

Title: 13/14 PSI - Quantum Gravity Review - Lecture 7

Date: Feb 26, 2014 10:15 AM

URL: <http://pirsa.org/14020069>

Abstract:

Quantization of Gauge Systems

- basis (kinematical) phase space

- "base set of ("point-separating") phase space fct.
 $\{Q_i, P_j\}_{i,j=1,\dots,N} \xrightarrow{\text{Poisson bracket}} \text{observable algebra}$

- Kinematical Hilbert space "carrying a representation of observable algebra":

Quantization of Gauge Systems

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- Kinematical Hilbert space "carrying a representation": $f \mapsto \hat{f}$
of observable algebra: $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$

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of observable algebra": $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$

- $\mathcal{H}_{\text{kin}}$

- find a quantization of constraints $C_I \mapsto \hat{C}_I$

- want to achieve

$$\{C_I, C_K\} = f_{IKL} C_L$$

in 4D gravity
phase space
dependent

$$\leadsto [\hat{C}_I, \hat{C}_K] = \widehat{f_{IKL}} \hat{C}_L$$

$\rightarrow \hat{f}$

Poisson bracket
algebra

on : $f \mapsto \hat{f}$
 $\{g\}$

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- find physical states: solution of the constraints

$$\hat{C}_I \psi_{\text{phys}} = 0$$

Poisson bracket
algebra

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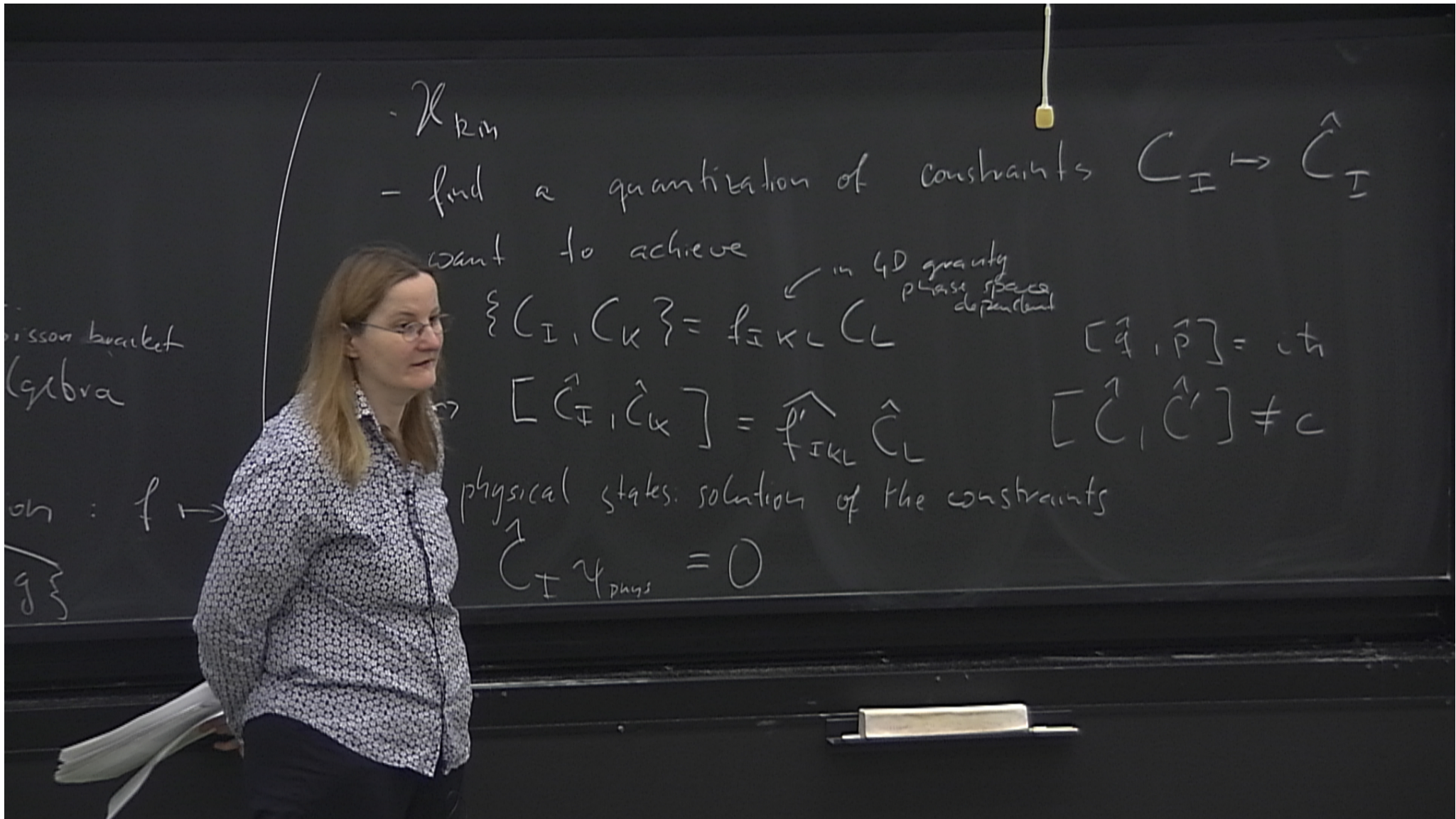
$$\{C_I, C_K\} = f_{IKL} C_L$$

