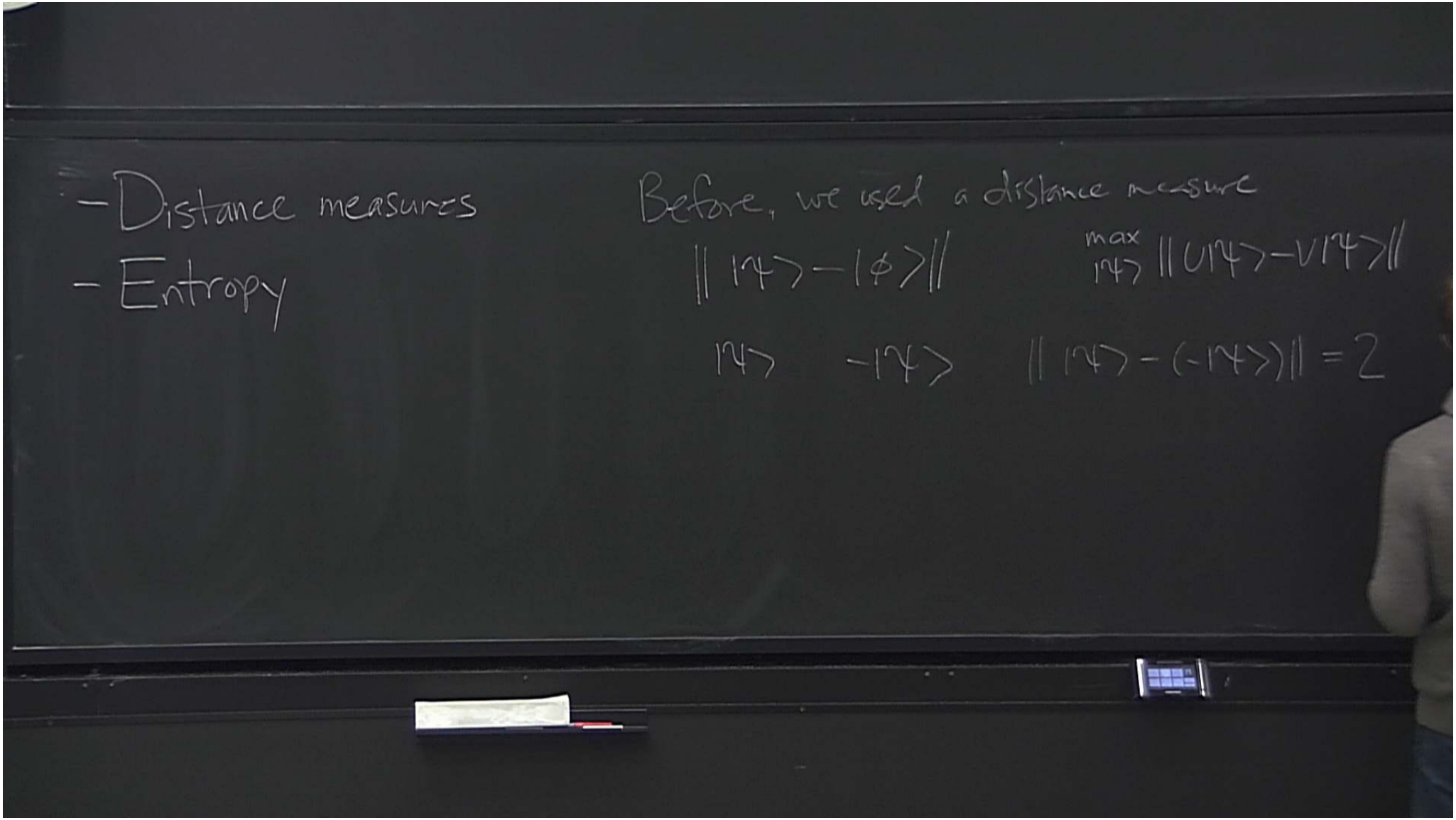
Title: 13/14 PSI - Quantum Information Review - Lecture 10

Date: Feb 28, 2014 11:30 AM

URL: http://pirsa.org/14020082

Abstract:


- Distance measures
- Entropy
Before, we used a distance measure
$\| |\psi\rangle - |\phi\rangle \|$
$\max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \|$
$|\psi\rangle \quad -|\psi\rangle \quad \| |\psi\rangle - (-|\psi\rangle) \| = 2$

