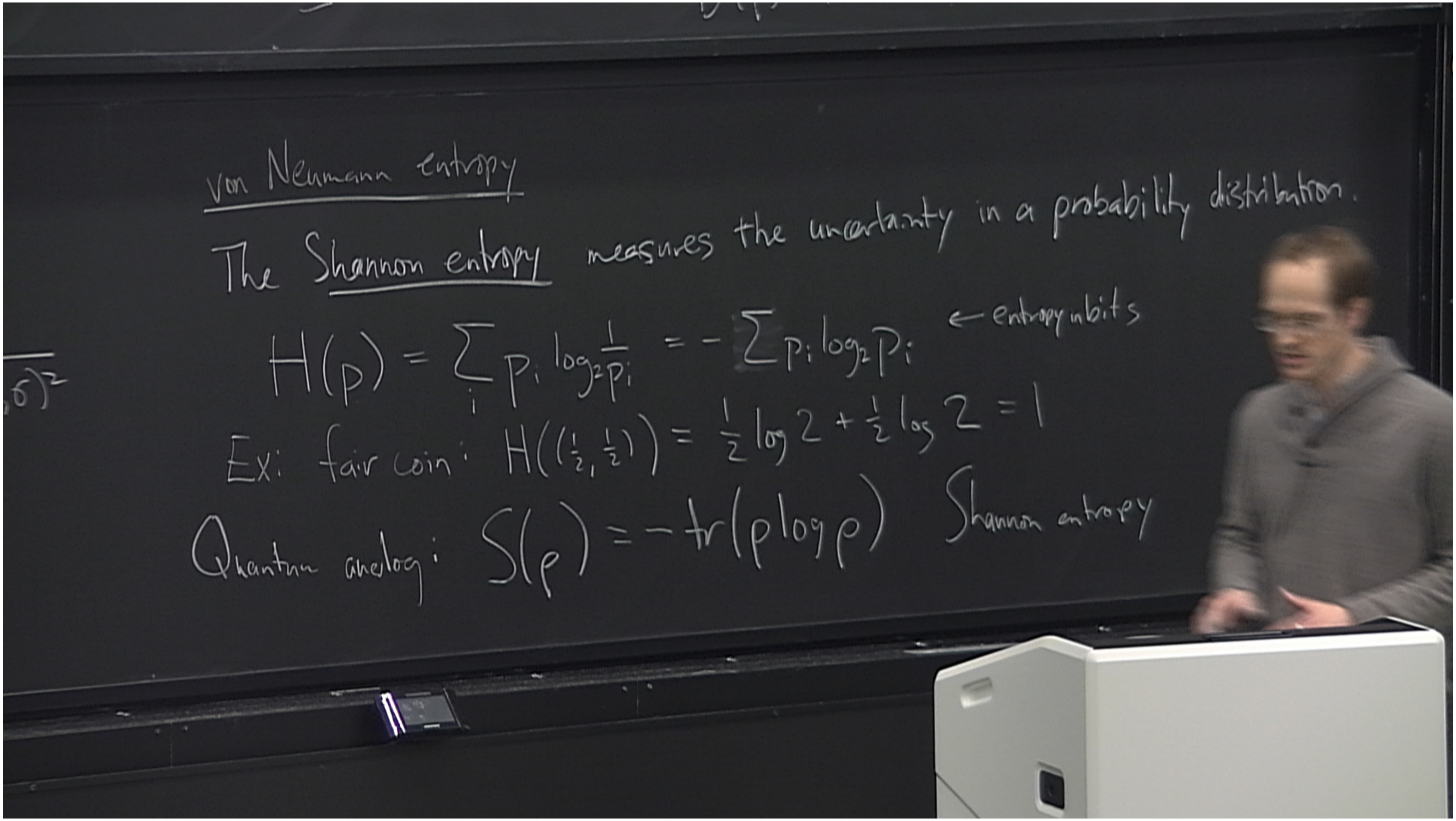
- $0 \le D(\rho,\sigma) \le 1$
- $D(\rho,\sigma) = 0 \iff \rho = \sigma$
- $D(\rho,\tau) \le D(\rho,\sigma) + D(\sigma,\tau)$

$\Bigg\}$ D is a metric

$$D(U\rho U^\dagger, U\sigma U^\dagger) = D(\rho,\sigma)$$

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho,\sigma)$$

<u>Theorem</u>: $1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2}$


von Neumann entropy
The Shannon entropy measures the uncertainty in a probability distribution.
$H(p) = \sum_i p_i \log_2 \frac{1}{p_i} = -\sum p_i \log_2 p_i$
← entropy in bits
Ex: fair coin: $H((\frac{1}{2}, \frac{1}{2})) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1$
Quantum analog: $S(\rho) = -\mathrm{tr}(\rho \log \rho)$
Shannon entropy