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- find a quantization of constraints $C_I \mapsto \hat{C}_I$

want to achieve

$\{C_I, C_K\} = f_{IKL} C_L$ ↖ in 4D gravity
phase space
dependent

$[\vec{q}, \vec{p}] = i\hbar$

$[\hat{C}_I, \hat{C}_K] = \widehat{f_{IKL}} \hat{C}_L$

$[\hat{C}, \hat{C}'] \neq c$

physical states: solution of the constraints

$\hat{C}_I \psi_{\text{phys}} = 0$

Poisson bracket
(algebra)

on : $f \mapsto$

$\overline{g\}$

Poisson bracket
algebra

on : $f \mapsto \hat{f}$
 $\{f, g\}$

- $\mathcal{H}_{\text{kin}}$

- find a quantization of constraints $C_I \mapsto \hat{C}_I$

- want to achieve

$$\{C_I, C_K\} = f_{IKL} C_L$$

in 4D gravity
phase space
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$$\leadsto [\hat{C}_I, \hat{C}_K] = \widehat{f_{IKL}} \hat{C}_L$$

- find physical states: solution of the constraints

$$\forall I \quad \hat{C}_I \psi_{\text{phys}} = 0$$

Anomalies

$$[\hat{q}, \hat{p}] = i\hbar$$

$$[\hat{C}, \hat{C}'] = c$$

algebra
 map: $f \mapsto \hat{f}$
 $\{ \hat{C}_I \}$

$\leadsto [\hat{C}_I, \hat{C}_K] = \hat{f}_{IKL} \hat{C}_L$

- find physical states: solution of the constraints
 $\forall I \quad \hat{C}_I \psi_{\text{phys}} = 0$

Find phys. inner product
 $[\hat{C}, \hat{C}'] = c$



algebra

on : $f \mapsto \hat{f}$

$\{g\}$

$\leadsto [\hat{C}_I, \hat{C}_K] = \hat{f}_{IKL} \hat{C}_L$

$[\hat{C}, \hat{C}'] = c$

- find physical states: solution of the constraints

$\forall I \quad \hat{C}_I \psi_{\text{phys}} = 0$ • Find phys. inner product

$$[f, g] = i\hbar \{f, g\}$$

Quantum geometry

1. Kinematics

- $A_a^i(x), E_k^b(y)$

standard field theory (canon)