istance measure

$$\max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \|$$

$$\| |\psi\rangle - (-|\psi\rangle) \| = 2$$

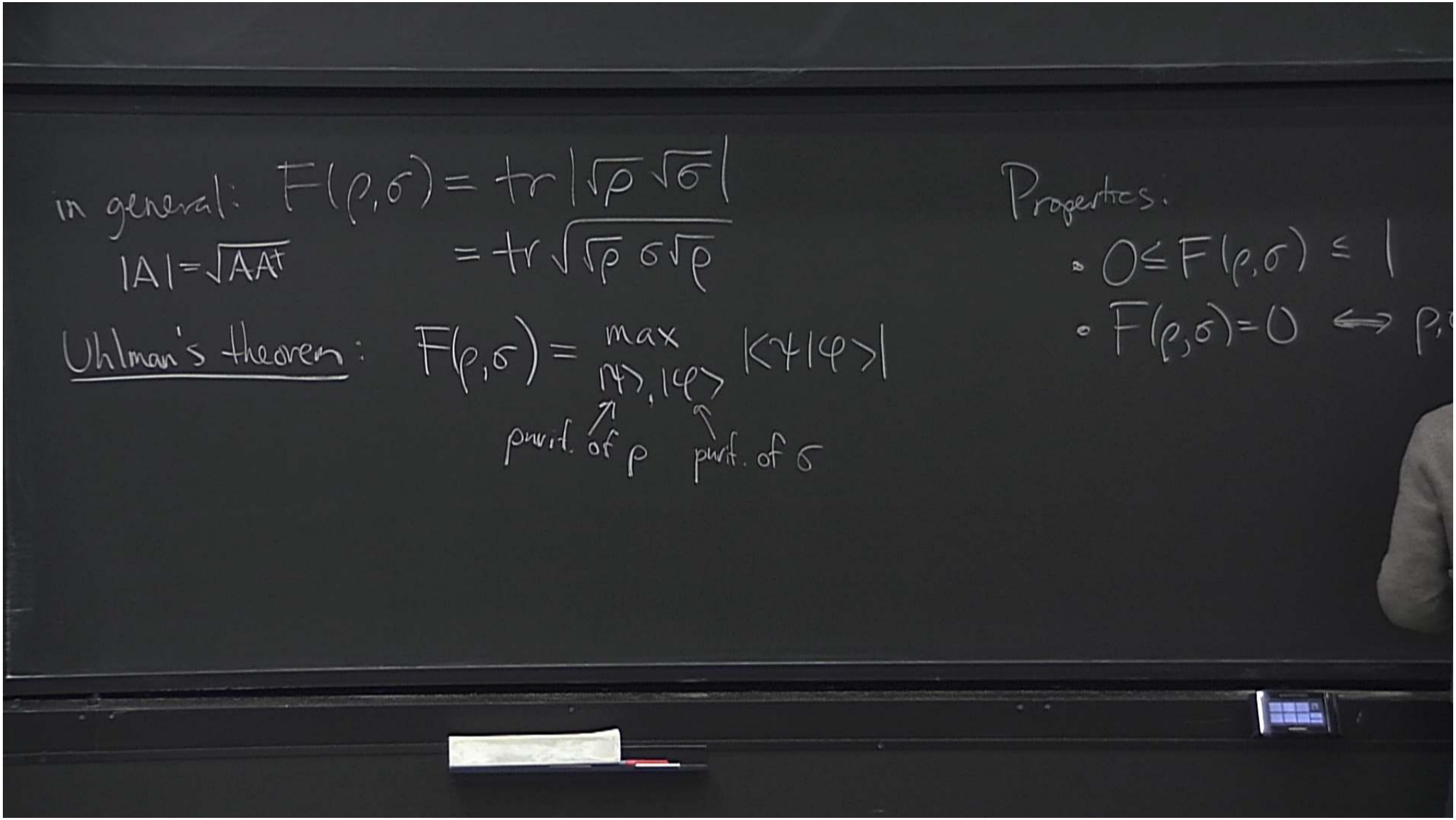
Fidelity

$F(|\psi\rangle, |\varphi\rangle) = |\langle\psi|\varphi\rangle|$ "overlap"

POVM $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$ on $|\varphi\rangle$

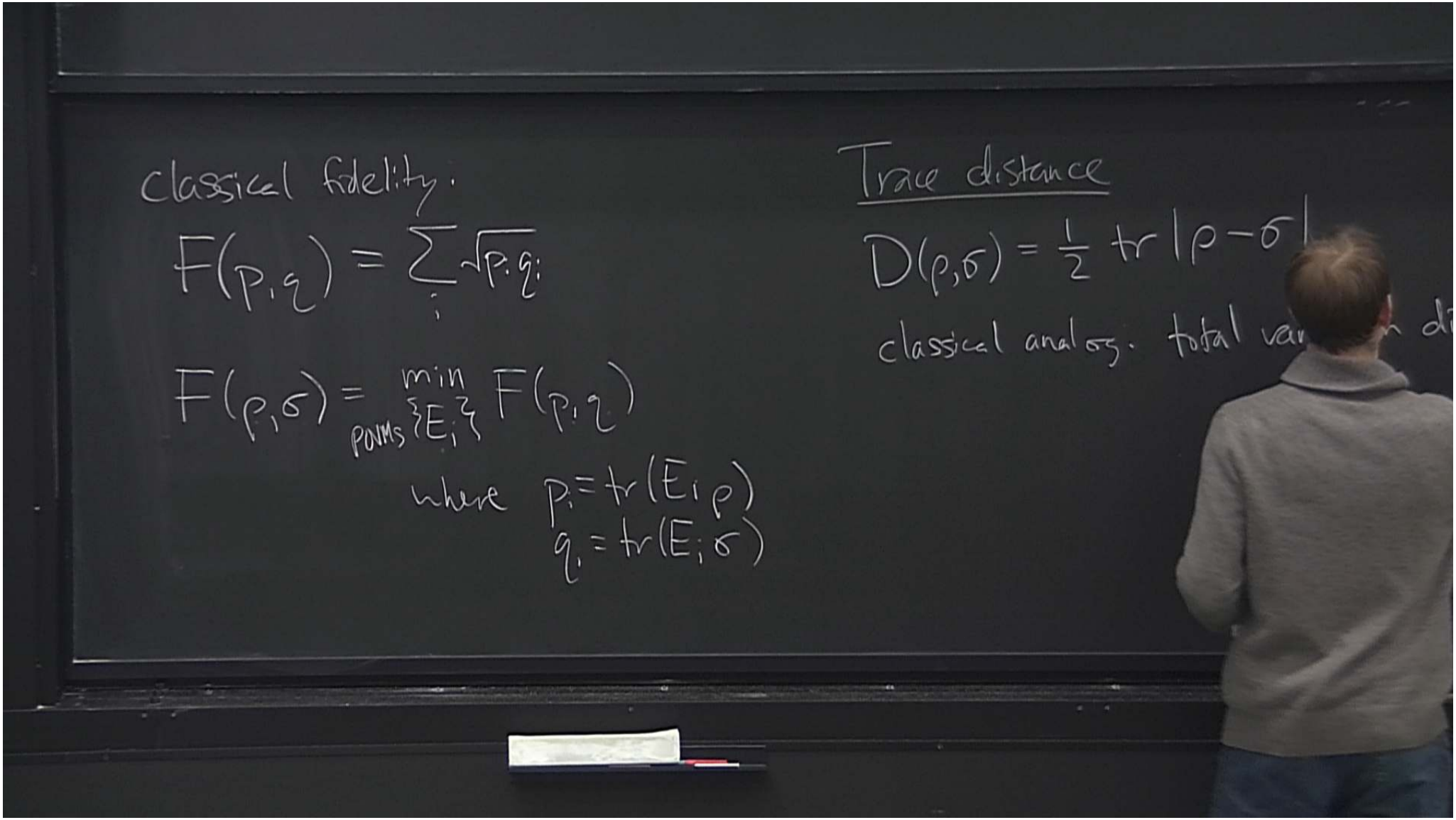
gives $|\psi\rangle\langle\psi|$ w.p. $F(|\psi\rangle, |\varphi\rangle)^2$