$$\hat{\psi}(x), \hat{\pi}(x), \hat{q}(x), \hat{p}(x)$$

$$[f, g] = i\hbar \{f, g\}$$

Quantum geometry

1. Kinematics

$$A_a^i(x), E_k^b(y)$$

standard field theory on (curved) spacetime

$$\hat{\pi}(x), \hat{Q}(x), \hat{Q}^\dagger(x)$$

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta(x, y)$$

$$[f, g] = i\hbar \{f, g\}$$

Quantum geometry

1. Kinematics

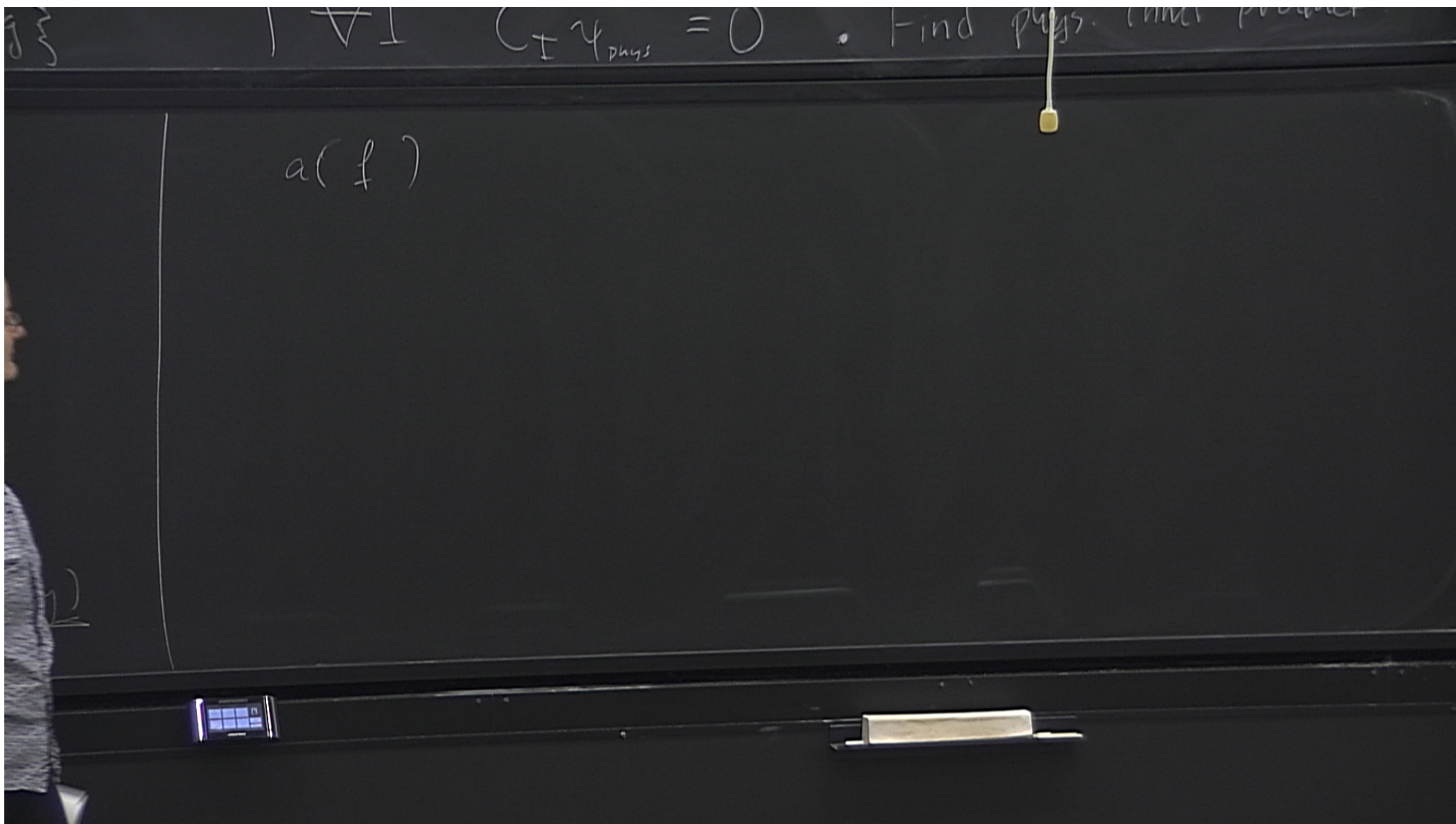
- $A_a^i(x), E_k^b(y)$

standard field theory on (curved) spacetime

$$\hat{\phi}(x), \hat{\pi}(x), \hat{Q}(x), \hat{Q}^\dagger(x)$$

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta(x, y)$$

→ cannot be quantized as s.a. operators.