$F(|\psi\rangle, \rho)^2 = \mathrm{tr}(|\psi\rangle\langle\psi|\rho) = \langle\psi|\rho|\psi\rangle$


in general: $F(\rho,\sigma) = \mathrm{tr}|\sqrt{\rho}\sqrt{\sigma}|$
$= \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$
$|A| = \sqrt{AA^\dagger}$
Properties.
• $0 \le F(\rho,\sigma) \le 1$
• $F(\rho,\sigma) = 0 \iff \rho,$
Uhlman's theorem: $F(\rho,\sigma) = \max_{|\psi\rangle,|\varphi\rangle} |\langle\psi|\varphi\rangle|$
purif. of ρ
purif. of σ

Properties:

- $0 \leq F(\rho, \sigma) \leq 1$
- $F(\rho, \sigma) = 0 \iff \rho, \sigma$ have $\perp$ support
- $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)$ $\forall$ unitaries U
- $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$ $\forall$ q. ops. $\mathcal{E}$


classical fidelity.
$F(p,q) = \sum_i \sqrt{p_i q_i}$
$F(\rho,\sigma) = \min_{\text{POVMs } \{E_i\}} F(p,q)$
where $p_i = \mathrm{tr}(E_i \rho)$
$q_i = \mathrm{tr}(E_i \sigma)$
Trace distance
$D(\rho,\sigma) = \frac{1}{2}\,\mathrm{tr}\,|\rho-\sigma|$
classical analog. total var

Trace distance

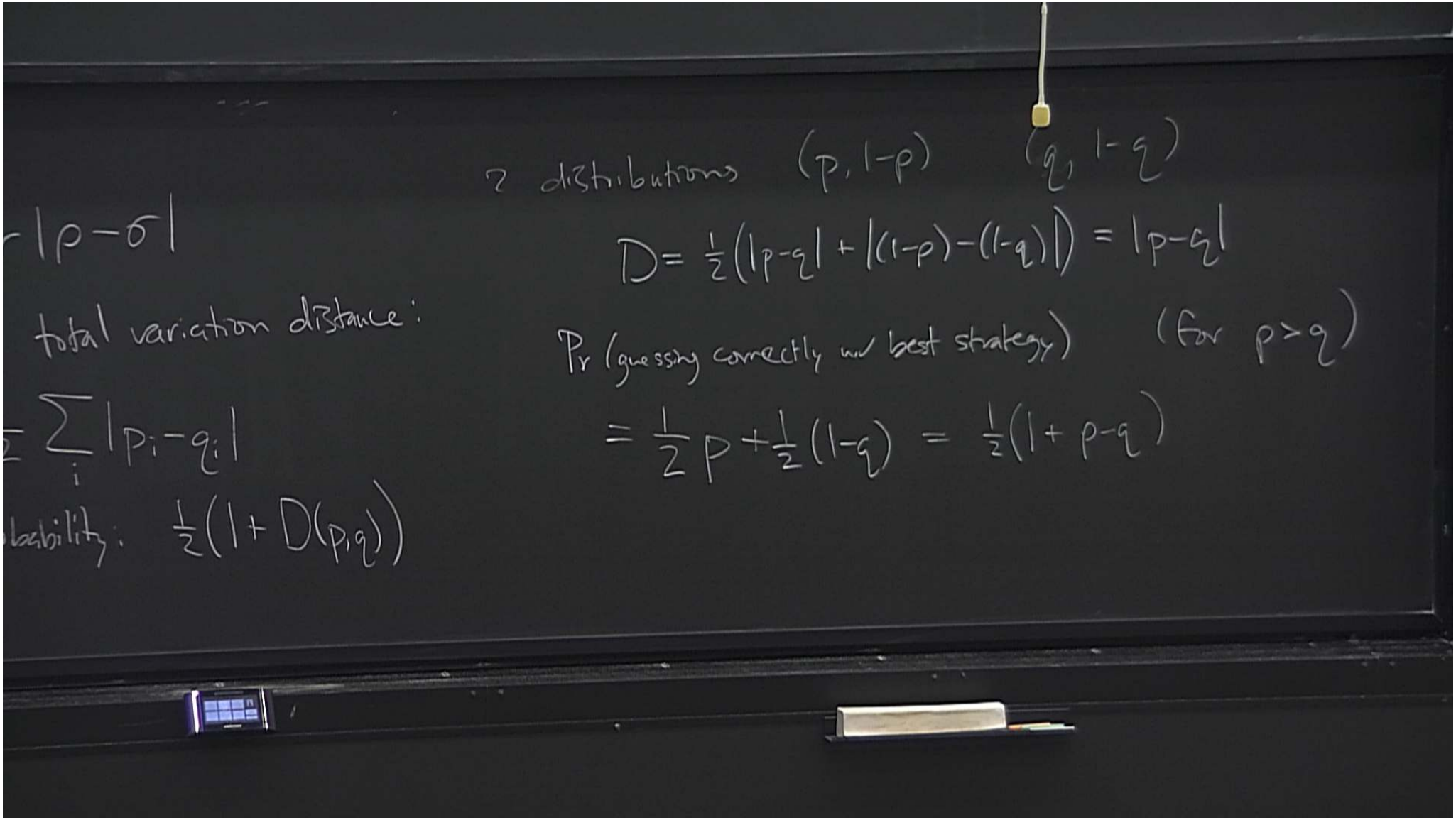
$$D(\rho,\sigma) = \frac{1}{2}\,\mathrm{tr}\,|\rho-\sigma|$$