$\nabla \cdot \psi_{\text{phys}} = 0$ • Find phys. inner product

$$\langle f | g \rangle = \int \psi(x) f(x) \sqrt{g} d^3x$$

$\nabla \cdot \vec{A} = 0$ • Find phys. inner product

$$\varphi(f) = \int \varphi(x) f(x) \sqrt{g} d^3x$$

density $\rightarrow \pi(g) = \int \pi(x) d^3x$

$$\{\varphi(f), \pi(g)\}_{\{f, g \in \mathcal{M}\}}$$

$$\{\varphi(f), \pi(g)\} = \int$$

$\nabla \cdot \vec{A} = 0$ $\psi_{\text{phys}} = 0$ • Find phys. inner product

$$\varphi(f) = \int \varphi(x) f(x) \sqrt{g} d^3x$$

density $\rightarrow \pi(f') = \int f'(x) \pi(x) d^3x$

$$\{\varphi(f), \pi(g)\}_{f, g \in \mathcal{A}(\Sigma)}$$

$$\{\varphi(f), \pi(f')\} = \int f(x) f'(x) \sqrt{g} d^3x$$

$\nabla \cdot \vec{A} = 0$ • Find phys. inner product

$$\varphi(f) = \int \varphi(x) f(x) \sqrt{g} d^3x$$

density $\rightarrow \pi(f') = \int f'(x) \pi(x) d^3x$

$$\{\varphi(f), \pi(g)\} \{f, g \in \mathcal{A}(\mathcal{M})\}$$

$$\left\{ \underset{\substack{\uparrow \\ \text{3D integral}}}{\varphi(f)}, \underset{\substack{\uparrow \\ \text{3D integral}}}{\pi(f')} \right\} = \int f(x) f'(x) \sqrt{g} d^3x$$

$\nabla \cdot \vec{A} = 0$ $\psi_{\text{phys}} = 0$ • Find phys. inner product

$$\langle f | g \rangle = \int \varphi(x) f(x) \sqrt{g} d^3x$$

density $\rightarrow \pi(f) = \int f'(x) \pi(x) d^3x$

$$\{\varphi(f), \pi(g)\} \quad \{f, g \in \mathcal{M}(\mathcal{M})\}$$

$$\left\{ \underset{\substack{\uparrow \\ \text{3D Integral}}}{\varphi(f)}, \underset{\substack{\uparrow \\ \text{3D Integral}}}{\pi(f')} \right\} = \int f(x) f'(x) \sqrt{g} d^3x$$

of "observable algebra": $[f, g] = i\hbar \widehat{\{f, g\}}$

Natural smearing

(2+1)

A_a^k

$\leadsto$

"holonomy"

path γ

E_b^b

$\leadsto$

e_b^k

path γ^*

of "observable algebra": $[f, g] = i\hbar \widehat{\{f, g\}}$

Natural smearing

$(2+1)D$ $A_a^k \rightsquigarrow$ "holonomy"

path γ

$E_b^k \rightsquigarrow e_b^k$

path γ^*

of "observable algebra": $[f, g] = i\hbar \widehat{\{f, g\}}$

Natural smearing

(2+1)D:

A_a^k

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"holonomy"

path γ

(3+1)

flux

$\downarrow$ (2D)

e_b^k

path γ^*

of observable algebra: $[f, g] = i\hbar \widehat{\{f, g\}}$

Natural smearing

(2+1)D: $A_a^k \rightsquigarrow$ "holonomy"

(3+1) flux $E_b^k \rightsquigarrow$ "fluxes"



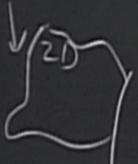
Holonomies

- "Lie algebra valued form" connection

of "observable algebra": $[f, g] = i\hbar \widehat{\{f, g\}}$

Natural smearing

(2+1)D: $A_a^k \rightsquigarrow$ "holonomy"

(3+1) flux
 $E_k^b \rightsquigarrow e_b^k$
 "fluxes"


Holonomies

- "Lie algebra valued form" connection

A_a^k
 E_k^b

of observable algebra:

$$[f, g] = i\hbar \widehat{\{f, g\}}$$

Natural smearing

$(2+1)$

A_a^k

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path γ

E_b^k

$\rightarrow e_b^k$

path γ^*

"fluxes"

Holonomies

- "Lie algebra valued form" connection

A_a^k

E_b^k

of observable algebra:

$$[f, g] = i\hbar \widehat{\{f, g\}}$$

Natural smearing

(2+1)D:

$\rightarrow$ "holonomy"

path γ

(3+1)

flux

$\downarrow \sqrt{2\pi}$

$\rightarrow e^k$

path γ^*

Holonomies

- "Lie algebra valued form" connection

$$A_a^k \rightarrow A_a = A_a^k T_k$$

E^{bk}

of observable algebra:

$$[f, g] = i\hbar \{f, g\}$$

Natural smearing

(2+1)D:

A^a

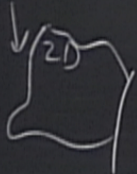
"hol"

"holonomy"

path γ

path γ^*

(3+1)
flux



Holonomies

- "Lie algebra valued form" connection

$$A_a^k \rightarrow A_a = A_a^k T_k$$

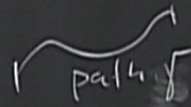
$$E^{bk} \rightarrow E^b = E^{bk} T_k$$

T_k basis of Lie Algebra

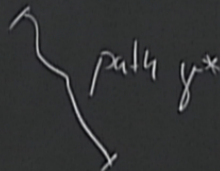
of observable algebra: $[f, g] = i\hbar \{f, g\}$

Natural smearing

(2+1)D: $A_a^k \rightarrow$ "holonomy"
"holonomy"



(3+1) flux $E_k^b \rightarrow e_b^k$ "fluxes"
2D



Holonomies

- "Lie algebra valued form" connection

$$A_a^k \rightarrow A_a = A_a^k T_k$$

$$E^{bk} \rightarrow E^b = E^{bk} T_k$$

- T_k basis of Lie Algebra $\mathfrak{su}(2)$

$\forall I \quad \oint_I \psi_{\text{phys}} = 0$ • Find phys. inner product

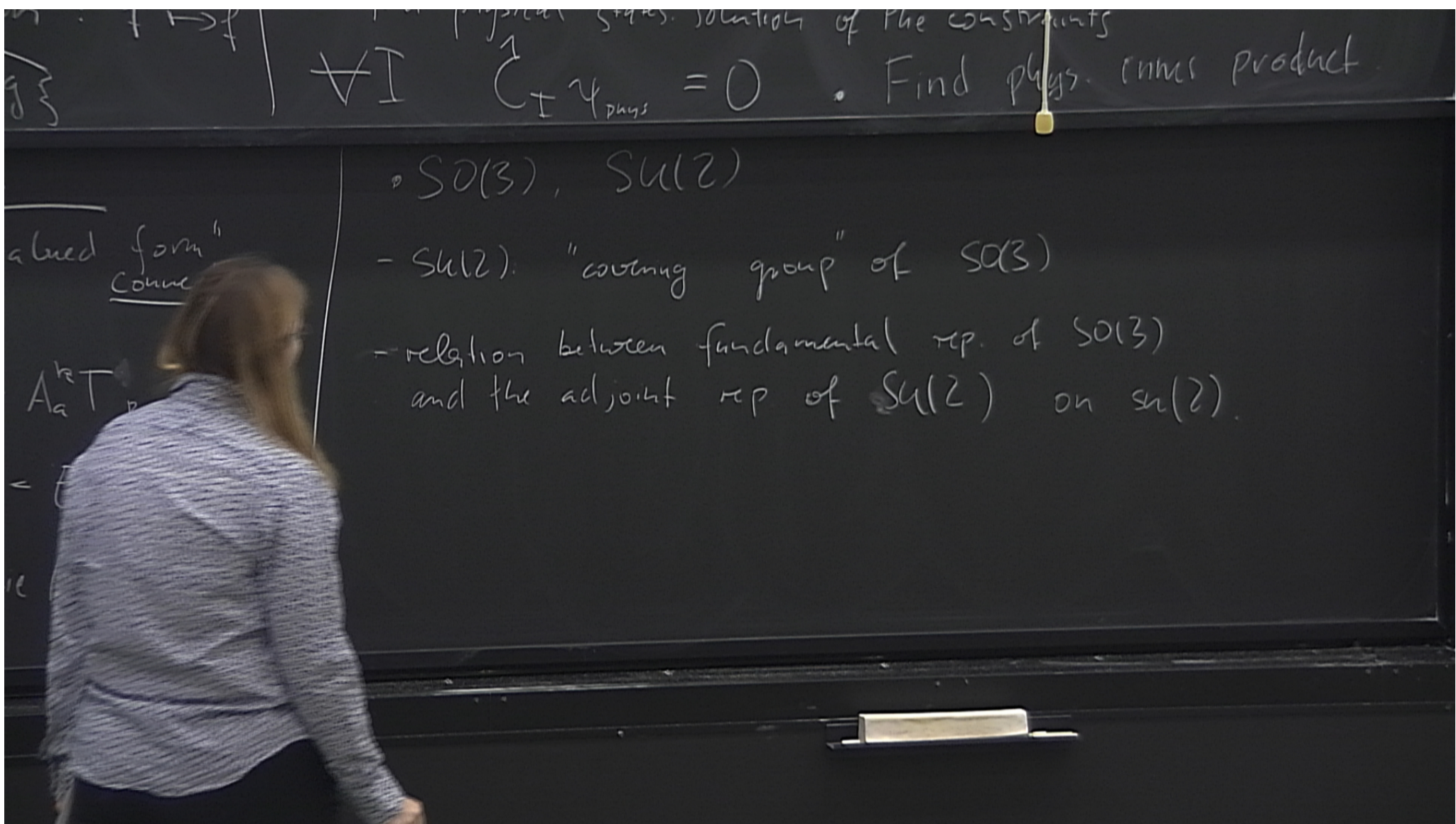
alued form
connection

$$A_a^k T_R$$

$$= E^k{}_a T_R$$

re Algebra $su(2)$

- $SO(3), SU(2)$
- $SU(2)$: "covering group" of $SO(3)$



physical states. solution of the constraints
 $\forall I \quad \oint_I \psi_{\text{phys}} = 0$ • Find phys. inner product

alued form"
Conne
 $A_a^k T_k$
= F
it

- $SO(3), SU(2)$
- $SU(2)$: "covering group" of $SO(3)$
- relation between fundamental rep. of $SO(3)$ and the adjoint rep of $SU(2)$ on $\mathfrak{su}(2)$.

physical states. solution of the constraints
 $\forall I \quad \int \psi_{\text{phys}} = 0$ • Find phys. inner product

• $SO(3)$, $SU(2)$

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$SO(3)$: R_i acting on $\mathbb{R}^3$

$\forall I \quad \oint_{\pm} \psi_{\text{phys}} = 0$ • Find phys. inner product

alued form
connection

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Algebra $su(2)$