classical analog: total variation distance:

$$D(p,q) = \frac{1}{2}\sum_i |p_i - q_i|$$

max guessing probability: $\frac{1}{2}(1 + D(p,q))$

2 distributions $(p, 1-p)$ $(q,$


$|p-\sigma|$
total variation distance:
$\sum_i |p_i - q_i|$
probability: $\frac{1}{2}(1 + D(p,q))$
2 distributions $(p, 1-p)$ $(q, 1-q)$
$D = \frac{1}{2}(|p-q| + |(1-p)-(1-q)|) = |p-q|$
P_r (guessing correctly w/ best strategy) (for $p > q$)
$= \frac{1}{2}p + \frac{1}{2}(1-q) = \frac{1}{2}(1 + p - q)$

Theorem (Helstrom):

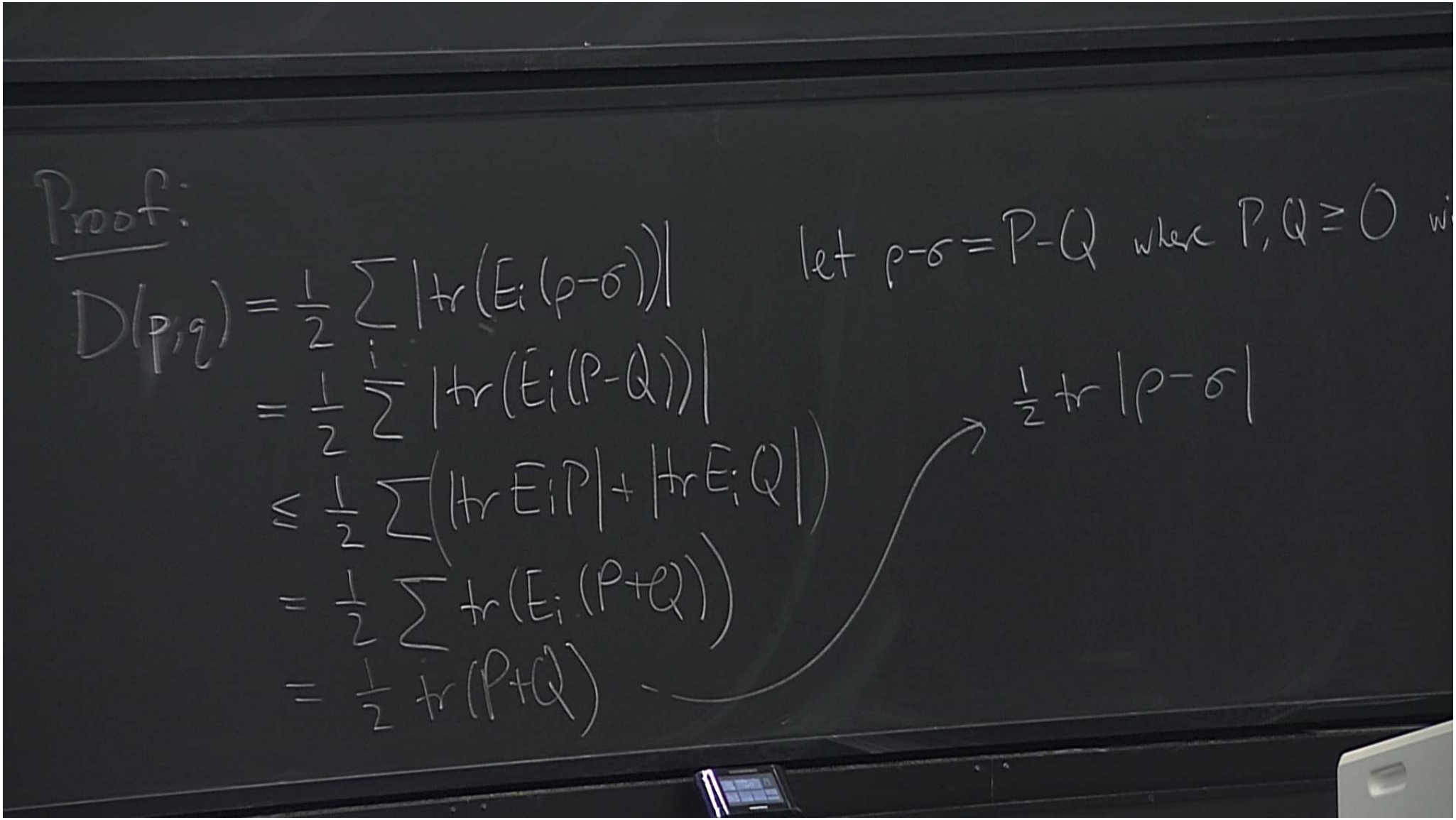
$$D(\rho,\sigma) = \max_{\text{POVMs } \{E_i\}} D(p,q)$$

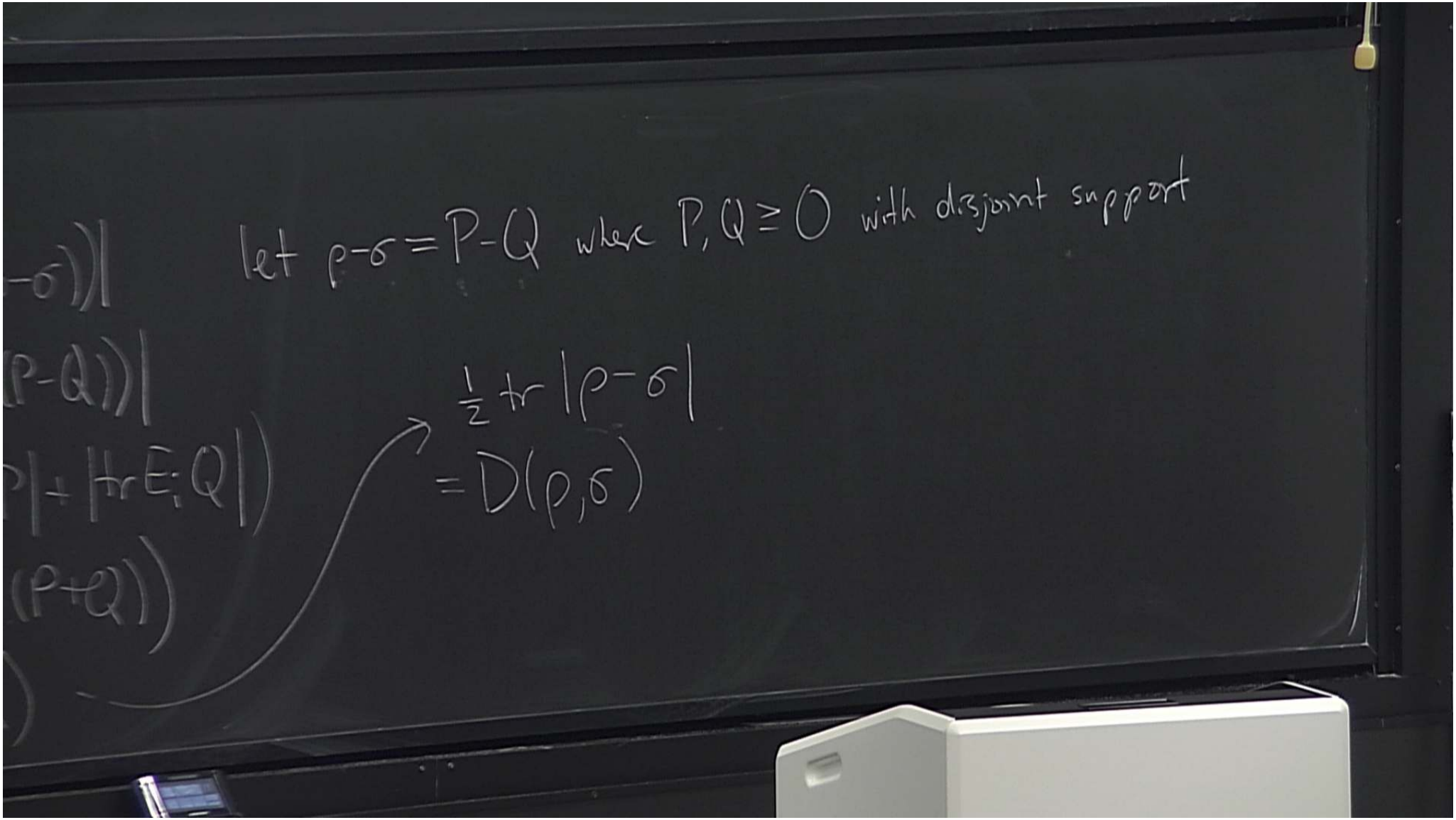
where $p_i = \mathrm{tr}(E_i \rho)$

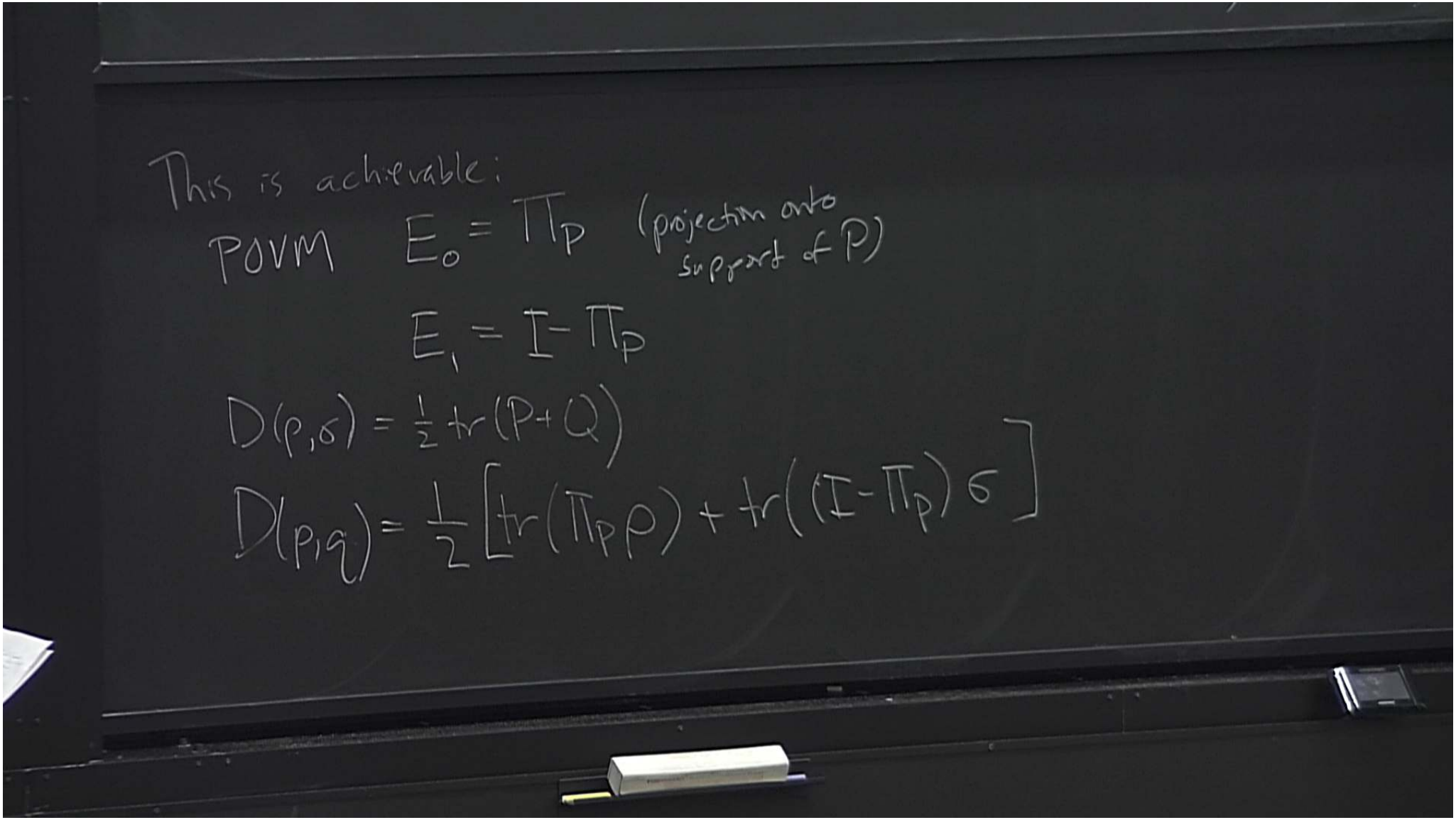