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relation between fundamental rep. of $SO(3)$
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$SO(3)$: R_i acting on $\mathbb{R}^3$, fundamental Rep.

$SU(2)$: adjoint on Lie-algebra $\mathfrak{su}(2) \sim \mathbb{R}^3$

$\forall I \quad \oint_I \psi_{\text{phys}} = 0$ • Find phys. inner product

algebra form
connection

$$A_a^k T_R$$

$$= E^k{}_a T_R$$

Lie Algebra $su(2)$

$SO(3), SU(2)$

$SU(2)$: "covering group" of $SO(3)$

relation between fundamental rep. of $SO(3)$
 and the adjoint rep of $SU(2)$ on $su(2)$.

$SO(3)$: R_i acting on $\mathbb{R}^3$, fundamental Rep.

$SU(2)$: adjoint on Lie-algebra $su(2) \simeq \mathbb{R}^3$

• $\mathfrak{su}(2)$ - basis

$$T_i = -\frac{i}{2} \sigma_i \leftarrow \text{Pauli matrices}$$

• $\mathfrak{su}(2)$ - basis

$T_i = -\frac{i}{2} \sigma_i$ ← Pauli matrices

$$[T_i, T_j] = \epsilon_{ij}^k T_k$$

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

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adjoint rep of $\mathfrak{su}(2)$:

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adjoint rep of $\mathfrak{SU}(2)$: $\mathcal{S}(g) T_i = T_i$

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$$[T_i, T_k] = \epsilon_{ij}^{\quad k} T_k$$

$$\text{tr}(T_i T_k) = -\frac{1}{2} \delta_{ik}$$

adjoint rep of $SU(2)$: $\mathcal{S}(g) T_i = g \cdot T_i$

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

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adjoint rep of $\mathfrak{SU}(2)$: $\mathcal{S}(g) T_i = g^{-1} T_i g$

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matrix product
(2x2 matrices)