$q_i = \mathrm{tr}(E_i \sigma)$

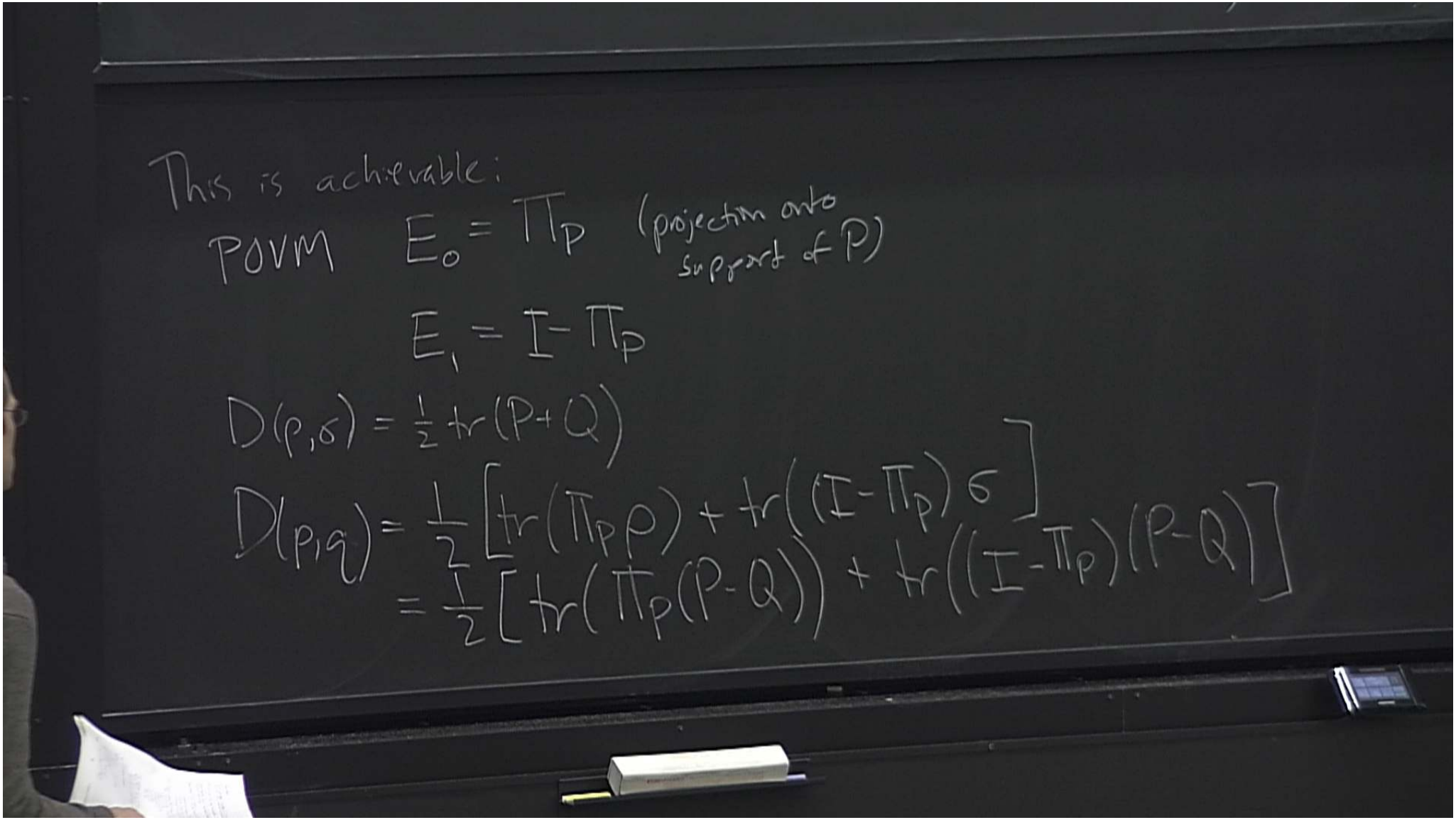
Proof:

$$D(p,q) = \frac{1}{2} \sum_i |\mathrm{tr}(E_i(\rho-\sigma))|$$


Proof:
$D(p,q) = \frac{1}{2}\sum_i |\mathrm{tr}(E_i(\rho-\sigma))|$
$= \frac{1}{2}\sum_i |\mathrm{tr}(E_i(P-Q))|$
$\leq \frac{1}{2}\sum_i (|\mathrm{tr}E_iP| + |\mathrm{tr}E_iQ|)$
$= \frac{1}{2}\sum_i \mathrm{tr}(E_i(P+Q))$
$= \frac{1}{2}\mathrm{tr}(P+Q)$
let $\rho-\sigma = P-Q$ where $P, Q \geq 0$ wi
$\frac{1}{2}\mathrm{tr}|\rho-\sigma|$


let ρ−σ = P−Q where P, Q ≥ 0 with disjoint support
−σ)|
P−Q))|
P|+|trE;Q|)
(P+Q))
½ tr|ρ−σ|
= D(ρ,σ)


This is achievable:
POVM $E_0 = \Pi_P$ (projection onto support of P)
$E_1 = I - \Pi_P$
$D(\rho,\sigma) = \frac{1}{2} tr(P+Q)$
$D(\rho,\sigma) = \frac{1}{2}\left[tr(\Pi_P \rho) + tr((I-\Pi_P)\sigma\right]$


This is achievable:
POVM $E_0 = \Pi_P$ (projection onto support of P)
$E_1 = I - \Pi_P$
$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}(P+Q)$
$D(\rho,\sigma) = \frac{1}{2}\left[\mathrm{tr}(\Pi_P \rho) + \mathrm{tr}\left((I-\Pi_P)\sigma\right)\right]$
$= \frac{1}{2}\left[\mathrm{tr}\left(\Pi_P(P-Q)\right) + \mathrm{tr}\left((I-\Pi_P)(P-Q)\right)\right]$

$\frac{1}{2}\left(\mathrm{tr}\left(\Pi_P(\underbrace{\rho-\sigma}_{P-Q})\right)+\mathrm{tr}(\sigma)\right)$

$=\frac{1}{2}(1+\mathrm{tr}\,P)$

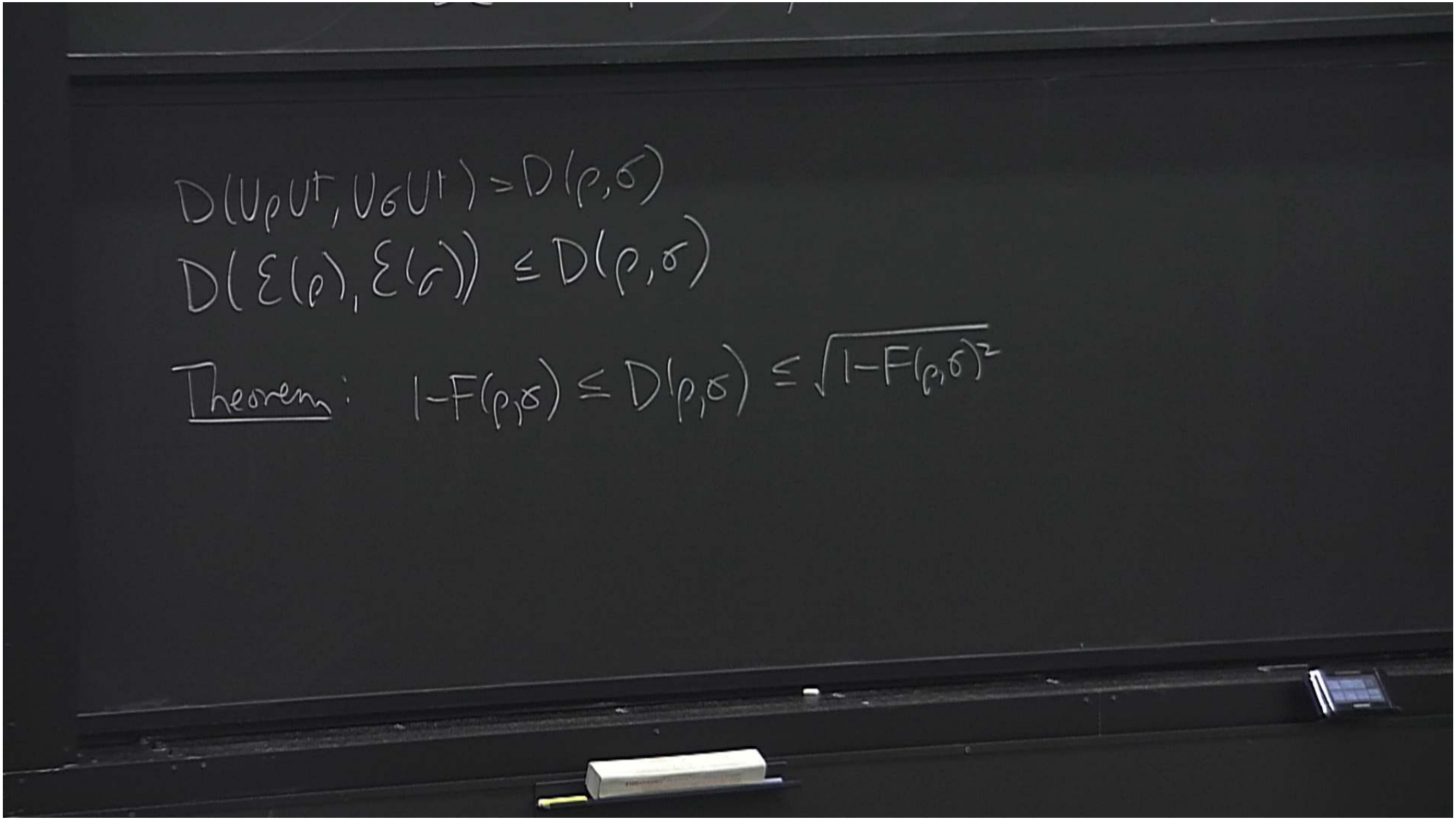
$\mathrm{tr}(\rho-\sigma)=0 \quad \underline{\mathrm{tr}(P)=\mathrm{tr}(Q)}$