• $su(2)$ - basis

$$T_i = -\frac{i}{2} \sigma_i \quad \leftarrow \text{Pauli matrices}$$

$$\{T_i, T_k\} = \epsilon_{ijk} T_k$$

$$[T_i, T_k] = -\frac{1}{2} \delta_{ik}$$

of $SU(2)$: $S(g) T_i = g^{-1} T_i g$

$$S(g_1) \circ S(g_2) = S(g_1 \cdot g_2)$$

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

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adjoint rep of $\mathfrak{SU}(2)$: $\mathcal{S}(g) T_i = g^{-1} T_i g \quad g \in \mathfrak{SU}(2)$

$$\mathcal{S}(g_1) \circ \mathcal{S}(g_2) = \mathcal{S}(g_1 \cdot g_2)$$

matrix product
(2x2 matrices)

$$R_k^l(g) T_l =$$

combination of T 's

$$\underbrace{R_k^l(g) T_l}_{\text{combination of } T's} = g^{-1} T_k g$$

combination of T 's

$$E^a = E^a_j T_j$$

$$E^a = E^a_j T_j \xrightarrow[\text{as}]{\text{transform}} g E^a g^{-1}$$

(one index:
transforming
in vector rep)

$$F_{ab} = F^j_{ab} T_j \xrightarrow{\quad} g F_{ab} g^{-1}$$

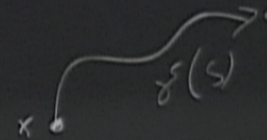
$$A_a = A^j_a T_j \xrightarrow[\text{as}]{\text{transforms}} g A_a g^{-1} + g \partial_a g^{-1}$$

Holonomies.

$$V = V^j T_j$$

Holonomies.

$$V(x) = V(x)^j T_j$$



Holonomies.

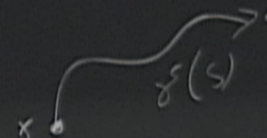
$$V(x) = V(x)^j T_j$$



$$V(s) = V(0)$$

Holonomies:

$$V(x) = V(x)^j T_j$$



- $V(s) = h(s) V(0) h(s)^{-1}$
- $h(s)$: will be the holonomy from $\gamma(0)$ to $\gamma(s)$
- $V(s)$ transforms in adjoint rep.

Holonomies.

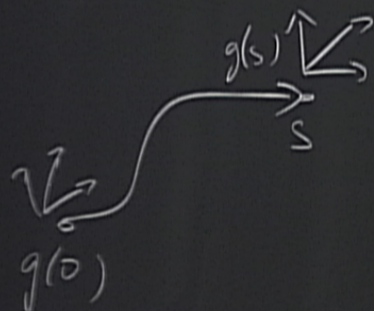
$$V(x) = V(x)^j T_j$$



- $V(s) = h(s) V(0) h(s)^{-1}$
- $h(s)$: will be the holonomy from $\gamma(0)$ to $\gamma(s)$
- $V(s)$ transforms in adjoint rep. $\Rightarrow$
 $h(s) \rightarrow g(\gamma(s)) h(s) g(\gamma(0))^{-1}$

Holonomies:

$$V(x) = V(x)^j T_j$$



$$V(s) = h(s) V(0) h(s)^{-1}$$

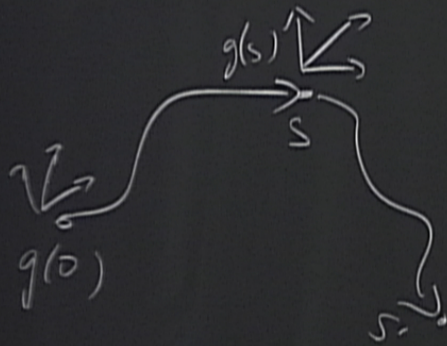
$h(s)$: will be the holonomy from $\gamma(0)$ to $\gamma(s)$

$V(s)$ transforms in adjoint rep. $\Rightarrow$

$$h(s) \rightarrow g(\gamma(s)) h(s) g(\gamma(0))^{-1}$$

Holonomies:

$$V(x) = V(x)^j T_j$$



$$V(s) = h(s) V(0) h(s)^{-1}$$

$h(s)$: will be the holonomy from $\gamma(0)$ to $\gamma(s)$

$$V(s) \text{ transforms in adjoint rep. } \Rightarrow$$

$$h(s) \rightarrow g(\gamma(s)) h(s) g(\gamma(0))^{-1}$$

$h(s) =$ hol'd of the connection
 $=$ path ordered (exp) integral

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 $=$ path ordered (exp) integral