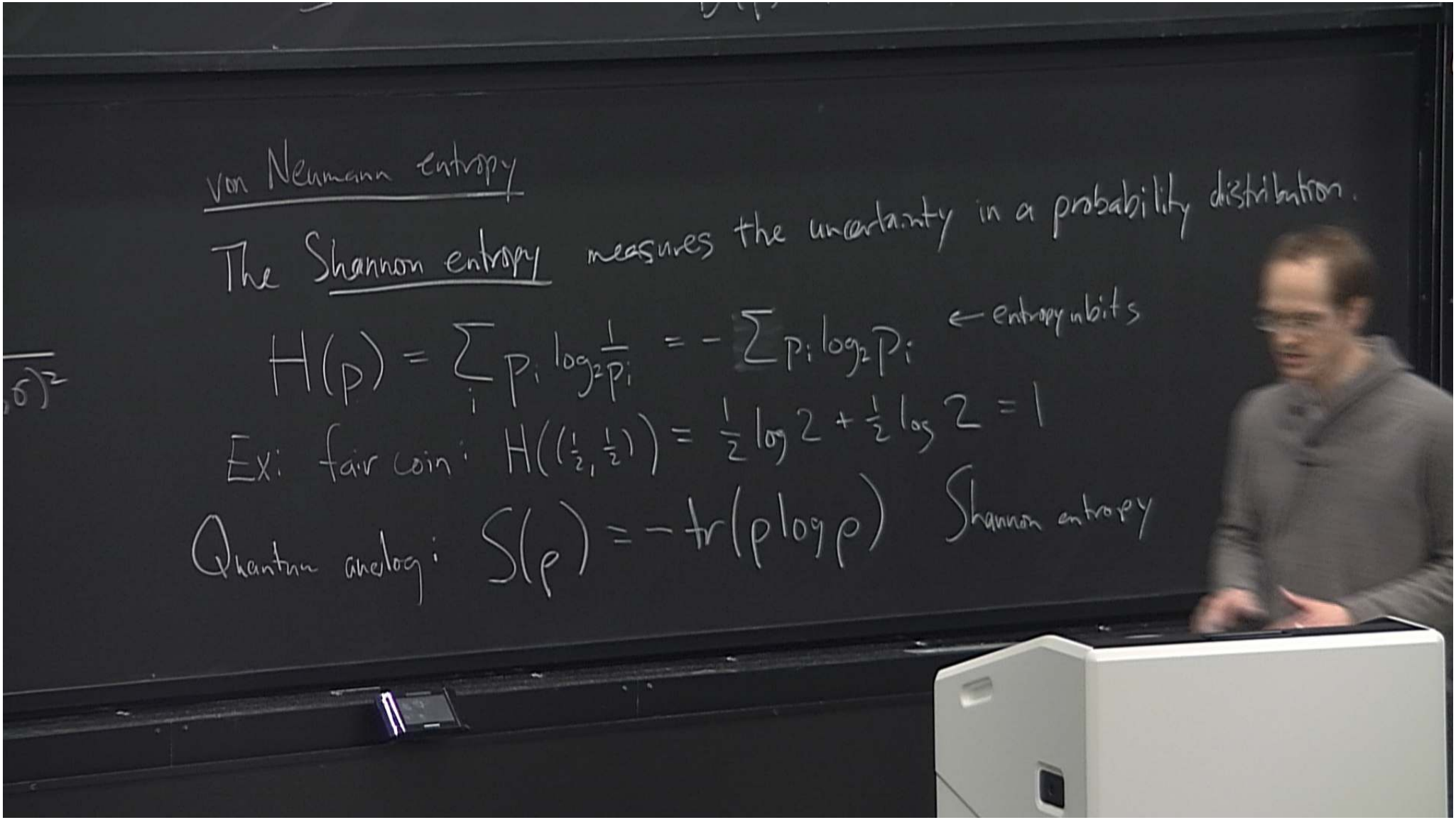
$P-Q)]$

Properties:

- $0 \le D(\rho,\sigma) \le 1$
- $D(\rho,\sigma)=0 \iff \rho=\sigma$
- $D(\rho,\tau) \le D(\rho,\sigma)+D(\sigma,\tau)$

} D is a metric


$D(U_\rho U^\dagger, U_\sigma U^\dagger) = D(\rho,\sigma)$
$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho,\sigma)$
Theorem: $1-F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1-F(\rho,\sigma)^2}$


von Neumann entropy
The Shannon entropy measures the uncertainty in a probability distribution.
$H(p) = \sum_i p_i \log_2 \frac{1}{p_i} = -\sum p_i \log_2 p_i$ ← entropy in bits
Ex: fair coin: $H((\frac{1}{2}, \frac{1}{2})) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1$
Quantum analog: $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ Shannon entropy